Werk

Titel: Mathematische Annalen

Verlag: Springer

Jahr: 1989

Kollektion: Mathematica

Werk Id: PPN235181684_0283

PURL: http://resolver.sub.uni-goettingen.de/purl?PID=PPN235181684_0283|LOG_0072

Without loss of generality (adding a large constant to f, $\underline{u}$, $\bar{u}$) we may assume that $\bar{u} \geqq \underline{u} \geqq 1$, $f \geqq 1$ a.e. in Ω. Therefore, there exists $\theta \in (0,1)$ small enough such that $\underline{u} \geqq \theta \bar{u}$ in Ω. Let then $\theta_0 = \sup\{\theta \in (0,1]/\underline{u} \geqq \theta \bar{u}$ in $\Omega\}$ – we follow a uniqueness argument which was introduced in a different context by Laetsch [14]. If $\theta_0 = 1$, we are done. We thus argue by contradiction and assume that $\theta_0 < 1$. Of course, we have $\underline{u} \geqq \theta_0 \bar{u}$ in Ω. We then consider $z = \varepsilon d^{-\alpha}$ and we observe that z satisfies

$$-\Delta z + |\nabla z|^p + \lambda z \leqq \varepsilon^p d^{-\beta} + C_\varepsilon d^{-\beta+1}$$

and this is less than f for ε small enough say $\varepsilon \leqq \varepsilon_0$. We choose $\varepsilon = \varepsilon_0$. In fact $z_\delta = \varepsilon(d+\delta)^{-\alpha}$ also satisfies

$$-\Delta z_\delta + |\nabla z_\delta|^p + \lambda z_\delta \leqq f \quad \text{in} \quad \Omega.$$

And we consider $w_{\gamma,\delta} = (\theta_0 - \gamma)\bar{u} + (1 - \theta_0 + \gamma) z_\delta$; $w_{\gamma,\delta}$ satisfies for $\gamma < \theta_0$

$$-\Delta w_{\gamma,\delta} + |\nabla w_{\gamma,\delta}|^p + \lambda w_{\gamma,\delta} \leqq (\theta_0 - \gamma) f + (1 - \theta_0 + \gamma) f \equiv f \quad \text{in} \quad \Omega$$

and since $\underline{u}$, $\bar{u}$ blow up near the boundary we have $w_{\gamma,\delta} \leqq \theta_0 \bar{u} \leqq \underline{u}$ near the boundary. Therefore, by the maximum principle, $w_{\gamma,\delta} \leqq \underline{u}$ in Ω. We now let γ go to 0_+ and then δ go to 0_+ to find

$$\theta_0 \bar{u} + (1 - \theta_0) z \leqq \underline{u} \quad \text{in} \quad \Omega$$

but we obviously have $z \geqq v\bar{u}$ for some $v > 0$. Hence,

$$(\theta_0 + (1 - \theta_0)v)\bar{u} \leqq \underline{u} \quad \text{in} \quad \Omega$$

and this contradicts the definition of θ_0. □

IV. Superquadratic Hamiltonians

IV.1. Interior Gradient Bounds and Maximum Solutions

We begin with a result which gives interior gradient bounds for solutions of (1): similar bounds were first derived in [16, 19] and the proofs are recalled in the appendix. We only remark here that a sharper form of these bounds may be obtained by a simple scaling argument.

Theorem IV.1. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy*

$$|f(x)| \leqq C_1 d(x)^{-\beta} \quad \textit{for some} \quad \beta \geqq 0, \qquad C_1 \geqq 0. \tag{41}$$

Let $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) be a solution of (1) satisfying

$$\lambda u \geqq -C_2 \quad \textit{for some} \quad C_2 \geqq 0. \tag{42}$$

Then, we set $\gamma = \dfrac{1}{p-1}$ if $\beta \leqq q$, γ arbitrary in $\left(\dfrac{\beta}{p}, 1\right)$ if $\beta > q$ and $\gamma = \dfrac{\beta}{p}$ if $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $|\nabla f(x)| d(x)^{-\beta-1} \in L^\infty(\Omega)$. With these notations and assumptions we have

$$|\nabla u(x)| \leqq C_3 d(x)^{-\gamma} \quad \textit{in} \quad \Omega, \quad \square \tag{43}$$

where C_3 only depends on C_1, C_2, γ, β and the diameter of Ω.

Remark IV.1. The bound is optimal as it may be easily checked on simple examples like $\frac{C_0}{d^\alpha}$ if $p \leqq 2$ ($-\mathrm{Log}\, d$ if $p=2$) with C_0, α given as in Theorem I.1 or Theorem I.3, or $-C_0 d^\alpha$ if $p>2$ with $\alpha = \frac{p-2}{p-1}$ if $\beta \leqq q$, $\alpha = 1-\beta/p$ if $\beta<p$ and C_0 is a convenient positive constant. □

Exactly as in [19], this implies of course the following result

Corollary IV.1. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy* (41). *Then, any solution $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) of* (1) *which is bounded from below belongs to $W^{1,s}(\Omega)$ with $s<p-1$ if $p>2$ and $\beta \leqq q$, $s<p/\beta$ if $p>\beta>q$ (and thus $p>2$). In addition, any such solution may be extended continuously on $\bar\Omega$ and $u \in C^{0,\theta}(\bar\Omega)$ with $\theta=(p-2)/(p-1)$ if $p>2$, $\beta \leqq q$; $\theta = 1-\beta'/p$ if $p>\beta>\beta' q$; and $\theta=1-\beta/p$ if $p>\beta>q$ and $f \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies $|\nabla f| d^{-\beta-1} \in L^\infty(\Omega)$.* □

We now just sketch the *proof of Theorem IV.1*: let $x_0 \in \Omega$, set $r=\frac{1}{2}d(x_0)$ and consider $v(x)=r^{-(1-\gamma)}u(x_0+rx)$ for $x \in B(0,1)$. One checks easily that v solves

$$-r^\sigma \Delta v + |\nabla v|^p + \lambda r^\nu u = r^{p\gamma} f(x_0+rx) \quad \text{in} \quad B(0,1) \tag{44}$$

with $\sigma=(p-1)\gamma-1$, $\nu=(p-1)\gamma+1$. Next, observe that

$$|r^{p\gamma} f(x_0+rx)| \leqq C_4 \quad \text{on} \quad B(0,1),$$

where C_4 depends only on C_R and β. And if $\beta=q$, then $\sigma=0$, $\nu=2$ while if $\beta>q$, $\nu=\sigma+2$ and $\sigma>0$. If $\beta \leqq q$ or if $\beta>q$ and $d \in W^{1,\infty}_{\mathrm{loc}}$, $|\nabla f| d^{-\beta-1} \in L^\infty(\Omega)$, interior estimates are available (see appendix) and we deduce from this

$$|\nabla v(0)| \leqq C_3$$

which of course yields (43). □

In the last case, we observe that

$$\|\Delta v\|_{L^m(0,0,k))} \leqq \frac{C'(m,k)}{r^\sigma} \quad \text{for all} \quad m \geqq 1, \quad k \in (0,1).$$

But then, recalling the following "standard" inequality for all $m>N$

$$\|\nabla v\|_{L^\infty(B(0,\frac{1}{2}))} \leqq C \|\nabla v\|^{1-\frac{N}{m}}_{L^m(B(0,\frac{1}{2}))} \{\|\Delta v\|_{L^m(B(0,\frac{1}{2}))} + \|\nabla v\|_{L^m(B(0,\frac{1}{2}))}\}^{\frac{N}{m}}$$

we finally obtain

$$|\nabla v(0)| \leqq C(m) r^{-N\sigma/m} \quad \text{for all} \quad m>N.$$

And this yields (43). □

Next, using these estimates and Corollary IV.1, we may now deduce easily the following

Corollary IV.2. *Let $p>2$, let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below on Ω and satisfy* (41) *with $\beta<p$. Then, there exist solutions $\underline{u}, \bar u$ of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) bounded from below such that if v is a solution of* (1) *in $W^{2,r}(\Omega)$ ($\forall r<\infty$), respectively $W^{2,r}_{\mathrm{loc}}(\Omega)$*

($\forall r<\infty$), then $\underline{u}\geqq v$ in Ω, respectively $\bar{u}\geqq v$ in Ω. Furthermore, if $v\in L^1_{\mathrm{loc}}(\Omega)$ satisfies (21) *then $v\leqq\bar{u}$ a.e. in Ω. And if $\bar{u}_\delta$, $\underline{u}_\delta$ denote the corresponding maximum solutions of* (1) *with Ω replaced by Ω_δ then*

$$\bar{u}_{\delta'}\geqq\underline{u}_{\delta'}\geqq\bar{u}_\delta\geqq\underline{u}_\delta\geqq\bar{u}\geqq\underline{u} \quad \text{in} \quad \Omega_{\delta'} \qquad \text{for} \quad 0<\delta<\delta' \tag{45}$$

and $\bar{u}_\delta$ decreases to $\bar{u}$ as δ goes to 0_+. □

Remark IV.2. A consequence of the results we will prove in the following sections is the following: assume that $f\in L^\infty(\mathbb{R}^N)$ and denote by $\bar{u}^\delta$, $\underline{u}^\delta$ the corresponding maximum solution of (1) with Ω replaced by Ω^δ then

$$\underline{u}_{\delta'}\leqq\bar{u}_{\delta'}\leqq\underline{u}_\delta\leqq\bar{u}_\delta\leqq\underline{u} \quad \text{in} \quad \Omega \quad \text{for} \quad 0<\delta<\delta' \tag{46}$$

and $\underline{u}_\delta$ increases to $\underline{u}$ as δ goes to 0_+.

Remark IV.3. We will show in Sect. V that if f behaves like $C_1 d^{-\beta}$ near the boundary with $0\leqq\beta<p$ [$\beta=0$ means $f\in L^\infty(\Omega)$] then $\underline{u}=\bar{u}$ in Ω.

Proof of Corollary IV.2. The existence of the maximum solutions $\underline{u}$, $\bar{u}$ is exactly the same as in Theorem III.2. Next, the string of inequalities in (45) follows from the definitions of $\bar{u}$, $\underline{u}$. Finally, $\bar{u}_\delta$ decreases to a solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$) in view of the estimates given by Theorem IV.1. Therefore, the limit is below $\bar{u}$. Since on the other hand, by (45), $\bar{u}_\delta\geqq\bar{u}$, we conclude easily. □

We conclude this section by a property of $\bar{u}$, $\underline{u}$ which will be useful later on.

Proposition IV.1. *Let ω be a bounded smooth domain such that $\bar{\omega}\subset\bar{\Omega}$. Let $v\in W^{2,r}(\omega)$ ($\forall r<\infty$) (resp. $W^{2,r}_{\mathrm{loc}}(\omega)\cap C(\bar{\omega})$ ($\forall r<\infty$)) be a subsolution of* (1) *with Ω replaced by ω. If $v\leqq\underline{u}$ (resp. $v\leqq\bar{u}$) on $\partial\omega\cap\Omega$ then $v\leqq\underline{u}$ (resp. $v\leqq\bar{u}$) in ω.* □

Proof. Let $\varepsilon>0$, $v_\varepsilon=v-\varepsilon$ satisfies the same properties than v. In one case, we just consider

$$w_\varepsilon=\bar{u} \quad \text{in} \quad \Omega-\omega, \qquad =\max(\bar{u},v-\varepsilon) \quad \text{in} \quad \omega$$

and we observe that w_ε is a subsolution of (1) [in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$]. Therefore, by Corollary IV.2, $w_\varepsilon\leqq\bar{u}$ and thus $v\leqq\bar{u}$ in ω by letting ε go to 0.

In the other case, the above construction has to be modified a bit since w_ε does not belong to $W^{2,r}(\Omega)$ ($\forall r<\infty$). We then consider $\beta_\varepsilon(t)=\varepsilon\beta\left(\dfrac{t}{\varepsilon}\right)$ where $\beta(t)=t$ if $t\geqq 0$, $\beta\in C^\infty(\mathbb{R})$, β is convex, $1\geqq\beta'(t)\geqq 0$ on $\mathbb{R}$, $\beta(t)\equiv -1$ if $t\leqq -2$. And we now introduce

$$z_\varepsilon=\bar{u} \quad \text{in} \quad \Omega-\omega, \qquad =\bar{u}+\beta_\varepsilon(v_\varepsilon-\bar{u}) \quad \text{in} \quad \omega.$$

Now, $z_\varepsilon\in W^{2,r}(\Omega)$ ($\forall r<\infty$) and we claim that z_ε is a subsolution of (1). We only have to check this claim inside ω where we find

$$\begin{aligned}\nabla z_\varepsilon&=\beta'_\varepsilon\nabla v_\varepsilon+(1-\beta'_\varepsilon)\nabla\bar{u},\\ -\Delta z_\varepsilon&=\beta'_\varepsilon(-\Delta v_\varepsilon)+(1-\beta'_\varepsilon)(-\Delta\bar{u})-\beta''_\varepsilon|\nabla v_\varepsilon-\nabla\bar{u}|^2\\ &\leqq\beta'_\varepsilon(-\Delta v_\varepsilon)+(1-\beta'_\varepsilon)(-\Delta\bar{u}),\\ z_\varepsilon&\leqq\bar{u}+\beta'_\varepsilon(v_\varepsilon-\bar{u})\end{aligned}$$

and our claim follows easily from these inequalities.

We may now complete the proof of Proposition IV.1 since, by definition, $z_\varepsilon \leqq \underline{u}$ in Ω and letting ε go to 0, remarking that β_ε converges uniformly to t^+, we obtain $v \leqq \underline{u}$ in ω. $\square$

Remark IV.4. The expert reader will notice that this result is one form of the dynamic programming principle for the associated stochastic control problem!

IV.2. An Estimate on the Boundary Behaviour

We want to show in this section some properties of $\bar{u}$, $\underline{u}$ like (17). We will be always dealing with the case $p > 2$, $f \in L^\infty_{\mathrm{loc}}(\Omega)$ bounded from below and satisfying (41) with $\beta < p$. Hence, Corollary IV.2 and Proposition IV.1 apply. In all the results which follow in this section and in Sect. IV.3, we will not recall these assumptions.

Theorem IV.2. *The maximum solutions $\bar{u}$, $\underline{u}$ satisfy* (17) *with* $\alpha = -\dfrac{p-2}{p-1}$. *In addition, if f satisfies*

$$\liminf \{ f(x) d(x)^\theta \mid d(x) \to 0_+ \} > 0 \quad \textit{for some} \quad \theta \in (q, \beta] \tag{47}$$

then $\bar{u}$, $\underline{u}$ satisfy (17) *with* $\alpha = 1 - \theta/p$. $\square$

Remark IV.5. Again, this result is rather optimal since if f satisfies (41) with $\beta \leqq q$, we already know that $u \in C^{0,\alpha}(\bar{\Omega})$ and $-C_0 d^\alpha$ gives a simple example (for the ad hoc $C_0 > 0$) which shows the sharpness of (17). Similarly if $f(x)$ behaves like $C_1 d(x)^{-\beta}$ for some $q < \beta < p$ then we already know that $u \in C^{0,\alpha}(\bar{\Omega})$ and again $-C_0 d^\alpha$ shows the sharpness of (17). The only improvement we could think of would be to show (and we were unable to do it)

$$\liminf_{y \in \Omega, y \to x} \{ u(y) - u(x) \} |y - x|^{-\alpha} = -C_0, \quad \text{for all} \quad x \in \partial\Omega,$$

where $C_0 = (p-2)^{-1}(p-1)^{\frac{p-2}{p-1}}$ if $\beta < q$, solves $C_0^p \alpha^p - C_0 \alpha(1-\alpha) = C_1$ if $\beta = q$, $C_0 = \dfrac{1}{\alpha} C_1^{1/p}$ if $q < \beta < p$ at least when f behaves like $C_1 d^{-\beta}$ near the boundary.

Proof of Theorem IV.2. The proof is rather delicate so we will begin with a simpler claim than (17). But let us first give the idea of the proof: we just observe that (17) is equivalent to say that for all $x_0 \in \partial\Omega$, $u\,(=\underline{u}, \bar{u})$ $-\varepsilon|x - x_0|^\alpha$ cannot have a local minimum in $\bar{\Omega}$ at x_0 for ε small enough. To prove this fact, we will argue by contradiction and we will do so by building a subsolution on a neighbourhood of x_0 such that on the boundary of the neighbourhood it is below u while it is above u at x_0. This will contradict Proposition IV.1 proving thus our claim.

To explain how this strategy works, we will begin proving that if $\varphi \in C^{1,1}(\bar{\Omega})$ then $u - \varphi$ cannot have a local minimum on $\bar{\Omega}$ at $x_0 \in \partial\Omega$ where $u = \underline{u}$ or $\bar{u}$. Assume by way of contradiction that x_0 is a local minimum of $u - \varphi$ for some $\varphi \in C^{1,1}(\bar{\Omega})$. Then, denoting by $\xi_0 = \nabla\varphi(x_0)$, there exists $C \geqq 0$ such that

$$u(x) \geqq u(x_0) + (\xi_0, x - x_0) - C|x - x_0|^2 \quad \text{for all} \quad x \in \bar{\Omega}. \tag{48}$$

We then consider the following function defined on $\bar{\omega}$ where $\omega = \{x \in \Omega, d(x) < \delta\}$ where $\delta > 0$ will be determined later on

$$w(x) = u(x_0) + (\xi_0, x - x_0) - C|x - x_0|^2 + \mu(\delta^\alpha - d^\alpha), \qquad \forall x \in \bar{\omega} \tag{49}$$

with $\alpha = \dfrac{p-2}{p-1}$, for some $\mu > 0$ to be determined. In view of (48) and (49), we have

$$w \leqq u \quad \text{on} \quad \partial\omega \cap \Omega, \qquad w(x_0) > u(x_0). \tag{50}$$

Hence, Proposition IV.1 will yield the desired contradiction if we show that w is a subsolution of (1) in ω. Therefore, we compute in ω

$$\begin{aligned} -\Delta w + |\nabla w|^p + \lambda w - f &= 2NC + \alpha\mu d^{\alpha-1}\Delta d - \mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}} \\ &\quad + \left|\xi_0 - C(x - x_0) - \alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p + \lambda w - f \\ &\leqq C\left(1 + \frac{1}{d^{1-\alpha}}\right) - \mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}} + \left|\xi_0 - C(x - x_0) - \alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p, \end{aligned}$$

where C denotes various constants independent of δ. Recalling that $|\nabla d| = 1$, $(1-\alpha)p = 2 - \alpha$, we see that if δ is small enough and $(\alpha\mu)^{p-1} < 1 - \alpha$ [depending only on μ, $|\xi_0|$, C in (49), a lower bound on f and Ω] w is a subsolution of (1) in ω.

We now show (17): it is enough to show that the following inequality cannot hold

$$u(x) \geqq u(x_0) - \varepsilon_0|x - x_0|^\alpha - C|x - x_0|^2, \qquad \forall x \in \bar{\Omega} \tag{51}$$

for small ε_0, $\delta > 0$ and for some $C \geqq 0$, where $\alpha = \dfrac{p-2}{p-1}$ or $\alpha = 1 - \theta/p$ if f satisfies (47). Indeed, if (51) holds, then we introduce

$$w(x) = u(x_0) - \varepsilon_0\beta_\varepsilon(|x - x_0|) - C|x - x_0|^2 + \mu(\delta^\alpha - d^\alpha) \quad \text{in} \quad \omega$$

where $\omega = \{x \in \Omega / d(x) < \delta\}$, $\beta_\varepsilon(t)$ is the function defined by

$$\beta_\varepsilon(t) = \frac{\alpha}{2}\frac{|t|^2}{\varepsilon} + \frac{2-\alpha}{2}\varepsilon^{\frac{\alpha}{2-\alpha}} \quad \text{if} \quad |t| \leqq \varepsilon^{\frac{1}{2-\alpha}}, \qquad = |t|^\alpha \quad \text{if} \quad |t| \geqq \varepsilon^{\frac{1}{2-\alpha}}.$$

In view of (51), (50) will hold if

$$\mu\delta^\alpha > \frac{2-\alpha}{2}\varepsilon_0\varepsilon^{\frac{\alpha}{2-\alpha}}. \tag{52}$$

Next, we compute for x in ω the following quantity

$$\begin{aligned} -\Delta w + |\nabla w|^p + \lambda w - f &= 2NC + (N-1)\varepsilon_0\beta'\frac{1}{|x - x_0|} + \varepsilon_0\beta'' - \mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}} \\ &\quad + \alpha\mu\frac{1}{d^{1-\alpha}}\Delta d + \left|-\varepsilon_0\beta'\frac{x - x_0}{|x - x_0|} - 2C(x - x_0) - \alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p + \lambda w - f \end{aligned}$$

(in fact this equality holds a.e. in ω), and this yields

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-f+\varepsilon_0 N\frac{\alpha}{\varepsilon}+C\mu\frac{1}{d^{1-\alpha}}-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}$$
$$+\left|\varepsilon_0\beta'\frac{x-x_0}{|x-x_0|}+2C(x-x_0)+\alpha\mu\frac{\nabla d}{d^{1-\alpha}}\right|^p.$$

Now, if we begin by the case where we do not assume (47), then we just bound f by a constant C and we deduce

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C+\varepsilon_0 N\frac{\alpha}{\varepsilon}+C\mu\frac{1}{d^{1-\alpha}}+\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}$$
$$+\left(\varepsilon_0\alpha\frac{1}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p.$$

We then choose $\varepsilon=(t\mu\delta^\alpha\varepsilon_0^{-1})^{\frac{2-\alpha}{\alpha}}$ with $0<t<\dfrac{2}{2-\alpha}$ so that (52) holds and we obtain

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C+C\mu\frac{1}{d^{1-\alpha}}+Nt^{-\frac{2-\alpha}{\alpha}}\alpha\mu^{-\frac{2-\alpha}{\alpha}}\delta^{-(2-\alpha)}\varepsilon_0^{2/\alpha}$$

$$+-\mu\alpha(1-\alpha)\frac{1}{d^{2-\alpha}}+\left(\alpha t^{-\frac{1-\alpha}{\alpha}}\delta^{-(1-\alpha)}\mu^{-\frac{1-\alpha}{\alpha}}\varepsilon_0^{1/\alpha}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p.$$

Next, if we fix t in $\left(0,\dfrac{2}{2-\alpha}\right)$ and μ in $\left(0,(t\alpha)^{\frac{1}{p-1}}\alpha^{-1}\right)$, recalling that $d(x)<\delta$, we see that for ε_0 small enough (depending only on N, t, μ, α) we may bound the above terms by

$$C+C\mu\frac{1}{d^{1-\alpha}}-K\frac{1}{d^{2-\alpha}}$$

for some $K>0$, and then we conclude choosing δ small enough.

In the other case, that is when we assume (47), we obtain

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-\frac{\nu}{d^\theta}+N\varepsilon_0\frac{\alpha}{\varepsilon}+\left(\varepsilon_0\alpha\frac{1}{\varepsilon^{\frac{1-\alpha}{2-\alpha}}}+C+\alpha\mu\frac{1}{d^{1-\alpha}}\right)^p$$
$$\leqq C-\frac{\nu}{d^\theta}+N\varepsilon_0\frac{\alpha}{\varepsilon}+\left(C+\alpha\varepsilon_0\varepsilon^{\frac{\theta}{\theta+p}}+\alpha\mu d^{-\frac{\theta}{p}}\right)^p$$

and again writing $\varepsilon=(t\mu\delta^\alpha\varepsilon_0^{-1})^{\frac{2-\alpha}{\alpha}}$ with $0<t<\dfrac{2}{2-\alpha}$ so that (52) holds we deduce

$$-\Delta w+|\nabla w|^p+\lambda w-f\leqq C-\frac{\nu}{d^\theta}+N\alpha t^{-\frac{2-\alpha}{\alpha}}\delta^{-(2-\alpha)}\mu^{-\frac{2-\alpha}{\alpha}}\varepsilon_0^{\frac{2}{\alpha}}$$

$$+\left(C+\alpha t^{-\frac{1-\alpha}{\alpha}}\mu^{-\frac{1-\alpha}{\alpha}}\delta^{-\frac{\theta}{p}}\varepsilon_0^{\frac{1}{\alpha}}+\alpha\mu d^{-\frac{\theta}{p}}\right)^p.$$

And, if we choose t in $\left(0, \frac{2}{2-\alpha}\right)$, μ in $(0, \nu^{1/p}\alpha^{-1})$ we see that for ε_0 small enough the above terms may be bounded by

$$C - \frac{\nu}{d^\theta}$$

therefore w is a subsolution in ω for δ small enough and we conclude. □

We, in fact, proved the

Corollary IV.3. *The maximum solutions $\underline{u}$, $\bar{u}$ satisfy for all $x \in \partial\Omega$*

$$\liminf_{x \in \Omega, x \to x_0} \{u(x) - u(x_0)\} |x - x_0|^{-\alpha} \leqq -K < 0 \tag{53}$$

where $K = K(p, N)$ and $\alpha = 1 - 1/(p-1)$.

And if f satisfies (47), *then* (53) *holds with $\alpha = 1 - \frac{\theta}{p}$ and $K = K(p, \theta, N, C_1)$ where $C_1 = \liminf \{f(x)d(x)^\theta / d(x) \to 0_+\}$.*

IV.3. Infinite Neumann Conditions

Our goal in this section is to investigate the behaviour of the maximum solutions near the boundary. We suspect that the results given in Proposition IV.3 hold in full generality but we were unable to prove it.

We will first sketch the proof of

Proposition IV.2. *Let $f \in W^{1,\infty}(\Omega)$, $p > 2$.*

i) *If Ω is a ball (or if Ω is an half-space), the maximum solutions of* (1) *are Lipschitz tangentially i.e. if $\Omega = B_R$ then*

$$|u(y) - u(x)| \leqq C|y - x| \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad |y| = |x| \tag{54}$$

and if $\Omega = \{x_N > 0\}$ then

$$|u(y) - u(x)| \leqq C|y - x| \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad y_N = x_N \tag{55}$$

for some $C \geqq 0$, where $u = \underline{u}$ or $\bar{u}$.

ii) *If Ω is convex, then $u = \underline{u}$ or $\bar{u}$ satisfies*

$$|\nabla u - (\nabla u(x) \cdot n(x))n(x)| \leqq C d^{-1/2} \quad \text{in} \quad \Omega \tag{56}$$

for some $C \geqq 0$, where n is any smooth vector-field equal to the unit outward normal near $\partial\Omega$ (i.e. $n = -\nabla d$ near $\partial\Omega$). And if $2 < p < 3$, this yields

$$|u(x) - u(y)| \leqq C|x - y|^{2(p-2)/(3p-5)} \quad \forall y, x \in \bar{\Omega} \quad \text{with} \quad d(x) = d(y). \tag{57}$$

Remark IV.6. It is proved in Lasry and Lions [15] that if Ω is convex, f is convex ($\in C(\Omega)$) and satisfies (41) then $\underline{u}$ and $\bar{u}$ are convex. In addition, if (41) holds then $\underline{u}, \bar{u} \in C^{0,1-\gamma}(\bar{\Omega})$ with $\gamma = \frac{1}{p-1}$ if $\beta \leqq q$ and $\gamma = \frac{\beta}{p}$ if $\beta > q$. This Hölder continuity combined with the convexity then implies

$$|\nabla u-(\nabla u\cdot n)n|\leqq Cd^{-\gamma/2} \quad \text{in} \quad \Omega. \tag{58}$$

This improved bound on the tangential gradient enables us, in that particular case, to follow the arguments given below. □

Proof of Proposition IV.2. i) In the case of the half-space, one simply remark that $u(\cdot+he_i)$ (for $1\leqq i\leqq N-1$) is the maximum solution of (1) with f replaced by $f(\cdot+he_i)$ hence, using as in Corollary II.1, the maximality

$$\|u(\cdot+he_i)-u(\cdot)\|_\infty\leqq \frac{1}{\lambda}\|f(\cdot+he_i)-f(\cdot)\|_\infty\leqq C|h|$$

and (55) is proved. One proves (54) similarly replacing the tangential translations by rotations.

ii) Let y be an arbitrary point in $\bar{\Omega}$, we set $u_t(x)=\frac{1}{t^\alpha}u(y+t(x-y))w$ for $0<t<1$, $x\in y+\frac{1}{t}(\Omega-y)=\Omega_t$ with $\alpha=\frac{p-2}{p-1}$. Observe that $\Omega\subset\Omega_t$ and that u_t solves

$$-\Delta u_t+|\nabla u_t|^p+\lambda t^2u_t=t^{2-\alpha}f(tx) \quad \text{in} \quad \Omega_t.$$

Therefore, we have for some $C\geqq 0$

$$-\Delta u_t+|\nabla u_t|^p+\lambda u_t\leqq f+C(1-t) \quad \text{in} \quad \Omega,$$

and $u_t-\frac{C}{\lambda}(1-t)$ is a subsolution of (1); hence $u_t\leqq u+\frac{C}{\lambda}(1-t)$. But this inequality immediately implies

$$(x-y,\nabla u(x))\geqq -C, \qquad \forall (x,y)\in\Omega\times\bar{\Omega}, \tag{59}$$

which in turn yields (56) and (57). □

The improved Hölder continuity of $\underline{u}$, $\bar{u}$ in the tangential directions enables us to obtain the

Proposition IV.3. *Let $f\in W^{1,\infty}(\Omega)$, $p>2$. Assume that either Ω is a ball, or Ω is an half-space, or Ω is convex and $p<3$, then $\alpha=\frac{p-2}{p-1}$ the maximum solutions $\underline{u}$, $\bar{u}$ satisfy*

$$t^{-\alpha}\{u(x_0-tn(x_0))-u(x_0)\}\to -C_0 \quad \textit{as} \quad t\to 0_+, \quad \textit{uniformly in } x_0\in\partial\Omega \tag{60}$$

$$\nabla u(x)d(x)^{1-\alpha}\to C_0\alpha n \quad \textit{as} \quad d(x)\to 0_+, \tag{61}$$

where $C_0=(1-\alpha)^{1/(p-1)}\alpha^{-1}$.

Proof. We just sketch it. Let $x_0\in\partial\Omega$, we introduce the blown-up-functions u_t defined by $u_t(x)=t^{-\alpha}\{u(x_0+tx)-u(x_0)\}$ defined on $Q_t=(\Omega-x_0)/t$. We want to let t go to 0_+. We first observe that by Theorem IV.1 and Corollary IV.1 u_t is bounded in $L^\infty(Q_t\cap B_R)$ ($\forall R<\infty$). In addition, u_t solves

$$-\Delta u_t+|\nabla u_t|^p=t^{2-\alpha}\{f(x_0+tx)-\lambda u(x_0+tx)\} \quad \text{in} \quad Q_t.$$

And we obtain easily a priori bounds from the interior gradient estimates: therefore u_t is relatively compact and any convergent subsequence u_{t_n} converges uniformly on compact sets, as t_n goes to 0, to a solution $v \in W^{2,r}_{\mathrm{loc}}(\Pi)$ $(\forall r < \infty)$ of

$$-\Delta v + |\nabla v|^p = 0 \quad \text{in} \quad \Pi, \qquad v(x_0) = 0, \qquad |v(x)| \leqq C|x|^\alpha \quad \text{in} \quad \Pi,$$

where $\Pi = \{x \in \mathbb{R}^N / n(x_0) \cdot x < 0\}$. In addition, using Proposition IV.1, we deduce that v is above any function $w \in W^{2,r}_{\mathrm{loc}}(\omega) \cap C(\bar{\omega})$ [resp. $W^{2,r}(\omega)$ $\forall r < \infty$ if we are dealing with $\underline{u}$] satisfying

$$-\Delta w + |\nabla w|^p \leqq 0 \quad \text{in} \quad \omega, \qquad w \leqq v \quad \text{on} \quad \partial\omega \cap \Pi.$$

Finally, the estimates (54), (55) or (57) imply that v depends only on the variable $x \cdot n(x_0)$ i.e. $v(x) = \varphi(-x \cdot n(x_0))$ where φ solves

$$-\varphi'' + |\varphi'|^p = 0 \quad \text{for} \quad t > 0, \qquad \varphi(0) = 0, \qquad \varphi \in C([0,\infty)) \cap C^2(0,\infty).$$

Hence, $\varphi(t) \leqq 0$ or $\varphi(t) = C_0 \lambda^\alpha - C_0(t+\lambda)^\alpha$ on $\mathbb{R}_+$ for some $\lambda \geqq 0$. But, since v is "a maximum solution" we deduce that $\varphi \geqq \psi$ on $[0, L]$ for all $L > 0$, $\psi \in C^2([0, L])$ satisfying

$$\psi(L) \leqq \varphi(L), \qquad -\psi'' + |\psi'|^p \leqq 0 \quad \text{in} \quad (0, L).$$

And this implies by Theorem IV.2 and its proof that

$$\liminf_{t \to 0_+} \varphi(t) t^{-\alpha} < 0$$

therefore $\varphi(t) \equiv -C_0 t^\alpha$ and we conclude easily. □

We conjecture that if $p > 2$, $f \in L^\infty(\Omega)$ then (60) and (61) always hold. Of course, in view of the preceding argument, it would be enough to prove that $|u(x) - u(y)| \leqq C|x-y|^\theta$ for some $\theta < \alpha$ if $x, y \in \Omega$, $d(x) = d(y)$ but this type of estimate seems rather difficult to obtain in general.

V. Viscosity Formulation of the Boundary Conditions

V.1. Uniqueness Results

If we accept the stochastic control interpretation of the solutions built in the preceding sections, one is led (see [20] for more details) to the following formulation of the boundary condition

$$\text{for all} \quad \varphi \in C^2(\bar{\Omega}), \qquad u - \varphi \quad \text{achieves its minimum over} \quad \Omega. \tag{62}$$

Or course, an equivalent formulation in the case when $u \in C(\bar{\Omega})$ is to impose that $u - \varphi$ never has a local minimum on $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$ for all $\varphi \in C^2(\bar{\Omega})$. It is quite clear that solutions considered in Sects. II and III satisfy (62), even with $\varphi \in C(\bar{\Omega})$, since they blow up at $\partial\Omega$. Similarly, the maximum solutions built in Sect. IV also satisfy (62), even with $\varphi \in C^{0,\theta}(\bar{\Omega})$ for $\theta > \alpha$, since they satisfy (17): indeed, assume for instance that $\bar{u} - \varphi$ does not achieve its minimum over Ω. Since $u, \varphi \in C(\bar{\Omega})$, there is a minimum point x_0 over $\bar{\Omega}$ of $u - \varphi$ and $x_0 \in \partial\Omega$. Then, we have for $x \in \Omega$

$$u(x) - u(x_0) \geqq \varphi(x) - \varphi(x_0)$$