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Continuous Fields of C^*-Algebras Coming from Group Cocycles and Actions

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Recently I have been attempting to formulate a suitable C^*-algebraic framework for the subject of deformation quantization [3, 19]. Continuous fields of C^*-algebras provide one of the key elements for this framework. The main examples of deformation quantizations which I have constructed up to now in this C^*-algebra framework come from letting either cocycles on groups, or actions of groups, vary. It has thus become necessary to show that one obtains in this way fields of C^*-algebras that are indeed continuous. Since this material is of a general nature, and can be useful in other situations [4–6, 11–13] it has seemed appropriate to give a separate exposition of it, in the present article.

Section 1 of this article contains a review of the published results on continuous fields which we will need, as well as a discussion of the fact that the approach which we will take involves treating upper and lower semi-continuity separately. In Sect. 2 we discuss the continuity of fields of C^*-algebras which arise from varying cocycles on groups, while in Sect. 3 we do the same for actions which vary.

1. Generalities About Continuous Fields

One slight novelty of our approach to continuous fields of C^*-algebras is that we view the continuity of fields as arising from the combination of upper and lower semi-continuity, and we observe that for various constructions these two forms of semi-continuity arise from fairly different considerations. Vaguely speaking, upper semi-continuity tends to arise from the universality of the constructions (as is evident in the extensive literature concerning upper semi-continuous fields, referenced in [9]); whereas lower semi-continuity tends to arise from the presence of natural fields of representations on Hilbert space.

In treating upper semi-continuity, we will find it very useful to use the relationship with disintegration over central subalgebras which has been widely discussed (see references in [9]). We now review briefly some of the facts which we

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need. For any C^*-algebra A we let $M(A)$ denote its multiplier algebra. For any locally compact space Ω we let $C_\infty(\Omega)$ denote the algebra of complex-valued continuous functions on Ω vanishing at infinity.

To phrase the results succinctly we make the following definition, motivated by terminology of Fell, as clarified by the second half of Proposition 1 of [18] and, for uper semi-continuous fields, by Proposition 2.3 of [9].

1.1. Definition. Let $\{A_\omega\}$ be a field of C^*-algebras over a locally compact space Ω, and let A be a C^*-algebra (for the supremum norm) of cross-sections of $\{A_\omega\}$. We will say that S is *maximal* if

1) The evaluation maps π_ω from A to each A_ω are surjective.

2) For each $a \in A$ the function $\omega \mapsto \|\pi_\omega(a)\|$ is upper semi-continuous, and vanishes at infinity on Ω.

3) The pointwise product of any element of A with any element of $C_\infty(\Omega)$ is again in A.

It follows from the results in Chap. 2 of [9], notably Proposition 2.3, that a maximal A determines a unique (upper semi-continuous) continuity structure on the field, and is in turn uniquely determined by this continuity structure. We will use the usual imprecise notation of referring to just $\{A_\omega\}$ as a (semi-) continuous field, where the continuity structure is to be inferred from the context. The next proposition is the slight extension of Theorem 2.4 of [9] from compact to locally compact Ω.

We include the brief proof, of the first part along lines due to Varela.

1.2. Proposition. *Let A be a C^*-algebra, and let C be a C^*-subalgebra of the center of $M(A)$. Let Ω denote the spectrum of C, so that we can view C as $C_\infty(\Omega)$. For each $\omega \in \Omega$ let I_ω denote the ideal of elements of C which vanish at ω, let $J_\omega = AI_\omega$ (closure of linear span), and let $A_\omega = A/J_\omega$. Let π_ω denote the homomorphism from A to A_ω. Then for any $a \in A$ the function*

$$\omega \mapsto \|\pi_\omega(a)\|$$

is upper semi-continuous, so that the field $\{A_\omega\}$ of C^*-algebras over Ω is upper semi-continuous. If A is an essential C-module, that is, $A = CA$ (closure of linear span), then A is identified with the maximal algebra of cross sections of $\{A_\omega\}$ by the map $a \mapsto \{\pi_\omega(a)\}$.

Proof. For given $a \in A$ let $\omega_0 \in \Omega$ and $\varepsilon > 0$ be given. Then by the definition of the quotient norm there is a finite sum $t = \sum f_i a_i$ with $f_i \in I_{\omega_0}$ and $a_i \in A$ such that

$$\|\pi_{\omega_0}(a)\| \geqq \|a - t\| - \varepsilon.$$

Choose $g \in C$ with $\|g\| \leqq 1$ such that $g \equiv 1$ in a neighborhood N of ω_0, but such that all the gf_i are small enough so that $\|gt\| < \varepsilon$. Then

$$\begin{aligned}\|\pi_{\omega_0}(a)\| &\geqq \|a-t\| - \varepsilon \geqq \|g(a-t)\| - \varepsilon \geqq \|ga\| - 2\varepsilon \\ &= \|a - (1-g)a\| - 2\varepsilon.\end{aligned}$$

Since $l - g$ is in I_ω for $\omega \in N$, it follows that

$$\|\pi_{\omega_0}(a)\| \geqq \|\pi_\omega(a)\| - 2\varepsilon$$

for $\omega \in N$, which shows the desired upper semi-continuity. The last statement of the proposition is clear from Theorem 2.4 of [9] if Ω is compact. But one reduces immediately to the compact case by adjoining the identity element of $M(A)$ to C (so forming the one-point compactification of Ω), and by noting that the condition $A = CA$ says exactly that the fibre over the point at infinity is $\{0\}$. Q.E.D.

We now give a converse to the above proposition which is a slightly specialized version of Proposition 1.2 of [9].

1.3. Proposition. *Let $\{A_\omega\}$ be a field of C^*-algebras over locally compact Ω, and let A be a maximal C^*-algebra of cross-sections. For each ω let J_ω denote the kernel of the evaluation map π_ω from A to A_ω. Then*

$$J_\omega = I_\omega A$$

(closure of linear span) where I_ω denotes the ideal of elements of $C_\infty(\Omega)$ which vanish at ω.

Proof. Let $\omega_0 \in \Omega$ and $a \in J_{\omega_0}$ be given. Since $\omega \mapsto \|\pi_\omega(a)\|$ is upper semi-continuous and $\pi_{\omega_0}(a) = 0$, there is, for any given $\varepsilon > 0$, a compact neighborhood N of ω_0 such that $\|\pi_\omega(a)\| < \varepsilon$ for $\omega \in N$. Let f be an element of I_{ω_0} taking values between 0 and 1 such that $f(\omega) = 1$ whenever $\|\pi_\omega(a)\| \geqq \varepsilon$ (which is possible since a vanishes at infinity). Let g be an element of $C_\infty(\Omega)$ taking values between 0 and $1/\varepsilon$ which on the union of N with the set where $f(\omega) \geqq \varepsilon$ agrees with $(\sup(f, \varepsilon))^{-1}$, so that $0 \leqq fg \leqq 1$, and $(fg)(\omega) = 1$ if $f(\omega) \geqq \varepsilon$. By hypothesis, $ga \in A$. But a quick calculation shows that

$$\|a - f(ga)\| \leqq \varepsilon .$$

Thus we have approximated a within ε by an element of $I_\omega A$ as desired. Q.E.D.

In contrast to the situation for upper semi-continuity, there appears to be little general discussion in the literature of structures which result in lower semi-continuity of fields of C^*-algebras; and indeed a precise formulation appears difficult to find. But, in vague terms, one can view the approach which we will use as follows. Consider again a subalgebra $C = C_\infty(\Omega)$ in the center of $M(A)$, and assume that $A = CA$. If we take a faithful representation, ϱ, of A, then ϱ extends to $M(A)$ and so to C. We can now try to disintegrate ϱ over C into a field $\{\varrho_\omega\}$ of representations of A. Under favorable circumstances, one can hope that the field $\{\varrho_\omega(A)\}$ of C^*-algebras will give a decomposition of A. Examples show that in this situation the continuity which one is likely to obtain is lower semi-continuity, that is, for each $a \in A$ the function $\omega \mapsto \|\varrho_\omega(a)\|$ will be lower semi-continuous.

The following example illustrates well the above ideas, and suggests the phenomena which may arise also in the non-commutative case.

1.4. Example. Let Ω denote the interval $[-1, 1]$ of the real line R, let S be the interval $[1, \infty)$, and let $M = \Omega \times S$. Let $A = C_b(M)$, the algebra of continuous bounded functions on M. We wish to examine how A can be expressed as a field of C^*-algebras over Ω. Let $C = C(\Omega)$, viewed as consisting of functions on M constant in the S direction. Thus $C \subset M(A) = A$. Following the method of Propositions 1.2 and 1.3, we form I_ω and $A_\omega = A/I_\omega A$, which gives a field which is upper semi-continuous. Let ϕ be the function in A which has value 0 for $\omega \leqq 0$, has value 1 if both $\omega > 0$ and $s \geqq 1/\omega$, and for each fixed $s > 0$ interpolates linearly for ω between 0 and $1/s$. Thus ϕ has value 0 everywhere on $\{0\} \times S$, but for any $\omega > 0$ there are values of s such that $\phi(\omega, s) = 1$. Then it is easily seen that $\omega \mapsto \|\pi_\omega(\phi)\|$ is not lower semi-continuous at $\omega = 0$. On the other hand, A has the evident faithful representation on $L^2(M)$, where planar Lebesgue measure is used on M. This representation can be disintegrated in a very natural way by the representations ϱ_ω on $L^2(S)$ defined by

$$(\varrho_\omega(\psi)\xi)(s) = \psi(\omega, s)\xi(s)$$

for $\psi \in A$ and $\xi \in L^2(S)$. It is easily seen that $\omega \mapsto \|\varrho_\omega(\psi)\|$ is lower semi-continuous, but for the special function ϕ defined above, $\omega \mapsto \|\varrho_\omega(\phi)\|$ is not upper semi-continuous at $\omega = 0$.

2. Fields from Cocycles

We will now discuss fields of C^*-algebras which arise when one varies two-cocycles on a locally compact group G, as this will be useful in [27]. The direction of my investigation of this topic was originally motivated by Theorem 1 of [2]. We will actually consider a slightly more general situation than needed for [27], namely that in which there is a fixed action, α, of G on a C^*-algebra A, and the cocycles take values in the group $UZM(A)$ of unitary elements of the center of the multiplier algebra, $M(A)$, of A. This extra generality is useful in treating certain C^*-algebras which have arisen recently in solid state physics [4], and in particular in the quantum Hall effect. (See [5] and Theorem 12 of [6], as well as [30]; also see [11, 13] for a different direction.)

For the case in which G is discrete, the corresponding twisted group C^*-algebras are defined and studied in thorough detail in [31]. On the other hand, the literature concerning the case in which G is not discrete is in a somewhat less satisfactory state. There is much discussion of the twisted L^1-algebras under various hypotheses. (See [23] and [28] and the references they give.) A treatment of twisted crossed product C^*-algebras has very recently been given for the separable case in [24], but I am not aware of any treatment of the twisted C^*-algebras in the general case. It seems fairly clear that all could be recast into Fell's elegant framework of C^*-algebraic bundles [14]. But this would require an extensive discussion, which is not appropriate in the present paper, both because the extra generality is not needed for our immediate purposes in [27], and because in applications one is usually presented with cocycles rather than a bundle, and the coyclces usually have sufficient continuity to make it apparent that all works smoothly (and the connection with Fell's theory is known by [8] and [20]). Consequently, we will here be somewhat cavalier about these technicalities, and will work as though we know that all works smoothly at the C^*-algebra level. However, for ease of exposition in dealing with questions of measurability, we will assume that G, if it is not discrete, is second countable. Again, with more work this hypothesis could probably be substantially eliminated by using the techniques in [16] and [20]. Frequently we will only discuss explicitly the second countable case, letting the reader check [31] for the general discrete case. Finally, let us mention that some treatments (see [20] and its references) do not even require the cocycles to be center-valued, but [31] does require this, so we restrict to this case.

We will throughout let Ω be a locally compact space which will serve as the base space for our fields, and also at this point as the parameter space for cocycles. We let $C_\infty(\Omega, A)$ denote the C^*-algebra of continuous A-valued functions on Ω vanishing at infinity. The following definition is more-or-less that made in Sect. 1 of [22], if we view the cocycle σ as having values in $UZM(C_\infty(\Omega, A))$.

2.1. Definition. By a *continuous field over Ω of α-cocycles* of G we will mean a function σ on $G \times G \times \Omega$ with values in $UZM(A)$ such that

1) If we fix $\omega \in \Omega$ then σ is a normalized α-cocycle on G, that is,

$$\sigma(x, yz; \omega)\alpha_x(\sigma(y, z; \omega)) = \sigma(xy, z; \omega)\sigma(x, y; \omega)$$

for x, y, $z \in G$ (where α here is the extension of α to $M(A)$), and

$$\sigma(x, e; \omega) = 1 = \sigma(e, x; \omega)$$

where e denotes the identity element of G.

2) If we fix x, $y \in G$, then σ is continuous on Ω.

3) For any $f \in C_\infty(\Omega, A)$ the function

$$(x, y) \mapsto f(\cdot)\sigma(x, y; \cdot)$$

from $G \times G$ to $C_\infty(\Omega, A)$ is Bochner measurable.

An example which is important for [27], consists of letting A be the complex numbers, letting $G = \mathbb{R} \times \mathbb{R}$ and $\Omega = \mathbb{R}$, and defining σ by

$$\sigma((r, s), (t, u); \omega) = \exp(i\omega(ru - st)).$$

Returning to the general case, we recall (Theorem 2.2 of [8]) that the corresponding twisted L^1-algebra is defined as follows. Choose a left Haar measure on G. For Φ, $\Psi \in L^1(G, C_\infty(\Omega, A))$ one defines the twisted convolution by

$$(\Phi * \Psi)(x) = \int \Phi(y)\alpha_y(\Psi(y^{-1}x))\sigma(y, y^{-1}x; \cdot)dy$$

and the involution by

$$\Phi^*(x) = \alpha_x(\Phi(x^{-1})^*)\sigma(x, x^{-1}; \cdot)^*\Delta(x^{-1}),$$

where Δ is the modular function of G. We will denote the resulting Banach $*$-algebra by $L^1(G, \Omega, A, \sigma)$, leaving the action α to be understood.

For any $\omega \in \Omega$ let I_ω denote the ideal in $C_\infty(\Omega)$ consisting of functions vanishing at ω, and let J_ω denote the corresponding ideal of $C_\infty(\Omega, A)$. Then α can be viewed as giving an action on J_ω, and σ can be viewed as having values on $UZM(J_\omega)$, so that exactly as above we can form $L^1(G, J_\omega, \sigma)$, which is a closed ideal of $L^1(G, \Omega, A, \sigma)$. Finally, let σ_ω denote $\sigma(\cdot, \cdot; \omega)$, a cocycle with values in $UZM(A)$, so that we can define $L^1(G, A, \sigma_\omega)$. There is an evident evaluation homomorphism from $L^1(G, \Omega, A, \sigma)$ to $L^1(G, A, \sigma_\omega)$ whose kernel is exactly $L^1(G, J_\omega, \sigma)$. From Satz 1 of [21] one sees that $L^1(G, A, \sigma_\omega)$ has the corresponding quotient norm. Thus from the exact sequence

$$0 \to J_\omega \to C_\infty(\Omega, A) \to A \to 0$$

we obtain, in the strongest sense, the exact sequence

$$0 \to L^1(G, J_\omega, \sigma) \to L^1(G, \Omega, A, \sigma) \to L^1(G, A, \sigma_\omega) \to 0,$$

for each fixed $\omega \in \Omega$.

We now define $C^*(G, \Omega, A, \sigma)$ to be the enveloping C^*-algebra of $L^1(G, \Omega, A, \sigma)$, and in the same way we define $C^*(G, J_\omega, \sigma)$ and $C^*(G, A, \sigma_\omega)$. Now from 2.29 of [31] we have:

2.2. Proposition. *Let B be a Banach $*$-algebra possessing a bounded approximate identity, and let J be a closed $*$-ideal (2-sided) of B also possessing a bounded approximate identity. Then, letting $C^*(\ )$ denote enveloping C^*-algebras, we obtain the short exact sequence of C^*-algebras*

$$0 \to C^*(J) \to C^*(B) \to C^*(B/J) \to 0.$$

However, the situation as to whether every $L^1(G, \Omega, A, \sigma)$ has a bounded approximate identity is somewhat murky. According to the discussion in the first paragraph on p. 311 of [28], it can be shown that if the action α is trivial this is always so. But it would be nice to have a proof of this in print, for one might well then be able to see that the proof also works for non-trivial α. (But careful – in the notation of [28] α is the cocycle, non-trivial.) At any rate, it seems highly likely that almost all specific examples of significant interest will be smooth at the identity as defined in Definition 2 of [7], and in this case the combination of Theorem 1 of [7] with the results of [14], notably Proposition 8.2, shows that approximate identities exist. (Alternately, [20] can be combined with [14]. See also Remark 2.6 of [24].) So we will now proceed as if this were always so. Thus from Proposition 2.2 we obtain:

2.3. Proposition. *With notation and caveats as above, the sequence*

$$0 \to C^*(G, J_\omega, \sigma) \to C^*(G, \Omega, A, \sigma) \to C^*(G, A, \sigma_\omega) \to 0$$

is exact for every $\omega \in \Omega$.

Now it is clear that $C_\infty(\Omega)$ acts as an algebra of (central) multipliers on $L^1(G, \Omega, A, \sigma)$, and that

$$L^1(G, J_\omega, \sigma) = L^1(G, \Omega, A, \sigma) I_\omega.$$

It is easily seen from this that $C_\infty(\Omega)$ also acts as an algebra of central multipliers of $C^*(G, \Omega, A, \sigma)$, and that

$$C^*(G, J_\omega, \sigma) = C^*(G, \Omega, A, \sigma) I_\omega.$$

We thus find that we are exactly in the right position to apply Proposition 1.2 to obtain:

2.4. Theorem. *With notation and caveats as above, let σ be a continuous field over Ω of α-cocycles on the discrete, or second countable locally compact, group G. Then the field of C^*-algebras $\{C^*(G, A, \sigma_\omega)\}$ over Ω, with the continuity structure coming from $C^*(G, \Omega, A, \sigma)$, is upper semi-continuous.*

We now turn to the question of lower semi-continuity. With notation as above, pick any faithful representation of A on a Hilbert space H, and any positive Borel measure on Ω of full support, and consider the corresponding faithful representation, p, of $C_\infty(\Omega, A)$ on $L^2(\Omega, H)$. We can then induce (see [23] or [8]) this to a representation of $L^1(G, \Omega, A, \sigma)$. The corresponding C^*-norm and completion give, by definition, the reduced C^*-algebra $C_r^*(G, \Omega, A, \sigma)$. Again, $C_\infty(\Omega)$ acts as an algebra of multipliers, and there is a natural choice of disintegration of the representation with respect to $C_\infty(\Omega)$ given by composing the evaluation homomorphism π_ω with the representations of $L^1(G, A, \sigma_\omega)$ on $L^2(G, H)$ obtained by inducing the representation of A on H. More specifically, define a twisted covariant pair (U, μ) of representations of $(G, C_\infty(\Omega, A))$ on $L^2(G, L^2(\Omega, H))$ by

$$(U_y\eta)(x) = p(\sigma(x^{-1}, y))\eta(y^{-1}x)$$
$$(\mu(a)\eta)(x) = p(\alpha_x^{-1}(a))\eta(x)$$

for $x, y \in G$ and $a \in A$. It is easily verified that (U, μ) satisfies Definition 2.1 of [8], aside from questions of measurability in the non-separable case. Consequently, by Theorem 3.3 of [8] the integrated form, ϱ, of (U, μ), defined by

$$\varrho(\Phi)\eta = \int \mu(\Phi(y))U_y\eta\, dy$$

for $\Phi \in L^1(G, \Omega, A, \sigma)$, defines a *-representation of $L^1(G, \Omega, A, \sigma)$, the induced representation. We let ϱ_ω denote the corresponding representation of $L^1(G, \Omega, A, \sigma)$ or $C_r^*(G, \Omega, A, \sigma)$ on $L^2(G, H)$ obtained by the same formulas as above except evaluating at ω. It is clear that the range of ϱ_ω on $C_r^*(G, \Omega, A, \sigma)$ will be essentially $C_r^*(G, A, \sigma_\omega)$, so that we are considering the field $\{C_r^*(G, A, \sigma_\omega)\}$. At this juncture the measure originally chosen on Ω is seen to be irrelevant – we only introduced it to put matters into the general framework sketched earlier.

2.5. Theorem. *With notation and caveats as above, let σ be a continuous field over Ω of α-cocycles on the discrete, or second countable locally compact, group G. Then the field of C^*-algebras $\{C_r^*(G, A, \sigma_\omega)\}$ over Ω, with the continuity structure coming from $C_r^*(G, \Omega, A, \sigma)$, is lower semi-continuous.*

Proof. We show first that for any $\Phi \in L^1(G, \Omega, A, \sigma)$, the function $\omega \mapsto \varrho_\omega(\Phi)$ is continuous for the strong operator topology from $L^2(G, H)$. By the usual 3ε argument, it suffices to show this for $\Phi \in L_c^\infty(G, \Omega, A)$, the space of bounded measurable functions of compact support on G with values in $C_\infty(\Omega, A)$, since it is dense in $L^1(G, \Omega, A, \sigma)$. Let $\xi \in L^2(G, H)$, and let $\omega_0 \in \Omega$. For fixed $x \in G$, as ω converges to ω_0 the function

$$y \mapsto \mu(\Phi(y, \omega))p(\sigma(x^{-1}, y, \omega))\xi(y^{-1}x)$$

converges pointwise to the corresponding function for ω_0, and is dominated by the function

$$y \mapsto \|\Phi(y)\|_\infty \, \|\xi(y^{-1}x)\|$$

in $L^2(G)$. Thus, by the dominated convergence theorem, $(\varrho_\omega(\Phi)\xi)(x)$ converges to $(\varrho_{\omega_0}(\Phi))\,\xi(x)$ for each $x \in G$. But these functions are in turn dominated by the function

$$\|\Phi(\cdot)\|_\infty * \|\xi\|,$$

which is in $L^2(G)$, where $*$ denotes here ordinary convolution on G. Thus, again by the dominated convergence theorem, $\varrho_\omega(\Phi)\xi$ converges in $L^2(G, H)$ to $\varrho_{\omega_0}(\Phi)\xi$. Thus

$\varrho_\omega(\Phi)$ converges to $\varrho_{\omega_0}(\Phi)$ in the strong operator topology. But it is easily seen that this strong operator topology convergence implies that $\omega \mapsto \|\varrho_\omega(\phi)\|$ is lower semi-continuous. Q.E.D.

Suppose now that G is amenable. For G discrete it is proven in 5.1 of [31] that reduced and full twisted crossed product C^*-algebras then coincide, while in Theorem 7.7.7 of [25] this is shown for crossed product algebras with arbitrary G. By combining the techniques used in the proofs of these two cases, it seems very likely that one can extend this result to various varieties of more general twisted group C^*-algebras. But since this matter is not of central importance to the present paper, and since there are many other situations where the reduced and full twisted crossed product C^*-algebras coincide, we will content ourselves with simply formulating:

2.6. The Amenability Hypothesis. *We will say that a collection $\{G, A, \sigma, \alpha\}$ consisting of an action α of a locally compact group G on a C^*-algebra A together with a corresponding cocycle σ, satisfies the amenability hypothesis if $C_r^*(G, A, \alpha, \sigma) = C^*(G, A, \alpha, \sigma)$.*

Combining this with Theorems 2.4 and 2.5, we obtain:

2.7. Corollary. *With notation and caveats as earlier, assume that each $\{G, A, \sigma_\omega\}$ satisfies the amenability hypothesis. Then the field $\{C^*(G, A, \sigma_\omega)\}$ over Ω is continuous.*

When G is discrete and $A = \mathbb{C}$ (so α is trivial) one has the following universal formulation of the above results. Let $\Gamma = \Gamma_G$ be the set of normalized cocycles on G with values in the set T of complex numbers of modulus one. Equip Γ with the topology of pointwise convergence. Since the conditions defining a normalized cocycle are closed conditions, Γ is compact by Tychonoff's theorem. Then the tautological field $\gamma \mapsto \gamma$ is a continuous field over Γ of cocycles on G. Combining Theorems 2.4 and 2.5 with Corollary 2.7 and the fact that, according to [31], the amenability hypothesis is always true for amenable discrete groups, one obtains:

2.8. Corollary. *Let G be discrete, and let Γ be the compact space of all normalized T-valued cocycles on G. Then*

a) *$\{C^*(G, \gamma)\}$ is an upper semi-continuous field on Γ.*

b) *$\{C_r^*(G, \gamma)\}$ is a lower semi-continuous field on Γ.*

c) *If G is amenable, then $\{C^*(G, \gamma)\} = \{C_r^*(G, \gamma)\}$ is a continuous field over Γ.*

It is clear that the above corollary remains true if Γ is replaced by any closed subset of Γ. Because the set of skew bicharacters of an Abelian discrete group is such a closed subset, one obtains immediately from the above corollary a proof of the statement in [12] that non-commutative tori form a continuous field (and a proof of the special case of this statement applying to irrational and rational rotation C^*-algebras, used in [13], which is also handled by [2]).

Actually, it seems reasonable to me to conjecture that $\{C_r^*(G, \gamma)\}$ is a continuous field even when G is not amenable (and a corresponding fact for non-discrete G). But an attempt to imitate for $\{C_r^*(G, \gamma)\}$ the proof of Theorem 2.4 founders on the fact that it seems not to be known whether the exact sequence used in that proof is also exact for the reduced algebras. Antony Wassermann pointed out to me that the results of [17] show that in a closely related situation such exactness actually does fail. Specifically, if G is the free group on two generators, then $C^*(G)$ is not an exact C^*-algebra, in the sense that there are exact sequences

$$0 \to I \to A \to Q \to 0$$

of C^*-algebras for which

$$0 \to I \underset{\min}{\otimes} C^*(G) \to A \underset{\min}{\otimes} C^*(G) \to Q \underset{\min}{\otimes} C^*(G) \to 0$$

is not exact. But for this to happen A must be non-commutative (and it deals with the *full* group C^*-algebra), while in the conjecture indicated above the relevant algebra $C(\Gamma)$ is commutative. We also mention that no example seems to be known of a discrete group G whose *reduced* algebra $C_r^*(G)$ is not exact, although it is likely that such examples exist. If G is again the free group on two generators, then Proposition 5.3 of [10] and the discussion following show that $C_r^*(G)$ is exact. (But see Proposition 4.5 of [32].) In general, it is, of course, also interesting to ask if $\{C^*(G, \gamma)\}$ also might always be continuous.

3. Fields from Actions

We now turn to similar questions for crossed product algebras [25], where now we let the action vary. There is some overlap here with the results in [29], but the techniques are quite different.

3.1. Definition. By an *upper semi-continuous field of actions on C^*-algebras* for a locally compact group G we mean an upper semi-continuous field $\{A_\omega\}$ of C^*-algebras over a locally compact space Ω, with maximal C^*-algebra of sections A, and evaluation maps π_ω, together with an action α of G on A which carries the kernel K_ω of each π_ω into itself, and so induces an action on each A_ω. If the field $\{A_\omega\}$ is actually continuous, then we will say that the field of actions is continuous.

3.2. Theorem. *Let α be an upper semi-continuous field of actions of G on the field $\{A_\omega\}$ of C^*-algebras. For each $\Phi \in L^1(G, A)$ and each $\omega \in \Omega$, let $\|\Phi\|_\omega$ denote the norm of the image of Φ in the crossed product algebra $C^*(G, A_\omega; \alpha)$. Then the function*

$$\omega \mapsto \|\Phi\|_\omega$$

is upper semi-continuous, so that $\{C^*(G, A_\omega; \alpha)\}$ is an upper semi-continuous field of C^*-algebras over Ω. Furthermore, $C^*(G, A, \alpha)$ is identified with this field's maximal C^*-algebra of sections.

Proof. Much as in the proof of Proposition 2.3, it follows from Proposition 2.2 together with Satz 1 of [21] that the sequence

$$0 \to C^*(G, K_\omega; \alpha) \to C^*(G, A; \alpha) \to C^*(G, A_\omega; \alpha) \to 0$$

is exact. As before, for $\omega \in \Omega$ let I_ω denote the ideal in $C_\infty(\Omega)$ of functions vanishing at ω. By Proposition 1.3 we have $K_\omega = I_\omega A$. Now it is clear from the hypotheses that the elements of $C_\infty(\Omega) \subseteq M(A)$ are invariant under α. It follows easily that

$$C^*(G, K_\omega; \alpha) = I_\omega C^*(G, A; \alpha),$$

so that

$$C^*(G, A_\omega; \alpha) = C^*(G, A; \alpha)/I_\omega C^*(G, A; \alpha).$$

But then the upper semi-continuity and the identification of $C^*(G, A, \alpha)$ as the maximal C^*-algebra of sections follow from Proposition 1.2. Q.E.D.

We need a corresponding lower semi-continuity result. I have not seen how to obtain this without imposing stronger hypotheses on the field involved. These can be motivated by observing that a key feature in the proof of Theorem 2.5 was that all the fiber algebras ended up acting on the same fixed Hilbert space, so that the strong operator topology could be employed.

3.3. Definition. Let $\{A_\omega\}$ be an upper semi-continuous field of C^*-algebras over Ω, with A its maximal C^*-algebra of sections. We will say that the field is *Hilbert continuous* if there is a fixed Hilbert space H and, for each $\omega \in \Omega$, a faithful representation σ_ω of A_ω on H (which thus also defines a representation of A) such that for all $a \in A$ the function

$$\omega \mapsto \sigma_\omega(\pi_\omega(a))$$

is continuous for the strong operator topology.

Since the norm on operators is lower semi-continuous with respect to the strong operator topology, it follows that a Hilbert-continuous field is, in fact, continuous (since the σ_ω are faithful).

3.4. Question. How can one characterize those continuous fields of C^*-algebras which are actually Hilbert-continuous?

In preparation for the last part of the next theorem, we remark that there are many situations where $C^*(G, A, \alpha) = C_r^*(G, A, \alpha)$ even when G is not amenable. (See Theorem 4.5 and Proposition 4.8 of [1].)

3.5. Theorem. *Let Ω, $\{A_\omega\}$, A, K_ω, G and α be as in Theorem 3.2, except assume in addition that the field is Hilbert-continuous. For each $\phi \in C_c(G, A)$ and each $\omega \in \Omega$ let $\|\phi\|_\omega^r$ denote the norm of the image of ϕ in $C_r^*(G, A_\omega, \alpha)$. Then*

a) *For each $\phi \in C_c(G, A_\omega, \alpha)$ the function $\omega \mapsto \|\phi\|_\omega^r$ is lower semi-continuous.*

b) *Suppose that $C_r^*(G, A_\omega, \alpha) = C^*(G, A_\omega, \omega)$ for all ω (for instance if G is amenable). Then the function*

$$\omega \mapsto \|f\|_\omega = \|f\|_\omega^r$$

is continuous, $C_r^*(G, A, \alpha) = C^*(G, A, \alpha)$, and the field $\{C^*(G, A_\omega, \alpha)\}$, which is again a Hilbert-continuous field, has $C^*(G, A, \alpha)$ as its maximal C^*-algebra of sections.

Proof. For each $\omega \in \Omega$ let $C_\omega = C_r^*(G, A_\omega, \alpha)$. Let H and $\{\sigma_\omega\}$ be as in the definition of a Hilbert-continuous field. For each $\omega \in \Omega$ let ϱ_ω be the representation of C_ω on $L^2(G, H)$ obtained by inducing the representation σ_ω to C_ω, so that, essentially by definition, $\|\phi\|_\omega^r = \|\varrho_\omega(\phi)\|$ for ϕ in $C_c(G, A)$. We will show that for any ϕ in $C_c(G, A)$ the function $\omega \mapsto \varrho_\omega(\phi)$ is strong operator continuous. Now for any $\xi \in L^2(G, H)$ we have, by definition (see [25]),

$$\varrho_\omega(\phi)\xi = \int_G \tilde{\sigma}_\omega(\phi(y)) L_y \xi \, dy,$$

where L denotes the operator of left translation by y, and where for $a \in A$ we define $\tilde{\sigma}_\omega(a)$ by

$$(\tilde{\sigma}_\omega(a)\xi)(x) = \sigma_\omega(\alpha_x^{-1}(a))\xi(x).$$

But

$$\|\tilde{\sigma}(\phi(y))L_y\xi\| \leqq \|\phi(y)\| \, \|\xi\|,$$

and the right-hand side is integrable since ϕ is continuous of compact support. Furthermore, it is easily seen by using the Lebesgue dominated convergence theorem that for each fixed y the integrands converge as ω goes to ω_0, by the assumption on the σ_ω's. Hence, again by the Lebesgue dominated convergence theorem, $\varrho_\omega(\phi)\xi$ converges to $\varrho_{\omega_0}(\phi)\xi$ as ω goes to ω_0. This shows the strong operator continuity of the representations ϱ_ω as functions of ω. But from this, the lower semi-continuity of $\|\varrho_\omega(\phi)\|$ for any given ϕ follows immediately. Now σ_ω is a faithful representation of A_ω, by assumption, and so ϱ_ω is a faithful representation of C_ω, essentially by the definition of the reduced crossed product [25]. It follows that $\|\phi\|_\omega$ is a lower semi-continuous function of ω for each ϕ, which proves a).

If $C_r^*(G, A_\omega, \alpha) = C^*(G, A_\omega, \alpha)$ for each ω, then, of course, $\|\phi\|_\omega^r = \|\phi\|_\omega$ for each ω, so that these are both continuous. Furthermore, the proof of part a) given above provides the fixed Hilbert space $L^2(G, H)$ and the representations ϱ_ω which play the role of the new σ_ω's to give the Hilbert-continuity of the field. We next show that $C_r^*(G, A, \alpha) = C^*(G, A, \alpha)$. Now $C_r^*(G, A, \alpha)$ is a quotient of $C^*(G, A, \alpha)$. Thus it suffices to show that every irreducible representation of $C^*(G, A, \alpha)$ factors through $C_r^*(G, A, \alpha)$. So let σ be an irreducible representation of $C^*(G, A, \alpha)$. Since $C_\infty(\Omega)$ is in the center of the multiplier algebra $M(A)$ of A and is α-invariant, it is in the center of $M(C^*(G, A, \alpha))$. The extension of σ to $M(C^*(G, A, \alpha))$ must carry $C_\infty(\Omega)$ to scalar multiples of the identity operator since σ is irreducible. Thus σ on $C_\infty(\Omega)$ corresponds to evaluation at some point, say ω of Ω. Let I_ω denote, as in the proof of Theorem 2.4, the ideal of functions in $C_\infty(\Omega)$ vanishing at ω. Then, as in the proof of that theorem, the kernel K_ω of the homomorphism of A onto A_ω is $I_\omega A$. Clearly $I_\omega A$ is in the kernel of σ (extended to $M(C^*(G, A, \alpha))$), and so $C^*(G, K_\omega, \alpha)$ is in the kernel of σ. Thus σ can be considered to be a representation of the quotient, that is, of $C^*(G, A_\omega, \alpha)$. But by assumption $C^*(G, A_\omega, \alpha) = C_r^*(G, A_\omega, \alpha)$. Now $C_r^*(G, A_\omega, \alpha)$ is a quotient of $C_r^*(G, A, \alpha)$, even if the corresponding kernel is in general hard to describe. Thus σ gives a representation of $C_r^*(G, A, \alpha)$. But it is easily seen that on the dense subalgebra $C_c(G, A)$, this σ, as representation of $C_r^*(G, A, \alpha)$, agrees with the original σ, so that the original σ factors through $C_r^*(G, A, \alpha)$ as desired. Thus $C_r^*(G, A, \alpha) = C^*(G, A, \alpha)$. Finally, the last part of Theorem 3.2 applies to show that $C^*(G, A, \alpha)$ is exactly the maximal C^*-algebra of sections of the field $\{C^*(G, A_\omega, \alpha)\}$. Q.E.D.

3.6. Corollary. *Let Ω be a locally compact space, and for each $\omega \in \Omega$ let α^ω be an action of the locally compact group G on a fixed C^*-algebra B, such that for each $x \in G$ and $b \in B$ the function $\omega \mapsto \alpha_x^\omega(b)$ is norm continuous on Ω. For each $\phi \in C_c(G, B)$ let $\|\phi\|_\omega$ and $\|\phi\|_\omega^r$ denote the norms of ϕ in $C^*(G, B, \alpha^\omega)$ and $C_r^*(G, B, \alpha^\omega)$ respectively.*

a) *For each* ϕ *the function* $\omega \mapsto \|\phi\|_\omega$ *is upper semi-continuous.*

b) *For each* ϕ *the function* $\omega \mapsto \|\phi\|_\omega^r$ *is lower semi-continuous.*

c) *Hence if* $C_r^*(G, B, \alpha^\omega) = C^*(G, B, \alpha^\omega)$ *for all* ω *(for instance if* G *is amenable), then the function* $\omega \mapsto \|\phi\|_\omega = \|\phi\|_\omega^r$ *is continuous. Furthermore, in this case* $\{C^*(G, B, \alpha^\omega)\}$ *is a Hilbert-continuous field of* C^**-algebras, and if* $A = C_\infty(\Omega, B)$*, then* $\{\alpha^\omega\}$ *defines an action* α *of* G *on* A *such that* $C_r^*(G, A, \alpha) = C^*(G, A, \alpha)$*, and* $C^*(G, A, \alpha)$ *can be identified with the maximal* C^**-algebra for* $\{C^*(G, B, \alpha^\omega)\}$.

Proof. Of course, the action α on A is defined by

$$(\alpha_x(f))(\omega) = \alpha_x^\omega(f(\omega))$$

for $f \in A$. A simple argument using the hypotheses on $\{\alpha^\omega\}$ shows that $\alpha_x(f) \in A$. A straightforward uniform continuity argument on compact subsets of Ω shows that α is strongly continuous, so defines an action of G on A. For each $\omega \in \Omega$ evaluation at ω gives a homomorphism of A onto B which carries α to α^ω. Thus we are exactly in the setting of Theorem 3.2, so that part a) follows immediately from Theorem 3.2.

We are dealing here essentially with the constant field over Ω with fiber B. This is clearly a Hilbert-continuous field. Thus, parts a) and b) of Theorem 3.4 immediately imply parts b) and c) here. Q.E.D.

We remark that Lemma 2 of [11] is an immediate consequence of the above corollary.

There seems little doubt that there is a joint generalization of the two main results of this section to the case in which one has cocycles and actions both of which are varying. Since we have no present use in mind for such a result, it seemed best to wait until a significant application arises before treating this situation, but presumably the proof can be modeled on the proofs given above.

As indicated in the introduction, interesting applications of the results of this section will be given in [27]. A relation between the results of this section and proper actions will be given in [26].

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