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Common Fixed Points of Commuting Holomorphic Maps

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0. Introduction

In 1964, Shields proved the following theorem:

Theorem 0.1 (Shields [S]). *Let Δ be the unit disk in $\mathbf{C}$ and $\mathscr{F}$ a family of holomorphic functions of Δ into itself, continuous up to the boundary of Δ. Assume that $f \circ g = g \circ f$ for every $f, g \in \mathscr{F}$. Then there exists a point $\tau \in \bar{\Delta}$ such that $f(\tau) = \tau$ for every $f \in \mathscr{F}$.*

In other words, a *commuting family* $\mathscr{F}$ of holomorphic functions (i.e., such that $f \circ g = g \circ f$ for every $f, g \in \mathscr{F}$) continuous up to the boundary always admits a *common fixed point* (i.e., a point τ such that $f(\tau) = \tau$ for all $f \in \mathscr{F}$) in $\bar{\Delta}$. This result is a feature of the holomorphic structure, and not some sort of consequence of Brouwer's theorem: indeed, there are examples of commuting continuous functions mapping the closed unit interval $[-1, 1] \subset \mathbf{R}$ into itself without common fixed points (see Boyce [Bo] and Huneke [Hu]).

The first generalization of Shields' result to several complex variables is due to Eustice [E], who in 1972 proved the same fact for holomorphic maps of the bidisk $\Delta^2 = \Delta \times \Delta \subset \mathbf{C}^2$. Shortly later, Suffridge [Su] in 1974 found a proof of the same result for the unit ball B^n of $\mathbf{C}^n$ (actually, Suffidge constructed a common fixed point for two commuting maps, but his proof can be adapted to a generic family).

Shields' proof of Theorem 0.1 relies on the main fact of iteration theory of holomorphic functions of Δ into itself, the Wolff-Denjoy theorem:

Theorem 0.2 (Wolff [W 1, 2, 3], Denjoy [D]). *Let $f: \Delta \to \Delta$ be a holomorphic function, $f \neq \mathrm{id}_\Delta$. Assume that f is not an automorphism of Δ with exactly one fixed point. Then the sequence $\{f^k\}$ of iterates of f converges, uniformly on compact sets, to a constant function $\tau \in \bar{\Delta}$.*

In their proofs, Eustice and Suffridge used some weak version of the iteration theory on the bidisk and the ball, respectively, together with Shields' theorem to build up an induction argument; in particular, in Suffridge's proof the fact that the fixed point set of a holomorphic map of B^n into itself is biholomorphic to a ball of smaller dimension plays a fundamental rôle. Furthermore, it should be remarked that the whole strength of the iteration theory in the ball (due to Hervé [H]) yields an easier proof of Suffridge's result.

Recently ([A 1, 2]), the iteration theory has been developed in strongly convex domains, and the aim of this paper is to use it to extend Shields' theorem to this situation. Actually, a common fixed point for two commuting maps was already constructed in [A 1]; here we shall prove the complete generalization of Theorem 0.1. As in Eustice's and Suffridge's proofs, we shall use in a fundamental way the structure of the fixed point set of a holomorphic map, as described by Vigué [Vi 1, 2]. Unfortunately, it is not clear whether the fixed point set of a holomorphic map of a convex domain into itself is biholomorphic to a convex domain of smaller dimension; we only know how to describe it by means of complex geodesics, a concept introduced by Vesentini [V 1, 2] and mainly studied by Lempert [L 1, 2] and Royden and Wong [RW]. So, the first section of this paper is devoted to a review of some facts about complex geodesics, probably already known but lacking in suitable bibliographical references. For the sake of simplicity, we shall prove several results for strongly convex domains with C^3 boundary, but they probably hold in domains with C^2 boundary too. The proof of the main theorem is the bulk of the second section of the paper, and it was already announced in [A 3].

We shall denote by $\mathrm{Hol}(M, N)$ the space of holomorphic maps from the complex manifold M into the complex manifold N, endowed with the compact-open topology; by $\|\cdot\|$ the usual euclidean norm on $\mathbf{C}^n$, and by ω the Poincaré distance on Δ. For every $z, w \in \mathbf{C}^n$ we set

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j w_j;$$

in particular, $\langle z, \bar{w}\rangle$ is the usual hermitian product on $\mathbf{C}^n$. Finally, if $D \subset\subset \mathbf{C}^n$ has at least C^2 boundary, we shall denote by $\mathbf{n}_x$ the exterior unit normal to D at $x \in \partial D$.

I would like to thank László Lempert for an illuminating conversation regarding Proposition 1.8.

1. Complex Geodesics

Let $D \subset\subset \mathbf{C}^n$ be a bounded domain in $\mathbf{C}^n$. For every $z \in D$, we identify the tangent space $T_z D$ with $\mathbf{C}^n$, as usual. The *Kobayashi metric* $\kappa_D: TD \to \mathbf{R}^+$ is defined by

$$\forall z \in D \quad \forall v \in T_z D \quad \kappa_D(z; v) = \inf\{|\xi| \mid \exists \varphi \in \mathrm{Hol}(\Delta, D): \varphi(0) = z,\ \varphi'(0) = v\}.$$

The *Kobayashi distance* $k_D: D \times D \to \mathbf{R}^+$ on D is the integrated form of κ_D:

$$\forall z_0, z_1 \in D \quad k_D(z_0, z_1) = \inf\left\{\int_0^1 \kappa_D(\gamma(t); \gamma'(t)) \,\middle|\, \gamma \text{ is a } C^1 \text{ curve in } D \text{ connecting } z_0 \text{ and } z_1\right\}.$$

For general properties of the Kobayashi metric and distance consult [K 1, 2] and references therein.

Lempert [L 1] has shown that if D is convex then the Kobayashi distance is given by

$$\forall z_0, z_1 \in D \quad k_D(z_0, z_1) = \inf\{\omega(0, \zeta) \mid \exists \varphi \in \mathrm{Hol}(\Delta, D) : \varphi(0) = z_0,\ \varphi(\zeta) = z_1\}.$$

A *complex geodesic* φ is a holomorphic map from Δ into D which is an isometry for the Poincaré distance on Δ and the Kobayashi distance on D, i.e.,

$$\forall \zeta_1, \zeta_2 \in \Delta \quad k_D(\varphi(\zeta_1), \varphi(\zeta_2)) = \omega(\zeta_1, \zeta_2).$$

Remark that if a complex geodesic φ extends continuously to $\partial\Delta$, then $\varphi(\partial\Delta) \subset \partial D$.

A *geodesic disk* is the image of a complex geodesic. Vesentini [V 2] has shown that a complex geodesic is determined up to an automorphism of Δ by its image. We shall say that a complex geodesic φ is *passing through* two given points $z_0, z_1 \in D$ if z_0 and z_1 belong to the image of φ; it is clear that, up to an automorphism of Δ, we can assume $\varphi(0) = z_0$ and $\varphi(\zeta_0) = z_1$ for some $\zeta_0 \in (0, 1)$. Analogously, we shall say that a complex geodesic φ is *tangent* to a direction $v \in \mathbf{C}^n$ at the point $z_0 \in D$ if $\varphi(0) = z_0$ and $\varphi'(0) = \lambda v$ for some $\lambda > 0$.

The main facts about complex geodesics in convex domains are summarized in:

Theorem 1.1. *Let $D \subset\subset \mathbf{C}^n$ be a bounded convex domain. Then:*

(i) (Lempert [L 1], Royden and Wong [RW]) *For any two points $z_0, z_1 \in D$ there exists a complex geodesic passing through z_0 and z_1;*

(ii) (Lempert [L 1], Royden and Wong [RW]) *For any point $z_0 \in D$ and any direction $v \in \mathbf{C}^n$ there exists a complex geodesic tangent to v at z_0;*

(iii) (Lempert [L 1]) *If D is strongly convex with C^2 boundary, every complex geodesic extends to a $C^{0,1/2}$ map of $\bar{\Delta}$ into $\bar{D}$;*

(iv) (Lempert [L 1]) *More generally, if D is strongly convex with C^r boundary ($r = 3, \ldots, \infty$), every complex geodesic extends to a C^{r-2} map of $\bar{\Delta}$ into $\bar{D}$;*

(v) (Vesentini [V 2], Royden and Wong [RW]) *A holomorphic map $\varphi : \Delta \to D$ is a complex geodesic iff there are $\zeta_0, \zeta_1 \in \Delta$ such that $k_D(\varphi(\zeta_0), \varphi(\zeta_1)) = \omega(\zeta_0, \zeta_1)$;*

(vi) (Vesentini [V 2], Royden and Wong [RW]) *A holomorphic map $\varphi : \Delta \to D$ is a complex geodesic iff $\kappa_D(\varphi(0); \varphi'(0)) = 1$.*

Lempert gave the following characterization of complex geodesics in convex domains with C^2 boundary:

Theorem 1.2 (Lempert [L 1]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^2 domain. Then a holomorphic map $\varphi : \Delta \to D$ is a complex geodesic iff it extends to a $C^{0,1/2}$ map of $\bar{\Delta}$ into $\bar{D}$ and there exists a continuous function $p : \partial\Delta \to \mathbf{R}^+$ such that the map $\zeta \mapsto \zeta p(\zeta)\overline{\mathbf{n}_{\varphi(\zeta)}}$ defined on $\partial\Delta$ extends continuously to a holomorphic map $\tilde{\varphi} : \Delta \to \mathbf{C}^n$. Furthermore, p can be chosen so that on Δ we have*

$$\langle \varphi', \tilde{\varphi} \rangle \equiv 1. \tag{1.1}$$

A holomorphic map $\varrho : D \to D$ is a *holomorphic retraction* if $\varrho^2 = \varrho$. In particular, $\varrho(D)$ is a complex submanifold of D (see Rossi [R] or Cartan [C]), and coincides with the set of fixed points of ϱ. There is a strong connection between complex geodesics and holomorphic retractions:

Theorem 1.3 (Lempert [L2]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^r domain, ($r=2,3,\ldots,\infty$), and $\varphi \in \mathrm{Hol}(\Delta, D)$ a complex geodesic. Then there exists a holomorphic function $F: D \to \Delta$ such that*

$$\forall z \in D \quad \langle z-\varphi(F(z)), \tilde{\varphi}(F(z))\rangle = 0\,. \tag{1.2}$$

In particular, $F \circ \varphi = \mathrm{id}_\Delta$, and $\varphi \circ F$ is a holomorphic retraction of D onto $\varphi(\Delta)$. Furthermore, F extends C^{r-2} to ∂D.

The function F is called the *left inverse* of the complex geodesic φ.

The rest of this section is devoted to the study of the uniqueness of complex geodesics. First of all, in strongly convex domains there exists a unique geodesic disk passing through two given points:

Proposition 1.4 (Lempert [L1], Royden and Wong [RW]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^2 domain. Then*

(i) *For any two distinct points z_0, $z_1 \in D$ there exists a unique geodesic disk passing through z_0 and z_1;*

(ii) *For any point $z_0 \in D$ and direction $v \in \mathbf{C}^n$, $v \neq 0$, there exists a unique geodesic disk tangent to v at z_0.*

Proof. Since the proof of (ii) is quite similar to the proof of (i), we shall describe in detail the latter only. The existence is Theorem 1.1 (i). Assume now that $\varphi_0, \varphi_1: \Delta \to D$ are two complex geodesics passing through z_0 and z_1; up to automorphisms of Δ we can assume that $\varphi_0(0)=\varphi_1(0)=z_0$ and $\varphi_0(\zeta_0)=\varphi_1(\zeta_0)=z_1$ for some $\zeta_0 \in \Delta$. For $\lambda \in [0,1]$ set $\varphi_\lambda=(1-\lambda)\varphi_0+\lambda\varphi_1$. Clearly, every φ_λ is a holomorphic map of Δ into D; moreover, $\varphi_\lambda(0)=z_0$ and $\varphi_\lambda(\zeta_0)=z_1$ for every $\lambda \in [0,1]$. Then, by Theorem 1.1 (v), every φ_λ is a complex geodesic; in particular, $\varphi_\lambda(\partial\Delta) \subset \partial D$ for every $\lambda \in [0,1]$. But D is strongly convex; hence $\varphi_\lambda|_{\partial\Delta}$ does not depend on λ. In particular, $\varphi_0|_{\partial\Delta}=\varphi_1|_{\partial\Delta}$ and, since bounded holomorphic maps are completely determined by their boundary values, $\varphi_0 \equiv \varphi_1$. q.e.d.

Given $z_0, z_1 \in D$, the unique complex geodesic φ such that $\varphi(0)=z_0$ and $\varphi(\zeta_0)=z_1$ for some $\zeta_0 \in (0,1)$ will be called the *normalized complex geodesic* passing through z_0 and z_1.

Now we want to discuss the uniqueness of complex geodesics passing through given points in the boundary of D. To do so, we shall need the concept of angular derivative. Let f be a holomorphic map of Δ into itself, with radial limit $\tau \in \bar{\Delta}$ at $\sigma \in \partial\Delta$; then the *angular derivative* of f at σ is defined by

$$f'(\sigma)=\lim_{t\to 1}\frac{\tau-f(t\sigma)}{(1-t)\sigma}\,. \tag{1.3}$$

It turns out (see for instance Burckel [Bu]) that the limit (1.3) always exists (and it can be $+\infty$); moreover, if $\tau=\sigma$ then $f'(\sigma) \in (0,+\infty]$ and, if it is finite, it coincides with the non-tangential limit of f' at σ. Conversely, if f extends C^1 to $\bar{\Delta}$, then the angular derivative is always finite and it is clearly given by the value of f' at σ.

The facts we need about the angular derivative are summarized in:

Theorem 1.5. *Let $f: \Delta \to \Delta$ be holomorphic. Then*

(i) (Herzich [He]) *If $f(0)=0$ and the radial limit of f at 1 is 1, then $f'(1) \geqq 1$. Moreover, $f'(1)=1$ iff $f=\mathrm{id}_\Delta$.*

(ii) (Behan [B]) *If the radial limit of f at* 1 *is* 1, *and at* -1 *is* -1, *then* $f'(1)f'(-1)\geqq 1$. *Moreover,* $f'(1)f'(-1)=1$ *iff* f *is an automorphism of* Δ *fixing* 1 *and* -1.

To compute angular derivatives, we shall use the

Lemma 1.6. *Let* $D\subset\subset \mathbf{C}^n$ *be a strongly convex* C^3 *domain,* $x\in\partial D$ *and* $\varphi:\Delta\to D$ *a complex geodesic such that* $\varphi(1)=x$. *Denote by* $F:D\to\Delta$ *the left inverse of* φ. *Then*

$$\forall v\in\mathbf{C}^n \quad dF_x(v)=\frac{\langle v,\overline{\mathbf{n}_x}\rangle}{\langle\varphi'(1),\overline{\mathbf{n}_x}\rangle}=\langle v,\tilde{\varphi}(1)\rangle. \tag{1.4}$$

Proof. Since for any $\zeta\in\Delta$ we have $dF_{\varphi(z)}(\varphi'(\zeta))=1$ and

$$\ker dF_{\varphi(\zeta)}=\{v\in\mathbf{C}^n|\langle v,\tilde{\varphi}(\zeta)\rangle=0\},$$

it is clear that [by (1.1)]

$$dF_{\varphi(\zeta)}(v)=\langle v,\tilde{\varphi}(\zeta)\rangle.$$

Then letting $\zeta\to 1$ and recalling Theorem 1.2, we get (1.4). q.e.d.

Now we can prove

Proposition 1.7. *Let* $D\subset\subset\mathbf{C}^n$ *be a strongly convex* C^3 *domain. Then for any* $z_0\in D$ *and* $x\in\partial D$ *there exists a unique complex geodesic* $\varphi:\Delta\to D$ *such that* $\varphi(0)=z_0$ *and* $\varphi(1)=x$.

Proof. First of all the existence. Let $\{z_k\}\subset D$ be a sequence converging to x; denote by φ_k the normalized complex geodesic passing through z_0 and z_k. Since D is taut, up to a subsequence we can assume that $\{\varphi_k\}$ converges to a holomorphic map $\varphi:\Delta\to D$. Clearly $\varphi(0)=z_0$; moreover for all $\zeta\in\Delta$

$$k_D(z_0,\varphi(\zeta))=\lim_{k\to\infty}k_D(z_0,\varphi_k(\zeta))=\omega(0,\zeta),$$

and φ is a complex geodesic. Then it extends $C^{0,1/2}$ to ∂D, and clearly $\varphi(1)=x$.

Assume now that ψ is another complex geodesic with $\psi(0)=z_0$ and $\psi(1)=x$. Denote by F (respectively G) the left inverse of φ (respectively ψ). We claim that

$$F\circ\psi=\mathrm{id}_\Delta. \tag{1.5}$$

In fact, by Lemma 1.6

$$(F\circ\psi)'(1)=dF_x(\psi'(1))=\frac{\langle\psi'(1),\overline{\mathbf{n}_x}\rangle}{\langle\varphi'(1),\overline{\mathbf{n}_x}\rangle}.$$

Analogously,

$$(G\circ\varphi)'(1)=\frac{\langle\varphi'(1),\overline{\mathbf{n}_x}\rangle}{\langle\psi'(1),\overline{\mathbf{n}_x}\rangle}=\frac{1}{(F\circ\psi)'(1)}.$$

Then, by Theorem 1.5 (i), $(F\circ\psi)'(1)=(G\circ\varphi)'(1)=1$ and therefore (1.5) is proven.

Now, (1.5) means that $\langle\psi-\varphi,\tilde{\varphi}\rangle\equiv 0$ on $\bar{\Delta}$. In particular,

$$\langle\psi(\sigma)-\varphi(\sigma),\overline{\mathbf{n}_{\varphi(\sigma)}}\rangle=0$$

for every $\sigma \in \partial\Delta$. Since D is strongly convex, this implies that $\psi \equiv \varphi$ on $\partial\Delta$, and hence everywhere. q.e.d.

Using a similar argument, we can prove the uniqueness of the geodesic disk passing through two given boundary points:

Proposition 1.8. *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain. Then for any pair of distinct points $x_1, x_2 \in \partial D$ there exists a unique (up to automorphisms of Δ) complex geodesic φ such that $\varphi(1) = x_1$ and $\varphi(-1) = x_2$.*

Proof. We begin with the existence. Let $\{z_k\} \subset D$ be a sequence converging to x_2, and denote by φ_k a complex geodesic such that $\varphi_k(1) = x_1$, $z_k \in \varphi_k((-1, 1))$ and

$$\|\varphi_k(0) - x_1\| < \frac{\|x_2 - x_1\|}{2}. \tag{1.6}$$

Since D is bounded, up to a subsequence we can assume that $\{\varphi_k\}$ converges to a holomorphic map $\varphi : \Delta \to \mathbf{C}^n$. Since D is strongly convex, either $\varphi(\Delta) \subset D$ or φ is a constant contained in ∂D. The last possibility cannot occur [by (1.6)]; so $\varphi \in \mathrm{Hol}(\Delta, D)$, and it is clear that φ is as desired.

Assume now that ψ is another complex geodesic with $\psi(1) = x_1$ and $\psi(-1) = x_2$, and denote again by F (respectively G) the left inverse of φ (respectively ψ). We claim that this time

$$F \circ \psi \in \mathrm{Aut}(\Delta). \tag{1.7}$$

Indeed, by Lemma 1.6

$$(F \circ \psi)'(1) \cdot (F \circ \psi)'(-1) = \frac{\langle \psi'(1), \overline{\mathbf{n}_{x_1}} \rangle}{\langle \varphi'(1), \overline{\mathbf{n}_{x_1}} \rangle} \cdot \frac{\langle \psi'(-1), \overline{\mathbf{n}_{x_2}} \rangle}{\langle \varphi'(-1), \overline{\mathbf{n}_{x_2}} \rangle}.$$

For the same reason,

$$(G \circ \varphi)'(1) \cdot (G \circ \varphi)'(-1) = \frac{1}{(F \circ \psi)'(1) \cdot (F \circ \psi)'(-1)}.$$

Then Theorem 1.5 (ii) yields

$$(F \circ \psi)'(1) \cdot (F \circ \psi)'(-1) = 1,$$

and then (1.7). Hence, up to compose ψ with an automorphism of Δ, we can assume that $F \circ \psi = \mathrm{id}_\Delta$, and the assertion follows as in the proof of Proposition 1.7. q.e.d.

Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain: for every $z_0 \in D$ and $w_0 \in \bar{D}$, with $z_0 \neq w_0$, let φ_{z_0, w_0} denote the unique complex geodesic such that

$$\varphi_{z_0, w_0}(0) = z_0 \quad \text{and} \quad \varphi_{z_0, w_0}(\tanh[k_D(z_0, w_0)]) = w_0,$$

where $\tanh[k_D(z_0, w_0)] = 1$ if $w_0 \in \partial D$.

Lemma 1.9. *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain. Then the map $(z_0, w_0) \mapsto \varphi_{z_0, w_0}$ from $D \times \bar{D}$ minus the diagonal to $\mathrm{Hol}(\Delta, D) \cap C^0(\bar{\Delta}, \bar{D})$ is continuous.*

Proof. Let $\{(z_k, w_k)\}$ be a sequence in $D \times \bar{D}$ (with $z_k \neq w_k$ for all $k \in \mathbf{N}$) converging to $(z_0, w_0) \in D \times \bar{D}$ with $z_0 \neq w_0$; we have to show that $\varphi_k = \varphi_{z_k, w_k}$ tends to φ_{z_0, w_0}. Since

D is taut and $\varphi_k(0)=z_k \to z_0 \in D$, it suffices to show that the only limit point of $\{\varphi_k\}$ is φ_{z_0,w_0}.

Let $\{\varphi_{k_\nu}\}$ be a subsequence of $\{\varphi_k\}$ converging to a map φ. Since D is taut, $\varphi \in \mathrm{Hol}(\Delta, D)$ is a complex geodesic with $\varphi(0)=z_0$. Put

$$s_k = \tanh(k_D(z_k, w_k));$$

obviously, $s_k \to s_0 = \tanh(k_D(z_0, w_0))$. Therefore

$$\varphi(s_0) = \lim_{\nu\to\infty} \varphi_{k_\nu}(s_{k_\nu}) = \lim_{\nu\to\infty} w_{k_\nu} = w_0,$$

and φ must be φ_{z_0,w_0}. q.e.d.

Now we can define the representation of D onto the unit ball B^n of $\mathbf{C}^n$ introduced by Lempert [L1]. Fix $z_0 \in D$, and define $\Phi_{z_0}: \bar{D} \to \overline{B^n}$ by setting $\Phi_{z_0}(z_0)=0$ and

$$\Phi_{z_0}(z) = \tanh(k_D(z_0, z)) \frac{\varphi'_{z_0,z}(0)}{\|\varphi'_{z_0,z}(0)\|}. \tag{1.8}$$

In other words, $\Phi_{z_0}(z)$ is the point w of B^n such that $w/\|w\|$ is parallel to $\varphi'_{z_0,z}$ and $k_{B^n}(0,w)=k_D(z_0,z)$. Then

Proposition 1.10. *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and $z_0 \in D$. Then the map $\Phi_{z_0}: \bar{D} \to \overline{B^n}$ defined in* (1.8) *is a homeomorphism of $\bar{D}$ with $\overline{B^n}$ such that*

(i) *$\Phi_{z_0}(z_0)=0$ and $\Phi_{z_0}(\partial D)=(\partial B^n)$;*

(ii) *for any $z \in D$ we have $k_D(z_0,z)=\omega(0, \|\Phi_{z_0}(z)\|)$;*

(iii) *for any $z \in \bar{D}\backslash\{z_0\}$ we have*

$$\varphi_{z_0,z}(\zeta) = \Phi_{z_0}^{-1}\left(\zeta \frac{\Phi_{z_0}(z)}{\|\Phi_{z_0}(z)\|}\right).$$

Furthermore, Φ_{z_0} depends continuously on z_0.

Proof. Since every $\varphi_{z_0,z}$ is holomorphic, by Lemma 1.9 $\varphi'_{z_0,z}(0)$ depends continuously on z_0 and z, and so Φ_{z_0} is continuous and depends continuously on z_0. Since $\bar{D}$ is compact, it suffices to show that Φ_{z_0} is bijective, and to verify (iii).

Φ_{z_0} is injective: assume $\Phi_{z_0}(z_1)=\Phi_{z_0}(z_2)$. Then $k_D(z_0,z_1)=k_D(z_0,z_2)$ and

$$\frac{\varphi'_{z_0,z_1}(0)}{\|\varphi'_{z_0,z_1}(0)\|} = \frac{\varphi'_{z_0,z_2}(0)}{\|\varphi'_{z_0,z_2}(0)\|}.$$

Now, φ_{z_0,z_1} and φ_{z_0,z_2} are complex geodesics; hence, by Theorem 1.1 (vi),

$$\kappa_D(z_0; \varphi'_{z_0,z_1}(0)) = 1 = \kappa_D(z_0; \varphi'_{z_0,z_2}(0)),$$

and so $\|\varphi'_{z_0,z_1}(0)\| = \|\varphi'_{z_0,z_2}(0)\|$. Thus $\varphi'_{z_0,z_1}(0)=\varphi'_{z_0,z_2}(0)$; hence, by Proposition 1.4 (ii), $\varphi_{z_0,z_1}=\varphi_{z_0,z_2}$ and then $z_1=z_2$.

Φ_{z_0} is surjective: let $w=rx \in \overline{B^n}$, where $r \in (0,1]$ and $x \in \partial B^n$. Choose a complex geodesic $\varphi \in \mathrm{Hol}(\Delta, D)$ such that $\varphi(0)=z_0$ and $\varphi'(0)=\lambda x$ for some $\lambda>0$. Clearly, $\varphi=\varphi_{z_0,z}$ for some $z \in \bar{D}$; we can also assume $\tanh(k_D(z_0,z))=r$. Then

$$\Phi_{z_0}(z) = \tanh(k_D(z_0,z)) \cdot \frac{\lambda x}{\lambda} = w.$$

Finally, we are left to show that

$$\Phi_{z_0}(\varphi_{z_0,z}(\zeta))=\zeta\frac{\Phi_{z_0}(z)}{\|\Phi_{z_0}(z)\|}$$

for all $\zeta\in\Delta^*$. Fix $\zeta_0\in\Delta^*$; by definition,

$$\tanh[k_D(z_0,\varphi_{z_0,z}(\zeta_0))]=|\zeta_0|$$

and

$$\frac{\Phi_{z_0}(z)}{\|\Phi_{z_0}(z)\|}=\frac{\varphi'_{z_0,z}(0)}{\|\varphi'_{z_0,z}(0)\|}.$$

Now let $\tau=\zeta_0/|\zeta_0|\in\partial\Delta$, and define $\psi\in\mathrm{Hol}(\Delta,D)$ by $\psi(\zeta)=\varphi_{z_0,z}(\tau\zeta)$. ψ is a complex geodesic such that $\psi(0)=z_0$ and $\psi(|\zeta_0|)=\varphi_{z_0,z}(\zeta_0)$; therefore

$$\Phi_{z_0}(\varphi_{z_0,z}(\zeta_0))=|\zeta_0|\frac{\psi'(0)}{\|\psi'(0)\|}=\zeta_0\frac{\varphi'_{z_0,z}(0)}{\|\varphi'_{z_0,z}(0)\|},$$

and we are done. q.e.d.

It should be remarked that Lempert [L 1], using different methods, proved that if D has C^r boundary ($r=4,\ldots,\infty$) then Φ_{z_0} is a C^{r-3}-diffeomorphism of $\bar{D}\setminus\{z_0\}$ onto $\bar{B}^n\setminus\{0\}$, depending C^{r-3} on z_0.

2. Common Fixed Points

We can now move toward the main theorem of this paper; we are left to recall few facts about iteration theory and the structure of fixed point sets. If f is a holomorphic map of a domain D into itself, we shall denote by $\mathrm{Fix}(f)$ the set of fixed points of f in D. Remark that, by definition, $\mathrm{Fix}(f)$ is always contained in D, even if f is continuous up to the boundary of D and has fixed points there. Moreover, the set of fixed points of f in $\bar{D}$ is in general strictly greater of the closure in $\bar{D}$ of $\mathrm{Fix}(f)$ [consider for instance the map $f: B^2\to B^2$ given by $f(z,w)=(z^3,w)$].

The structure of $\mathrm{Fix}(f)$ in convex domains is quite well understood:

Theorem 2.1 (Vigué [Vi 1, 2]). *Let $D\subset\subset\mathbf{C}^n$ be a bounded convex domain, and $f\in\mathrm{Hol}(D,D)$. Then:*

(i) $\mathrm{Fix}(f)$ *is a (possibly empty) closed connected submanifold of D;*

(ii) *for any pair of distinct points $z_1,z_2\in\mathrm{Fix}(f)$ there exists a geodesic disk passing through z_1 and z_2 contained in* $\mathrm{Fix}(f)$.

It should be remarked that if z_1 and z_2 are two distinct points in the topological closure $\overline{\mathrm{Fix}(f)}\subset\bar{D}$ of $\mathrm{Fix}(f)$, then again we can find a geodesic disk passing through z_1 and z_2; it suffices to use the construction of geodesic disks passing through given boundary points described in the proofs of Propositions 1.7 and 1.8. Combining this with the results of the previous section we get:

Corollary 2.2. *Let $D\subset\subset\mathbf{C}^n$ be a strongly convex C^2 domain, and take $f_1,\ldots,f_\mu\in\mathrm{Hol}(D,D)$ such that $F=\mathrm{Fix}(f_1)\cap\ldots\cap\mathrm{Fix}(f_\mu)\neq\emptyset$. Then $\bar{F}=\overline{\mathrm{Fix}(f_1)\cap\ldots\cap\mathrm{Fix}(f_\mu)}\subset\bar{D}$ is homeomorphic to a compact convex subset of $\mathbf{C}^n$.*

Proof. Take $z_0 \in F$. If $F=\{z_0\}$, the assertion is obvious. Otherwise, Proposition 1.10 and Theorem 2.1 show that

$$\bar{F}=\Phi_{z_0}^{-1}(\overline{B^r}\cap V),$$

for a suitable linear subspace V of $\mathbf{C}^n$, and we are done. q.e.d.

The main facts of iteration theory we need are summarized in:

Theorem 2.3 (Abate [A 1, 2]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^2 domain, and $f \in \mathrm{Hol}(D, D)$. Then:*

(i) *If* $\mathrm{Fix}(f)=\emptyset$, *then the sequence of iterates $\{f^k\}$ of f converges, uniformly on compact sets, to a constant map $x \in \partial D$.*

(ii) *If* $\mathrm{Fix}(f)\neq\emptyset$, *then there is a subsequence $\{f^{k_\nu}\}$ of iterates of f converging, uniformly on compact sets, to a holomorphic retraction ϱ_f of D onto a submanifold M_f of D. Moreover, ϱ_f does not depend on the particular subsequence, but only on f, and $f|_{M_f}$ is an automorphism of M_f.*

If $\mathrm{Fix}(f)\neq\emptyset$, the holomorphic retraction ϱ_f is called the *limit retraction* of f.
Finally we can prove our main theorem:

Theorem 2.4. *Let $D \subset\subset \mathbf{C}^n$ be a strongly convex C^3 domain, and $\mathscr{F} \subset \mathrm{Hol}(D, D) \cap C^0(\bar{D})$ a family of commuting holomorphic maps. Then there exists a point $x \in \bar{D}$ such that $f(x)=x$ for all $f \in \mathscr{F}$.*

Proof. Assume first that there is $f \in \mathscr{F}$ without fixed points in D. Then, by Theorem 2.3 (i), there exists a point $x \in \partial D$ such that the sequence of iterates of f converges to x. Hence for any $g \in \mathscr{F}$ we have

$$g(x)=\lim_{k\to\infty} g(f^k(z))=\lim_{k\to\infty} f^k(g(z))=x,$$

where z is any point of D, and we are done.

So assume that $\mathrm{Fix}(f)\neq\emptyset$ for every $f \in \mathscr{F}$. The first key observation now is that if $f, g \in \mathrm{Hol}(D, D)$ commute, then g sends $\mathrm{Fix}(f)$ into itself. In fact, if $z \in \mathrm{Fix}(f)$ then

$$f(g(z))=g(f(z))=g(z).$$

The second key observation is that if $f, g \in \mathscr{F}$ then ϱ_f and ϱ_g commute. In particular, $\varrho_f \circ \varrho_g$ is a holomorphic retraction, and moreover

$$\varrho_f \circ \varrho_g(D)=\varrho_f(D)\cap\varrho_g(D).$$

Indeed, $\varrho_f \circ \varrho_g(D)$ is contained in $\varrho_f(D)\cap\varrho_g(D)$ for ϱ_f and ϱ_g commute; on the other hand, $\varrho_f(D)\cap\varrho_g(D)$ is clearly contained in $\mathrm{Fix}(\varrho_f \circ \varrho_g)=\varrho_f \circ \varrho_g(D)$. An induction argument then shows that for any $f_1, \ldots, f_\mu \in \mathscr{F}$ the map $\varrho_{f_1}\circ\ldots\circ\varrho_{f_\mu}$ is a holomorphic retraction of D onto

$$\varrho_{f_1}\circ\ldots\circ\varrho_{f_\mu}(D)=\varrho_{f_1}(D)\cap\ldots\cap\varrho_{f_\mu}(D).$$

In particular, the intersection on the right-hand side is always a non-empty closed connected submanifold of D. Choose $f_1, \ldots, f_\mu \in \mathscr{F}$ so that the dimension of $\varrho_{f_1}(D)\cap\ldots\cap\varrho_{f_\mu}(D)$ is minimal; then for any $f \in \mathscr{F}$ we should have

$$\varrho_{f_1}(D)\cap\ldots\cap\varrho_{f_\mu}(D)\subset\varrho_f(D). \tag{2.1}$$

Set $\varrho=\varrho_{f_1}\circ\ldots\circ\varrho_{f_\mu}$ and $M=\varrho(D)$. Then ϱ is a holomorphic retraction of D onto M commuting with every $f\in\mathscr{F}$. Moreover, for every $f\in\mathscr{F}$ we have $M\subset\varrho_f(D)$, by (2.1), and $f(M)\subset M$, for $M=\mathrm{Fix}(\varrho)$. Now, $f|_{\varrho_f(D)}$ is an automorphism of $\varrho_f(D)$; hence $f(M)=M$ and $f|_M\in\mathrm{Aut}(M)$. Finally, for any $f\in\mathscr{F}$ we have $\mathrm{Fix}(f)\cap M\neq\emptyset$; indeed, since ϱ commutes with f, $\varrho(\mathrm{Fix}(f))\subset\mathrm{Fix}(f)$ and so

$$\mathrm{Fix}(f)\cap M=\varrho(\mathrm{Fix}(f))\neq\emptyset\,.$$

We shall denote $\mathrm{Fix}(f)\cap M$ by F_f; since it is easy to check that $F_f=\mathrm{Fix}(\varrho\circ f)$, every F_f is again a closed connected submanifold of D, invariant under the action of any $g\in\mathscr{F}$.

Now, $k_M=k_D|_{M\times M}$; in fact

$$\forall z_1,z_2\in M\quad k_D(z_1,z_2)\leqq k_M(z_1,z_2)=k_M(\varrho(z_1),\varrho(z_2))\leqq k_D(z_1,z_2)\,.$$

In particular, every complex geodesic in M is a complex geodesic in D; moreover, since $M=\mathrm{Fix}(\varrho)$, for every two distinct points in $\bar{M}$ passes a unique complex geodesic.

If $\mathscr{F}$ has a common fixed point in ∂D, our work is clearly finished. So assume that there are no common fixed points of $\mathscr{F}$ in the boundary of D; we want to construct a common fixed point in the interior of D. First of all we claim that

$$\forall f_1,\ldots,f_\mu\in\mathscr{F}\quad F_{f_1}\cap\ldots\cap F_{f_\mu}\neq\emptyset\,. \tag{2.2}$$

We argue by induction on μ. For $\mu=1$ is clear. Assume $F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}\neq\emptyset$, and take $f_\mu\in\mathscr{F}$. By Corollary 2.2, $\overline{F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}}$ is homeomorphic to a compact convex subset of $\mathbf{C}^n$. Since it is also invariant under f_μ, by Brouwer's theorem f_μ has a fixed point in $\overline{F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}}$. We have to show that f_μ actually has a fixed point in $F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}$. Assume it does not. If f_μ had a unique fixed point $x\in\overline{F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}}\cap\partial D$, then $\{x\}$ should be invariant under any $f\in\mathscr{F}$, that is x should be a common fixed point of $\mathscr{F}$ contained in ∂D, against our assumption. So f_μ should have at least two distinct fixed points $x,y\in\overline{F_{f_1}\cap\ldots\cap F_{f_{\mu-1}}}\cap\partial D$. Let $\varphi:\bar{\Delta}\to\bar{D}$ be the unique complex geodesic passing through x and y; clearly, $\varphi(\bar{\Delta})\subset\bar{M}$. But $f_\mu|_M$ is an automorphism of M; hence $f_\mu\circ\varphi$ is again a complex geodesic passing through x and y. By Proposition 1.8, $f_\mu\circ\varphi(\Delta)=\varphi(\Delta)$, that is f_μ sends $\varphi(\Delta)$ into itself, and without fixed points, for $\varphi(\Delta)\subset F_{f_2}\cap\ldots\cap F_{f_{\mu-1}}$. But then the sequence of iterates of $f_\mu|_{\varphi(\Delta)}$ should converge to a point of the boundary, by Theorem 0.2, and this is impossible, for f_μ has fixed points in D, by assumption. The contradiction arises from assuming $F_{f_1}\cap\ldots\cap F_{f_\mu}=\emptyset$; hence $F_{f_1}\cap\ldots\cap F_{f_\mu}\neq\emptyset$, as claimed.

So we have proven that, if $\mathscr{F}$ has no common fixed points in ∂D, then (2.2) holds. In particular, the intersection of every finite subset of the family $\{\overline{F_f}\,|\,f\in\mathscr{F}\}$ is not empty; since $\bar{D}$ is compact, this implies that the intersection of the whole family is not empty, and every element in this intersection is a common fixed point of $\mathscr{F}$. q.e.d.

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