We may now adapt without any real modification the proof of Theorem V.1 to deduce

$$\sup_{\Omega-\Omega_\delta}(w-v)=\max_{\partial\Omega_\delta}(w-v)$$

and we conclude letting θ go to 1. $\square$

VII. Optimal Stochastic Control with State Constraints

We now want to use the results obtained in the preceding sections in order to solve some optimal stochastic control problems with state constraints: a rather vague way to formulate our problem is to say that we want to "constrain a Brownian motion in a given domain Ω by controlling its drift". More precisely, we consider a system whose state is given by the solution of the following stochastic differential equation

$$dX_t=a_t dt+\sqrt{2}dB_t, \quad X_0=x\in\Omega, \tag{3}$$

where B_t is a Brownian motion on a standard probability space (Ω, F, F_t, P) and where a_t is the control process i.e. a progressively measurable stochastic process taking values in $\mathbb{R}^N$ for instance. In other words

$$X_t=x+\int_0^t a_s ds+\sqrt{2}B_t$$

and we assume (at least) that $\int_0^T |a_s|ds<\infty$ a.s. $(\forall r<\infty)$.

We will say that this control a is admissible if $X_t\in\Omega\ \forall t\geqq 0$ a.s. Even if we could work with general controls of the above form, we will restrict ourselves to feedbacks or Markovian controls which are defined as follows. Let $a\in C(\Omega;\mathbb{R}^N)$, we may solve the stochastic differential equation

$$dX_t=a(X_t)dt+\sqrt{2}dB_t \quad \text{for} \quad 0\leqq t<\tau_x, \qquad X_0=x\in\Omega, \tag{85}$$

where τ_x is the first exit time of X_t from Ω i.e., $\tau_x=\inf\{t\geqq 0, X_t\notin\Omega\}$ ($\tau=+\infty$ if $X_t\in\Omega$ for all $t\geqq 0$). Thus, $a(X_t)$ is really the control but we will ignore this minor point of terminology and we will say that $a(\cdot)$ is the control (or control policy). Next, we define an admissible control as a control $a(\cdot)$ such that

$$P(\tau_x<\infty)=0 \quad \text{for all} \quad x\in\Omega. \tag{86}$$

And we will denote by $\mathscr{A}$ the class of all admissible controls.

For each a, we define a cost function

$$J_\lambda(x,a)=E\int_0^\infty \{f(X_t)+c|a(X_t)|^q\}e^{-\lambda t}dt, \tag{87}$$

where $c=c(p,q)=q^{-1}p^{-1/(p-1)}$, $\lambda>0$ is the discount factor. Observe that the running cost $f(x)+c|a|^q$ contains two terms: one which measures the cost for the state to be at x, and the other measuring the cost for using the control a. All throughout this section $f\in L^\infty_{\mathrm{loc}}(\Omega)$ is bounded from below so that $J_\lambda(x,a)$ makes sense even if it may be infinite. This is of course a very special example but we will come back on part 2 on much more general problems for which similar results to those which follow still hold.

Finally, we want to minimize J_λ. We introduce the value function

$$u_\lambda(x) = \inf_{a \in \mathscr{A}} J_\lambda(x, a), \qquad \forall x \in \Omega. \tag{88}$$

The typical questions that one wants to solve in such problems is to determine u_λ and possibly an optimal control (here an optimal Markovian control or an optimal feedback), i.e. some a in $\mathscr{A}$ such that

$$u_\lambda(x) = J_\lambda(x, a), \qquad \forall x \in \Omega.$$

And this is precisely what we will achieve using the results of the preceding sections. Let us also observe that it is not completely obvious that $\mathscr{A} \neq \emptyset$, let alone that there exists $a \in \mathscr{A}$ such that $J(x, a) < \infty$ for $x \in \Omega$.

Finally, we will also consider the case of ergodic control which consists, roughly speaking, in taking $\lambda = 0$.

VII.1. Subquadratic Hamiltonians

Theorem VII.1. *Let $1 < p < 2$, let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (78). *Then, the value function u_λ given by* (88) *is the unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) such that $u_\lambda \to +\infty$ as $d \to 0_+$. Furthermore, $a_0(x) = p|\nabla u_\lambda|^{p-2}\nabla u_\lambda(x)$ is the unique optimal markovian control.*

Proof. We denote by $\tilde{u}_\lambda$ the unique solution of (1) in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) such that $\tilde{u}_\lambda \to +\infty$ as $d \to 0_+$. We are first going to show that $\tilde{u}_\lambda \geqq u_\lambda$ and that $a_0 \in \mathscr{A}$. Indeed, for $\delta > 0$ let $\tau^\delta_x = \inf(t \geqq 0, X_t \notin \Omega_\delta)$, we apply Itô's formula on $[0, \tau_\delta]$ with the process X_t corresponding to the control $a_0(x) = -p|\nabla \tilde{u}_\lambda|^{p-2}\nabla u_\lambda$ and we find for all x Ω_δ

$$\tilde{u}_\lambda(x) = E\int_0^{\tau^\delta_x} \{-\Delta\tilde{u}_\lambda(X_t) + p|\nabla\tilde{u}_\lambda|^p(X_t) + \lambda\tilde{u}_\lambda(X_t)\}e^{-\lambda t}dt + E\tilde{u}_\lambda(X_{\tau^\delta_x})e^{-\lambda\tau^\delta_x}$$

hence from the equation (1) this yields

$$\tilde{u}_\lambda(x) = E\int_0^{\tau^\delta_x} \{f(X_t) + c|a_0(X_t)|^q\}e^{-\lambda t}dt + E\tilde{u}_\lambda(X_{\tau^\delta_x})e^{-\lambda\tau^\delta_x}, \qquad \forall x \in \Omega_\delta. \tag{89}$$

In particular, we may deduce from this quantity

$$\left(\inf_{\partial\Omega_\delta} \tilde{u}_\lambda\right) E[e^{-\lambda\tau^\delta_x}] \leqq C, \qquad \forall x \in \Omega_\delta$$

for some C independent of δ.

Now, since $\tilde{u}_\lambda \to +\infty$ as $d \to 0_+$, $\left(\inf_{\partial\Omega_\delta} \tilde{u}_\lambda\right) \to +\infty$ as $\delta \to 0_+$, therefore $E[e^{-\lambda\tau^\delta_x}] \to 0$ as $\delta \to 0_+$. Hence $E[e^{-\lambda\tau_x}] = 0$ for all $x \in \Omega$ and this precisely means that $a_0 \in \mathscr{A}$.

In addition, we may also deduce from (89) that for δ small enough and for $x \in \Omega_\delta$

$$\tilde{u}_\lambda(x) \geqq E\int_0^{\tau^\delta_x} \{f(X_t) + c|a_0(X_t)|^q\}e^{-\lambda t}dt$$

since $\tilde{u}_\lambda \geqq 0$ for $x \in \Omega - \Omega_\delta$. Now, if we let δ go to and if we use the fact that $a_0 \in \mathscr{A}$ and thus $\tau_x^\delta \to +\infty$ a.s. as $\delta \to 0_+$ for all $x \in \Omega$, this yields

$$\tilde{u}_\lambda(x) \geqq J(x, a_0), \quad \forall x \in \Omega. \tag{90}$$

We next show that $\tilde{u}_\lambda \equiv u_\lambda$. If this is the case, (90) then implies that a_0 is optimal. To show that $\tilde{u}_\lambda \equiv u_\lambda$, we first recall from section that there exist w_n subsolutions of (1) in $W^{2,r}(\Omega)$ ($\forall r < \infty$) such that $w_n \underset{n}{\to} \tilde{u}_\lambda$ uniformly on compact subsets of Ω. Therefore, if $a \in \mathscr{A}$, we find using again Itô's rule for all $x \in \Omega_\delta$

$$w_n(x) \leqq E \int_0^{\tau_x^\delta} \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt + E[w_n(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}].$$

Now, if $J(x, a) = +\infty$, we obviously have $w_n(x) \leqq J(x, a)$, while if $J(x, a) < \infty$, we deduce from the above inequality letting δ go to 0_+

$$w_n \leqq E \int_0^\infty \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} dt = J(x, a)$$

since $\tau_x^\delta \to +\infty$ a.s. as $\delta \to 0_+$ and thus

$$|E[w_n(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}]| \leqq \sup_{\bar{\Omega}} |w_n| E[e^{-\lambda \tau_x^\delta}] \to 0 \quad \text{as} \quad \delta \to 0_+.$$

Therefore, letting n go to $+\infty$, we finally deduce for all $x \in \Omega$, $a \in \mathscr{A}$

$$\tilde{u}_\lambda(x) \leqq J(x, a)$$

and our claim is proved. □

The uniqueness of the optimal control is a bit technical but very simple to understand so we just sketch the argument: assume that a is optimal then applying Itô's rule we find for all $\delta > 0$, $x \in \Omega_\delta$

$$\begin{aligned} u_\lambda(x) &= E \int_0^{\tau_x^\delta} \{f(X_t) - a(X_t) \cdot \nabla u_\lambda(X_t) - |\nabla u_\lambda(X_t)|^p\} e^{-\lambda t} dt + E u_\lambda(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta} \\ &\leqq E \int_0^{\tau_x^\delta} \{f(X_t) + c|a(X_t)|^q\} e^{-\lambda t} + E u_\lambda(X_{\tau_x^\delta}) e^{-\lambda \tau_x^\delta}. \end{aligned}$$

But recalling that $u_\lambda(x) = J(x, a)$ for all $x \in \Omega$ and using the Markov property of X_t, we deduce that the above right-hand side is also equal to $u_\lambda(x)$. Therefore, the equality yields that for all $x \in \Omega_\delta$

$$a(X_t) = a_0(X_t) \quad \text{for all} \quad t \in (0, \tau_x^\delta) \quad \text{a.s.},$$

(where X_t is the solution corresponding to a) and letting δ go to 0_+ we finally find that for all $x \in \Omega$, $a(x) = a_0(x)$ (recall that a, a_0 are continuous on Ω). □

VII.2. Superquadratic Hamiltonians

Theorem VII.2. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below, satisfy* (41) *for some $\beta < p$ and let $p > 2$. Then, the value function u_λ given by* (88) *is the unique solution of* (1) *in $W^{2,r}_{\mathrm{loc}}(\Omega)$ ($\forall r < \infty$) satisfying* (62).

Proof. We will approximate the Hamiltonian $|\xi|^p$ as follows: let $R \geqq 1$, consider some Hamiltonian H_R such that H_R is convex on $\mathbb{R}^N$, $H_R(\xi)=|\xi|^p$ if $|\xi| \leqq R$, $H_R(\xi)|\xi|^{-\alpha}$ is constant for $|\xi|$ large where $\frac{\alpha}{\alpha-1}>\beta$ and $1<\alpha<2$, H_R increases uniformly on compact subsets to $|\xi|^p$ as R goes to $+\infty$. And we denote by $L_R(\eta)$ the following convex function

$$-L_R(\eta)=\inf_{\xi\in\mathbb{R}^N}(\eta\cdot\xi+H_R(\xi))$$

so that $L_R(\eta)\geqq c|\eta|^q$ and L_R decreases uniformly on compact subsets to $c|\eta|^q$.

Then, because $H_R=C_R|\xi|^\alpha$ for $|\xi|$ large, it is not difficult to adapt the results and the proofs of Sect. II to show that there exists a unique solution u_R of

$$-\Delta u_R+H_R(\nabla u_R)+\lambda u_R=f(x) \quad \text{in} \quad \Omega, \qquad u_R\in W^{2,r}_{\text{loc}}(\Omega) \quad (\forall r<\infty)$$

such that $u_R\to+\infty$ as $d\to 0_+$.

And exactly as in the preceding section, we can check that

$$u_R(x)=\inf_{a\in\mathscr{A}} E\int_0^\infty \{f(X_t)+L_R(a(X_t))\}e^{-\lambda t}dt, \qquad \forall x\in\Omega.$$

Of course, u_R decreases to the value function u_λ given by (88). On the other hand, we remark that we may choose $H_R\in C^2(\mathbb{R}^N)$ such that

$$|D^2H_R(\xi)|\,|\xi|^2, \qquad |DH_R(\xi)|\,|\xi|\leqq C_0(H_R(\xi)+1), \qquad \forall\xi\in\mathbb{R}^N,$$

and

$$H_R(\xi)\geqq|\xi|^\alpha, \qquad \forall\xi\in\mathbb{R}^N$$

for some C_0 independent of R. And we may adapt the a priori estimates in the appendix (see also part 2) to deduce that u_R is bounded in $W^{1,\infty}_{\text{loc}}$ and thus in $W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$. Hence, $u_\lambda\in W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ and solves (1). But then by Corollary IV.1, u_λ extends continuously to $\bar{\Omega}$ and we may apply Theorem V.2 to deduce that u_λ is the unique solution of (1) satisfying (62). □

The question of the optimality of the control $a_0=-p|\nabla u_\lambda|^{p-2}\nabla u_\lambda$ is much more delicate: in fact, if an optimal control exists, by a similar proof to the one made in the preceding section, it has to be a_0 and if we know that $a_0\in\mathscr{A}$ then a_0 is the optimal Markovian control. Hence, the main problem is whether $a_0\in\mathscr{A}$. We know how to prove that $a_0\in\mathscr{A}$ only when (61) (or some easy variants) holds and we refer the reader to Sect. IV.3 where a few cases when (61) holds are given. Indeed, if (61) holds then we deal with a diffusion X_t satisfying

$$dX_t=\sqrt{2}\,dB_t+a_0(X_t)dt,$$

where a_0 satisfies

$$a_0(x)d(x)\to-\mu n \quad \text{as} \quad d(x)\to 0_+, \tag{91}$$

with $\mu=q>1$. Then, we claim that for any diffusion process of the above form, if (91) holds and $\mu>1$, then X_t never leaves Ω with probability 1, while if (91) holds and $\mu<1$, X_t hots $\partial\Omega$ in finite time with propability 1. Indeed, if $\mu>1$, we apply Itô's rule with $-\log d(x)$ and we find for all $T<\infty$

$$-\log d(x)=E\left[-\log d(X_{\tau\wedge T})+\int_0^{\tau\wedge T}\left\{\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}-\frac{a\cdot\Delta d}{d}\right\}(X_s)ds\right]$$

(in fact we should replace τ by τ_δ for $\delta>0\ldots$). And we observe that

$$\frac{\Delta d}{d}-\frac{|\nabla d|^2}{d^2}-\frac{a\cdot\nabla d}{d}$$

behaves like $(\mu-1)\frac{1}{d^2}$ near $\partial\Omega$ and so this quantity is bounded from below on Ω. Hence, we obtain

$$E[-\log d(X_{\tau\wedge T})]\leqq CT-\log d(x)$$

therefore for all $x\in\Omega$, $P(\tau<T)=0$ and we conclude since this holds for all $T<\infty$.

On the other hand if $\mu<1$ by a simple argument, showing that $E[\tau_x]\leqq C$ for all $x\in\Omega$ is easily done if we prove the existence of a supersolution of

$$-\Delta z-a\cdot\nabla z\geqq\varepsilon \quad \text{in} \quad \Omega, \qquad \text{for some} \quad \varepsilon>0, \quad z\in C(\bar\Omega), \quad z=0 \quad \text{on} \quad \partial\Omega.$$

But this is achieved by considering for $\mu_0\in(\mu,1)$ the function

$$z_1=d^{1-\mu_0}(1-\mu_0)^{-1}-d^2\{2(\mu_0+1)\}^{-1}$$

which satisfies in $\Omega-\Omega_\delta$ for some δ small enough

$$-\Delta z_1-a\cdot\nabla z_1=\mu_0 d^{-\mu_0-1}-a\cdot\nabla d d^{-\mu_0}+\frac{1}{\mu_0+1}\{1+a\cdot\nabla d d\}$$
$$+\left\{\frac{1}{\mu_0+1}-d^{-\mu_0}\right\}\Delta d\geqq K>0$$

for some $K>0$. Then, we consider the solution z_2 of

$$-\Delta z_2-a\cdot\nabla z_2=1 \quad \text{in} \quad \Omega_\delta, \qquad z_2=0 \quad \text{on} \quad \partial\Omega_\delta.$$

Finally, we set $z=z_1$ in $\Omega-\Omega_\delta$, $=z_1|_{\partial\Omega_\delta}+\gamma z_2$ in $\bar\Omega_\delta$ where γ is small enough so that $\gamma\frac{\partial z_2}{\partial n}\geqq\frac{\partial z_1}{\partial n}$ on $\partial\Omega_\delta$. It is then easy to check that z satisfies the desired inequality with $\varepsilon=\min(K,\gamma)$.

VII.3. Forced Constraints

We first observe that by the results and methods of the preceding sections (and the interior estimates given in the Appendix), for any $p>1$ and for any $f\in L^\infty_{\text{loc}}(\Omega)$ bounded from below there exists a solution $\underline{u}\in W^{2,r}_{\text{loc}}(\Omega)$ $(\forall r<\infty)$ of (1) such that for all $v\in W^{2,r}(\Omega)$ $(\forall r<\infty)$ satisfying

$$-\Delta v+|\nabla v|^p+\lambda v\leqq f \quad \text{in} \quad \Omega \tag{92}$$

then $v \leqq \underline{u}$ in Ω. Of course, if $1 < p \leqq 2$ and $f(x) \leqq C d(x)^{-1}$ then (see Sect. II) $\underline{u} \to +\infty$ as $d \to 0_+$ and $\underline{u}$ is the minimum such solution, if $p > 2$ and $f(x) \leqq C d(x)^{-\beta}$ for some $\beta < p$ then $\underline{u}$ is the unique solution of (1) satisfying (62) (see Sects. IV and V), while if $f(x) \geqq c d(x)^{-\beta} - C$ for some $c > 0$, $\beta \geqq \max(p, q)$ then $\underline{u} \to +\infty$ as $d \to 0_+$ and $\underline{u}$ is the minimum solution of (1) bounded from below [and we have uniqueness if f behaves like $C_1 d(x)^{-\beta}$]. In fact, if $1 < p \leqq 2$, then $\underline{u} \to +\infty$ as $d \to 0_+$ and $\underline{u}$ is the minimum such solution.

We then have the following

Proposition VII.1. *Let $1 < p \leqq 2$, or let $p > 2$ and $f \geqq c d^{-\beta} - C$ for some $c > 0$, $C \geqq 0$, $\beta \geqq p$. Then, the value function u_λ given by* (88) *is the above ("minimum explosive") solution $\underline{u}$. In addition, $a_0(x) = -p|\nabla u|^{p-2}\nabla u$ is the unique optimal Markovian control.*

In fact, since $\underline{u} \to \infty$ as $d \to 0_+$, the proof is exactly the same as the proof of Theorem VI.1: one shows that $\underline{u} \geqq u_\lambda$ and $a_0 \in \mathscr{A}$, then $\underline{u} \leqq u_\lambda$ and a_0 is the unique optimal Markovian control.

Remark VII.1. These results show that for any f bounded from below the formula (88) yields a finite function (locally bounded) on Ω. This may be proved directly by a tedious probabilistic construction of a control $\bar{a} \in \mathscr{A}$ such that $J(x, \bar{a}) < \infty$ for all $x \in \Omega$.

VII.4. Ergodic Control

We now want to explain in this section the control problems associated with the asymptotic problems solved in Sect. VI. We begin with the cases when solutions go to $+\infty$ as $d(x)$ goes to 0_+.

Theorem VII.3. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (78), *let $1 < p \leqq 2$. We denote by (v, u_0) the solutions given by Theorem VI.1. Then, we have the following equalities: for any $a \in \mathscr{A}$, let θ_a be a stopping time bounded by some arbitrary $T \geqq 0$ (independent of a), then*

$$v(x) = \inf_{a \in \mathscr{A}} E \int_0^{\theta_a} \{f(X_t) + c|a(X_t)|^q\} dt + v(X_{\theta_a}) - \theta_a u_0\,, \quad \forall x \in \Omega\,, \tag{93}$$

$$u_0 = \lim_{T \to \infty} \inf_{a \in \mathscr{A}} E \frac{1}{T} \int_0^T \{f(X_t) + c|a(X_t)|^q\} dt\,, \quad \forall x \in \Omega \tag{94}$$

and the control $a_0 = -p|\nabla v|^{p-2}\nabla v$ belongs to $\mathscr{A}$ and is the unique optimal Markovian control where optimal means that (93)–(94) *are equalities when we choose $a = a_0$.*

Theorem VII.4. *Let $f \in L^\infty_{\mathrm{loc}}(\Omega)$ satisfy* (40) *and let $p > 1$. Denoting by (v, u_0) the solutions given by Theorem VI.2, Theorem VII.3 still holds.*

Since the proof of Theorem VII.4 is very similar to the one of Theorem VII.3 we will only prove the latter.

We first deduce from Itô's formula that if X_t^0 denotes the process corresponding to the choice a_0 then for all $\delta > 0$, $x \in \Omega$

$$v(x) = E \int_0^{\theta_0 \wedge \tau_x^\delta} \{f(X_t^0) + c|a_0(X_t^0)|^q\} dt + v(X^0_{\theta_0 \wedge \tau_x^\delta}) - \theta_0 \wedge \tau_x^\delta u_0\,, \tag{95}$$

where θ_0 stands for θ_{a_0} and τ_x^δ is the first exit time from Ω_δ. In particular for $\theta_0=T$, we deduce

$$E(v(X^0_{T\wedge\tau_x^\delta}))\leqq v(x)+CT, \quad \text{for some} \quad C\geqq 0.$$

Therefore, recalling that v is bounded from below, we obtain

$$\left(\inf_{\partial\Omega_\delta} v\right) P[\tau_x^\delta\leqq T]\leqq v(x)+C(1+T)$$

and since $v\to+\infty$ as $d\to 0_+$, we deduce that $a_0\in\mathscr{A}$.

In addition, if we pass to the limit in (95) as δ goes to 0_+, we find for all $x\in\Omega$

$$v(x)=E\int_0^{\theta_0}\{f(X_t^0)+c|a_0(X_t^0)|^q\}dt-\theta_0u_0+\lim_{\delta\to 0_+}E[v(X^0_{\theta_0\wedge\tau_x^\delta})]$$

and

$$\lim_{\delta\to 0_+}E[v(X^0_{\theta_0\wedge\tau_x^\delta})]\geqq\lim_{\delta\to 0_+}\{E[(v+C)(X^0_{\theta_0})\mathbb{1}_{\theta_0\leqq\tau_x^\delta}]-C\},$$

where $C\leqq\inf_\Omega v$, and this last expectation increases to $E[v(X^0_{\theta_0})]$. Hence, we finally obtain for all $x\in\Omega$

$$v(x)\geqq E\int_0^{\theta_0}\{f(X_t^0)+c|a_0(X_t^0)|^q\}dt+v(X^0_{\theta_0})-\theta_0u_0. \tag{96}$$

And taking $\theta_0=T$, we also deduce for all $x\in\Omega$

$$u_0\geqq\varlimsup_{T\to\infty}\frac{1}{T}E\int_0^T\{f(X_T^0)+c|a_0(X_t^0)|^q\}dt \tag{97}$$

since $E\frac{1}{T}v(X_T^0)\geqq-\frac{C}{T}\to 0$ as $T\to\infty$.

To complete the proof of Theorem VII.3, we basically need to prove the complementary inequalities in (93)–(94). This will be achieved by first introducing some approximated problem: let (v^δ, u_0^δ) be the solution in $W^{2,r}_{\text{loc}}(\Omega^\delta)\times\mathbb{R}$ $(\forall r<\infty)$ of (79) with Ω replaced by Ω^δ such that $v^\delta(x_0)=0$, $v^\delta\to+\infty$ as $d(x)\to 0_+$. With the techniques of Sect. VI one can show that $v^\delta\uparrow v$ as δ goes to 0_+ and converges uniformly on compact subsets of Ω, while $u_0^\delta\downarrow u_0$ as $\delta\downarrow 0_+$.

Using Itô's formula, we immediately obtain for all $x\in\Omega$

$$v^\delta(x)\leqq\inf_{a\in\mathscr{A}}E\int_0^{\theta_a}\{f(X_t)+c|a(X_t)|^q\}dt+v^\delta(X_{\theta_a})-\theta_a u_0^\delta$$

and letting δ go to 0_+ we deduce since $v^\delta\uparrow v$ as $\delta\downarrow 0_+$

$$v(x)\leqq\inf_{a\in\mathscr{A}}E\int_0^{\theta_a}\{f(X_t)+c|a(X_t)|^q\}dt+v(X_{\theta_a})-\theta_a u_0$$

and this combined with (96) yields (93). In addition taking $\theta_a=T$, we also deduce

$$u_0^\delta\leqq\inf_{a\in\mathscr{A}}E\frac{1}{T}\int_0^T\{f(X_t)+c|a(X_t)|^q\}dt+\frac{2}{T}\sup_{\bar\Omega}|v^\delta|$$