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Laws of Large Numbers of Hypergroups on $\mathbb{R}_+$

Hansmartin Zeuner*

Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Federal Republic of Germany

1. Introduction

Being aware of the difficulties in formulating and proving a law of large numbers for random walks on an arbitrary locally compact group, it seems hopeless to try the same attempt on the even more general structure of a hypergroup. However, since these difficulties arise in part from the complicated geometric structure of many of the groups considered, one might expect that it is possible to obtain theorems on hypergroups which are of particularly simple geometry. This has successfully been done by Eymard, Roynette, Gallardo, and Ries (see [8, 15, 14]) in the case of the hypergroups on $\mathbb{N}$ related to the Gegenbauer polynomials. Whereas on the real line (with the usual topology) there is exactly one structure as a topological group, there is an abundant collection of hypergroups on the half line $\mathbb{R}_+$ (see Chébli [5], Zeuner [28]). In the case of Chébli-Trimèche hypergroups enough analytical tools are developed to prove the law of large numbers and the central limit theorem. The latter will be studied in the forthcoming paper [29].

If $X_1, X_2, \ldots$ are i.i.d. random variables with values in a group, the corresponding random walk is the sequence $S_1, S_2, \ldots$ defined by $S_n := X_n X_{n-1} \ldots X_1$. Since the operation on a hypergroup is only defined in terms of the convolution of measures, the random walk $(S_n : n \geqq 1)$ can only be defined by its distribution and not as a function of $(X_n : n \geqq 1)$. The notion of *concretization* and a randomized multiplication is introduced in 3.3 in order to obtain an explicit construction of $(S_n : n \geqq 1)$ in terms of $(X_n : n \geqq 1)$ for every 2^{nd} countable locally compact hypergroup.

As in the classical case, the moments of a random variable are introduced, both to formulate the conditions under which a particular limit theorem holds, and to calculate the actual value of the limit. This has to be done by a modified definition to fit with the hypergroup operation. The first and second moments are in very close connection with the notion of the dispersion of a probability measure used in Faraut [9] and Trimèche [24]. As to be expected by the results of Guivarc'h [16],

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two different situations occur depending on the parameter ϱ (see 2.2) which determines the growth of the hypergroup. If $\varrho > 0$ the hypergroup is of exponential growth and the expectation of every nonzero random variable is strictly positive. This result includes the symmetric spaces of rank one of non-compact type and corresponds to Guivarc'h [16], corollaire on page 77. If $\varrho = 0$ the expectation of every random variable is 0 and so is the limit of $\frac{1}{n} S_n$ in probability. This result should be compared with [16, théorème 3, p. 72].

In both cases a strong law of large numbers will be proved. Apart from the different and less general situation in this article the main difference with the results in [16] is the fact that for hypergroups on $\mathbb{R}_+$ the law of large numbers can be formulated without the use of a gauge function and the a.s. limit of $\frac{1}{n} S_n$ can be calculated explicitly.

2. Preliminaries

2.1. Let K be a *hypergroup* in the sense of Jewett [20]; this means that K is a locally compact space with an associative convolution $(x, y) \mapsto \varepsilon_x * \varepsilon_y \in \mathcal{M}^1(K)$ (the space of probability measures on K) such that there exist a neutral element $e \in K$ and an inversion $x \mapsto x^\vee$ satisfying certain conditions (see [19, 20, 23] for details). In the cases considered in this article (except in the third paragraph) K will be *Hermitian* (i.e. $x^\vee = x$ for all $x \in K$); in particular this implies the commutativity of K.

The *dual* $\hat{K}$ of the Hermitian hypergroup K is the space of all real valued multiplicative functions φ on K with $\varphi(e) = \|\varphi\|_\infty = 1$ [20, 6.3]. For every probability measure P on K the Fourier transform $\mathcal{F}P$ is the continuous real-valued function $\varphi \mapsto \mathcal{F}P(\varphi) := \int \varphi \, dP$ on $\hat{K}$. It is a well known fact that the uniqueness theorem and the continuity theorem for the Fourier transform are valid for many commutative hypergroups [3, 19, 20].

2.2. In the sequel we consider the class of *Chébli-Trimèche hypergroups* on $K := \mathbb{R}_+$: For every function A on $\mathbb{R}_+$ (which turns out to be the Lebesgue density of a Haar measure of K) satisfying $A(0) = 0$, A strictly increasing and unbounded, $\frac{A'}{A}$ decreasing on $\mathbb{R}_+^*$, and $\frac{A'(x)}{A(x)} = \frac{\alpha}{x} + B(x)$ on a neighbourhood of 0 (where $\alpha > 0$ and B is an odd $\mathscr{C}^\infty$-function on $\mathbb{R}$), there exists a unique hypergroup structure on $\mathbb{R}_+$ such that

$$\frac{\partial}{\partial x}\left(A(x)A(y) \frac{\partial}{\partial x} \left(\int f \, d\varepsilon_x * \varepsilon_y\right)\right) = \frac{\partial}{\partial y}\left(A(x)A(y) \frac{\partial}{\partial y} \left(\int f \, d\varepsilon_x * \varepsilon_y\right)\right)$$

for every even $\mathscr{C}^\infty$-function f on $\mathbb{R}$ and $x, y \in \mathbb{R}_+$. The neutral element of this hypergroup is 0 and the inversion is the identity mapping.

The growth of this hypergroup is determined by the number $\varrho := \frac{1}{2} \lim_{x \to \infty} \frac{A'(x)}{A(x)} \geqq 0$: If $\varrho > 0$ then we obtain $A(x) \geqq A(1) \cdot e^{2\varrho(x-1)}$ for $x \geqq 1$ and so the hypergroup is of exponential growth; if $\varrho = 0$ then $(\mathbb{R}_+, *)$ is exponentially bounded. The multiplicative functions are precisely the solutions φ_λ ($\lambda \in \mathbb{C}$) of the differential equation

$$\varphi_\lambda'' + \frac{A'}{A}\varphi_\lambda' + (\varrho^2 + \lambda^2)\varphi_\lambda = 0, \qquad \varphi_\lambda(0) = 1,\ \varphi_\lambda'(0) = 0$$

and the dual is $\hat{K} = \{\varphi_\lambda : \lambda \in \mathbb{R}_+ \cup i[0, \varrho]\}$. In the following we will identify $\hat{K}$ with the set of parameters $\mathbb{R}_+ \cup i[0, \varrho]$. The proof of the preceding results can be found in [5, 28].

2.3. The most important technical tool used in this article is the Laplace representation for the multiplicative functions φ_λ $(\lambda \in \mathbb{C})$ proved in [5, Proposition I–IV]: For every $x \in \mathbb{R}_+$ there exists a probability measure ν_x on $[-x, x]$ such that

$$\varphi_\lambda(x) = \int e^{-t(\varrho + i\lambda)} \nu_x(dt) \quad \text{for} \quad x \in \mathbb{R}_+,\ \lambda \in \mathbb{C}.$$

Furthermore the measure τ_x with the density $t \mapsto e^{-\varrho t}$ with respect to ν_x is a symmetric subprobability measure on $\mathbb{R}$ which depends continuously on x [in the weak topology on $\mathscr{M}^b(\mathbb{R})$]. Therefore τ may be considered as a sub-Markovian kernel from $\mathbb{R}_+$ into $\mathbb{R}$ and it follows from the Laplace representation for φ_λ that for every $P \in \mathscr{M}^1(\mathbb{R}_+)$ we have

$$\mathscr{F}P(\lambda) = \widehat{\tau P}(\lambda) \quad \text{for all } \lambda \in \mathbb{R}_+$$

where $\tau P(A) = \int \tau_x(A) P(dx)$ for every Borel measurable subset A of $\mathbb{R}$, and $\hat{}$ denotes the usual Fourier transform on the real line (which should be well distinguished from the Fourier transform $\mathscr{F}$ of $\mathbb{R}_+$ considered as a hypergroup).

3. Concretizations of Hypergroups

The main problem which makes it difficult to state probabilistic results on a hypergroup K is the fact that the definition does not allow us to define the "product" of two independent random variables X and Y with values in K as a K-valued random variable $X \cdot Y$ directly. It is clear, however, that the distribution of this product – if it exists – should be $P_X * P_Y$. It is the purpose of this paragraph to construct such a random variable, unifying the different approaches which have been made in concrete examples.

3.1. Definition. Let $(K, *)$ be a hypergroup (not necessarily commutative), μ a probability measure on a compact set M and $\Phi: K \times K \times M \to K$ be Borel-measurable. (M, μ, Φ) is called a *concretization* of $(K, *)$ if

$$\mu\{\Phi(x, y, \cdot) \in A\} = (\varepsilon_x * \varepsilon_y)(A) \quad \text{for} \quad x, y \in K,\ A \in \mathfrak{B}(K).$$

Here $\mathfrak{B}(K)$ denotes the Borel σ-field of K.

3.2. Examples

3.2.1. Let G be a locally compact group and $*$ the convolution defined by the group operation. If we define $\Phi(x, y, 1) := xy$ for $x, y \in G$ then $(\{1\}, \varepsilon_1, \Phi)$ is a concretization of $(G, *)$.

3.2.2. More generally let H be a compact subgroup of G and $(G//H, *)$ the double coset hypergroup (see Jewett [20]). Then (H, ω_H, Φ) is a concretization of $(G//H, *)$ if we define $\Phi(x,y,h) := H\varphi(x)h\varphi(y)H$ where $\varphi: G//H \to G$ is measurable and satisfies $x = H\varphi(x)H$ for every $x \in G//H$ (if G is locally compact, metrizable, and separable the existence of φ follows from [4]).

3.2.3. Let $K := \mathbb{R}_+$, $\varepsilon_x * \varepsilon_y := \frac{1}{2}\varepsilon_{|x-y|} + \frac{1}{2}\varepsilon_{x+y}$, $M := \{-1, 1\}$, $\mu := \frac{1}{2}\varepsilon_{-1} + \frac{1}{2}\varepsilon_1$, and $\Phi(x,y,\lambda) := |x + \lambda y|$. Then (M, μ, Φ) is a concretization of $(\mathbb{R}_+, *)$.

3.2.4. Let $\alpha > -\frac{1}{2}$ and $(\mathbb{R}_+, *)$ be the hypergroup defined in [22, 18] (see also [11]) by

$$\varepsilon_x * \varepsilon_y := c_\alpha \int_{-1}^{1} \varepsilon_{\sqrt{x^2+y^2-2\lambda xy}} (1-\lambda^2)^{\alpha-1/2} d\lambda \qquad (x, y \in \mathbb{R}_+)$$

with $c_\alpha := \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2}) \cdot \sqrt{\pi}}$. A concretization of this hypergroup is given by $M := [-1, 1]$, $\mu := g \cdot \lambda_{[-1,1]}$ (where $g(\eta) := c_\alpha \cdot (1-\eta^2)^{\alpha-1/2}$ and $\lambda_{[-1,1]}$ denotes the Lebesgue measure on $[-1, 1]$) and $\Phi(x,y,\lambda) := \sqrt{x^2+y^2-2\lambda xy}$.

3.2.5. Let $\alpha > -\frac{1}{2}$ and $([0, \pi], *)$ the hypergroup defined in [2] by

$$\varepsilon_x * \varepsilon_y := c_\alpha \int_{-1}^{1} \varepsilon_{\arccos(\cos x \cos y + \lambda \sin x \sin y)} (1-\lambda^2)^{\alpha-1/2} d\lambda \quad \text{for} \quad x, y \in [0, \pi].$$

A concretization of this hypergroup is given by $M := [-1, 1]$, $\mu := g \cdot \lambda_{[-1,1]}$ as in 3.2.4 and

$$\Phi(x,y,\lambda) := \arccos(\cos x \cos y + \lambda \sin x \sin y).$$

3.2.6. If we choose M and μ as in 3.2.4 and

$$\Phi(x,y,\lambda) := \operatorname{arch}(\cosh x \cosh y + \lambda \sinh x \sinh x) \quad \text{for} \quad x, y, \in \mathbb{R}_+,\ \lambda \in [-1, 1],$$

we obtain a concretization of the hyperbolic hypergroup [21, 26, 27].

3.2.7. Let $\alpha > \beta > -\frac{1}{2}$ and consider the hypergroup operation on $\mathbb{R}_+$ defined in [12]. In this case a concretization is given by

$$M := [0, 1] \times [0, \pi], \qquad \mu := g \cdot \lambda_M^2$$

[with $g(r, \vartheta) := c_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin \vartheta)^{2\beta}$], and

$$\Phi(x, y, (r, \vartheta)) := \operatorname{arch}\left[\frac{1}{2}(1+\cosh x)(1+\cosh y) + \frac{r^2}{2}(1-\cosh x)(1-\cosh y) + r \cos\vartheta \sinh x \sinh y\right].$$

3.2.8. Consider the convolution

$$\varepsilon_x * \varepsilon_y := \sum_{s=0}^{x \wedge y} \frac{|x-y|+2s+1}{(x+1)(y+1)} \varepsilon_{|x-y|+2s} (x, y \in \mathbb{N})$$

on the nonnegative integers $\mathbb{N}$ (compare [8] and [15]). In [15] a concretization of this hypergroup is given by $M := \mathbb{S}^2$ (the sphere in $\mathbb{R}^3$), μ the uniform distribution on $\mathbb{S}^2$ and

$$\Phi(x,y,D):=|x-y|+2\lfloor\tfrac{1}{2}\{\|(x+1)e_0+(y+1)D\|-|x-y|\}\rfloor$$

(where e_0 is a fixed unit vector in $\mathbb{R}^3$ and $\|\ \|$ the Euclidean norm).

3.3. Definition. In the sequel let (M,μ,Φ) be a concretization of the hypergroup $(K,*)$ and $(\Omega,\mathfrak{A},P)$ be a probability space. If X and Y are K-valued random variables and if Λ is an M-valued random variable, independent from (X,Y) and satisfying $P_\Lambda=\mu$ we define

$$X \overset{\Lambda}{\cdot} Y:=\Phi(X,Y,\Lambda).$$

This is a K-valued random variable.

More generally let $(X_n:n\geqq 1)$ be a sequence of K-valued random variables and $(\Lambda_n:n\geqq 1)$ be a sequence of M-valued random variables with $P_{\Lambda_n}=\mu$ for $n\geqq 1$ and such that $X_1,\Lambda_1,X_2,\Lambda_2,\dots$ are independent. Then we define ${}_\Lambda\prod_{j=1}^{n}$ recursively by

$${}_\Lambda\prod_{j=1}^{0} X_j:=e \quad\text{and}\quad {}_\Lambda\prod_{j=1}^{n} X_j:=X_n \overset{\Lambda_n}{\cdot} {}_\Lambda\prod_{j=1}^{n-1} X_j$$

for $n\geqq 1$.

It is clear that $\left({}_\Lambda\prod_{j=1}^{n} X_j:n\in\mathbb{N}\right)$ is a (non homogeneous) Markov chain, the transition kernel being

$$P\left\{{}_\Lambda\prod_{j=1}^{n} X_j\in A \,\middle|\, {}_\Lambda\prod_{j=1}^{n-1} X_j=x\right\}=(P_{X_n}*\varepsilon_x)(A)\quad P\text{-a.s.}$$

If the hypergroup is commutative we will write $X\overset{\Lambda}{+}Y$ instead of $X\overset{\Lambda}{\cdot}Y$ and ${}_\Lambda\sum_{j=1}^{n}$ instead of ${}_\Lambda\prod_{j=1}^{n}$.

3.4. Proposition. *Let (X,Y,Λ) be independent and $P_\Lambda=\mu$. Then $P_{X\overset{\Lambda}{\cdot}Y}=P_X*P_Y$.*

Proof.

$$\begin{aligned}
P_{X\overset{\Lambda}{\cdot}Y}(A)&=P\{\Phi(X,Y,\Lambda)\in A\}\\
&=\int\int P\{\Phi(x,y,\Lambda)\in A\}P_X(dx)P_Y(dy)\\
&=\int\int \mu\{\Phi(x,y,\cdot)\in A\}P_X(dx)P_Y(dy)\\
&=\int\int \varepsilon_x*\varepsilon_y(A)P_X(dx)P_Y(dy)\\
&=(P_X*P_Y)(A)\quad\text{for}\quad A\in\mathfrak{B}(K).\ \square
\end{aligned}$$

3.5. *Remark.* It is clear that a concretization is not uniquely determined by the hypergroup and hence the same is true for $X\overset{\Lambda}{\cdot}Y$. However, the following proposition shows that the joint distribution of X, Y, and $X\overset{\Lambda}{\cdot}Y$ does not depend on the choice of the concretization.

3.6. Proposition. *Let X, Y, X', and Y' be K-valued random variables with $P_{(X,Y)}=P_{(X',Y')}$. Furthermore let (M,μ,Φ) and (M',μ',Φ') be concretizations of $(K,*)$ and*

Λ, Λ' be M-valued resp. M'-valued random variables such that Λ is independent of (X, Y) and Λ' is independent of (X', Y') with $P_\Lambda = \mu$ and $P_{\Lambda'} = \mu'$. Then $P_{(X,Y,X\overset{\Lambda}{*}Y)} = P_{(X',Y',X'\overset{\Lambda'}{*}Y')}$.

Proof. For every $A, B, C \in \mathfrak{B}(K)$ we have

$$\begin{aligned} P_{(X,Y,X\overset{\Lambda}{*}Y)}(A \times B \times C) &= \int_{A\times B} P\{\Phi(x, y, \Lambda) \in C\} P_{(X,Y)}(d(x, y)) \\ &= \int_{A\times B} \varepsilon_x * \varepsilon_y(C) P_{(X',Y')}(d(x, y)) \\ &= \int_{A\times B} P\{\Phi'(x, y, \Lambda') \in C\} P_{(X',Y')}(d(x, y)) \\ &= P_{(X',Y',X'\overset{\Lambda'}{*}Y')}(A \times B \times C). \quad \square \end{aligned}$$

3.7. Corollary. *If $(X_n, \Lambda_n : n \in \mathbb{N})$ are independent with $P_{\Lambda_n} = \mu$ then the distribution of $\left(X_n, \overset{\Lambda}{\prod_{j=1}^{n}} X_j : n \geqq 1\right)$ does not depend on the concretization (M, μ, Φ) or on the choice of $(\Lambda_n : n \geqq 1)$.*

3.8. Proposition. *Let $(K, *)$ be a (locally compact) hypergroup with countable base of topology. Then there exists a mapping $\Phi : K \times K \times [0,1] \to K$ such that $([0,1], \lambda_{[0,1]}, \Phi)$ is a concretization of $(K, *)$.*

Proof. We will only treat the case that K is not countable (if K is at most countable we may construct Φ in the same way as below without worrying about measurability). It follows from the assumptions that there exists a bimeasurable bijection $\psi : K \to [0,1]$. The induced mapping $\mu \mapsto \psi(\mu)$ from $\mathscr{M}^1(K)$ onto $\mathscr{M}^1([0,1])$ is Borel measurable and hence so is the mapping $p : [0,1]^2 \to \mathscr{M}^1([0,1])$ defined by

$$p(x, y) := \psi(\varepsilon_{\psi^{-1}(x)} * \varepsilon_{\psi^{-1}(y)}).$$

It is a well known fact that for every $\mu \in \mathscr{M}^1[0,1]$ there is a unique left continuous increasing function $\varphi_\mu : [0,1] \to [0,1]$ such that $\varphi_\mu(\lambda_{[0,1]}) = \mu$, namely

$$\varphi_\mu(\eta) = 0 \vee \sup\{z \in [0,1] : \mu([0, z]) < \lambda\}.$$

We will prove that the mapping $\Phi_0 : [0,1]^3 \to [0,1]$ defined by $\Phi_0(x, y, \lambda) := \varphi_{p(x,y)}(\lambda)$ for all $x, y, \lambda \in [0,1]$ is Borel measurable. It follows from the left sided continuity of φ_μ that it suffices to show that for every $\lambda \in [0,1]$ the mapping

$$\mu \mapsto \varphi_\mu(\lambda) = 0 \vee \sup\{z \in [0,1] : \mu([0, z[) < \lambda\}$$

is Borel measurable. This, however, is a consequence of

$$\{\mu \in \mathscr{M}^1([0,1]) : \varphi_\mu(\lambda) \leqq \varepsilon\} = \{\mu \in \mathscr{M}^1([0,1]) : \textstyle\int 1_{[0,\varepsilon]} d\mu \geqq \lambda\}.$$

The mapping $\Phi : K \times K \times [0,1] \to K$ can therefore be defined by

$$\Phi(h, k, \lambda) := \psi^{-1}(\Phi_0(\psi(h), \psi(k), \lambda)). \quad \square$$

3.9. We are now considering the special cases $K = \mathbb{R}_+$ and $K = [0,1]$ – see [1, 5, 28]. It follows from [28] that we may suppose without loss of generality that

$$\min \operatorname{supp}(\varepsilon_x * \varepsilon_y) = |x - y| \quad \text{for} \quad x, y \in K$$

and

$$\max \operatorname{supp}(\varepsilon_x * \varepsilon_y) = x + y \quad \text{if} \quad x, y \in K \text{ (and } x + y \leqq 1 \text{ in the case } K = [0,1]).$$

Since in these cases there is no need for a Borel isomorphism ψ in the proof of 3.8, we get the following additional properties of Φ:

$$\Phi(x, y, 0) = |x - y| \quad \text{for} \quad x, y \in K$$

and

$$\Phi(x, y, 1) = x + y \quad \text{for} \quad x, y \in K \text{ (and } x + y \leqq 1 \text{ in the case } K = [0,1]).$$

Furthermore $\Phi(x, 0, \lambda) = \Phi(0, x, \lambda) = x$. Every hypergroup on $\mathbb{R}_+$ or $[0,1]$ is commutative [28, Corollary 2.4] and hence

$$\Phi(x, y, \lambda) = \Phi(y, x, \lambda) \quad \text{for} \quad x, y \in K, \ \lambda \in [0,1].$$

It is easy to prove that for every $\lambda \in [0,1]$ the mapping $\Phi(\cdot, \cdot, \lambda): K \times K \to K$ is lower semicontinuous. Under the additional assumptions that $\varepsilon_x * \varepsilon_y$ is diffuse for $x, y > 0$ and $\operatorname{supp}(\varepsilon_x * \varepsilon_y) = [|x - y|, x + y]$ (which happens to be true if $(\mathbb{R}_+, *)$ is a Chébli-Trimèche hypergroup as shown by Trimèche [25, Sect. 8]), $\Phi(\cdot, \cdot, \lambda)$ is continuous for every $\lambda \in [0,1]$.

4. Moments

The usual definition of moments of higher order on a locally compact group [16, 18] depends on the choice of the gauge function $\delta_V(x) := \inf\{n \geqq 1 : x \in V^n\}$ and essentially only states whether the moment of order α ($\alpha > 0$) exists or not. On Chébli-Trimèche hypergroups however, the moment of order n of a probability measure can be defined for every integer $n \geqq 1$ in a unique way fitting (in the sense of 4.14) with the convolution structure of the hypergroup.

From now on let $(K, *)$ be a Chébli-Trimèche hypergroup on $\mathbb{R}_+$ (see 2.2). It is proved in [5] that $\varphi_\lambda(x)$ is an analytic function of λ. The derivations of $\varphi_\lambda(x)$ with respect to λ will be the most important tool to define moments for each probability measure on $\mathbb{R}_+$ in a way which is consistent with the convolution structure.

4.1. Definition. For every $\lambda \in \mathbb{C}$, $x \in \mathbb{R}_+$, and $n \geqq 0$ let

$$\varphi_{n,\lambda}(x) := \left(\frac{\partial}{\partial \mu}\right)^n \varphi_{\lambda + i\mu}(x)|_{\mu = 0} \quad \text{and} \quad m_n(x) := \varphi_{n, i\varrho}(x).$$

Some elementary properties of the functions $\varphi_{n,\lambda}$ and m_n will be proved first.

4.2. Let L be the differential operator on $\mathbb{R}_+$ defined by $Lf(x) = -f''(x) - \frac{A'(x)}{A(x)} f'(x)$ for $x > 0$ and $f \in \mathscr{C}^2(\mathbb{R}_+)$ with $f'(0) = 0$. By differentiating the differential equation $\varphi_\lambda = (\varrho^2 + \lambda^2)\varphi_\lambda$, $\varphi_\lambda(0) = 1$, $\varphi'_\lambda(0) = 0$ with respect to λ we obtain

$$L\varphi_{n,\lambda} = (\varrho^2 + \lambda^2)\varphi_{n,\lambda} + 2in\lambda\varphi_{n-1,\lambda} - n(n-1)\varphi_{n-2,\lambda}, \quad \varphi_{n,\lambda}(0) = \varphi'_{n,\lambda}(0) = 0$$

and especially

$$Lm_n = -2n\varrho m_{n-1} - n(n-1)m_{n-2}, \quad m_n(0) = m_n'(0) = 0 \quad \text{for} \quad n \geqq 1$$

(with $m_0(x) = 1$ for every $x \in \mathbb{R}_+$).

4.3. It follows from the Laplace representation (2.3) that

$$\varphi_{n,\lambda}(x) = \int_{-x}^{x} t^n e^{-t(\varrho + i\lambda)} v_x(dt) = \int_{-x}^{x} t^n e^{-it\lambda} \tau_x(dt)$$

and

$$m_n(x) = \int_{-x}^{x} t^n v_x(dt) \quad \text{for} \quad x \in \mathbb{R}_+,\ \lambda \in \mathbb{C},\ n \geqq 1\,.$$

4.4. If $\lambda \in i[0, \varrho]$ $\varphi_{n,\lambda}$ is real valued for every $n \geqq 1$ since φ_λ is real valued. For $\lambda \in \mathbb{R}_+$ $\varphi_{n,\lambda}$ is real valued if n is even and $i\varphi_{n,\lambda}$ is real if n is odd. This follows since $\varphi_\lambda(x)$ is an analytic function of λ and φ_λ is real for $\lambda \in \mathbb{R}_+$.

It follows from $m_n(x) = \int_0^x t^n(e^{\varrho t} + (-1)^n e^{-\varrho t})\tau_x(dt)$ that $m_n \geqq 0$ for every $n \geqq 1$.

4.5. We now have to study the two cases $\varrho = 0$ and $\varrho > 0$ separately. We begin with the case $\varrho = 0$. It is clear that $m_n = 0$ if n is odd.

4.6. Lemma. *Let $\varrho = 0$, $\lambda \in \mathbb{R}_+$, and $n \in \mathbb{N}$. Then*

a) $m_{2k} \leqq 1 + m_{2n}$ *for every* $k < n$,
b) $|\varphi_{2n,\lambda}| \leqq m_{2n}$, *and*
c) $|\varphi_{2n-1,\lambda}| \leqq 1 + m_{2n}$.

Proof.

a) $$m_{2k}(x) = \int t^{2k} v_x(dt) \leqq \int (1 + t^{2n}) v_x(dt) = 1 + m_{2n}(x),$$

b) $$|\varphi_{2n,\lambda}(x)| \leqq \int |t^{2n} e^{-it\lambda}| v_x(dt) = \int t^{2n} v_x(dt) = m_{2n}(x),$$

c) $$\begin{aligned} |\varphi_{2n-1,\lambda}(x)| &\leqq \int |t^{2n-1} e^{-it\lambda}| v_x(dt) = \int |t|^{2n-1} v_x(dt) \\ &\leqq \int (1 + t^{2n}) v_x(dt) = 1 + m_{2n}(x)\,. \quad \square \end{aligned}$$

4.7. Theorem. *Let P be a probability measure on $\mathbb{R}_+$ and $n \geqq 1$. Then the following conditions are equivalent:*

(i) $\int m_{2n} dP$ *is finite,*
(ii) $\mathscr{F}P$ *is* $2n-1$ *times differentiable,* $\mathscr{F}P^{(2n-1)}(0) = 0$ *and* $\mathscr{F}P^{(2n)}(0)$ *exists.*

In both cases $\mathscr{F}P^{(k)}(\lambda) = i^k \int \varphi_{k,\lambda} dP$ *for all* $k \leqq 2n$, $\lambda \in \mathbb{R}_+$. *In particular* $\mathscr{F}P^{(2k)}(0) = \int m_{2k} dP$.

Proof. "i ⇒ ii": By 4.6 a) and induction $\mathscr{F}P^{(2n-1)}$ exists. From 4.6 c) and b) we obtain

$$\left| \frac{\varphi_{2n-2,\lambda} - \varphi_{2n-2,\mu}}{\lambda - \mu} \right| \leqq m_{2n} + 1$$

and

$$\left| \frac{\varphi_{2n-1,\lambda} - \varphi_{2n-1,\mu}}{\lambda - \mu} \right| \leqq m_{2n}$$

and it follows from the dominated convergence theorem that $\mathscr{F}P^{(2n-1)}(\lambda)$ and $\mathscr{F}P^{(2n)}(\lambda)$ exist and equal $i^{2n-1}\cdot\int\varphi_{2n-1,\lambda}dP$ and $i^{2n}\cdot\int\varphi_{2n,\lambda}dP$ respectively. In particular

$$\mathscr{F}P^{(2n-1)}(0)=\int\varphi_{2n-1,0}dP=\int m_{2n-1}dP=0.$$

"ii $\Rightarrow$ i": Because of $\mathscr{F}P^{(2n-1)}(0)=0$ the $2n$-th derivative $\mathscr{F}P^{(2n)}(0)$ equals

$$2\lim_{h\to 0}\frac{1}{h^2}(\mathscr{F}P^{(2n-2)}(h)-\mathscr{F}P^{(2n-2)}(0)).$$

Since $\varphi_{2n-2,\lambda}\leqq m_{2n-2}$ by 4.6 b) we may apply Fatou's lemma to obtain

$$\begin{aligned}0&\leqq\int m_{2n}dP\\&=-\int\left(\frac{\partial}{\partial h}\right)^2\varphi_{2n-2,h}|_{h=0}dP\\&=2\int\lim_{h\to 0}\frac{1}{h^2}(m_{2n-2}-\varphi_{2n-2,h})dP\\&\leqq 2\liminf_{h\to 0}\frac{1}{h^2}\int m_{2n-2}-\varphi_{2n-2,h}dP\\&=2\liminf_{h\to 0}\frac{1}{h^2}(\mathscr{F}P^{(2n-2)}(0)-\mathscr{F}P^{(2n-2)}(h))\\&=-2\mathscr{F}P^{(2n)}(0)<\infty.\quad\square\end{aligned}$$

4.8. *Remark.* The condition $\mathscr{F}P^{(2n-1)}(0)=0$ – which does not occur in the usual formulation of this theorem on $\mathbb{R}$ – cannot be dismissed. For example in the case of Kingman's hypergroups [$A(x)=x^{2\alpha+1}$, $\alpha>-\frac{1}{2}$, see 3.2.4] the (Cauchy type) distribution P with density

$$x\mapsto\frac{2\Gamma(\alpha+\frac{3}{2})}{\sqrt{\pi}\,\Gamma(\alpha+1)}\frac{x^{2\alpha+1}}{(1+x^2)^{\alpha+3/2}}$$

has the Fourier transform $\lambda\mapsto e^{-\lambda}$ (see [7, 8.6(4)]) which is infinitely often differentiable on $\mathbb{R}_+$ but $\int m_2 dP=\infty$.

Let us now suppose that $\varrho>0$. Then m_n is strictly positive on $\mathbb{R}_+^*$ for every $n\geqq 0$. In view of (7.6) and (5.4) this fact corresponds to the corollary on p. 77 in [16].

4.9. Lemma. *Let $\varrho>0$ and $n\geqq 0$. Then*

a) $\varphi_{n,i\lambda}>0$ *if* $\lambda\in[0,\varrho]$,

b) $\varphi_{n,i\lambda}\leqq\varphi_{n,i\mu}$ *if* $0\leqq\lambda\leqq\mu\leqq\varrho$, *and*

c) $\dfrac{m_n-\varphi_{n,i\lambda}}{\varrho-\lambda}\leqq\dfrac{m_n-\varphi_{n,i\mu}}{\varrho-\mu}$ *if* $0\leqq\lambda\leqq\mu<\varrho$.

Proof. a) $\varphi_{n,i\lambda}(x)=\int t^n e^{-t(\varrho-\lambda)}\nu_x(dt)>0$.

b)
$$\begin{aligned}\varphi_{n,i\lambda}(x)&=\int_{-x}^{x}t^n e^{-t(\varrho-\lambda)}\nu_x(dt)=\int_0^x t^n(e^{t\lambda}+(-1)^n e^{-t\lambda})\tau_x(dt)\\&\leqq\int_0^x t^n\cdot(e^{t\mu}+(-1)^n e^{-t\mu})\tau_x(dt)\\&=\varphi_{n,i\mu}(x)\end{aligned}$$

since sinh and cosh are increasing functions.

c) $$\frac{m_n(x)-\varphi_{n,i\lambda}(x)}{\varrho-\lambda}=\int_0^x t^n\frac{(e^{t\varrho}+(-1)^n e^{-t\varrho})-(e^{t\lambda}+(-1)^n\cdot e^{-t\lambda})}{\varrho-\lambda}\tau_x(dt)$$
$$\leqq\int_0^x t^n\frac{(e^{t\varrho}+(-1)^n e^{-t\varrho})-(e^{t\mu}+(-1)^n e^{-t\mu})}{\varrho-\mu}\tau_x(dt)$$
$$=\frac{m_n(x)-\varphi_{n,i\mu}(x)}{\varrho-\mu}$$

since sinh and cosh are both convex functions. □

4.10. Lemma. *Let* $\varrho>0$. *Then* $m_n(x)\leqq\left(\frac{2}{\varrho}\right)^n+\varrho m_{n+1}(x)$ for all $x\geqq 0$, $n\in\mathbb{N}$.

Proof. If n is odd we conclude from $\sinh y\leqq y\cosh y$ for all $y\geqq 0$ that

$$m_n(x)=\int_0^x t^n\cdot\sinh(t\varrho)\tau_x(dt)$$
$$\leqq\varrho\int_0^x t^{n+1}\cosh(t\varrho)\tau_x(dt)$$
$$=\varrho m_{n+1}(x)\quad\text{for all } x\geqq 0.$$

If n is even we use the inequality $\cosh y\leqq y\sinh y+1_{[0,2[}(y)$ to obtain $y^n\cosh y\leqq y^{n+1}\sinh y+2^n$ which implies

$$m_n(x)=\int_0^x t^n\cosh(t\varrho)\tau_x(dt)$$
$$\leqq\int_0^x(\varrho t^{n+1}\sinh(t\varrho)+(2/\varrho)^n)\tau_x(dt)$$
$$\leqq\varrho m_{n+1}(x)+(2/\varrho)^n\quad\text{for all } x\geqq 0.\quad\square$$

4.11. Theorem. *Let* $\varrho>0$, $n\geqq 1$, *and* P *be a probability measure on* $\mathbb{R}_+$. *Then the following conditions are equivalent:*

(i) $\int m_n dP$ *is finite,*

(ii) $\lambda\mapsto\mathscr{F}P(i\lambda)$ *is* n *times differentiable on* $[0,\varrho]$.

In both cases $\left(\frac{\partial}{\partial\lambda}\right)^k\mathscr{F}P(i\lambda)=\int\varphi_{k,i\lambda}dP$ *for all* $k\leqq n$, $\lambda\in[0,\varrho]$. *In particular* $\mathscr{F}P^{(k)}(0)=\int m_k dP$.

Proof. "i ⇒ ii": It follows from 4.10 and induction that $f:\lambda\mapsto\int\varphi_{i\lambda}dP$ is $(n-1)$ times differentiable on $[0,\varrho]$ and $f^{(n-1)}(\lambda)=\int\varphi_{n-1,i\lambda}dP$. The mean value theorem and 4.9 b) imply that

$$\varphi_{n-1,i\lambda}(x)-\varphi_{n-1,i\mu}(x)|\leqq|\lambda-\mu|\cdot m_n(x).$$

By Lebesgue's theorem we obtain that $f^{(n-1)}$ is differentiable and

$$\left(\frac{\partial}{\partial\lambda}\right)^n\mathscr{F}P(i\lambda)=f^{(n)}(\lambda)=\int\varphi_{n,i\lambda}dP.$$

"ii $\Rightarrow$ i": By induction the first $n-1$ moments $\int m_k dP$ $(k \leqq n-1)$ are finite and $f^{(n-1)}(\lambda) = \int \varphi_{n-1,i\lambda} dP$. It follows from 4.9 c) that the difference quotients $\frac{m_{n-1} - \varphi_{n-1,i\lambda}}{\varrho - \lambda}$ $(\lambda < \varrho)$ approach m_n increasingly as $\lambda \nearrow \varrho$ and hence by the theorem of monotone convergence

$$\int m_n dP = \frac{\partial}{\partial \lambda} \int \varphi_{n-1,i\lambda} dP|_{\lambda=\varrho}$$

exists and equals $f^{(n)}(\varrho)$ which is finite. $\square$

4.12. *Remark.* Let $\varrho > 0$ and $\int m_n dP$ be finite. Then it follows from 4.3 and 4.9 b) that $|\varphi_{n,\lambda}(x)| \leqq m_n(x)$ for all $x \geqq 0$ and $\lambda \in \mathbb{C}$ such that $|\Im \lambda| \leqq \varrho$. Therefore the function $\lambda \mapsto \int \varphi_\lambda dP$ is n times differentiable in this strip and in particular $\eta \mapsto \int \varphi_{\eta + i\varrho} dP$ is n times differentiable on $\mathbb{R}$. This fact will be used later.

4.13. *Remark.* Let $\varrho \geqq 0$. Then a sufficient condition for $E(m_n(X))$ being finite (where X is a $\mathbb{R}_+$-valued random variable) is $E(X^n) < \infty$. This follows from the inequality $m_n(x) \leqq x^n$ for all $x \geqq 0$ which is a consequence of the fact that the measure ν_x in 4.3 is supported by $[-x, x]$.

4.14. Theorem. *Let $\varrho > 0$, X and Y be independent $\mathbb{R}_+$-valued random variables such that $E(m_n(X))$ and $E(m_n(Y))$ are finite. Then $E(m_n(X \overset{A}{+} Y))$ is finite and*

$$E(m_n(X \overset{A}{+} Y)) = \sum_{k=0}^{n} \binom{n}{k} E(m_k(X)) E(m_{n-k}(Y)).$$

Proof. This follows from the fact that the product of two n times differentiable functions is again n times differentiable, Theorem 4.11, and Leibniz's rule. $\square$

5. Expectation

5.1. In this paragraph the special properties of the function m_1 will be considered. We will assume $\varrho > 0$ throughout the whole paragraph. The function m_1 has already been defined in [9, 24] under the name "forme quadratique généralisée". It will be used to define a modified expectation for every $\mathbb{R}_+$-valued random variable consistent with the hypergroup structure (see 5.6).

5.2. *Examples.* If A is of the form $A(x) = (\sinh x)^\alpha$ for some $\alpha > 0$, the function m_1 can be written down in closed form for some values of α. According to Faraut [9], $m_1(x) = 2 \ln \cosh \frac{x}{2}$ if $\alpha = 1$ and $m_1(x) = x \coth x - 1$ if $\alpha = 2$. If $\alpha = 3/2$ one calculates $m_1(x) = 2 \ln \cosh \frac{x}{2} + \frac{1}{2}\left(\tanh \frac{x}{2}\right)^2$ $(x \geqq 0)$. If $A(x) = (\cosh x)^2$ then $m_1(x) = x \tanh x$.

5.3. *Remark.* By integrating the differential equation for m_1 (see 4.2), one obtains

$$m_1(x) = 2\varrho \int_0^x \frac{1}{A(y)} \int_0^y A(z) dz\, dy \quad \text{for} \quad x \geqq 0.$$

This formula has been used in [13].

5.4. Definition. Let $(\mathbb{R}_+, *)$ be a Chébli-Trimèche hypergroup with the corresponding function m_1. Then for every $\mathbb{R}_+$-valued random variable X, $E_*(X) := E(m_1(X))$ is called the *$*$-expectation* of X.

5.5. *Remark.* Although $E_*(X) = 0$ holds for every random variable in the case $\varrho = 0$, the $*$-expectation does not loose its entire sense as can be seen from Theorem 8.4. The notions of "dispersion" and "variance" are also used for $E_*(X)$ by some authors (see [9, 13, 24]). Theorems 7.6, 7.7, and 8.4, as well as the central limit theorems in the forthcoming paper [29] are the motivation to call $E_*(X)$ the "expectation" of X and to reserve the expression "variance" to the corresponding number related with the second moment function m_2.

5.6. Proposition. *Let A have the distribution μ and be independent from (X, Y). Then $E_*(X \overset{A}{+} Y) = E_*(X) + E_*(Y)$.*

The proof follows from 4.14. □

5.7. Lemma. *If $\varrho > 0$ then* $\lim\limits_{x \to \infty} \dfrac{m_1(x)}{x} = 1$.

Proof. Suppose that m_1'' takes negative values. Since $m_1''(0) = \dfrac{2\varrho}{\alpha+1} > 0$ there exists $x_0 > 0$ with $m_1'(x_0) > 0$, $m_1''(x_0) < 0$, and $m_1'''(x_0) < 0$. This would imply that $m_1' \cdot \dfrac{A'}{A}$ and m_1'' are strictly decreasing in a neighbourhood of x_0. But this is impossible since $m_1'' + m_1' \dfrac{A'}{A} = 2\varrho$ by 4.2. From this contradiction we conclude that $m_1'' \geqq 0$ and m_1' is increasing. Suppose now that $\beta := \lim\limits_{x \to \infty} m_1'(x) < 1$. This implies $m_1'' = 2\varrho - \dfrac{A'}{A} \cdot m_1' > 2\varrho - \dfrac{A'}{A} \beta$. When $\dfrac{A'}{A}(x)$ is close enough to 2ϱ the last number becomes strictly positive and hence $m_1''(x)$ is bounded away from 0 for large enough x. This is a contradiction with $\sup\{m_1'(x) : x \geqq 0\} = \beta < 1$. On the other hand, from $2\varrho m_1' \leqq m_1'' + \dfrac{A'}{A} m_1' = 2\varrho$ we obtain $m_1' \leqq 1$ and hence $m_1'(x) \nearrow 1$ as $x \to \infty$. This implies $\dfrac{m_1(x)}{x} \nearrow 1$. □

5.8. Corollary. *Let $\varrho > 0$ and X be a $\mathbb{R}_+$-valued random variable. Then $E_*(X)$ is finite if and only if $E(X)$ (the expectation of X in the usual sense) is finite.*

5.9. Proposition. *Let $\varrho > 0$ and X be a $\mathbb{R}_+$-valued random variable with $0 \leqq E_*(X) \leqq +\infty$. Then*

$$\frac{\partial}{\partial \lambda} E(\varphi_{i\lambda}(X))|_{\lambda=\varrho} = E_*(X).$$

Proof. If $E_*(X)$ is finite this is Theorem 4.11. If $E_*(X) = \infty$ this follows from 4.9 and the theorem of monotone convergence. □

6. Variance

We will now explore the properties of the function m_2 to obtain a modification of the notion of variance in a similar way as for the expectation in the last paragraph.

6.1. *Examples.* If $A(x)=x^\alpha$ ($\alpha \geqq 0$) we obtain $m_2(x)=\frac{1}{\alpha+1}x^2$. If $A(x)=(\sinh x)^2$, $m_2(x)=x^2+2-2x\cosh x$ [26, p. 191] and in the case $A(x)=(\cosh x)^2$ we have $m_2(x)=x^2$.

6.2. Lemma. *$m_2(x)^2 \leqq m_2(x) \leqq x^2$ for every $x \geqq 0$.*

Proof. The first inequality follows from 4.3 and Jensen's inequality; the second has already been proved in 4.13. □

6.3. Corollary. *Let $\varrho>0$. Then $\lim_{x\to\infty}\frac{1}{x^2}m_2(x)=1$.*

6.4. Lemma. *If $\varrho=0$ then m_2 is a convex function and $\lim_{x\to\infty}\frac{m_2(x)}{x}=+\infty$.*

Proof. The convexity of m_2 follows in the same way as the convexity of m_1 in the first part of the proof of 5.7. The assumption that m_2' is bounded leads to a contradiction since it implies

$$\lim_{x\to\infty} m_2''(x)=2-\lim_{x\to\infty} m_2'(x)\frac{A'(x)}{A(x)}=2.$$

Hence m_2' is unbounded. But from $m_2'(x)\nearrow\infty$ as $x\to\infty$ follows $\lim_{x\to\infty}\frac{m_2(x)}{x}=+\infty$ by the mean value theorem. □

6.5. Lemma. *Suppose that $\varrho=0$ and $\left\{x\frac{A'(x)}{A(x)}: x>0\right\}$ is bounded. Then there exists $\gamma>0$ such that $m_2(x)\geqq\gamma x^2$ for $x\geqq 0$.*

Proof. It follows from the differential equation $m_2''+\frac{A'}{A}m_2'=2$ that the function ψ defined on $\mathbb{R}_+$ by

$$\psi(x)=\begin{cases} m_2'(x)/x & \text{for } x>0 \\ m_2''(0)=\dfrac{2}{\alpha+1} & \text{for } x=0 \end{cases}$$

satisfies the differential equation

$$x\psi'(x)+\left(x\frac{A'(x)}{A(x)}+1\right)\psi(x)=2, \qquad \psi(0)=\frac{2}{\alpha+1}.$$

Let b be an upper bound for $x\frac{A'(x)}{A(x)}$ $(x>0)$. Then for every x such that $\psi(x)<\frac{2}{b+1}$ we obtain $\psi'(x)>0$ and so ψ is certainly bounded from below by 2γ where $\gamma:=\min\left(\frac{1}{b+1},\frac{1}{\alpha+1}\right)$. But this implies $m_2'(x)\geqq 2\gamma x$ and hence the result. □

6.6. Definition. In order to define the $*$-*variance* for every Chébli-Trimèche hypergroup $(\mathbb{R}_+, *)$ with corresponding functions m_1 and m_2 we introduce the function $v: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $v(x, \xi) := m_2(x) - 2\xi m_1(x) + \xi^2$ $(x, \xi \geqq 0)$ which is non-negative by 6.2. For every $\mathbb{R}_+$-valued random variable X such that $E(m_2(X)) < \infty$ the function $\xi \mapsto E(v(X, \xi))$ on $\mathbb{R}$ takes its minimum at $\xi = E_*(X)$, this value being

$$V_*(X) := E(v(X, E_*(X))) = E(m_2(X)) - E(m_1(X))^2 \geqq 0 .$$

If $E(m_2(X)) = \infty$ we define $V_*(X) = \infty$. $V_*(X)$ is called the $*$-*variance* of X.

6.7. *Remark.* In the case $\varrho = 0$ $V_*(X)$ equals $E(m_2(X))$; this number is called the "dispersion" of X in [24].

6.8. *Remark.* $V_*(X)$ is strictly positive unless $X = 0$ P-a.s. At the first look it is surprising that the $*$-variance of a constant $X \neq 0$ does not equal zero. But it reflects the fact that $X \overset{\Lambda}{+} Y$ is random even if $X > 0$ and $Y > 0$ are deterministic.

6.9. Proposition. *If* $\varrho > 0$ *or if* $\left\{ x \cdot \dfrac{A'(x)}{A(x)} : x > 0 \right\}$ *is bounded,* $V_*(X)$ *exists if and only if* $E(X^2) < \infty$.

Proof. This is a consequence of 6.3 in the first case, and 6.2 and 6.5 in the second. □

6.10. Proposition. *Let X and Y be independent* $\mathbb{R}_+$*-valued random variables. Then*

$$V_*(X \overset{\Lambda}{+} Y) = V_*(X) + V_*(Y) .$$

Both sides of this equation may be infinite.

Proof. If $E(m_2(X))$ or $E(m_2(Y))$ are infinite it follows from 4.14 that $E(m_2(X \overset{\Lambda}{+} Y))$ and hence $V_*(X \overset{\Lambda}{+} Y)$ equals $+\infty$. Let us therefore suppose that $V_*(X) < \infty$ and $V_*(Y) < \infty$. It follows from 4.14 that

$$\begin{aligned} V_*(X \overset{\Lambda}{+} Y) &= E(m_2(X \overset{\Lambda}{+} Y)) - E(m_1(X \overset{\Lambda}{+} Y))^2 \\ &= E(m_2(X)) + 2E(m_1(X))E(m_1(Y)) + E(m_2(Y)) \\ &\quad - E(m_1(X))^2 - 2E(m_1(X))E(m_1(Y)) - E(m_1(Y))^2 \\ &= V_*(X) + V_*(Y) . \quad \square \end{aligned}$$

6.11. *Remark.* If X and Y are not independent but only $*$-*uncorrelated* (in the sense that

$$E(m_1(X) m_1(Y)) = E_*(X) E_*(Y))$$

then the assertion of Proposition 6.10 remains valid.

7. Laws of Large Numbers in the Case of Exponential Growth

Recall that (M, μ, Φ) denotes a fixed concretization of a Chébli-Trimèche hypergroup $(\mathbb{R}_+, *)$.

7.1. Proposition. *Let X, Y, and Λ be independent* $\mathbb{R}_+$-, $\mathbb{R}_+$-, *and M-valued random variables such that* $P_\Lambda = \mu$.

a) *If* $E_*(X)$ *and* $E_*(Y)$ *are finite, then* $E(m_1(X \overset{A}{+} Y)|X) = m_1(X) + E_*(Y)$ *P-almost surely.*

b) *If* $V_*(X)$ *and* $V_*(Y)$ *are finite, then*

$$E(v((X \overset{A}{+} Y),\, E_*(X \overset{A}{+} Y))|X) = v(X, E_*(X)) + V_*(Y) \quad P\text{-a.s.}$$

c) *If* $|\Im\lambda| \leqq \varrho$, *then*

$$E(\varphi_\lambda(X \overset{A}{+} Y)|X) = \varphi_\lambda(X) \cdot E(\varphi_\lambda(Y)) \quad P\text{-a.s.}$$

Proof. a) Let $A \in \mathfrak{B}(\mathbb{R}_+)$. Then by 5.6 it follows

$$\begin{aligned} E(1_{\{X\in A\}} \cdot m_1(X \overset{A}{+} Y)) &= E(m_1((1_{\{X\in A\}}X) \overset{A}{+} Y)) - E(1_{\{X\notin A\}} m_1(Y)) \\ &= E(m_1(1_{\{X\notin A\}}X)) + E_*(Y) - P\{X \notin A\} E_*(Y) \\ &= E(1_{\{X\in A\}} \cdot [m_1(X) + E_*(Y)]) . \end{aligned}$$

b) For every $A \in \mathfrak{B}(\mathbb{R}_+)$ we conclude from 4.14 that

$$\begin{aligned} E(1_{\{X\in A\}} \cdot m_2(X \overset{A}{+} Y)) &= E(m_2(1_{\{X\in A\}}X) \overset{A}{+} Y)) - E(1_{\{X\notin A\}} m_2(Y)) \\ &= E(m_2(1_{\{X\in A\}}X)) + 2E(m_1(1_{\{X\in A\}}X))E_*(Y) \\ &\quad + E(m_2(Y)) - P\{X \notin A\} E(m_2(Y)) \\ &= E(1_{\{X\in A\}} \cdot [m_2(X) + 2m_1(X)E_*(Y) + E(m_2(Y))]) . \end{aligned}$$

Therefore $E(m_2(X \overset{A}{+} Y)|X) = m_2(X) + 2m_1(X)E_*(Y) + E(m_2(Y))$ P-almost surely and hence

$$\begin{aligned} E(v(X \overset{A}{+} Y, E_*(X \overset{A}{+} Y))|X) &= E(m_2(X \overset{A}{+} Y)|X) \\ &\quad - 2E_*(X \overset{A}{+} Y) E(m_1(X \overset{A}{+} Y)|X) + E_*(X \overset{A}{+} Y)^2 \\ &= m_2(X) + 2m_1(X)E_*(Y) + E(m_2(Y)) \\ &\quad - 2(E_*(X) + E_*(Y))\,(m_1(X) + E_*(Y)) \\ &\quad + (E_*(X) + E_*(Y))^2 \\ &= m_2(X) - 2m_1(X)E_*(X) + E_*(X)^2 \\ &\quad + E(m_2(Y)) - E_*(Y)^2 \\ &= v(X, E_*(X)) + V_*(Y) \quad P\text{-a.s.} \end{aligned}$$

c) Since φ_λ is a bounded multiplicative function we obtain for every $A \in \mathfrak{B}(\mathbb{R}_+)$

$$\begin{aligned} E(1_{\{X\in A\}} \cdot \varphi_\lambda(X \overset{A}{+} Y)) &= E(\varphi_\lambda((1_{\{X\in A\}}X) \overset{A}{+} Y)) - E(1_{\{X\notin A\}} \varphi_\lambda(Y)) \\ &= E(\varphi_\lambda(1_{\{X\in A\}}X))E(\varphi_\lambda(Y)) - P\{X \notin A\} E(\varphi_\lambda(Y)) \\ &= (E(1_{\{X\in A\}}\varphi_\lambda(X)) + P\{X \notin A\})E(\varphi_\lambda(Y)) \\ &\quad - P\{X \notin A\} E(\varphi_\lambda(Y)) \\ &= E(1_{\{X\in A\}} \cdot \varphi_\lambda(X) E(\varphi_\lambda(Y))) . \quad \square \end{aligned}$$

7.2. *Notation.* For the rest of this article we suppose that $X_1, X_2, \ldots, A_1, A_2, \ldots$ are independent $\mathbb{R}_+$- resp. M-valued random variables such that $P_{A_n} = \mu$ for every $n \geqq 1$. It follows from 3.3 that the process $(S_n : n \geqq 0)$ where $S_n := {}^{A}\sum_{j=1}^{n} X_j$ is a (non homogeneous) Markov chain.

7.3. Corollary. a) *If $E_*(X_n) < \infty$ resp. $V_*(X_n) < \infty$ for $n \geqq 1$ then $(m_1(S_n) : n \in \mathbb{N})$ resp. $(v(S_n, E_*(S_n)) : n \in \mathbb{N})$ are submartingales with respect to the canonical filtration.*

b) *If $\lambda \in i[0, \varrho]$ then $(\varphi_\lambda(S_n) : n \in \mathbb{N})$ is a supermartingale.*

Proof. From 7.1 a) we obtain for every $n \geqq 1$

$$\begin{aligned} E(m_1(S_n)|S_{n-1}) &= E(m_1(X_n \overset{A}{+} S_{n-1})|S_{n-1}) \\ &= m_1(S_{n-1}) + E_*(X_n) \geqq m_1(S_{n-1}) \quad P\text{-a.s.} \end{aligned}$$

The other assertions can be proved in the same way. □

Note that this corollary holds for any hypergroup K and $\mathbb{R}_+$-valued functions m_1 and m_2 on K such that 4.14 and $m_2 \geqq m_1^2$ hold.

For the rest of this paragraph we suppose $\varrho > 0$. In view of $A(x) \geqq A(1) \cdot e^{2\varrho(x-1)}$ for $x \geqq 1$ this implies that $(\mathbb{R}_+, *)$ is of exponential growth.

7.4. Theorem. *Let $(X_n : n \geqq 1)$ be an independent series of $\mathbb{R}_+$-valued random variables such that $\sum_{n=1}^{\infty} \frac{1}{n^2} V_*(X_n) < \infty$. Then*

$$\frac{1}{n}(S_n - m_1^{-1}(E_*(S_n))) \to 0 \quad P\text{-a.s.}$$

Proof. Let $s_n := E_*(S_n)$. It follows from 6.2 and 7.3 a), that $(v(S_n, s_n) : n \geqq 1)$ is a positive submartingale. Furthermore, the assumptions and 6.10 imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E(v(S_n, s_n) - v(S_{n-1}, s_{n-1})) < \infty .$$

Hence by Chow's law of large numbers [6] and 6.2 we obtain

$$\left(\frac{m_1(S_n) - s_n}{n}\right)^2 \leqq \frac{1}{n^2} v(S_n, s_n) \to 0 \quad P\text{-a.s.}$$

Since $(m_1^{-1})'(t) \searrow 1$ as $t \to \infty$ there is a number $a > 0$ such that $|m_1^{-1}(x) - m_1^{-1}(y)| \leqq 2|x - y| + a$ for all $x, y \in \mathbb{R}_+$. Therefore $\frac{m_1(S_n) - s_n}{n} \to 0$ implies $\lim_{n \to \infty} \frac{1}{n}(S_n - m_1^{-1}(s_n)) = 0$. □

7.5. *Remarks.*

7.5.1. If in the situation of the preceding theorem we assume additionally that $\frac{1}{n} E_*(S_n)$ is bounded we obtain

$$\lim_{n \to \infty} \frac{1}{n}(S_n - E_*(S_n)) = 0 \quad P\text{-a.s.}$$

This is a consequence of

$$\frac{1}{n}(m_1^{-1}(E_*(S_n))-E_*(S_n))=\frac{1}{n}E_*(S_n)\cdot\left(\frac{m_1^{-1}(E_*(S_n))}{E_*(S_n)}-1\right)\to 0$$

(compare 5.7).

7.5.2. If in the situation of the preceding theorem $\eta := \lim_{n\to\infty}\frac{1}{n}E_*(S_n)$ exists, then

$$\lim_{n\to\infty}\frac{1}{n}S_n=\eta \quad P\text{-a.s.}$$

7.5.3. Under additional assumptions on the function A we can obtain $m_1(x)=x+O(1)$ for $x\to\infty$. Then the conclusion of Theorem 7.4 may be written as $\lim_{n\to\infty}\frac{1}{n}(S_n-E_*(S_n))=0$ P-a.s.

7.6. Corollary. *Let $(X_n : n\geqq 1)$ be an i.i.d. sequence of integrable random variables. Then*

$$\frac{1}{n}S_n\to E_*(X_1) \quad P\text{-a.s.}$$

Proof. Let $a>0$ be arbitrary, consider the truncated variables $X_n^a := 1_{\{X_n<na\}}\cdot X_n$, and define $S_0^a := 0$, $S_n^a := S_{n-1}^a \overset{A_n}{+} X_n^a$, $s_n^a := E_*(S_n^a)$ for $n\geqq 1$, using the same A_n's as in the definition of S_n. By Lemma 6.2 we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2}V_*(X_n^a) &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{j=0}^{\infty}E(1_{\{aj\leqq X_n<a(j+1)\}}v(X_n^a, E_*(X_n^a)))\\
&\leqq \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\sum_{j=0}^{n-1}P\{aj\leqq X_n<a(j+1)\}\,(E_*(X_n)^2+a^2(j+1)^2)\right.\\
&\qquad \left.+\sum_{j=n}^{\infty}P\{aj\leqq X_n<a(j+1)\}\cdot E_*(X_n)^2\right\}\\
&\leqq \sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\sum_{j=0}^{n-1}P\{aj\leqq X_1<a(j+1)\}\cdot a^2(j+1)^2\right.\\
&\qquad \left.+\sum_{j=0}^{\infty}P\{aj\leqq X_1<a(j+1)\}E_*(X_1)^2\right\}\\
&= \sum_{j=0}^{\infty}P\{aj\leqq X_1<a(j+1)\}\\
&\qquad \times\left\{a^2(j+1)^2\cdot\sum_{n=j+1}^{\infty}\frac{1}{n^2}+E_*(X_1)^2\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}\right\}\\
&\leqq \sum_{j=0}^{\infty}2a^2(j+1)P\{aj\leqq X_1<a(j+1)\}+\frac{\pi^2}{6}\cdot E_*(X_1)^2\\
&\leqq 2a^2+2aE(X_1)+\frac{\pi^2}{6}E_*(X_1)^2<\infty\,.
\end{aligned}$$

On the other hand it follows from $\lim_{n\to\infty} E_*(X_n^a)=E_*(X_1)$ that

$$\lim_{n\to\infty}\frac{1}{n}s_n^a=E_*(X_1).$$

Hence 5.5.2 implies $\lim_{n\to\infty}\frac{1}{n}S_n^a=E_*(X_1)$ P-a.s. for every $a>0$.

The probability of $\Omega_a:=\{X_n<na \text{ for all } n\geqq 1\}$ is

$$\begin{aligned}P(\Omega_a)&=1-P\{X_n\geqq na \text{ for some } n\geqq 1\}\\&\geqq 1-\sum_{n=1}^{\infty}P\{X_n\geqq na\}\\&\geqq 1-\frac{1}{a}E(X_1).\end{aligned}$$

Since in the definition of S_n^a the same A_n's were used as in the construction of S_n we obtain that $S_n=S_n^a$ on Ω_a and hence $\lim_{n\to\infty}\frac{1}{n}S_n=E_*(X_1)$ P-a.s. on Ω_a. Since $P(\Omega_a)\to 1$ as $a\to\infty$ the corollary is proved. □

7.7. Theorem. *Let $(X_n: n\geqq 1)$ be an i.i.d. sequence of random variables with $E_*(X_1)=+\infty$. Then $\frac{1}{n}S_n\to\infty$ P-a.s.*

Proof. Let a be an arbitrary positive number. We will show that $P\left\{\frac{1}{n}S_n<a \text{ i.o.}\right\}=0$. This can be done by proving

$$\sum_{n\geqq 1}P\left\{\frac{1}{n}S_n<a\right\}<\infty$$

and using the Borel-Cantelli lemma.

Consider the functions $\lambda\mapsto E(\varphi_{i(\varrho-\lambda)}(X_1))$ and $\lambda\mapsto e^{-a\lambda}$. Since the derivations at 0 of these functions are $-\infty$ (by 5.9) and $-a$ there exists $\lambda\in]0,\varrho[$ such that

$$0<e^{a\lambda}\cdot E(\varphi_{i(\varrho-\lambda)}(X_1))<1.$$

Therefore from $\varphi_{i(\varrho-\lambda)}(x)=\int_{-x}^{x}e^{-t\lambda}v_x(dt)\geqq e^{-\lambda x}$ it follows

$$\begin{aligned}P\left\{\frac{1}{n}S_n<a\right\}&=P\{\varphi_{i(\varrho-\lambda)}(S_n)>\varphi_{i(\varrho-\lambda)}(an)\}\\&\leqq P\{\varphi_{i(\varrho-\lambda)}(S_n)>e^{-\lambda an}\}\\&\leqq e^{\lambda an}\cdot E(\varphi_{i(\varrho-\lambda)}(S_n))\\&=(e^{\lambda a}\cdot E(\varphi_{i(\varrho-\lambda)}(X_1))^n\end{aligned}$$

and finally

$$\sum_{n\geqq 1}P\left\{\frac{1}{n}S_n<a\right\}\leqq\sum_{n\geqq 1}(e^{\lambda a}E(\varphi_{i(\varrho-\lambda)}(X_1)))^n<\infty. \quad \square$$

7.8. *Remark.* In 7.2 we have only considered the case of a random walk starting at the neutral element 0 of $(\mathbb{R}_+, *)$. However, 7.6 and 7.7 (and clearly 7.4) remain valid if the starting point is arbitrarily distributed. A short look at the proofs of 7.6 and 7.7 shows that it suffices to suppose that $(X_2, X_3, \ldots)$ are identically distributed (X_1 even does not need to be integrable): $X_1 = S_1$ can then be considered as the starting point of the random walk $(S_n : n \geqq 1)$.

7.9. *Remark.* Let $(\mathbb{R}_+, *)$ be the Sturm-Liouville hypergroup with $A(x) = (\cosh x)^2$ (see [28, Example 2.5c]). This is not a Chébli-Trimèche hypergroup in the sense of 2.2 since $A(0) \neq 0$. However, the assertions of 7.4, 7.5.3, 7.6, and 7.7 remain valid. In this case $\varrho = 1$, $m_1(x) = x \tanh x$, $m_2(x) = x^2$, and $\varphi_\lambda(x) = \dfrac{\cos \lambda x}{\cosh x}$ $(x \geqq 0, \lambda \in \mathbb{C})$ and it is easily checked that the facts used in the proofs of 7.4, 7.6, and 7.7 also hold in this situation.

8. Laws of Large Numbers in the Case of Exponential Boundedness

8.1. In this paragraph we suppose that $\varrho = 0$. This implies $E_*(X) = 0$ for every random variable and therefore we expect the law of large numbers to be of a particularly simple form. For example if (in the terminology of 6.11) the variances $V_*(X_j) = E(m_2(X_j))$ are bounded by some constant $b > 0$ and the variables X_j are pairwise $*$-uncorrelated we obtain for every $\varepsilon > 0$

$$\begin{aligned} P\left\{\frac{1}{n} S_n \geqq \varepsilon\right\} &= P\{m_2(S_n) \geqq m_2(n\varepsilon)\} \\ &\leqq \frac{V_*(S_n)}{m_2(n\varepsilon)} \\ &\leqq \frac{nb}{m_2(n\varepsilon)} \to 0 \end{aligned}$$

by 6.4 and hence

$$\frac{1}{n} S_n \to 0 = E_*(X_j) \quad \text{in probability}.$$

8.2. However, the proof of a *strong* law becomes more difficult and requires some restrictions concerning the function m_2. For the rest of this paragraph we have to suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $m_2(\varepsilon x) \geqq \delta m_2(x)$ for every $x \geqq 0$.

8.3. *Examples*

8.3.1. If $\left\{x \dfrac{A'(x)}{A(x)} : x > 0\right\}$ is bounded, then 8.2 holds. This follows from 6.5. This criterium is useful if $\dfrac{A'}{A}$ decreases fast. The opposite case is considered in the following example.

8.3.2. Suppose that there is a $c > 0$ with $\dfrac{A'(2x)}{A(2x)} \geqq c \cdot \dfrac{A'(x)}{A(x)}$ for $x > 0$. Then 8.2 holds.

Proof. We consider the function $\varphi: \mathbb{R}_+ \to \mathbb{R}$ defined by $\varphi(x) := 4m_2\left(\frac{x}{2}\right) - 2cm_2(x)$ for $x \geqq 0$. From the convexity of m_2 (6.4) and 4.2 we obtain $m_2'\left(\frac{x}{2}\right) \cdot \frac{A'(x/2)}{A(x/2)} \leqq 2$ and hence

$$\left[2\frac{A'(x)}{A(x)} - \frac{A'(x/2)}{A(x/2)}\right] m_2'\left(\frac{x}{2}\right) \geqq (2c-1)\frac{A'(x/2)}{A(x/2)} m_2'\left(\frac{x}{2}\right) \geqq 4c-2.$$

An easy calculation yields

$$L\varphi(x) = 4c - 2 + \left[\frac{A'(x/2)}{A(x/2)} - 2\frac{A'(x)}{A(x)}\right] m_2'\left(\frac{x}{2}\right) \leqq 0.$$

Therefore the assumption $\varphi'(x_0) < 0$ leads to $\varphi''(x_0) > 0$ and hence $\varphi'(0) < 0$ which is a contradiction to $\varphi'(0) = 2m_2'(0) - 2cm_2'(0) = 0$. This implies $m_2\left(\frac{x}{2}\right) > \frac{c}{2} m_2(x)$ for every $x \geqq 0$.

Now let $\varepsilon > 0$. Then there is an $n \in \mathbb{N}$ with $2^{-n} < \varepsilon$ and we obtain

$$m_2(\varepsilon x) \geqq m_2(2^{-n}x) \geqq \left(\frac{c}{2}\right)^n m_2(x)$$

for every $x \geqq 0$. □

8.4. Theorem. *Suppose that* 8.2 *holds. Let* $(X_n : n \geqq 1)$ *be a series of independent random variables such that*

$$\sum_{n \geqq 1} \frac{1}{m_2(n)} V_*(X_n) < \infty.$$

Then $\frac{1}{n} S_n \to 0$ *P-almost surely.*

Proof. Since it follows from the assumption and 6.10 that

$$\sum_{n=1}^{\infty} \frac{1}{m_2(n)} E(m_2(S_n) - m_2(S_{n-1})) < \infty$$

we may apply Chow's law of large numbers [6] in order to obtain

$$\lim_{n \to \infty} \frac{1}{m_2(n)} m_2(S_n) = 0 \quad P\text{-a.s.}$$

But if for some $\varepsilon > 0$ $S_n > \varepsilon n$ happens infinitely often with positive probability and if $\delta(\varepsilon)$ is chosen according to 8.2 this would imply $\frac{m_2(S_n)}{m_2(n)} \geqq \frac{m_2(\varepsilon n)}{m_2(n)} \geqq \delta(\varepsilon)$ infinitely often with positive probability. Hence the assertion of the theorem. □

8.5. *Remark.* Even if the sequence $(X_n : n \geqq 1)$ is i.i.d. with $V_*(X_n) < \infty$ the condition of the preceding theorem not necessarily holds since $\sum_{n \geqq 1} \frac{1}{m_2(n)}$ may be infinite. Since it follows from 6.4 that $\frac{A'}{A} m_2' \leqq 2$ and hence $m_2(x) \leqq 2x \frac{A(x)}{A'(x)}$ this happens for example if $\frac{A'(x)}{A(x)} \sim \frac{1}{\ln x}$ for $x \to \infty$.

8.6. Corollary. *Suppose that* $\left\{x\cdot\frac{A'(x)}{A(x)}: x>0\right\}$ *is bounded. Then for every i.i.d. sequence* $(X_n: n\geqq 1)$ *of integrable* $\mathbb{R}_+$*-valued random variables* $\frac{1}{n}S_n\to 0$ *P-almost surely.*

Proof. Let $a>0$. As in the proof of 7.6 we consider the truncated variables $X_n^a := 1_{\{X_n<an\}}\cdot X_n$ and define $S_0^a := 0$, $S_n^a := S_{n-1}^a \overset{A_n}{+} X_n^a$ for $n\geqq 1$. Let γ be defined as in Lemma 6.5. Since X_1 is integrable,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{m_2(n)}V_*(X_n^a) &\leqq \frac{1}{\gamma}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{j=0}^{\infty}E(1_{\{aj\leqq X_n^a<a(j+1)\}}m_2(X_n^a))\\ &\leqq \frac{1}{\gamma}\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{j=0}^{n-1}P\{aj\leqq X_1<a(j+1)\}a^2(j+1)^2\\ &=\frac{1}{\gamma}\sum_{j=0}^{\infty}P\{aj\leqq X_1<a(j+1)\}\cdot a^2(j+1)^2\sum_{n=j+1}^{\infty}\frac{1}{n^2}\\ &\leqq \frac{2}{\gamma}\sum_{j=0}^{\infty}P\{aj\leqq X_1<a(j+1)\}\cdot a^2(j+1)\\ &\leqq \frac{2}{\gamma}(aE(X_1)+a^2)<\infty\end{aligned}$$

we obtain from 8.3.1 and the preceding theorem that $\frac{1}{n}S_n^a\to 0$ P-a.s. for every $a>0$. The rest of the proof is identical with 7.6. □

8.7. *Remark.* By the same argument as in 7.8. we see that 8.6 is valid even if the starting point of the random walk $(S_n: n\in\mathbb{N})$ is not 0 but arbitrary.

8.8. *Remark.* Suppose that $\left\{x\cdot\frac{A'(x)}{A(x)}: x>0\right\}$ is bounded and let $0<\beta<2$. Then for every i.i.d. sequence $(X_n: n\geqq 1)$ of $\mathbb{R}_+$-valued random variables such that $E(X_1^\beta)$ is finite, $\frac{1}{n^{1/\beta}}S_n\to 0$ P-almost surely. A similar result has been proved by Gallardo and Ries [15].

Proof. It is a straightforward generalization of Theorem 8.4 that for every independent sequence $(Y_n: n\geqq 1)$ such that $\sum_{n=1}^{\infty}\frac{1}{n^{2/\beta}}V_*(Y_n)<\infty$ we obtain $\frac{1}{n^{1/\beta}}\overset{A}{\sum_{j=1}^{n}}Y_j\to 0$ P-almost surely. If we choose Y_n to be the truncated variable $X_n 1_{\{X_n\leqq an^{1/\beta}\}}$ it follows as in the proof of 8.6 that $\frac{1}{n^{1/\beta}}S_n$ and $\frac{1}{n^{1/\beta}}\overset{A}{\sum_{j=1}^{n}}Y_j$ tend to the same limit almost surely. □

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