while

$$\min_{\Omega} (u_n - \varphi) \geqq \min_{\bar{\Omega}} (u - \varphi) - \|(u_n - u)^-\|_\infty,$$

hence $u_n(x_n) - \varphi(x_n)$ converges to $u(x_0) - \varphi(x_0)$. Now if x_n (or a subsequence) converges to $\bar{x} \in \bar{\Omega}$, then

$$u_n(x_n) - \varphi(x_n) \geqq u(x_n) - \varphi(x_n) - \|(u_n - u)^-\|_\infty \underset{n}{\to} u(\bar{x}) - \varphi(\bar{x})$$

and thus $\bar{x} = x_0$ by the uniqueness of the minimum.

Next, by maximum principle, we have

$$F_n(D^2\varphi(x_n), D\varphi(x_n), u_n(x_n), x_n) \geqq -C$$

and passing to the limit we find

$$-\Delta\varphi(x_0) + |\nabla\varphi(x_0)|^p + \lambda u(x_0) \geqq -C. \tag{75}$$

and we observe that we may replace φ by $\varphi + c(\delta^\alpha - (d+\delta)^\alpha)$ where $\delta > 0$, $\alpha = \dfrac{p-2}{p-1}$, $c > 0$, since $u - \varphi + c((d+\delta)^\alpha - \delta^\alpha)$ admits also a unique minimum at x_0. Therefore, we deduce from (75)

$$-\Delta\varphi(x_0) + c\alpha\delta^{-(1-\alpha)}\Delta d(x_0) - c\alpha(1-\alpha)\delta^{-(2-\alpha)} + |\nabla\varphi(x_0) - c\alpha d^{1-\alpha}\nabla d|^p \geqq -C$$

and if we choose c in such a way that $(c\alpha)^{p-1} < (1-\alpha)$, we easily reach a contradiction letting δ go to 0. □

From this stability result, we deduce the

Corollary V.2. *Let $p > 2$, let $f_n \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy*

$$f_n \geqq -C, \quad f_n \leqq C d^{-\beta} \quad \text{a.e. in} \quad \Omega, \quad \text{for some} \quad C \geqq 0, \quad \beta \in (0, p). \tag{76}$$

We denote by u_n the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ ($\forall r < \infty$) of (1) *satisfying* (62) *and we assume that f_n converges to f weakly in $L^\infty_{\mathrm{loc}} - *$. We denote by u the unique solution in $W^{2,r}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ ($\forall r < \infty$) of* (1) *satisfying* (62). *Then, u_n converges uniformly on $\bar{\Omega}$ to u.*

Proof. By Theorem IV.1, u_n is bounded in $C^{0,\gamma}(\bar{\Omega})$ for some $\gamma > 0$ and in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$). Then, we may assume (up to subsequences) that u_n converges uniformly on $\bar{\Omega}$ to a solution u of (1) [in $W^{2,r}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ for all $r < \infty$]. By Theorem V.2, u satisfies (62) and thus $u \equiv \tilde{u}$ by Theorem V.1. □

VI. The Ergodic Problem

In this section, we want to study the questions associated with the so-called ergodic stochastic control problems with state constraints. From the p.d.e.'s viewpoint this amounts to study the behaviour of λu and u as λ goes to 0 where u is the solution of (1) considered in the preceding sections. We will perform such an analysis in the three different cases studied above. The typical result we will obtain is that λu, $u - u(x_0)$ converge uniformly on compact subsets of Ω to $u_0 \in \mathbb{R}$, v solution of

$$-\Delta v + |\nabla v|^p + u_0 = f \quad \text{in} \quad \Omega, \qquad v(x_0) = 0 \tag{77}$$

with the same boundary conditions for v than for u. And these will uniquely determine (u_0, v). In the preceding statements and below, x_0 is any fixed point in Ω and we assume that Ω is connected.

VI.1. Subquadratic Hamiltonians

Whenever it exists, we will denote by u_λ the solution of (1) with appropriate boundary conditions and if x_0 is any fixed point in Ω we will denote by $v_\lambda(\cdot)=u_\lambda(\cdot) - u_\lambda(x_0)$. We assume throughout this section that $1<p<2$.

Theorem VI.1. *Let $f \in L^\infty_{\text{loc}}(\Omega)$ be bounded from below and satisfy*

$$\lim\{f(x)d(x)^{-q}/d(x)\to 0_+\}=0\,. \tag{78}$$

Let u_λ be the unique solution of (1) in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ such that $u_\lambda \to +\infty$ as $d\to 0_+$. Then, ∇u_λ and λu_λ are bounded in $L^\infty_{\text{loc}}(\Omega)$ and λu_λ, v_λ converge uniformly on compact subsets of Ω to $u_0\in\mathbb{R}$, $v\in W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ such that $v(x_0)=0$, v satisfies (15) *and*

$$-\Delta v+|\nabla v|^p+u_0=f \quad in \quad \Omega\,. \tag{79}$$

In addition, if $(\tilde u_0, \tilde v)\in\mathbb{R}\times W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ satisfies (79) *and $\tilde v$ goes to $+\infty$ as d goes to 0_+, then $\tilde u_0=u_0$, $\tilde v=v+C$ for some $C\in\mathbb{R}$.*

Proof. The proof involves several steps, we first obtain some bounds and we pass to the limit (Step 1). Then, we show that for any solution $(\tilde u_0, \tilde v)$ as above $\tilde v$ blows up at the boundary like $C_0 d^{-\alpha}$ (Step 2). Next, we show the uniqueness of u_0 (Step 3). Finally, we conclude with the uniqueness (up to constants) of $\tilde v$ (Step 4).

Step 1. Going through the proofs of Theorems II.1 and II.2, we see that u_λ satisfies for all $\varepsilon>0$, $\lambda\in(0,1]$

$$\frac{C_0-\varepsilon}{d^\alpha}-\frac{C_\varepsilon}{\lambda}\leqq u_\lambda\leqq\frac{C_0+\varepsilon}{d^\alpha}+\frac{C_\varepsilon}{\lambda} \tag{80}$$

for some $C_\varepsilon\geqq 0$, with the usual modifications if $\alpha=0$ (i.e. $p=2$). In particular, λu_λ is bounded from below and in L^∞_{loc}. Then, using Theorem IV.1, we deduce that ∇u_λ is bounded from below. Therefore, v_λ is bounded in $W^{1,\infty}_{\text{loc}}$.

We next want to show that v_λ satisfies

$$\frac{C_1}{d^\alpha}-C\leqq v_\lambda \quad \text{in} \quad \Omega, \quad \text{for some} \quad C_1\in(0,C_0), \quad C\geqq 0\,. \tag{81}$$

Observe first that v_λ satisfies

$$-\Delta v_\lambda+|\nabla v_\lambda|^p+\lambda v_\lambda+\lambda u_\lambda(x_0)=f \quad \text{in} \quad \Omega\,.$$

And if we choose C_1 in $(0, C_0)$, we obtain denoting by $z=\dfrac{C_1}{d^\alpha}$

$$-\Delta z+|\nabla z|^p-\lambda z\leqq f-\lambda u_\lambda(x_0) \quad \text{on} \quad \Omega-\Omega_\delta$$

if δ is small enough, say $\delta\leqq\delta_0$. Now, there exists a constant $M\geqq 0$ such that

$$v_\lambda\geqq M \quad \text{on} \quad \Omega_{\delta_0}\,.$$

Hence, adapting the comparison results proved in Sect. II, we deduce

$$v_\lambda \geqq -M + \frac{C_1}{d^\alpha} \quad \text{on} \quad \Omega.$$

Extracting subsequences if necessary – the convergence of the whole sequence will follow from the uniqueness –, we may now pass to the limit $\lambda u_\lambda(x_0)$ converges to a constant u_0, v_λ converges to a solution v of (79) satisfying (81) and such that $v(x_0)=0$.

Step 2. Let $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ be a solution of (79) such that $\tilde{v}$ goes to $+\infty$ as d goes to 0_+. We want to prove that $\tilde{v}$ satisfies (15). To this end, we recall that $\bar{w}_{\varepsilon,\delta} = \dfrac{C_0+\varepsilon}{(d-\delta)^\alpha}$ satisfies

$$-\Delta \bar{w}_{\varepsilon,\delta} + |\nabla \bar{w}_{\varepsilon,\delta}|^p \geqq f - \tilde{u}_0 \quad \text{in} \quad \Omega_\delta - \Omega_{\delta_0} \quad \text{if} \quad 0<\delta<\delta_0=\delta_0(\varepsilon).$$

Then, let $M_\varepsilon = \sup\{|\tilde{v}(x)|/x\in\Omega,\ d(x)=\delta_0(\varepsilon)\}$, we deduce from the maximum principle that

$$\tilde{v} \leqq \bar{w}_{\varepsilon,\delta} + M_\varepsilon \quad \text{on} \quad \Omega_\delta - \Omega_{\delta_0}$$

and letting δ go to 0, we deduce

$$-C \leqq \tilde{v} \leqq (C_0+\varepsilon) d^{-\alpha} + M_\varepsilon \quad \text{on} \quad \Omega. \tag{82}$$

Next, we simply observe that $\tilde{v}$ satisfies

$$-\Delta \tilde{v} + |\nabla \tilde{v}|^p + \tilde{v} = g \quad \text{in} \quad \Omega, \qquad \tilde{v} \to +\infty \quad \text{as} \quad d \to 0_+$$

where $g = f - \tilde{u}_0 + \tilde{v} \in L^\infty_{\text{loc}}(\Omega)$ satisfies (78) because of (82). Therefore, Theorem II.2 yields the desired behaviour of $\tilde{v}$ near $\partial\Omega$.

Step 3. We first show that if (u_0, v), $(\tilde{u}_0, \tilde{v})$ are two solutions of (79) such that $v, \tilde{v} \to +\infty$ as $d \to 0_+$ then $u_0 = \tilde{u}_0$. To do so, we adapt an argument from Lions [16, 19]. Assume for instance that $u_0 < \tilde{u}_0$ and let $\varepsilon > 0$, $\theta \in (0,1)$. Obviously, we have

$$\begin{aligned} -\Delta(\theta\tilde{v}) + |\nabla(\theta\tilde{v})|^p + \varepsilon\theta\tilde{v} &\leqq \theta f + \varepsilon\theta\tilde{v} - \theta\tilde{u}_0 \\ &\leqq f + C(1-\theta) + \varepsilon\theta\tilde{v} - \theta\tilde{u}_0 . \end{aligned}$$

Next, since v, $\tilde{v}$ behave like $C_0 d^{-\alpha}$ near $\partial\Omega$, $\theta\tilde{v} \leqq v + C_\theta$ in Ω; hence

$$\begin{aligned} -\Delta(\theta\tilde{v}) + |\nabla(\theta\tilde{v})|^p + \varepsilon\theta\tilde{v} &\leqq f + C(1-\theta) + \varepsilon v + \varepsilon C_\theta - \theta\tilde{u}_0 \\ &\leqq f + \varepsilon v - u_0 + (u_0 - \theta\tilde{u}_0) + \varepsilon C_\theta + C(1-\theta) \end{aligned}$$

while v satisfies of course

$$-\Delta v + |\nabla v|^p + \varepsilon v = f + \varepsilon v - u_0 \quad \text{in} \quad \Omega.$$

But $u_0 < \tilde{u}_0$. Therefore, for θ near 1 and ε small enough (depending on θ) $\theta\tilde{v}$ is a subsolution of the equation satisfied by v. By Theorem II.2, this implies $\theta\tilde{v} \leqq v$. Letting θ go to 1, we find $\tilde{v} \leqq v$. But, $v + C_1$, $\tilde{v} + C_2$ satisfy the same problems for arbitrary constants C_1, C_2 and we reach a contradiction.

Step 4. Uniqueness of v_0 up to a constant.

Let $C_1 \in (0, C_0)$, again we observe that

$$-\Delta\left(\frac{C_1}{d^\alpha}\right) + \left|\nabla\left(\frac{C_1}{d^\alpha}\right)\right|^p \leqq f - u_0 \quad \text{in} \quad \Omega - \Omega_\delta$$

for some small enough $\delta > 0$. Therefore, if $\theta \in (0,1)$, $w = \theta\tilde{v} + (1-\theta)\frac{C_1}{d^\alpha}$ satisfies

$$-\Delta w + |\nabla w|^p \leqq \theta(f - u_0) + (1-\theta)(f - u_0) = f \quad \text{in} \quad \Omega - \Omega_\delta .$$

And since $v, \tilde{v}$ behave like $\frac{C_0}{d^\alpha}$, $(w - v) \to -\infty$ as $d \to 0_+$. Therefore, by the maximum principle,

$$\max_{\Omega - \Omega_\delta} (w - v) = \max_{\partial\Omega_\delta} (w - v) .$$

Hence, if we let θ go to 1, we find that

$$\sup_{\Omega - \Omega_\delta} (\tilde{v} - v) = \max_{\partial\Omega_\delta} (\tilde{v} - v) .$$

On the other hand, we also deduce from the maximum principle that

$$\max_{\bar{\Omega}_\delta} (\tilde{v} - v) = \max_{\partial\Omega_\delta} (\tilde{v} - v) .$$

Therefore, any maximum point $\bar{x}$ of $\tilde{v} - v$ on $\partial\Omega_\delta$ is in fact a global maximum point of $\tilde{v} - v$ on Ω. But, since $\tilde{v} - v = \psi$ satisfies the equation

$$-\Delta\psi + B \cdot \nabla\psi = 0 \quad \text{in} \quad \Omega$$

for some B $L^\infty_{\text{loc}}(\Omega; \mathbb{R}^N)$, the strong maximum principle then yields

$$\tilde{v} - v \equiv (\tilde{v} - v)(\bar{x}) \quad \text{in} \quad \Omega$$

and we conclude. □

We would like to conclude this section with a few remarks on the case $p = 2$ which make a connection between our results and the interpretation of first eigenvalues in terms of optimal stochastic control that was considered by Holland [9, 10]. Indeed, if $p = 2$ and if v solves (79) with $v \to \infty$ as $d \to 0_+$ then we may perform the well known logarithmic transformation i.e. $v = -\operatorname{Log}\varphi$ and we find

$$-\Delta\varphi + f\varphi = u_0\varphi \quad \text{in} \quad \Omega, \qquad \varphi > 0 \quad \text{in} \quad \Omega, \qquad \varphi \to 0 \quad \text{as} \quad d \to 0_+ \tag{83}$$

or in other words u_0 is the minimum eigenvalue of the operator $(-\Delta + f)$ with Dirichlet boundary conditions. And the uniqueness of u_0 corresponds to the uniqueness of an eigenvalue with a positive eigenfunction, while the uniqueness of v up to an additive constant corresponds to the uniqueness of φ up to a multiplicative constant.

VI.2. Forced Infinite Boundary Conditions

We will be now concerned with the case when f grows so fast at the boundary that u_λ automatically has to blow up at $\partial\Omega$. To simplify the presentation, we will only consider the case when f satisfies (40) and therefore, by Theorem III.4, u_λ is the unique solution of (1) which is bounded from below.

Theorem VI.2. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy* (40) *and let $p > 1$, we denote by u_λ the unique solution of* (1) *which is bounded from below. Then, ∇u_λ and λu_λ are bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ and λu_λ, v_λ converge uniformly on compact subsets of Ω to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ such that $v(x_0) = 0$, v satisfies* (79) *and*

$$M^{-1} d^{-\alpha} - M \leqq v \leqq M d^{-\alpha} + M \quad \text{in} \quad \Omega, \quad \text{for some} \quad M \geqq 1,$$

$$\text{where} \quad \alpha = \frac{\beta}{p} - 1 \quad \text{if} \quad \beta > p \tag{84}$$

and $d^{-\alpha}$ is replaced by $|\mathrm{Log}\, d|$ if $\beta = p \geqq q$. In addition, if f satisfies (18′), *then v satisfies* (19). *Furthermore, if $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ satisfies* (79) *and $\tilde{v}$ is bounded from below, then $\tilde{u}_0 = u_0$, $\tilde{v} = v + C$ for some $C \in \mathbb{R}$.*

Remark VI.1. If we consider the special case $p = 2$ and if we perform the same logarithmic transformation as in the preceding section $v = -\mathrm{Log}\,\varphi$, we see that we are dealing with bounded, positive solutions of (83) and that the very fact that f blows up fast enough at $\partial\Omega$ forces φ to vanish on the boundary. Again, the uniqueness part of the above result may be interpreted as a uniqueness for first eigenvalues and eigenfunctions of the operator $-\Delta + f$ where no boundary condition on φ is imposed except "φ is bounded".

Proof of Theorem VI.2. Most of the proof of Theorem VI.1 goes through in this case except for the uniqueness arguments which use the precise behaviours of v, $\tilde{v}$ near the boundary. Of course, if we assume (18′) then the proof of Theorem VI.1 applies with some rather easy adaptations. In the general case, however, we have to involve slightly more elaborate arguments to show the uniqueness part of the above result. We only prove as in Theorem VI.1 that v, $\tilde{v}$ both satisfy (84). Next, we prove that $u_0 = \tilde{u}_0$. We see that the corresponding proof (Step 3) in the proof of Theorem VI.1 only uses the fact that $\theta\tilde{v} \leqq v + C_\theta$ for any two solutions (u_0, v), $(\tilde{u}_0, \tilde{v})$ and for all $\theta \in (0, 1)$. But this can be deduced from Theorem III.4: indeed $w = \theta\tilde{v}$ is a subsolution of the equation

$$-\Delta w + |\nabla w|^p + w \leqq g = \theta f + \theta\tilde{v} - \tilde{u}_0 \quad \text{in} \quad \Omega .$$

But in view of (40) and (84) $\theta f + \theta\tilde{v} - \tilde{u}_0 \leqq f + C'_\theta \leqq f + v - u_0 + C_\theta$ for some constants C'_θ, $C_\theta \geqq 0$. Therefore, by Theorem III.4, we deduce

$$w \leqq v + C_\theta \quad \text{in} \quad \Omega$$

and our claim is proved.

Finally, we have to show the uniqueness of v up to a constant. Then, we observe that the proof given in Step 4 of the proof of Theorem VI.1 still applies provided we take C_1 small enough, indeed the only difference comes into the verification that $w - v \to -\infty$ as $d \to 0_+$ where $w = \theta\tilde{v} + (1-\theta)\dfrac{C_1}{d^\alpha}$ with $0 < \theta < 1$. But the inequality we just proved shows that

$$w \leqq \frac{1+\theta}{2}\, v + C_\theta + (1-\theta)\frac{C_1}{d^\alpha}$$

and taking C_1 small enough so that $\frac{C_1}{d^\alpha} \leqq \frac{1}{4} v + C$ we find

$$w \leqq \left(\frac{1+\theta}{2} + \frac{1-\theta}{4} \right) v + C_\theta + C = \frac{3+\theta}{4} v + C_\theta + C$$

therefore $w - v \to -\infty$ as $d \to 0_+$ since $v \to +\infty$ as $d \to 0_+$.

Then, the proof of Theorem VI.1 applies and we may conclude the proof of Theorem VI.2. □

VI.3. Superquadratic Hamiltonians

We now conclude this section by examining the remaining case namely the case when $p > 2$ and $f \in L^\infty_{\mathrm{loc}}(\Omega)$, is bounded from below and satisfies (41) with $\beta < p$. Then, we know there exists a unique solution u_λ of (1) satisfying (17) or (62). As λ goes to 0_+, we obtain the

Theorem VI.3. *Let $p > 2$, let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (41) *for some $\beta < p$. We denote by u_λ the unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ satisfying* (62). *Then, ∇u_λ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$ and λu_λ is bounded in $L^\infty(\Omega)$. And λu_λ, v_λ converge uniformly on $\bar{\Omega}$ to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ $(\forall r < \infty)$ such that $v(x_0) = 0$, v satisfies* (17) *and* (79). *In addition, if $(\tilde{u}_0, \tilde{v}) \in \mathbb{R} \times W^{2,r}_{\mathrm{loc}}(\Omega)$ $(\forall r < \infty)$ satisfies* (79) *and if $\tilde{v}$ satisfies* (62), *then $\tilde{u}_0 = u_0$, $\tilde{v} = v + C$ for some $c \in \mathbb{R}$.*

Proof. Clearly, λu_λ is bounded from below. Then, we may apply the local gradient bound given in Theorem IV.1: hence, ∇u_λ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$. But this bound (see Corollary IV.1) also implies that u_λ is bounded in $C^{0,\theta}(\bar{\Omega})$ for some $\theta \in (0,1)$ independent of λ. Therefore, up to subsequences, λu_λ and u_λ converge uniformly on $\bar{\Omega}$ to $u_0 \in \mathbb{R}$, $v \in W^{2,r}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ $(\forall r < \infty)$ such that $v(x_0) = 0$, v satisfies (79). Next, by Theorem V.2, v satisfies (62) and therefore, by Proposition V.1, v satisfies (17).

Notice also that v is the unique solution satisfying (62) of

$$-\Delta v - |\nabla v|^p + \lambda v = g_\lambda \quad \text{in} \quad \Omega,$$

where $g_\lambda = f - u_0 + \lambda v$.

Next, using Theorem V.1, the uniqueness of u_0 follows immediately as in Step 3 of the proof of Theorem VI.1 ($\theta = 1$ is enough in this case).

Finally, we want to prove the uniqueness of v_0 up to a constant. Again, the only fact we have to prove is the following

$$\sup_{\Omega - \Omega_\delta} (\tilde{v} - v) = \max_{\partial \Omega_\delta} (\tilde{v} - v).$$

To this end, we set $\alpha = \frac{p-2}{p-1}$ and we consider for $\theta \in (0,1)$, $w = \theta \tilde{v} - (1-\theta) C_1 d^\alpha$ then for $C_1 > 0$, δ small enough (independent of θ) we have

$$-\Delta w + |\nabla w|^p \leqq f - u_0 - 1 \quad \text{in} \quad \Omega - \Omega_\delta.$$

In particular, for λ small enough, we have

$$-\Delta w + |\nabla w|^p + \lambda w \leqq g_\lambda \quad \text{in} \quad \Omega - \Omega_\delta.$$

We may now adapt without any real modification the proof of Theorem V.1 to deduce

$$\sup_{\Omega - \Omega_\delta} (w-v) = \max_{\partial\Omega_\delta} (w-v)$$

and we conclude letting θ go to 1. □

VII. Optimal Stochastic Control with State Constraints

We now want to use the results obtained in the preceding sections in order to solve some optimal stochastic control problems with state constraints: a rather vague way to formulate our problem is to say that we want to "constrain a Brownian motion in a given domain Ω by controlling its drift". More precisely, we consider a system whose state is given by the solution of the following stochastic differential equation

$$dX_t = a_t dt + \sqrt{2}\, dB_t, \quad X_0 = x \in \Omega, \tag{3}$$

where B_t is a Brownian motion on a standard probability space (Ω, F, F_t, P) and where a_t is the control process i.e. a progressively measurable stochastic process taking values in $\mathbb{R}^N$ for instance. In other words

$$X_t = x + \int_0^t a_s ds + \sqrt{2} B_t$$

and we assume (at least) that $\int_0^T |a_s| ds < \infty$ a.s. $(\forall r < \infty)$.

We will say that this control a is admissible if $X_t \in \Omega\ \forall t \geqq 0$ a.s. Even if we could work with general controls of the above form, we will restrict ourselves to feedbacks or Markovian controls which are defined as follows. Let $a \in C(\Omega; \mathbb{R}^N)$, we may solve the stochastic differential equation

$$dX_t = a(X_t)dt + \sqrt{2}\, dB_t \quad \text{for} \quad 0 \leqq t < \tau_x, \qquad X_0 = x \in \Omega, \tag{85}$$

where τ_x is the first exit time of X_t from Ω i.e., $\tau_x = \inf\{t \geqq 0, X_t \notin \Omega\}$ ($\tau = +\infty$ if $X_t \in \Omega$ for all $t \geqq 0$). Thus, $a(X_t)$ is really the control but we will ignore this minor point of terminology and we will say that $a(\cdot)$ is the control (or control policy). Next, we define an admissible control as a control $a(\cdot)$ such that

$$P(\tau_x < \infty) = 0 \quad \text{for all} \quad x \in \Omega. \tag{86}$$

And we will denote by $\mathscr{A}$ the class of all admissible controls.

For each a, we define a cost function

$$J_\lambda(x, a) = E \int_0^\infty \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt, \tag{87}$$

where $c = c(p,q) = q^{-1} p^{-1/(p-1)}$, $\lambda > 0$ is the discount factor. Observe that the running cost $f(x) + c|a|^q$ contains two terms: one which measures the cost for the state to be at x, and the other measuring the cost for using the control a. All throughout this section $f \in L^\infty_{\mathrm{loc}}(\Omega)$ is bounded from below so that $J_\lambda(x, a)$ makes