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A $W^{1,p}$-Estimate for Solutions to Mixed Boundary Value Problems for Second Order Elliptic Differential Equations

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1. Introduction

In this paper we shall prove that, under rather weak hypotheses, any solution to a mixed boundary value problem for a second order elliptic differential equation is in the Sobolev space $W^{1,p}$ for some $p>2$. Our starting point is the following result due to Meyers [6]:

If for some $q>2$ it holds the implication

$$u \in W_0^{1,2}(G), \qquad \Delta u \in W^{-1,q}(G) \Rightarrow u \in W_0^{1,q}(G), \tag{1.1}$$

then for every positive definite matrix (a_{ij}) of bounded measurable functions on G there exists a $p>2$ such that

$$u \in W_0^{1,2}(G), \qquad \sum_{i,j=1}^{N} D_i(a_{ij}D_j u) \in W^{-1,p}(G) \Rightarrow u \in W_0^{1,p}(G). \tag{1.2}$$

Here G is a bounded domain in $\mathbb{R}^N$, D_i denotes the derivative with respect to the coordinate x_i of $x=(x_1,\ldots,x_N)\in\mathbb{R}^N$, and $W_0^{1,p}(G)$, $W^{-1,p}(G)$ are the usual Sobolev spaces.

The hypothesis (1.1) is satisfied for every $q\in\,]2,\infty[$ provided that G is a bounded domain of class C^1 (see Simader [8, Theorem 4.6]). We shall show that an analogue of Meyers' result holds if the homogeneous Dirichlet boundary condition [which is included in the requirement $u\in W_0^{1,2}(G)$] is replaced by a mixed boundary condition. Moreover, we are going to prove that results of the type (1.2) can be obtained not only for differential operators of the form

$$u \mapsto \sum_{i,j=1}^{N} D_i(a_{ij}D_j u)$$

but also for (generally nonlinear) operators of the form

$$u \mapsto \sum_{i=1}^{N} D_i b_i(\cdot, u, Du) + b_0(\cdot, u, Du);$$

here and later on Du denotes the gradient of u, and the dot indicates the dependence on the spatial variable. For precise assumptions with respect to $b=(b_0, \ldots, b_N)$ see Sect. 4. It is well known that in the case of mixed boundary conditions in general one cannot expect an analogue of (1.1) to hold for *every* $q>2$. We are able to prove, however, that under weak hypotheses there exists *some* $q>2$ such that an analogue of (1.1) holds also in that case. For this proof it is essential that we deal with second order differential operators only. ($W^{k,p}$-estimates for solutions to elliptic equations of order $2k$ in case of smooth boundary conditions were presented, for example by Nečas [7] and Krbec [5].)

Let us mention that local L^p-estimates for gradients of solutions to nonlinear problems were obtained also by other authors (see Giaquinta and Giusti [3] and the papers quoted therein). However, to our knowledge these authors made no attempt to prove global estimates for solutions to mixed boundary value problems.

The paper is organized as follows. In Sect. 2 we shall introduce the notation and some notions needed later on. In particular, we shall define a class of subsets of $\mathbb{R}^N$ called *regular*. This class turns out to be quite useful for the formulation of results on general boundary value problems. For any regular $G \subset \mathbb{R}^N$ we shall introduce spaces $W_0^{1,p}(G)$ and $W^{-1,p}(G)$ coinciding with the usual Sobolev spaces provided that G is open. We shall denote by R_q the class of all regular subsets of $\mathbb{R}^N$ for which an analogue of (1.1) holds. Section 3 contains some preliminary results. In particular, it will be shown that the validity of the relation $G \in R_q$ depends on the local properties of G only. In Sect. 4 we shall prove that the relation $G \in R_q$ implies a regularity result of type (1.2). We shall take advantage of an iteration procedure which has widely been used by Koševev in order to prove other regularity results (see [4]). In Sect. 5 we shall show that if $G \in R_q$, $q>2$, and if $\tilde{G}$ is obtained from G by a Lipschitzian transformation, then there exists a $p>2$ such that $\tilde{G} \in R_p$. From this fact it will follow that for every regular subset G of $\mathbb{R}^N$ there exists a $q>2$ such that $G \in R_q$.

2. Notations and Notation

If G is any subset of the Euclidean N-space $\mathbb{R}^N$, then we denote by $\mathring{G}$, ∂G and $\bar{G}$ the interior, the boundary, and the closure of G, respectively. We write $|x|$ for the Euclidean norm of $x \in \mathbb{R}^N$.

Assume that u is a solution to a second order elliptic differential equation in a domain $\Omega \subset \mathbb{R}^N$. Let u satisfy a Dirichlet condition on $\tilde{\Gamma} \subset \partial\Omega$ and natural boundary conditions on $\Gamma := \partial\Omega \setminus \tilde{\Gamma}$. If one wants to prove a regularity result for u, then one has to impose an appropriate "regularity condition" on Γ and $\tilde{\Gamma}$. We are going to show that it is useful to formulate all conditions and results in terms of $G := \Omega \cup \Gamma$.

Definition 1. Let G and $\tilde{G}$ be subsets of $\mathbb{R}^N$. A bijection $\Phi: G \to \tilde{G}$ will be called a *Lipschitz-transformation*, if Φ and Φ^{-1} are Lipschitzian with respect to the standard metrics of G and $\tilde{G}$.

Definition 2. We shall call $G \subset \mathbb{R}^N$ *regular*, if G is bounded and if for every $y \in \partial G$ there exist subsets U and $\tilde{U}$ of $\mathbb{R}^N$ and a Lipschitz-transformation $\Phi: U \to \tilde{U}$ such that U is an open neighbourhood of y in $\mathbb{R}^N$ and that $\Phi(U \cap G)$ is one of the following sets:

$$E_1 := \{x \in \mathbb{R}^N : |x| < 1,\ x_N < 0\},$$
$$E_2 := \{x \in \mathbb{R}^N : |x| < 1,\ x_N \leqq 0\},$$
$$E_3 := \{x \in E_2 : x_N < 0 \text{ or } x_1 > 0\}.$$

Remark 1. Apart from boundedness regularity of G means, roughly speaking, that the parts $\Gamma := G \setminus \mathring{G}$ and $\tilde{\Gamma} := \bar{G} \setminus G$ of the boundary ∂G are separated by a Lipschitzian hypersurface of ∂G.

Remark 2. In the following we shall assume always – even if this is not mentioned explicitly – that G is a regular subset of $\mathbb{R}^N$. Then G is of finite Lebesgue measure. The boundary $\partial G = \partial \mathring{G}$ is of N-dimensional Lebesgue measure 0. Therefore we are allowed to identify the spaces $L^p(G)$ and $L^p(\mathring{G})$.

Definition 3. For $1 \leqq p \leqq \infty$ we denote by $W_0^{1,p}(G)$ the closure of the set

$$\{u \,|\, \mathring{G} : u \in C_0^\infty(\mathbb{R}^N),\ \operatorname{supp} u \cap (\bar{G} \setminus G) = \emptyset\}$$

in the Sobolev space $W^{1,p}(\mathring{G})$, equipped with the standard norm of that space. The space dual to $W_0^{1,p'}(G)$ will be denoted by $W^{-1,p}(G)$; here (and later on) p' denotes the exponent conjugate to p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. For the norms in $W_0^{1,p}(G)$ and $W^{-1,p}(G)$ we write $\|\cdot\|_{1,p}$ and $\|\cdot\|_{-1,p}$, respectively. If necessary we indicate the dependence of these norms on G by an additional index. By J_G we denote the duality map of the Hilbert space $W_0^{1,2}(G)$.

Remark 3. If G is open, then our definition of $W_0^{1,p}(G)$ coincides with the usual one. If G is closed then $W_0^{1,p}(G) = W^{1,p}(\mathring{G})$.

Remark 4. If this should not lead to misunderstandings we write $W_0^{1,p}$, $W^{-1,p}$, and J instead of $W_0^{1,p}(G)$, $W^{-1,p}(G)$, and J_G, respectively.

Remark 5. Let $1 \leqq p \leqq q \leqq \infty$. Then $W_0^{1,q} \hookrightarrow W_0^{1,p}$, and $W_0^{1,q}$ is dense in $W_0^{1,p}$ (the sign $\hookrightarrow$ means that the imbedding is continuous). Therefore we have $W^{-1,p'} \hookrightarrow W^{-1,q'}$.

Remark 6. From the formula

$$\langle Ju, v \rangle = \int_G (uv + Du \cdot Dv)\,dx, \quad \text{for} \quad u, v \in W_0^{1,2},$$

it follows easily that J maps $W_0^{1,p}$, $p > 2$, into $W^{-1,p}$ and that $J \,|\, W_0^{1,p}$ is continuous as a map from $W_0^{1,p}$ into $W^{-1,p}$. Throughout this paper, for $p \geqq 2$, we shall use M_p as an abbreviation for

$$\sup\{\|u\|_{1,p} : u \in W_0^{1,p},\ \|Ju\|_{-1,p} \leqq 1\}.$$

Note that $M_2 = 1$.

Definition 4. For $2 \leqq q < \infty$ we denote by R_q the class of all regular subsets G of $\mathbb{R}^N$ for which J_G maps $W_0^{1,q}(G)$ *onto* $W^{-1,q}(G)$.

Remark 7. If G is a bounded domain of class C^1 then $G \in \bigcap_{q \geqq 2} R_q$. This follows easily from a result stated by Simader (see [8, Theorem 4.6]). As mentioned in the introduction we shall show in Sect. 5 that for every regular G there exists a $q>2$ such that $G \in R_q$.

Remark 8. In view of the Open Mapping Theorem the relation $G \in R_q$ implies that $M_q < \infty$.

Let us introduce some further notation. By Y_p, $1<p<\infty$, we denote the space $L^p(G; \mathbb{R}^{N+1})$, equipped with its standard norm. The space dual to Y_p will be identified with $Y_{p'}$. Moreover, let $L \in \mathscr{L}(W_0^{1,2}; Y_2)$ be defined by $Lu := (u, Du)$, $u \in W_0^{1,2}$. Obviously, $J = L^*L$. It is easy to check that L maps $W_0^{1,p}$, $p>2$, continuously into Y_p and that L^* maps Y_p, $p>2$, continuously into $W^{-1,p}$.

3. Preliminary Results

Lemma 1. *Let $G \in R_q$ for some $q>2$. Then $G \in R_p$ for $2 \leqq p \leqq q$ and $M_p \leqq M_q^\theta$ if $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$.*

Proof. 1. Let $P \in \mathscr{L}(Y_2; Y_2)$ be defined by $P := LJ^{-1}L^*$. Since $G \in R_q$ the operator P maps Y_q continuously into itself. It is easy to check that $\|P\|_{\mathscr{L}(Y_2;Y_2)} = 1$ and that $\|P\|_{\mathscr{L}(Y_q;Y_q)} \leqq M_q$. In view of the well-known Riesz-Thorin Interpolation Theorem (see, e.g., Bergh and Löfström [1]) this implies that P maps Y_p continuously into itself and that $\|P\|_{\mathscr{L}(Y_p;Y_p)} \leqq M_2^{1-\theta} M_q^\theta$ provided that $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$, $\theta \in [0,1]$.

2. Let $2 \leqq p \leqq q$, and let $f \in W^{-1,p}$ be fixed. We define

$$\forall v \in W_0^{1,2} : z(Lv) := \langle f, v \rangle .$$

Because v is uniquely determined by Lv, this definition makes sense. Since $z(Lv) \leqq \|f\|_{-1,p} \|v\|_{1,p'}$, z is a continuous linear functional on a subspace of $Y_{p'}$. By the Hahn-Banach Theorem z can be extended to a functional on $Y_{p'}$ (again denoted by z) with the same norm. Thus, $z \in Y_p$ and $\|z\|_{Y_p} = \|f\|_{-1,p}$. Moreover, $L^*z = f$, because

$$\forall v \in W_0^{1,2} : \langle L^*z, v \rangle = \langle z, Lv \rangle = \langle f, v \rangle .$$

For $u := J^{-1}f$ we have $Lu = LJ^{-1}L^*z = Pz \in Y_p$ and $\|Lu\|_{Y_p} \leqq M_q^\theta \|z\|_{Y_p} = M_q^\theta \|f\|_{-1,p}$. Consequently, $u \in W^{1,p}(\mathring{G})$ and $\|u\|_{1,p} \leqq M_q^\theta \|f\|_{-1,p}$. To show that u is in $W_0^{1,p}$ we proceed as follows. We choose a sequence (f_n) from $W^{-1,q}$ converging in $W^{-1,p}$ to f. (Note that $W_0^{1,p'} \hookrightarrow W_0^{1,q'}$ with dense imbedding and that $W_0^{1,p'}$ is reflexive as a subspace of a reflexive space. This implies that $W^{-1,q}$ is dense in $W^{-1,p}$.) Let $u_n := J^{-1}f_n$. Then $u_n \in W_0^{1,q} \subset W_0^{1,p}$ and $\|u_n - u_m\|_{1,p} \leqq M_q^\theta \|f_n - f_m\|_{-1,p}$. Hence (u_n) converges in $W_0^{1,p}$. Its limit must be u since $J^{-1} : W^{-1,2} \to W_0^{1,2}$ is continuous. Thus, J^{-1} maps $W^{-1,p}$ continuously into $W_0^{1,p}$ and

$$M_p = \sup\{\|u\|_{1,p} : u \in W_0^{1,p},\ \|Ju\|_{-1,p} \leqq 1\} \leqq M_q^\theta .$$

Lemma 2. *Let $\{U_0, \dots, U_r\}$ be an open covering of G, and let $q \geqq 2$. If $U_i \cap G \in R_q$, $i=0, \dots, r$, then $G \in R_q$.*

Proof. Let $I_0 := \{\frac{1}{2}\}$, $I_k := \left[\frac{1}{2} - \frac{k}{N}, \frac{1}{2} - \frac{k-1}{N}\right[$, $k=1, \dots, l$, where l denotes the largest integer such that $\frac{1}{2} > \frac{l}{N}$, and let $I_{l+1} := \left]0, \frac{1}{2} - \frac{l}{N}\right[$. We shall prove the assertion for $\frac{1}{q} \in I_k$ by induction with respect to k. For $k=0$, i.e. $q=2$, the assertion is trivial. Now let the assertion be proved for all q such that $\frac{1}{q} \in I_k$ for some $k \leqq l$. We want to prove the assertion under the hypothesis $\frac{1}{q} \in I_{k+1}$. Then we can choose p such that $\frac{1}{p} \in I_k$ and $\frac{1}{p} \leqq \frac{1}{q} + \frac{1}{N}$. In view of Lemma 1 we have $U_i \cap G \in R_p$, $i=0, \dots, r$. By our induction hypothesis this implies that $G \in R_p$. Let $f \in W^{-1,q}(G)$ and $u := J^{-1} f$. Because of $G \in R_p$ we obtain $u \in W_0^{1,p}(G) \subset L^q(G)$ (the inclusion follows from Sobolev's Imbedding Theorem). We choose a partition of unity $\{\varphi_0, \dots, \varphi_r\}$ subordinate to the covering $\{U_0, \dots, U_r\}$. We want to show that each of the functions $\varphi_i u$, $i=0, \dots, r$, is an element of $W_0^{1,q}(G)$. We have $\varphi_i u \in W_0^{1,p}(U_i \cap G)$ and (in view of the choice of p)

$$\begin{aligned}
\forall v \in W_0^{1,2}(U_i \cap G)\colon & \int_{U_i \cap G} (\varphi_i u v + D(\varphi_i u) \cdot Dv) dx \\
&= \int_{U_i \cap G} (u \varphi_i v + Du \cdot D(\varphi_i v) + (uDv - vDu) \cdot D\varphi_i) dx \\
&= \langle f, \varphi_i v \rangle + \int_{U_i \cap G} (uDv - vDu) \cdot D\varphi_i dx \\
&\leqq c(\|f\|_{-1,q} + \|u\|_{1,p}) \|v\|_{1,q', U_i \cap G}.
\end{aligned}$$

In the dual pairing $\langle f, \varphi_i v \rangle$ the function $\varphi_i v$ is to be interpreted in the usual way as a function defined on G vanishing on $G \setminus U_i$. The estimate shows that $J_{U_i \cap G}(\varphi_i u) \in W^{-1,q}(U_i \cap G)$. Consequently, $\varphi_i u \in W_0^{1,q}(U_i \cap G)$ and $u = \varphi_0 u + \dots + \varphi_r u \in W_0^{1,q}(G)$. This result completes the proof of Lemma 2.

Lemma 3. *For every $q \geqq 2$ the sets E_1 and E_2 are in the class R_q.*

Proof. Let $E := \{x \in \mathbb{R}^N : |x| < 1\}$, and let $q \geqq 2$ and $i \in \{1, 2\}$ be fixed. For any $u \in W_0^{1,2}(E_i)$ we define

$$(Su)(x) := \begin{cases} u(x) & \text{for} \quad x \in E_i, \\ (-1)^i u(x', -x_N) & \text{for} \quad x = (x', x_N) \in E \setminus E_i. \end{cases}$$

Clearly, if $u \in W_0^{1,q}(E_i)$ then $Su \in W_0^{1,q}(E)$. We fix $f \in W^{-1,q}(E_i)$ and set $u := J_{E_i}^{-1} f$. Then $u \in W_0^{1,2}(E_i)$ and

$$\begin{aligned}
\forall v \in W_0^{1,2}(E)\colon \langle J_E Su, v \rangle &= \int_E ((Su)v + DSu \cdot Dv) dx \\
&= \int_{E_i} (uw + Du \cdot Dw) dx = \langle J_{E_i} u, w \rangle = \langle f, w \rangle,
\end{aligned}$$

where

$$w(x) := v(x) + (-1)^i v(x', -x_N) \quad \text{for} \quad x = (x', x_N) \in E_i .$$

[Note that $w \in W_0^{1,2}(E_i)$.] Since $\|w\|_{1,q',E_i} \leqq 2\|v\|_{1,q',E}$ we have $\langle f, w\rangle = \langle g, v\rangle$ for some $g \in W^{-1,q}(E)$. Because $E \in R_q$ this implies that $Su \in W_0^{1,q}(E)$. In view of the definition of $W_0^{1,q}(E_i)$ we obtain $u = Su|E_i \in W_0^{1,q}(E_i)$. This completes the proof.

Remark 9. In the same manner one can prove that $E_4 := \{x \in E_2 : x_1 > 0\}$ is in R_q for every $q \geqq 2$.

4. Boundary Value Problems

Let b be a function satisfying the following hypotheses:

$$\begin{gathered} b : G \times \mathbb{R}^{N+1} \to \mathbb{R}^{N+1},\ b(\cdot, 0) \in L^q(G; \mathbb{R}^{N+1}) \text{ for some } q > 2, \\ b(\cdot, \xi) \text{ is measurable for every } \xi \in \mathbb{R}^{N+1}; \\ (b(x, \xi) - b(x, \eta)) \cdot (\xi - \eta) \geqq m|\xi - \eta|^2,\ m > 0, \\ |b(x, \xi) - b(x, \eta)| \leqq M|\xi - \eta|,\ M < \infty, \text{ for } x \in G,\ \xi,\ \eta \in \mathbb{R}^{N+1}. \end{gathered} \tag{4.1}$$

Of course, here the dot indicates the Euclidean scalar product in $\mathbb{R}^{N+1}$, and $|\xi|$ is the Euclidean norm of $\xi \in \mathbb{R}^{N+1}$. We define $A : W_0^{1,2} \to W^{-1,2}$ setting

$$\forall v \in W_0^{1,2} : \langle Au, v\rangle := \int_G b(\cdot, Lu) \cdot Lv\, dx, \tag{4.2}$$

where L is the operator introduced at the end of Sect. 2.

Remark 10. The operator A is strongly monotone and Lipschitzian (cf. [2, Chap. III]).

Remark 11. The hypotheses (4.1) are satisfied in particular, if

$$\begin{gathered} b_j(\cdot, \xi) = \sum_{i=1}^{N} a_{ij} \xi_i, \qquad j = 1, \dots, N, \\ b_0(\cdot, \xi) = a_0 \xi_0, \quad \text{for} \quad \xi = (\xi_0, \dots, \xi_N) \in \mathbb{R}^{N+1}, \end{gathered}$$

provided that $a_{ij} \in L^\infty(G)$, $i, j = 1, \dots, N$, and $a_0 \in L^\infty(G)$ are such that

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geqq m|\xi|^2,\ a_0(x) \geqq m,\ m > 0, \quad \text{for} \quad x \in G,\ \xi \in \mathbb{R}^N.$$

In that case (4.2) reads as follows:

$$\forall v \in W_0^{1,2} : \langle Au, v\rangle = \int_G \left(\sum_{i,j=1}^{N} a_{ij} D_i u D_j v + a_0 uv \right) dx.$$

Remark 12. For $p \in [2, q]$ the operator A maps $W_0^{1,p}$ continuously into $W^{-1,p}$. Indeed, if u, $\bar{u} \in W_0^{1,p}$, then

$$\forall v \in W_0^{1,2} : \langle Au, v\rangle \leqq \int_G (M|Lu| + |b(\cdot, 0)|)\, |Lv| dx \leqq c\|v\|_{1,p'},$$

$$\langle Au - A\bar{u}, v\rangle \leqq \int_G M|L(u - \bar{u})|\, |Lv| dx \leqq M\|u - \bar{u}\|_{1,p} \|v\|_{1,p'}.$$

The next theorem deals with the question whether A maps $W_0^{1,p}$ *onto* $W^{-1,p}$.

Theorem 1. *Let $G \in R_q$. Suppose that (4.1) holds and that A is defined by (4.2). Then A maps $W_0^{1,p}$ onto $W^{-1,p}$ provided that $p \in [2,q]$ and $M_p k < 1$, where $k := (1-m^2/M^2)^{1/2}$. If $p \in [2,q]$ and $M_p k < 1$, then*

$$\|A^{-1}f - A^{-1}g\|_{1,p} \leqq mM^{-2}M_p(1-M_pk)^{-1}\|f-g\|_{-1,p} \quad \text{for} \quad f, g \in W^{-1,p}.$$

For $p \in [2,q]$ the inequality $M_p k < 1$ is satisfied, if

$$\frac{1}{p} > \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{q}\right)\frac{|\log k|}{\log M_q}.$$

Proof. Let $t := mM^{-2}$ and let $(By)(x) := y(x) - tb(x, y(x))$ for $y \in Y_2$. It is easy to check that (4.1) implies that B, restricted to Y_p, $p \in [2,q]$, is a Lipschitzian mapping from Y_p into itself, where k is a Lipschitz constant of this mapping (cf. [2, Chap. III, Lemma 3.1]). Let $f \in W^{-1,p}$, $p \in [2,q]$, and let

$$Q_f u := J^{-1}(L^* BLu + tf) = u - tJ^{-1}(Au - f), \qquad u \in W_0^{1,p}.$$

Then Q_f is a Lipschitzian mapping from $W_0^{1,p}$ into itself, and $M_p k$ is a Lipschitz constant of Q_f. This follows from the fact that $G \in R_p$ (cf. Lemma 1) and from the properties of L and B. Thus, the requirement $M_p k < 1$ guarantees $Q_f : W_0^{1,p} \to W_0^{1,p}$ to be strictly contractive. By definition of Q_f the fixed point $u \in W_0^{1,p}$ of Q_f is a solution to $Au = f$. Hence A maps $W_0^{1,p}$ onto $W^{-1,p}$. Since $A : W_0^{1,2} \to W^{-1,2}$ is invertible, the fixed point u of Q_f is the unique solution to $Au = f$. If $f, g \in W^{-1,p}$ are given and u, v are the fixed points of Q_f, Q_g, respectively, then

$$\begin{aligned}\|u-v\|_{1,p} = \|Q_f u - Q_g v\|_{1,p} &\leqq M_p k\|u-v\|_{1,p} + \|Q_f v - Q_g v\|_{1,p} \\ &\leqq M_p k\|u-v\|_{1,p} + M_p t\|f-g\|_{-1,p}.\end{aligned}$$

Hence

$$\|u-v\|_{1,p} \leqq tM_p(1-M_pk)^{-1}\|f-g\|_{-1,p}.$$

The last assertion of the theorem follows from $M_p \leqq M_q^\theta$, where θ is defined by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$ (cf. Lemma 1).

Remark 13. Let the hypotheses of Theorem 1 be satisfied, and let $p \in [2,q]$ be such that $M_p k < 1$. Furthermore, let F be any mapping from $W_0^{1,2}$ into $W^{-1,p}$. Then from $Au = Fu$, $u \in W_0^{1,2}$, it follows that $u \in W_0^{1,p}$. This is an immediate consequence of Theorem 1. An example for F is given by

$$\langle Fu, v\rangle := \int_G d_0(\cdot, u)v\,dx + \int_\Gamma d_1(\cdot, u)v\,d\sigma \quad \text{for} \quad v \in W_0^{1,2},$$

where $\Gamma := G \backslash \mathring{G}$, and $d_0 : G \times \mathbb{R} \to \mathbb{R}$, $d_1 : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying appropriate growth conditions. (Γ is to be equipped with the standard surface measure.) In this case $Au = Fu$ means that

$$-\sum_{i=1}^N D_i b_i(\cdot, Lu) + b_0(\cdot, Lu) = d_0(\cdot, u) \quad \text{in} \quad \mathring{G},$$

$$\sum_{i=1}^N b_i(\cdot, Lu)\nu_i = d_1(\cdot, u) \quad \text{on} \quad \Gamma, \qquad u = 0 \quad \text{on} \quad \partial G \backslash \Gamma,$$

where $\nu=(\nu_1,\dots,\nu_N)$ is the outer unit normal at a point of Γ. The term $\sum_{i=1}^{N} b_i(\cdot, Lu)\nu_i$ is defined as an element of the Sobolev space $W^{-1/2,2}(\Gamma)$. This example shows that Theorem 1 can be used to prove $W^{1,p}$-estimates for solutions to rather general boundary value problems for second order elliptic differential equations.

Remark 14. An analogue of Theorem 1 holds for *systems* of second order equations. This follows easily from the fact that, for any $n\in\mathbb{N}$, the duality map of $W_0^{1,2}(G;\mathbb{R}^n)$ is "diagonal".

Remark 15. If there exists a function $\varphi: G\times\mathbb{R}^{N+1}\to\mathbb{R}$ such that

$$b(x,\xi)\cdot\eta=\frac{d}{dt}\varphi(x,\xi+t\eta)\Big|_{t=0}\quad\text{for}\quad \xi,\eta\in\mathbb{R}^{N+1},\ x\in G,$$

then the number k in Theorem 1 may be replaced by the strictly smaller number $(M-m)/(M+m)$ (cf. [2, Chap. III, Lemma 4.14]).

5. Lipschitz-Transformations of Sets $G\in R_q$

Let Φ be a Lipschitz-transformation from $G\subset\mathbb{R}^N$ onto $\tilde{G}\subset\mathbb{R}^N$. If G is regular then $\tilde{G}$ is regular as well, but from $G\in R_q$ it does not necessarily follow that $\tilde{G}\in R_q$. It holds, however, the following

Theorem 2. *Suppose that $G\in R_q$ for some $q>2$ and that Φ is a Lipschitz-transformation from G onto $\tilde{G}\subset\mathbb{R}^N$. Then there exists a $p>2$ such that $\tilde{G}\in R_p$.*

Proof. Let $Tu:=u\circ\Phi^{-1}$, for $u\in W_0^{1,2}(G)$. By means of the chain rule one can easily prove that T maps $W_0^{1,p}(G)$, $p\geqq 2$, continuously onto $W_0^{1,p}(\tilde{G})$. Let T^* be the adjoint operator of $T: W_0^{1,2}(G)\to W_0^{1,2}(\tilde{G})$. Standard calculations show that Theorem 1 is applicable to the operator $A:=T^*J_{\tilde{G}}T: W_0^{1,2}(G)\to W^{-1,2}(G)$ (cf. Remark 11). Consequently, there exists a $p>2$ such that A maps $W_0^{1,p}(G)$ onto $W^{-1,p}(G)$. Since T^* maps $W^{-1,p}(\tilde{G})$ into $W^{-1,p}(G)$ (this follows from the properties of T), the operator $J_{\tilde{G}}^{-1}=TA^{-1}T^*$ maps $W^{-1,p}(\tilde{G})$ into $W_0^{1,p}(\tilde{G})$. This shows that $\tilde{G}\in R_p$.

Theorem 3. *If $G\subset\mathbb{R}^N$ is regular, then $G\in\bigcup_{q>2} R_q$.*

Proof. In view of Lemma 2 it suffices to find an open covering $\{U_0,\dots,U_r\}$ of G such that $U_i\cap G\in\bigcup_{q>2} R_q$, $i=0,\dots,r$. Since ∂G is compact there exist open sets $U_1,\dots,U_r$ and Lipschitz-transformations $\Phi_1,\dots,\Phi_r$ such that $\partial G\subset\bigcup_{i=1}^{r} U_i$ and $\Phi_i(U_i\cap G)\in\{E_1,E_2,E_3\}$ (cf. Definition 2). One can find an open set of class C^1 such that $U_0\subset\mathring{G}$ and $G\subset\bigcup_{i=0}^{r} U_i$. Then $U_0\cap G=U_0\in\bigcap_{q\geqq 2} R_q$ (cf. Remark 7). Theorem 2 shows that $U_i\cap G\in\bigcup_{q>2} R_q$, $i=1,\dots,r$, if $E_i\in\bigcup_{q>2} R_q$, $i=1,2,3$. From Lemma 3 we know already that $E_i\in\bigcap_{q\geqq 2} R_q$, $i=1,2$. Moreover, elementary considerations show that there exists a Lipschitz-transformation mapping E_3 onto E_2. (One can also show that there exists a Lipschitz-transformation from E_3 onto E_4, cf. Remark 9.) Therefore, once more using Theorem 2, we find that $E_3\in\bigcup_{q>2} R_q$.

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