hence

$$u_0^\delta \leqq \varliminf_{T\to\infty} \inf_{a\in\mathscr{A}} E\frac{1}{T}\int_0^T \{f(X_t)+c|a(X_t)|^q\}dt$$

and letting δ go to 0, the resulting inequality combined with (97) yields (94). This also shows the optimality of a_0 and the uniqueness is easy to prove as in the preceding sections.

By the same truncation argument as in the proof of Theorem VII.2, one deduces the

Theorem VII.5. *Let $p>2$, let $f\in L^\infty_{\mathrm{loc}}(\Omega)$ be bounded from below and satisfy* (40). *We denote by (v,u_0) the solutions given by Theorem VI.2. Then, the identities* (93)–(94) *still hold.*

Appendix: On Some Local Gradient Bounds

We want to show here some local gradient bounds for solutions of

$$-\varepsilon\Delta u+|\nabla u|^p+\lambda u=f \quad \text{in} \quad \Omega, \qquad u\in W^{2,r}_{\mathrm{loc}}(\Omega) \quad (\forall r<\infty), \tag{A.1}$$

where $0<\varepsilon\leqq 1$, $0\leqq\lambda\leqq 1$, $1<p<\infty$, and $f\in L^\infty(\Omega)$ or even $f\in W^{1,\infty}(\Omega)$ and Ω is a bounded open set in $\mathbb{R}^n$. These bounds are obtained by the method introduced in [16, 19]. For related local bounds concerning different equations, we refer to Bombieri et al. [4], Ladyzhenskaya and Ural'tseva [13], Simon [26–28]. Our main result is the

Theorem A.1. *For any $\delta>0$, we set $\Omega_\delta=\{x\in\Omega/\mathrm{dist}(x,\partial\Omega)>\delta\}$.*

1) *Let $f\in W^{1,\infty}(\Omega)$, then we have for all $\delta>0$*

$$|\nabla u(x)|\leqq C_\delta \quad \text{if} \quad x\in\Omega_\delta, \tag{A.2}$$

where C_δ depends only on bounds on $|\nabla f|$, lower bounds on $\lambda u-f$, δ, and p.

2) *Let $f\in L^\infty(\Omega)$, then we have for all $r<\infty$, $\delta>0$*

$$\|\nabla u\|_{L^r(\Omega_\delta)}\leqq C_\delta, \tag{A.3}$$

where C_δ depends only on bounds on f, lower bounds on $f-\lambda u$, δ, p, and r.

Proof. We begin with case 1) i.e. when $f\in W^{1,\infty}(\Omega)$. In both cases, we will ignore the fact that u is not assumed to be smooth and we will thus skip the tedious approximation argument required to make the proof below complete. Then, let $\theta\in(0,1)$ to be determined later on and let $\varphi\in\mathscr{D}(\Omega)$, $0\leqq\varphi\leqq 1$ in Ω, $\varphi\equiv 1$ on Ω_δ, be such that

$$|\Delta\varphi|\leqq C\varphi^\theta, \qquad |\nabla\varphi|^2\leqq C\varphi^{1+\theta} \quad \text{in} \quad \Omega$$

for some C (depending only on δ, θ).

We next consider $w=|\nabla u|^2$ and we compute easily on $\mathrm{Supp}\,\varphi$

$$\left.\begin{aligned}&-\varepsilon\Delta(\varphi w)+p|\nabla u|^{p-2}\nabla u\cdot\nabla(\varphi w)+2\lambda\varphi w+2\varepsilon\varphi|D^2u|^2+2\varepsilon\frac{\nabla\varphi}{\varphi}\cdot\nabla(\varphi w)\\&=2\varphi\nabla f\cdot\nabla u+p|\nabla u|^{p-2}(\nabla u\cdot\nabla\varphi)w-\varepsilon(\nabla\varphi)w+2\varepsilon\frac{|\nabla\varphi|^2}{\varphi}w.\end{aligned}\right\} \tag{A.4}$$

Then, let $x_0 \in \Omega$ be a maximum point of φw: we may assume that $x_0 \in \operatorname{Supp}\varphi$ and by the classical maximum principle we deduce from (A.4) the following inequality where all functions are taken at x_0

$$2\varepsilon\varphi|D^2u|^2 \leqq C\varphi w^{1/2} + C\varphi^\theta w^{\frac{p+1}{2}} + C\varepsilon\varphi^\theta w\,. \tag{A.5}$$

Now, from Cauchy-Schwarz inequality and (A.1)

$$|D^2u|^2 \geqq \frac{1}{N}(\Delta u)^2 \geqq \frac{1}{N\varepsilon^2}(|\nabla u|^p + \lambda u - f)^2 \geqq \frac{1}{N\varepsilon^2}(|\nabla u|^p - C)^{+2}$$

and this combined with (A.5) yields

$$\varphi w^p \leqq C + C\varepsilon\varphi w^{1/2} + C\varepsilon\varphi^\theta w^{\frac{p+1}{2}} + C\varepsilon^2\varphi^\theta w\,. \tag{A.6}$$

Now, choosing $\theta \geqq \frac{3-p}{2}$, we deduce easily

$$\max_{\Omega} \varphi w = \varphi w(x_0) \leqq C\,.$$

In case 2), i.e. when $f \in L^\infty(\Omega)$ we use integral estimates as follows: let $m \geqq 1$, we multiply (A.4) by $(\varphi w)^m$ and we find

$$\begin{aligned}&\varepsilon m \int |\nabla(\varphi w)|^2(\varphi w)^{m-1}dx + p\int |\nabla u|^{p-2}\nabla u\cdot\nabla(\varphi w)(\varphi w)^m dx\\ &\quad + \varepsilon\int \varphi|D^2u|^2(\varphi w)^m dx + \frac{1}{N\varepsilon}\int \varphi(|\nabla u|^p - C)^{+2}\varphi^m w^m dx\\ &\quad + 2\varepsilon\int \varphi^{m-1}\nabla\varphi\cdot\nabla(\varphi^{m+1}w^{m+1})(m+1)^{-1}dx\\ &\leqq 2\int \varphi^{m+1}w^m\nabla f\cdot\nabla u\,dx + C_p\int w^{\frac{p+1}{2}}\varphi^{\theta m}w^m dx + C\varepsilon\int \varphi^{m+\theta}w^m dx\,.\end{aligned}$$

We now want to bound the following terms

$$\begin{aligned}&2\int \varphi^{m+1}w^m\nabla f\cdot\nabla u\,dx \leqq 2C\int \varphi^{m+1}w^m|D^2u|dx + m\int \varphi|\nabla(\varphi w)|(\varphi w)^{m+1}\\ &\quad\times|\nabla u|dx + C\int \varphi^{m+\theta}w^{m+1/2}dx\\ &\leqq \varepsilon\int \varphi^{m+1}w^m|D^2u|^2dx + \frac{C}{\varepsilon}\int \varphi^{m+1}w^m dx + \varepsilon\frac{m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1}dx\\ &\quad + C\frac{m}{\varepsilon}\int \varphi^{m+1}w^m dx + C\int \varphi^{m+\theta}w^{m+1/2}dx\,;\end{aligned}$$

$$\begin{aligned}&p\int |\nabla u|^{p-2}\nabla u\cdot\nabla(\varphi w)(\varphi w)^m dx \leqq \frac{\varepsilon m}{2}\int |\nabla(\varphi w)|^2(\varphi w)^{m-1}dx\\ &\quad + \frac{p^2}{\varepsilon m}\int \varphi^{m+1}w^{m+p}dx\,;\end{aligned}$$

$$2\varepsilon(m+1)^{-1}\int \varphi^{-1}\nabla\varphi\cdot\nabla(\varphi^{m+1}w^{m+1})dx \leqq \frac{C\varepsilon}{m+1}\int \varphi^{m+\theta}w^{m+1}dx\,.$$

And collecting all these bounds, we finally deduce

$$\frac{1}{N\varepsilon}\int \varphi(|\nabla u|^p - C)^{+2}\varphi^m w^m dx \leqq C_p \int w^{m+\frac{p+1}{2}}\varphi^{m+\theta}dx + C\varepsilon\int\varphi^{m+\theta}w^m dx$$

$$+C\int\varphi^{m+\theta}w^{m+1/2}dx + \frac{C}{\varepsilon}\int\varphi^{m+1}w^m dx + C\frac{m}{\varepsilon}\int\varphi^{m+1}w^m dx$$

$$+\frac{C\varepsilon}{m+1}\int\varphi^{m+\theta}w^{m+1}dx + \frac{p^2}{\varepsilon m}\int\varphi^{m+1}w^{m+p}dx.$$

To get rid of the last term, we choose m in $\left]\frac{p^2}{N},\infty\right[$ and we find

$$\int\varphi^{m+1}w^{m+p}dx \leqq C + C\int w^{m+\frac{p+1}{2}}\varphi^{m+\theta}dx.$$

And we conclude choosing $\theta \geqq (m+p)^{-1}\{(p+1)/2 + m(3-p)/2\}$.

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