And we obtain easily a priori bounds from the interior gradient estimates: therefore u_t is relatively compact and any convergent subsequence u_{t_n} converges uniformly on compact sets, as t_n goes to 0, to a solution $v \in W^{2,r}_{\mathrm{loc}}(\Pi)$ $(\forall r < \infty)$ of

$$-\Delta v + |\nabla v|^p = 0 \quad \text{in} \quad \Pi, \qquad v(x_0) = 0, \qquad |v(x)| \leqq C|x|^\alpha \quad \text{in} \quad \Pi,$$

where $\Pi = \{x \in \mathbb{R}^N / n(x_0) \cdot x < 0\}$. In addition, using Proposition IV.1, we deduce that v is above any function $w \in W^{2,r}_{\mathrm{loc}}(\omega) \cap C(\bar{\omega})$ [resp. $W^{2,r}(\omega)$ $\forall r < \infty$ if we are dealing with $\underline{u}$] satisfying

$$-\Delta w + |\nabla w|^p \leqq 0 \quad \text{in} \quad \omega, \qquad w \leqq v \quad \text{on} \quad \partial\omega \cap \Pi.$$

Finally, the estimates (54), (55) or (57) imply that v depends only on the variable $x \cdot n(x_0)$ i.e. $v(x) = \varphi(-x \cdot n(x_0))$ where φ solves

$$-\varphi'' + |\varphi'|^p = 0 \quad \text{for} \quad t > 0, \qquad \varphi(0) = 0, \qquad \varphi \in C([0,\infty)) \cap C^2(0,\infty).$$

Hence, $\varphi(t) \leqq 0$ or $\varphi(t) = C_0 \lambda^\alpha - C_0(t+\lambda)^\alpha$ on $\mathbb{R}_+$ for some $\lambda \geqq 0$. But, since v is "a maximum solution" we deduce that $\varphi \geqq \psi$ on $[0, L]$ for all $L > 0$, $\psi \in C^2([0, L])$ satisfying

$$\psi(L) \leqq \varphi(L), \qquad -\psi'' + |\psi'|^p \leqq 0 \quad \text{in} \quad (0, L).$$

And this implies by Theorem IV.2 and its proof that

$$\liminf_{t \to 0_+} \varphi(t) t^{-\alpha} < 0$$

therefore $\varphi(t) \equiv -C_0 t^\alpha$ and we conclude easily. □

We conjecture that if $p > 2$, $f \in L^\infty(\Omega)$ then (60) and (61) always hold. Of course, in view of the preceding argument, it would be enough to prove that $|u(x) - u(y)| \leqq C|x - y|^\theta$ for some $\theta < \alpha$ if $x, y \in \Omega$, $d(x) = d(y)$ but this type of estimate seems rather difficult to obtain in general.

V. Viscosity Formulation of the Boundary Conditions

V.1. Uniqueness Results

If we accept the stochastic control interpretation of the solutions built in the preceding sections, one is led (see [20] for more details) to the following formulation of the boundary condition

$$\text{for all} \quad \varphi \in C^2(\bar{\Omega}), \qquad u - \varphi \quad \text{achieves its minimum over} \quad \Omega. \tag{62}$$

Or course, an equivalent formulation in the case when $u \in C(\bar{\Omega})$ is to impose that $u - \varphi$ never has a local minimum on $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$ for all $\varphi \in C^2(\bar{\Omega})$. It is quite clear that solutions considered in Sects. II and III satisfy (62), even with $\varphi \in C(\bar{\Omega})$, since they blow up at $\partial\Omega$. Similarly, the maximum solutions built in Sect. IV also satisfy (62), even with $\varphi \in C^{0,\theta}(\bar{\Omega})$ for $\theta > \alpha$, since they satisfy (17): indeed, assume for instance that $\bar{u} - \varphi$ does not achieve its minimum over Ω. Since $u, \varphi \in C(\bar{\Omega})$, there is a minimum point x_0 over $\bar{\Omega}$ of $u - \varphi$ and $x_0 \in \partial\Omega$. Then, we have for $x \in \Omega$

$$u(x) - u(x_0) \geqq \varphi(x) - \varphi(x_0)$$

therefore

$$\liminf_{x\in\Omega, x\to x_0} \{u(x)-u(x_0)\}\,|x-x_0|^{-\alpha} \geqq \liminf_{x\in\Omega, x\to x_0} \{\varphi(x)-\varphi(x_0)\}\,|x-x_0|^{-\alpha}$$

and the right-hand side is 0 since φ is smooth and $\alpha\in(0,1)$. And we reach a contradiction with (17). In other words, any solution of (1) satisfying (17) does satisfy the boundary condition in "viscosity form" given by (62).

Our goal in this section is to prove, under quite general assumptions, that there is a unique solution of (1) satisfying (62). In particular, when this holds, this will imply that with the notations of Sect. IV the maximum solutions $\bar{u}$ and $\underline{u}$ are equal.

We may now state our main result.

Theorem V.I. *There exists a unique solution of* (1) *in* $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ *satisfying* (62) *under one of the following three sets of assumptions*

i) $1<p\leqq 2$, $f\in L^{\infty}_{\mathrm{loc}}(\Omega)$ *satisfies* (14) *and is bounded from below.*

ii) $1<p$, $f\in L^{\infty}_{\mathrm{loc}}(\Omega)$ *satisfies* (40).

iii) $2<p$, $f\in L^{\infty}_{\mathrm{loc}}(\Omega)$ *satisfies* (41) *with* $\beta<p$ *and is bounded from below.*

Corollary V.1. *Let* $p>2$. *Let* $f\in L^{\infty}_{\mathrm{loc}}(\Omega)$ *satisfy* (41) *with* $\beta<p$ *and be bounded from below. Then, the maximum solutions built in Sect. IV are equal. Furthermore, if* $f\in C(\bar{\Omega})$ *or if* $f\in C(\Omega)$ *and* $f(x)d(x)^{-\theta}\to C_1$ *as* $d\to 0_+$ *with* $0<\theta<p$, *they coincide also with the envelope of all* $C^2(\bar{\Omega})$ *subsolutions of* (1).

Remark V.1. Actually, (62) can be proved to be equivalent to

$$u-\varphi \text{ achieves its minimum in } \Omega \text{ for all quadratic functions } \varphi. \qquad (62')$$

Indeed, suppose (62′) holds and let $\varphi\in C^2(\bar{\Omega})$. Let $x_n\in\Omega$ be a minimizing sequence for $u-\varphi$ converging to some $x_0\in\bar{\Omega}$. For C large enough we have $\psi(x)<\varphi(x)\ \forall x\neq x_0$, where ψ is defined by $\psi(x)=\varphi(x_0)+\nabla\varphi(x_0)\cdot(x-x_0)-C|x-x_0|^2\ \forall x\in\Omega$. Hence, ψ is quadratic and $u(x)-\psi(x)>u(x)-\varphi(x)=\min(u-\varphi)=\min(u-\psi)$ for all $x\neq x_0$. Hence from (62′) x_0 lies in Ω.

Proof of Corollary V.1. As we already said, $\bar{u}$, $\underline{u}$ satisfy (62) and so are equal by Theorem V.1. In addition, if we denote by $\tilde{u}$ the envelope of all $C^2(\bar{\Omega})$ subsolutions of (1); we first claim that by the same arguments as in Sects. III and IV u is a solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar{\Omega})$ $(\forall r<\infty)$. If $f\in C^{0,\gamma}(\bar{\Omega})$ for some $\gamma>0$, the arguments indeed adapt without changes. If $f\in C(\bar{\Omega})$, we just approximate f by $f_n\in C^1(\bar{\Omega})$ such that $f_n\leqq f$, $f_n \underset{n}{\to} f$ uniformly on $\bar{\Omega}$. If $f\in C(\Omega)$ satisfies (63), we first observe that $g(x)=f(x)d(x)^{\theta}$ may be extended continuously to $\bar{\Omega}$ by giving it the value C_1 on $\partial\Omega$ then we approximate g by $g_n\in C^1(\Omega)$ such that $g_n\leqq g$, $g_n \underset{n}{\to} g$ uniformly on $\bar{\Omega}$, g_n is constant on $\partial\Omega$ and we consider $f_{n,R}=g_n(d(x)^{-\theta}\wedge R)$. And these approximations easily yield our claim on $\tilde{u}$.

Next, we observe that Proposition IV.1 and Theorem IV.2 may be applied or more precisely that their proofs are immediately adapted to the case of $\tilde{u}\leqq 0$ that $\tilde{u}$ satisfies (17). Hence, $\tilde{u}$ satisfies (62) and Corollary V.I is proved. □

V.2. Proofs

We begin with the *proof of Theorem V.1 in the case i).* We denote by $\bar{u}$ the unique solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ which blows up at (see Theorems II.1 and II.2) and we consider another solution u of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ satisfying (62). We obviously have $u \leqq \bar{u}$ by Theorems II.1 and II.2 and we want to show that the reverse inequality also holds. The strategy of proof is quite simple: we just observe that if $v \in C^2(\bar{\Omega})$ is a subsolution of (1) then $u-v$ achieves its minimum over Ω at some point x_0 because of (62) and by the maximum principle we deduce $(u-v)(x_0) \geqq 0$ hence $u \geqq v$. Therefore, if we are able to approximate $\bar{u}$ by $C^2(\bar{\Omega})$ subsolutions of (1), we complete the uniqueness proof. Now, if $f \in C^1(\bar{\Omega})$ such that $f_n \leqq f$, $f_n \geqq -C$ for some C independent of n and f_n converges uniformly to f on compact subsets of Ω. We next denote by $\bar{u}_n$ the corresponding unique solutions of (1) (with f replaced by f_n) which blow up near the boundary (Theorem II.1) and as remarked in Sect. II we know that $\bar{u}_n$ converges uniformly on compact subsets of Ω to $\bar{u}$ and of course $\bar{u}_n \leqq \bar{u}$. Since we know now by the proof of Theorem II.1 that $\bar{u}_n$ is an increasing limit of $C^2(\bar{\Omega})$ (use the smoothness of f_n) solutions of (1) (with f replaced by f_n), the desired sequence of subsolutions of (1) in $C^2(\bar{\Omega})$ is built.

However, this argument does not work as well if we only assume (as we did in Theorem V.1) that $f \in L^\infty_{\text{loc}}(\Omega)$, satisfies (14) and is bounded from below. In this general case, we approximate f by f_n given by

$$f_n = f \quad \text{in} \quad \Omega_{1/n}, \qquad = -C_0 \quad \text{in} \quad \Omega - \Omega_{1/n}, \tag{63}$$

where $C_0 \geqq 0$ is any constant such that $f \geqq -C_0$ in Ω.

Again, we consider the solutions $\bar{u}_n$ of (1) (with f replaced by f_n) which blow up near $\partial\Omega$. We know from the proof of Theorem II.1 that there exists for each $n \geqq 1$, a sequence $(\bar{u}_{n,m})_{m \geqq 1}$ satisfying

$$-\Delta \bar{u}_{n,m} + |\nabla \bar{u}_{n,m}|^p + \lambda \bar{u}_{n,m} = f_n \quad \text{in} \quad \Omega, \qquad \bar{u}_{n,m} \in W^{2,r}(\Omega)\,(\forall r < \infty),$$
$$\bar{u}_{n,m} = m \quad \text{on} \quad \partial\Omega \tag{64}$$

and $\bar{u}_{n,m} \underset{m}{\uparrow} \bar{u}_n$ uniformly on compact subsets of Ω.

Therefore, we obtain: $u \geqq \bar{u}_{n,m}$ and, passing to the limit in m, $u \geqq \bar{u}_n$. And we recall from the results and arguments of Sect. II that $\bar{u}_n$ increases to $\bar{u}$ and thus $u \geqq \bar{u}$, completing the proof of Theorem V.1 in case i).

The proof of case ii) is almost trivial: indeed, we apply (62) with $\varphi \equiv 0$ to deduce that any solution of (1) satisfying (62) is bounded from below on Ω. Thus, by Theorems III.3 and III.4, the uniqueness is proved.

Unfortunately, *the proof of case iii)* is much more complicated; in order to keep the ideas clear (or to try at least) we will begin with the case when $f \in C^1(\bar{\Omega})$ and Ω is starshaped (step 1), then we will treat the case when $f \in C^1(\bar{\Omega})$ but Ω is arbitrary (step 2) and we will conclude with the general case (step 3).

Step 1. $f \in C^1(\bar{\Omega})$, Ω is starshaped.

Without loss of generality we may assume that Ω is starshaped with respect to 0. Again, we denote by $\bar{u}$ the maximum solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r < \infty)$ or equivalently in $C^2(\Omega)$ in view of the smoothness of f – see Sect. IV. And we denote by u any other solution of (1) [in $C^2(\Omega)$] satisfying (62). Recall that $\bar{u} \in C(\bar{\Omega})$ (cf. Sect. IV) and observe that applying (62) with $\varphi = 0$, one deduces that u is bounded from below and thus u may be extended continuously to $\bar{\Omega}$ (cf. Sect. IV). Finally, $u \leqq \bar{u}$ and thus we want to show the reverse inequality.

We then introduce for $t \in (0,1)$

$$v_t(x) = t^{-\frac{p-2}{p-1}} u(tx) \quad \text{for} \quad x \in \Omega/t \supset \bar{\Omega}. \tag{65}$$

Obviously, v_t satisfies

$$-\Delta v_t + |\nabla v_t|^p + \lambda t^2 v_t = t^q f(tx) \quad \text{in} \quad \Omega/t, \qquad v_t \in C^2(\Omega/t) \cap C(\bar{\Omega}/t) \tag{66}$$

and thus in particular $v_t \in C^2(\bar{\Omega})$ and satisfies

$$-\Delta v_t + |\nabla v_t|^p + \lambda v_t \leqq f(x) + C(1-t) \quad \text{in} \quad \Omega \tag{67}$$

for some $C \geqq 0$ independent of t. In other words, $v_t - \frac{C}{\lambda}(1-t)$ is a $C^2(\bar{\Omega})$ subsolution of (1) and our strategy applies: $u \geqq v_t$ in $\bar{\Omega}$ and thus passing to the limit as t goes to 1 we conclude $u \geqq \bar{u}$ in $\bar{\Omega}$.

Step 2. $f \in C^1(\bar{\Omega})$, Ω arbitrary.

We first observe that by the maximum principle the minimum of $u - \bar{u}$ is achieved at the boundary. Furthermore, if $\theta \in (0,1)$, we may still assume that the minimum of $u - \theta\bar{u}$ is still achieved at the boundary. Indeed, if $u - \theta_n \bar{u}$ has an interior minimum over $\bar{\Omega}$ say at $x_n \in \Omega$ for some sequence $\theta_n \underset{n}{\to} 1$, then observing that $\theta_n \bar{u}$ satisfies

$$-\Delta(\theta_n \bar{u}) + |\nabla(\theta_n \bar{u})|^p + \lambda \theta_n \bar{u} \leqq \theta_n f \leqq f + C(1-\theta_n)$$

we deduce from the maximum principle

$$\min_{\bar{\Omega}} (u - \theta_n \bar{u}) \geqq -\frac{C}{\lambda}(1-\theta_n)$$

and we conclude letting n to to $+\infty$.

Therefore, let fix $\theta \in (0,1)$, we assume that $u - \theta\bar{u}$ has a minimum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$. Then, we remark that $u - \theta\bar{u} + (1-\theta)|x - x_0|^2$ has a *unique* maximum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$ and, denoting by $\tilde{u} = \theta\bar{u} + (1-\theta)|x-x_0|^2$, that $\tilde{u}$ satisfies

$$-\Delta\tilde{u} + |\nabla\tilde{u}|^p + \lambda\tilde{u} \leqq \theta f + C(1-\theta) \leqq f + C(1-\theta), \tag{68}$$

where C denotes various nonnegative constants independent of θ.

We next observe that for some small $\delta > 0$, the open set $Q = (x_0, \delta) \cap \Omega$ is starshaped with respect to a point that we denote by 0 such that $d(0) \geqq \gamma > 0$ where γ, δ are independent of x_0 and θ. We then consider as in step 1 the functions

$$v_t(x) = t^{-\frac{p-2}{p-1}} \tilde{u}(tx) \quad \text{for} \quad x \in Q/t, \qquad t \in (0,1)$$

and we obtain exactly as in step 1 using now (68) instead of (1)

$$-\Delta v_t + |\nabla v_t|^p + \lambda v_t \leqq f + C(1-\theta) + C(1-t) \quad \text{in} \quad Q, \qquad v_t \in C^2(\bar{\Omega}). \tag{69}$$

Let $\bar{x}$ be a minimum point of $u - v_t$ on $\bar{Q}$: because of (62), $\bar{x} \in Q$ or $\bar{x} \in \partial Q \cap \Omega$. If $\bar{x} \in Q$, we use maximum principle to deduce

$$\min_{\bar{\Omega}} (u - v_t) \geqq -\frac{C}{\lambda}(1-\theta) - \frac{C}{\lambda}(1-t)$$

and thus in particular

$$u(x_0) - t^{-\frac{p-2}{p-1}} \theta\bar{u}(x_0) \geqq -\frac{C}{\lambda}(1-\theta) - \frac{C}{\lambda}(1-t)$$

and we deduce letting t go to 1

$$\min_{\bar{\Omega}} (u-\theta\bar{u})=(u-\theta\bar{u})(x_0)\geqq -\frac{C}{\lambda}(1-\theta).$$

The conclusion follows upon letting θ go to 1.

In the other case i.e. if $\bar{x}\in\partial Q\cap\Omega$, we obtain letting t go to 1

$$(u-\tilde{u})(x_0)\geqq \min_{\partial Q\cap\Omega}(u-\tilde{u})$$

and this yields a contradiction with the fact that x_0 is the unique minimum point of $u-\tilde{u}$ over $\bar{\Omega}$.

Step 3. The general case.

We begin by observing that if $f\in C(\bar{\Omega})$ or even if f is continuous near $\partial\Omega$ the above proof is easily adapted: the only difficulty lies with the fact that $\bar{u}$, $\tilde{u}$, v_t do not belong to C^2 in general. But this can be taken care of by observing that $v_t*\varrho_\delta=v_{t,\delta}\left[\text{where } \varrho_\delta=\frac{1}{\delta^N}\varrho\left(\frac{\cdot}{\delta}\right),\ \varrho\geqq 0,\ \varrho\in\mathscr{D}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\varrho dx=1,\ \operatorname{Supp}\varrho\subset B(0,1)\right]$ satisfies

$$\begin{aligned}&-\Delta v_{t,\delta}+|\nabla v_{t,\delta}|^p+\lambda v_{t,\delta}\\&\leqq f*\varrho_\delta+C(1-\theta)+\omega(1-t)\quad\text{in}\quad\{x\in Q/t,\ \operatorname{dist}(x,\partial Q/t)>\delta\},\end{aligned}$$

where ω is a modulus of continuity of f near the boundary. Taking δ small enough, we find that $v_{t,\delta}\in C^2(\bar{Q})$ and

$$-\Delta v_{t,\delta}+|\nabla v_{t,\delta}|^p+\lambda v_{t,\delta}\leqq f+C(1-\theta)+\omega(1-t)+\omega(\delta)\quad\text{in}\quad Q$$

and we conclude as before letting δ go to 0, then t go to 1 and then θ go to 1.

To obtain the uniqueness in the case of a general f, we approximate f by f_n given by (63) where $C_0\geqq 0$ is any constant such that $f\geqq -C_0$ in Ω. The above arguments show that $u\geqq u_n$ where u_n is the unique solution of (1) satisfying (62) (with f replaced by f_n). Obviously, $\bar{u}\geqq u_n$ and u_n increases to a solution in $W^{2,r}(\Omega)$ $(\forall r<\infty)\cap C(\bar{\Omega})$ of (1) and we just have to show that $\bar{u}\leqq\hat{u}$, where $\hat{u}$ denotes the limit of u_n. To this end let $\theta\in(0,1)$, let $\gamma\in(\beta,p)$, $\sigma=1-\gamma/p$ and choose $K>0$, $C>0$ so that $w=-C-Kd^\sigma$ satisfies

$$-\Delta w+|\nabla w|^p+\lambda w\leqq -C_0-\nu d^{-\gamma}\quad\text{in}\quad\Omega,\quad\text{for some}\quad \nu>0. \tag{70}$$

Then, we remark that $z=\theta\bar{u}+(1-\theta)w$ satisfies

$$-\Delta z+|\nabla z|^p+\lambda z\leqq\theta f-(1-\theta)(C_0+\nu d^{-\gamma})=g.$$

But on $\Omega_{1/n}$,

$$g=f-(1-\theta)(f+C_0+\nu d^{-\gamma})\leqq f$$

while on $\Omega-\Omega_{1/n}$

$$g\leqq\theta C(1+d^{-\beta})-(1-\theta)(C_0+\nu d^{-\gamma})$$

and thus, $g\leqq f_n$ in Ω provided n is large enough say $n\geqq n_0(\theta)$. Hence,

$$\theta\bar{u}+(1-\theta)w\leqq u_n\quad\text{if}\quad n\geqq n_0(\theta)$$

and passing to the limit in n, we deduce

$$\theta\bar{u}+(1-\theta)u \leqq \hat{u}.$$

We may now conclude letting θ go to 1. □

V.3. Applications

We want in this section to show that (17) is equivalent to (62) when $p>2$ (with appropriate conditions on f) and that (62) is stable under some passages to the limit. This together with the uniqueness proved in Theorem V.1 will yield a rather powerful stability result. We begin with the relations between (17) and (62). Recall that (17) implies trivially (62).

Theorem V.1. *Let $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$, let $p>2$, let $f \in L^{\infty}_{\mathrm{loc}}(\Omega)$ be bounded from below and let $x_0 \in \Omega$. Assume that $u \in C(\bar{\Omega})$ satisfies* (62) *and*

$$-\Delta u+|\nabla u|^p+\lambda u \geqq f \quad \text{in} \quad \Omega, \quad \text{for some} \quad \lambda, C \geqq 0. \tag{72}$$

Then, u satisfies

$$\liminf_{x\in\Omega, x\to x_0} \{u(x)-u(x_0)\}\,|x-x_0|^{-\alpha}<0, \quad \text{where} \quad \alpha=(p-2)/(p-1). \tag{73}$$

Proof. The proof is almost the same as the one of Theorem IV.2: with the notations of Theorem IV.2, we just have to replace ω by w^n defined exactly as w with d replaced by $d+\frac{1}{n}$. Then, $w^n \in C^2(\bar{\omega})$ is a subsolution of (1) and $u \geqq w^n$ on $\partial\omega\cap\Omega$. Therefore, by maximum principle, $u-w^n$ achieves its minimum on $\partial\Omega$ and we reach a contradiction. □

Remark V.1. Many variants of the above result and of its proof exist that we will skip here.

We now present a stability result.

Theorem V.2. *Let $(F_n)_n$ be a sequence of continuous functions on $\mathbb{M}^N \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$ where $\mathbb{M}^N$ denotes the space of $N \times N$ symmetric matrices, let $u_n \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r<\infty)$ satisfy for some $C \geqq 0$ independent of n*

$$F_n(D^2u_n, Du_n, u_n, x) \geqq -C \quad \text{a.e. in} \quad \Omega. \tag{74}$$

We assume that $F_n(A,\xi,t,x) \leqq F_n(B,\xi,t,x)$ for all $\xi \in \mathbb{R}^N$, $t \in \mathbb{R}$, $x \in \Omega$, $A \geqq B$ (in the sense of symmetric matrices), F_n converges uniformly on compact subsets to $-\operatorname{tr}(A)+|\xi|^p+\lambda t$, for some $p>2$, $\lambda \geqq 0$, u_n satisfies (62) *and converges uniformly on compact subsets of Ω to some function $u \in C(\bar{\Omega})$, and that $(u_n-u)^-$ converges uniformly to 0 on Ω. Then, u satisfies* (62).

Proof. Assume by way of contradiction that $u-\varphi$ admits a minimum at $x_0 \in \partial\Omega$ for some $\varphi \in C^2(\bar{\Omega})$, without loss of generality we may assume that x_0 is the unique minimum point of $u-\varphi$. By assumption, $u_n-\varphi$ achieves its minimum over Ω at some point $x_n \in \Omega$. We remark that

$$\min_{\bar{\Omega}} (u_n-\varphi) \leqq u_n(x)-\varphi(x) \underset{n}{\rightarrow} u(x)-\varphi(x) \quad \text{for all} \quad x \in \Omega$$

while

$$\min_{\Omega}(u_n-\varphi)\geqq \min_{\bar{\Omega}}(u-\varphi)-\|(u_n-u)^-\|_\infty,$$

hence $u_n(x_n)-\varphi(x_n)$ converges to $u(x_0)-\varphi(x_0)$. Now if x_n (or a subsequence) converges to $\bar{x}\in\bar{\Omega}$, then

$$u_n(x_n)-\varphi(x_n)\geqq u(x_n)-\varphi(x_n)-\|(u_n-u)^-\|_\infty \xrightarrow[n]{} u(\bar{x})-\varphi(\bar{x})$$

and thus $\bar{x}=x_0$ by the uniqueness of the minimum.

Next, by maximum principle, we have

$$F_n(D^2\varphi(x_n), D\varphi(x_n), u_n(x_n), x_n)\geqq -C$$

and passing to the limit we find

$$-\Delta\varphi(x_0)+|\nabla\varphi(x_0)|^p+\lambda u(x_0)\geqq -C. \tag{75}$$

and we observe that we may replace φ by $\varphi+c(\delta^\alpha-(d+\delta)^\alpha)$ where $\delta>0$, $\alpha=\dfrac{p-2}{p-1}$, $c>0$, since $u-\varphi+c((d+\delta)^\alpha-\delta^\alpha)$ admits also a unique minimum at x_0. Therefore, we deduce from (75)

$$-\Delta\varphi(x_0)+c\alpha\delta^{-(1-\alpha)}\Delta d(x_0)-c\alpha(1-\alpha)\delta^{-(2-\alpha)}+|\nabla\varphi(x_0)-c\alpha d^{1-\alpha}\nabla d|^p\geqq -C$$

and if we choose c in such a way that $(c\alpha)^{p-1}<(1-\alpha)$, we easily reach a contradiction letting δ go to 0. □

From this stability result, we deduce the

Corollary V.2. *Let $p>2$, let $f_n\in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy*

$$f_n\geqq -C,\quad f_n\leqq Cd^{-\beta}\quad a.e.\ in\quad \Omega,\quad for\ some\quad C\geqq 0,\quad \beta\in(0,p). \tag{76}$$

We denote by u_n the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar{\Omega})$ ($\forall r<\infty$) of (1) *satisfying* (62) *and we assume that f_n converges to f weakly in $L^\infty_{\mathrm{loc}}-*$. We denote by u the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar{\Omega})$ ($\forall r<\infty$) of* (1) *satisfying* (62). *Then, u_n converges uniformly on $\bar{\Omega}$ to u.*

Proof. By Theorem IV.1, u_n is bounded in $C^{0,\gamma}(\bar{\Omega})$ for some $\gamma>0$ and in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r<\infty$). Then, we may assume (up to subsequences) that u_n converges uniformly on $\bar{\Omega}$ to a solution u of (1) [in $W^{2,r}_{\mathrm{loc}}(\Omega)\cap C(\bar{\Omega})$ for all $r<\infty$]. By Theorem V.2, u satisfies (62) and thus $u\equiv\tilde{u}$ by Theorem V.1. □

VI. The Ergodic Problem

In this section, we want to study the questions associated with the so-called ergodic stochastic control problems with state constraints. From the p.d.e.'s viewpoint this amounts to study the behaviour of λu and u as λ goes to 0 where u is the solution of (1) considered in the preceding sections. We will perform such an analysis in the three different cases studied above. The typical result we will obtain is that λu, $u-u(x_0)$ converge uniformly on compact subsets of Ω to $u_0\in\mathbb{R}$, v solution of

$$-\Delta v+|\nabla v|^p+u_0=f\quad\text{in}\quad\Omega,\qquad v(x_0)=0 \tag{77}$$