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NACHLASS.

THEORIA INTERPOLATIONIS

METHODO NOVA TRACTATA.

1.

PROBLEMA. *Invenire summam seriei*

$$\frac{a^n}{(a-b)(a-c)(a-d)(a-e)\ldots}+\frac{b^n}{(b-a)(b-c)(b-d)(b-e)\ldots}$$
$$+\frac{c^n}{(c-a)(c-b)(c-d)(c-e)\ldots}+\frac{d^n}{(d-a)(d-b)(d-c)(d-e)\ldots}$$
$$+\frac{e^n}{(e-a)(e-b)(e-c)(e-d)\ldots}+\text{etc.}$$

ubi a, b, c, d, e *sunt* m *quantitates diversae*, *atque* n *numerus integer quicunque positivus*, *negativus sive etiam* 0.

Solutio. Faciendo brevitatis caussa

$$\frac{1}{(a-b)(a-c)(a-d)(a-e)\ldots}=\alpha$$
$$\frac{1}{(b-a)(b-c)(b-d)(b-e)\ldots}=\beta$$
$$\frac{1}{(c-a)(c-b)(c-d)(c-e)\ldots}=\gamma$$
$$\frac{1}{(d-a)(d-b)(d-c)(d-e)\ldots}=\delta,\ \text{etc.}$$

ita ut summa quaesita, quam per S^n denotabimus fiat $=\alpha a^n+\beta b^n+\gamma c^n+\delta d^n+$etc.: manifestum est, si x exprimat quantitatem indeterminatam, ex evolutione aggregati

$$P = \frac{\alpha}{1-ax} + \frac{\beta}{1-bx} + \frac{\gamma}{1-cx} + \frac{\delta}{1-dx} + \text{ etc.}$$

in seriem secundum potestates ipsius x ascendentem, prodire

$$S^0 + S^1x + S^2xx + S^3x^3 + \text{ etc. in infin.}$$

Statuatur $(1-ax)(1-bx)(1-cx)(1-dx)\ldots = Q$, eritque Q functio integra indeterminatae x, ad ordinem m^{tum} ascendens; PQ autem fiet functio integra ordinis $m-1^{\text{ti}}$ puta $=$

$$\begin{aligned}&\alpha(1-bx)(1-cx)(1-dx)\ldots\\ &+\beta(1-ax)(1-cx)(1-dx)\ldots\\ &+\gamma(1-ax)(1-bx)(1-dx)\ldots\\ &+\delta(1-ax)(1-bx)(1-cx)\ldots\\ &+\text{ etc.}\end{aligned}$$

Qua propius considerata, patebit, per substitutionem $x = \frac{1}{a}$ omnes partes praeter primam evanescere, hanc vero abire in

$$\alpha\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)\left(1-\frac{d}{a}\right)\ldots = \frac{1}{a^{m-1}}$$

Simili modo per substitutionem $x = \frac{1}{b}$ evanescent omnes partes praeter secundam, quae fit $= \frac{1}{b^{m-1}}$. Perinde per substitutiones $x = \frac{1}{c}$, $x = \frac{1}{d}$ etc. transit PQ in $\frac{1}{c^{m-1}}$, $\frac{1}{d^{m-1}}$ etc. Hinc vero sequitur, $PQ - x^{m-1}$ per omnes has substitutiones valorem 0 obtinere, quod fieri nequit, nisi fuerit identice $= 0$, sive $PQ = x^{m-1}$; alioquin enim aequatio $PQ - x^{m-1} = 0$, quae non maioris quam $m-1^{\text{ti}}$ ordinis est, m radices diversas $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\ldots$ haberet.

Iam sit

$$Q = 1 - Ax + Bxx - Cx^3 + Dx^4 - \text{ etc.}$$

nempe A summa quantitatum $a, b, c, d\ldots$; B summa productorum e binis; C summa productorum e ternis etc., patetque, quum ex evolutione fractionis $\frac{x^{m-1}}{Q} = P$ prodire debeat

$$S^0 + S^1x + S^2xx + S^3x^3 + \text{ etc.}$$

primo: esse debere $S^0 = 0$, $S^1 = 0$, $S^2 = 0$ etc. usque ad $S^{m-2} = 0$*), tunc

*) Haecce solutionis pars iam ab ill. EULERO tradita est, per methodum a nostra aliquantum discrepantem. *Inst. Calc. Integr.* T. II pag. 432.

vero fieri $S^{m-1}=1$, $S^m=A$, tandemque terminos ulteriores tamquam membra seriei recurrentis per legem sequentem determinari:

$$\begin{aligned} S^m &= A \\ S^{m+1} &= AS^m - BS^{m-1} &&= AA-B \\ S^{m+2} &= AS^{m+1} - BS^m + CS^{m-1} &&= A^3 - 2BA + C \\ &\text{etc.} \end{aligned}$$

Facile quidem hinc colligitur S^{m+1} esse $= aa+bb+cc+\ldots+B$ sive summam quadratorum cum summa omnium productorum e binis diversis quantitatum $a, b, c, d\ldots$; sed quo clarius perspiciatur, quonam modo termini sequentes ex elementis $a, b, c, d\ldots$ formentur, observamus $\frac{1}{Q}$ esse productum e seriebus

$$\begin{aligned} &1+ax+aaxx+a^3x^3+\text{ etc.} \\ &1+bx+bbxx+b^3x^3+\text{ etc} \\ &1+cx+ccxx+c^3x^3+\text{ etc.} \\ &1+dx+ddxx+d^3x^3+\text{ etc.} \\ &\text{etc.} \end{aligned}$$

Hoc vero productum est $=\Sigma a^\lambda b^\mu c^\nu \ldots \times x^{\lambda+\mu+\nu\ldots}$, ubi exponentibus $\lambda, \mu, \nu\ldots$ omnes valores integri a 0 usque in infin. tribuendi omnibusque quibus fieri potest modis combinandi sunt. Quocirca ut in serie, in quam $\frac{1}{Q}$ evolvitur, eius termini, qui continet x^{n+1-m}, coëfficientem obtineamus, numerum $n+1-m$ omnibus quibus fieri potest modis in n partes integras $\lambda+\mu+\nu+\ldots$ (inter quas etiam pars 0 admittitur) discerpere oportet, omnibus quoque permutationibus harum partium permissis; atque tunc singula producta $a^\lambda b^\mu c^\nu\ldots$ in summam colligere, quae erit coëfficiens quaesitus, simulque $= S^n$. Levi attentione adhibita patebit, huic regulae prorsus aequivalere sequentem: Ex m quantitatibus $a, b, c, d\ldots$ omnes combinationes $n+1-m$ elementorum colligendae, admissis repetitionibus, et singulae tamquam producta considerandae, quorum aggregatum erit $=S^n$. Quare erit ut supra S^{m+1} summa omnium productorum e binis quantitatum $a, b, c, d\ldots$ tum diversis tum identicis; S^{m+2} summa omnium productorum e ternis diversis seu identicis etc.

Nihil iam superest, nisi ut summam progressionis nostrae pro valoribus negativis ipsius n definire doceamus. Ad quem finem partem primam summae S^{-n}, puta $\frac{a^{-n}}{(a-b)(a-c)(a-d)\ldots}$ sub hanc formam ponemus

$$\pm\frac{1}{abcd\ldots}\times\frac{\left(\frac{1}{a}\right)^{m+n-2}}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)\left(\frac{1}{a}-\frac{1}{d}\right)\ldots}$$

ubi signum superius vel inferius adoptandum est, prout m impar est vel par, similisque transformatio etiam ad partes reliquas applicari poterit. Quamobrem si per characterem T designetur expressio, quae perinde ex $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\ldots$ oritur, ut S ex $a, b, c, d\ldots$, manifesto fiet $S^{-n}=\pm\frac{T^{m+n-2}}{abcd\ldots}$. Hoc itaque modo hic casus ad praecedentem reductus est, fitque

$$S^{-1}=\pm\frac{1}{abcd\ldots},\qquad S^{-2}=\pm\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\ldots}{abcd\ldots},$$

S^{-3} aequalis producto ex $\pm\frac{1}{abcd\ldots}$ in summam omnium productorum e binis quantitatum $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\ldots$ diversis aut identicis etc.

2.

Applicabimus disquisitionem praecedentem ad eum casum, ubi quantitatibus a, b, c, d valores imaginarii tribuuntur: hac ratione ad quasdam insignes relationes perfacile perveniemus, quae alia methodo tractatae maiores difficultates obiicerent. Sit E basis logarithmorum naturalium, i quantitas imaginaria $\sqrt{-1}$; consideremus loco quantitatum realium $a, b, c\ldots$ imaginarias $E^{ia}, E^{ib}, E^{ic}, E^{id}\ldots$ et $E^{-ia}, E^{-ib}, E^{-ic}, E^{-id}\ldots$, statuamusque

$$\begin{aligned}&\frac{E^{ina}}{(E^{ia}-E^{ib})(E^{ia}-E^{ic})(E^{ia}-E^{id})\ldots}\\&+\frac{E^{inb}}{(E^{ib}-E^{ia})(E^{ib}-E^{ic})(E^{ib}-E^{id})\ldots}\\&+\frac{E^{inc}}{(E^{ic}-E^{ia})(E^{ic}-E^{ib})(E^{ic}-E^{id})\ldots}\\&+\frac{E^{ind}}{(E^{id}-E^{ia})(E^{id}-E^{ib})(E^{id}-E^{ic})\ldots}+\text{etc.}=S^n,\quad\text{atque}\end{aligned}$$

$$\begin{aligned}&\frac{E^{-ina}}{(E^{-ia}-E^{-ib})(E^{-ia}-E^{-ic})(E^{-ia}-E^{-id})\ldots}\\&+\frac{E^{-inb}}{(E^{-ib}-E^{-ia})(E^{-ib}-E^{-ic})(E^{-ib}-E^{-id})\ldots}\\&+\frac{E^{-inc}}{(E^{-ic}-E^{-ia})(E^{-ic}-E^{-ib})(E^{-ic}-E^{-id})\ldots}\\&+\frac{E^{-ind}}{(E^{-id}-E^{-ia})(E^{-id}-E^{-ib})(E^{-id}-E^{-ic})\ldots}+\text{etc.}=T^n\end{aligned}$$

Designando itaque multitudinem quantitatum $a, b, c, d \ldots$ per m, erunt $S^0, S^1, S^2 \ldots S^{m-2}$, nec non $T^0, T^1, T^2 \ldots T^{m-2}$ omnes $= 0$; porro $S^{m-1} = T^{m-1} = 1$; S^m summa quantitatum $E^{ia}, E^{ib}, E^{ic}, E^{id} \ldots$; S^{m+1} summa productorum omnium e binis diversis seu identicis; S^{m+2} summa productorum e ternis etc.; et perinde valores summarum T^m, T^{m+1}, T^{m+2} etc. e quantitatibus $E^{-ia}, E^{-ib}, E^{-ic}, E^{-id} \ldots$ formandi erunt.

Iam quum constet, esse $E^{ix} + E^{-ix} = 2\cos x$, $E^{ix} - E^{-ix} = 2i\sin x$, facile perspicietur, valorem expressionis $\frac{1}{2}(S^n + T^n)$, qui pro $n = 0, 1, 2 \ldots m-2$ fit $= 0$, pro $n = m-1$ autem $= 1$, pro $n = m$ fieri

$$= \cos a + \cos b + \cos c + \cos d \ldots.$$

similiterque fieri $\frac{1}{2}(S^{m+1} + T^{m+1})$ summam cosinuum omnium angulorum, qui oriuntur addendo *binos* ex his $a, b, c, d \ldots$ diversos seu identicos; $\frac{1}{2}(S^{m+2} + T^{m+2})$ summam cosinuum omnium angulorum, qui oriuntur addendo ex iisdem *ternos* etc. Perinde erit $\frac{S^n - T^n}{2i} = 0$ pro $n = 0, 1, 2 \ldots m-1$; porro $= \sin a + \sin b + \sin c + \sin d + \ldots$ pro $n = m$; et similiter $\frac{S^{m+1} - T^{m+1}}{2i}$ erit summa sinuum omnium angulorum, qui oriuntur addendo ex his $a, b, c, d \ldots$ binos diversos seu identicos; $\frac{S^{m+2} - T^{m+2}}{2i}$ summa sinuum omnium angulorum, qui oriuntur ex iisdem, ternos combinando etc.

Summarum S^n, T^n partes nunc propius considerabimus. Est

$$E^{ia} - E^{ib} = E^{\frac{1}{2}i(a+b)}\left(E^{\frac{1}{2}i(a-b)} - E^{\frac{1}{2}i(b-a)}\right) = 2iE^{\frac{1}{2}i(a+b)}\sin\tfrac{1}{2}(a-b)$$

perinde

$$E^{ia} - E^{ic} = 2iE^{\frac{1}{2}i(a+c)}\sin\tfrac{1}{2}(a-c) \text{ etc.}$$

Quamobrem in S^n partis primae denominator fit

$$= (2i)^{m-1}E^{\frac{1}{2}i(b+c+d\ldots+(m-1)a)}\sin\tfrac{1}{2}(a-b)\sin\tfrac{1}{2}(a-c)\sin\tfrac{1}{2}(a-d)\ldots$$

statuendoque $a + b + c + d + \ldots = s$, haec pars ipsa

$$= \frac{E^{i((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}}{(2i)^{m-1}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

Simili modo habetur

$$E^{-ia} - E^{-ib} = -2iE^{-\frac{1}{2}i(a+b)}\sin\tfrac{1}{2}(a-b)$$

unde tandem pars prima summae T^n provenit

$$= \frac{\pm E^{-i((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}}{(2i)^{m-1}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

ubi signum superius vel inferius valet, prout m impar est vel par. Quam cum parte prima summae S^n addendo, sive ab eadem subtrahendo, concludimus fieri *primo* pro valore impari ipsius m partem primam summae $\frac{1}{2}(S^n+T^n)$

$$= \frac{\cos((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}{(2i)^{m-1}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

partem primam summae $\frac{S^n-T^n}{2i}$

$$= \frac{\sin((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}{(2i)^{m-1}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

secundo pro valore pari ipsius m partem primam summae $\frac{1}{2}(S^n+T^n)$

$$= \frac{\sin((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}{2^{m-1}i^{m-2}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

partem primam summae $\frac{S^n-T^n}{2i}$

$$= \frac{-\cos((n+1-\frac{1}{2}m)a-\frac{1}{2}s)}{2^{m-1}i^{m-2}\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$

Manifesto partes sequentes expressionum $\frac{1}{2}(S^n+T^n)$, $\frac{S^n-T^n}{2i}$ primae prorsus analogae erunt, atque inde per solam commutationem characteris a cum $b, c, d\ldots$ orientur. Iam designando per k angulum arbitrarium ponendoque $n+1-\frac{1}{2}m=\lambda$, adiumento formularum

$$\cos(\lambda a+k) = \cos(k+\tfrac{1}{2}s)\cos(\lambda a-\tfrac{1}{2}s) - \sin(k+\tfrac{1}{2}s)\sin(\lambda a-\tfrac{1}{2}s)$$
$$\sin(\lambda a+k) = \cos(k+\tfrac{1}{2}s)\sin(\lambda a-\tfrac{1}{2}s) + \sin(k+\tfrac{1}{2}s)\cos(\lambda a-\tfrac{1}{2}s)$$

haud difficile perveniemus ad summationem serierum sequentium

$$\frac{\cos(\lambda a+k)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$
$$+\frac{\cos(\lambda b+k)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots}$$
$$+\frac{\cos(\lambda c+k)}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots}$$
$$+\frac{\cos(\lambda d+k)}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots} + \text{etc.} = U^\lambda$$

atque

$$\frac{\sin(\lambda a+k)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots}$$
$$+\frac{\sin(\lambda b+k)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots}$$
$$+\frac{\sin(\lambda c+k)}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots}$$
$$+\frac{\sin(\lambda d+k)}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots}+\text{etc.}=V^\lambda.$$

Casus primus, si m est impar.

In hoc casu λ debet esse non numerus integer sed fractione $\frac{1}{2}$ affectus sive numeri imparis semissis. Ad valores *negativos* ipsius λ non opus est respicere, quum habeatur $\cos(-\lambda a+k)=\cos(\lambda a-k)$ atque $\sin(-\lambda a+k)=-\sin(\lambda a-k)$, adeoque summatio pro valore negativo e summatione pro opposito positivo per solam mutationem ipsius k in $-k$ sponte demanat: haec observatio manifesto etiam in casu sequenti valebit. Iam patet facile, pro $n=0,1,\ldots m-2$, sive pro $\lambda=-\frac{1}{2}m+1, -\frac{1}{2}m+2, \ldots \frac{1}{2}m-1$, sive ut valores negativos omittamus, pro $\lambda=\frac{1}{2},\frac{3}{2},\frac{5}{2}\ldots\frac{1}{2}m-1$ fieri tum $U^\lambda=0$, tum $V^\lambda=0$; porro

$$U^{\frac{1}{2}m}=(2i)^{m-1}\cos(\tfrac{1}{2}s+k),\qquad V^{\frac{1}{2}m}=(2i)^{m-1}\sin(\tfrac{1}{2}s+k)$$
$$U^{\frac{1}{2}m+1}=(2i)^{m-1}\{\cos(\tfrac{1}{2}s+k+a)+\cos(\tfrac{1}{2}s+k+b)+\cos(\tfrac{1}{2}s+k+c)+\cos(\tfrac{1}{2}s+k+d)+\text{etc.}\}$$
$$V^{\frac{1}{2}m+1}=(2i)^{m-1}\{\sin(\tfrac{1}{2}s+k+a)+\sin(\tfrac{1}{2}s+k+b)+\sin(\tfrac{1}{2}s+k+c)+\sin(\tfrac{1}{2}s+k+d)+\text{etc.}\}$$

$U^{\frac{1}{2}m+2}$ vel $V^{\frac{1}{2}m+2}$ aequalem producto ex $(2i)^{m-1}$ in summam cosinuum vel sinuum omnium angulorum, qui oriuntur addendo $\frac{1}{2}s+k$ cum binis angulorum $a, b, c, d\ldots$ diversis seu identicis; $U^{\frac{1}{2}m+3}$ vel $V^{\frac{1}{2}m+3}$ aequalem producto ex $(2i)^{m-1}$ in summam cosinuum vel sinuum omnium angulorum, qui oriuntur addendo $\frac{1}{2}s+k$ cum ternis ex iisdem etc. Ceterum vix opus erit admonere, potestatem quantitatis imaginariae i cum exponente pariter pari fieri $=+1$, cum impariter pari $=-1$.

Casus secundus, si m est par.

In hoc casu λ debet esse numerus integer, fitque pro $\lambda=0,1,2\ldots\frac{1}{2}m-1$ tum $U^\lambda=0$, tum $V^\lambda=0$. Porro erit

$$U^{\frac{1}{2}m} = 2^{m-1}i^{m-2}\sin(\tfrac{1}{2}s+k),$$
$$V^{\frac{1}{2}m} = -2^{m-1}i^{m-2}\cos(\tfrac{1}{2}s+k);$$
$$U^{\frac{1}{2}m+1} = 2^{m-1}i^{m-2}\{\sin(\tfrac{1}{2}s+k+a)+\sin(\tfrac{1}{2}s+k+b)+\sin(\tfrac{1}{2}s+k+c)$$
$$+\sin(\tfrac{1}{2}s+k+d)+\text{etc.}\}$$
$$V^{\frac{1}{2}m+1} = -2^{m-1}i^{m-2}\{\cos(\tfrac{1}{2}s+k+a)+\cos(\tfrac{1}{2}s+k+b)+\cos(\tfrac{1}{2}s+k+c)$$
$$+\cos(\tfrac{1}{2}s+k+d)+\text{etc.}\}$$

$U^{\frac{1}{2}m+2}$, $U^{\frac{1}{2}m+3}$ etc. aequalis producto ex $2^{m-1}i^{m-2}$ in summam sinuum omnium angulorum, qui oriuntur addendo $\frac{1}{2}s+k$ cum binis, ternis etc. angulorum $a, b, c\ldots.$ diversis seu identicis; $V^{\frac{1}{2}m+2}$, $U^{\frac{1}{2}m+3}$ etc. aequalis producto ex $-2^{m-1}i^{m-2}$ in summam cosinuum omnium angulorum, qui oriuntur addendo $\frac{1}{2}s+k$ cum binis, ternis etc. eorundem angulorum.

3.

Sit X functio indeterminatae x huius formae

$$\alpha+\beta x+\gamma xx+\delta x^3+\text{etc.}$$

quae non excurrat in infinitum, sed abrumpatur, neque ultra terminum, qui continet x^{m-1} egrediatur, ita ut multitudo coëfficientium non sit maior quam m. Tunc si pro m valoribus diversis ipsius x, puta $a, b, c, d\ldots$ valores correspondentes ipsius X sunt cogniti, puta $= A, B, C, D\ldots$ ex his valor ipsius X valori alicui alii ipsius x respondens sequenti modo concinne eruetur. Sit t valor novus ipsius x, atque T valor respondens ipsius X, ita ut sequentes $m+1$ aequationes locum habeant

$$A = \alpha+\beta a+\gamma aa+\delta a^3+\ldots$$
$$B = \alpha+\beta b+\gamma bb+\delta b^3+\ldots$$
$$C = \alpha+\beta c+\gamma cc+\delta c^3+\ldots$$
$$D = \alpha+\beta d+\gamma dd+\delta d^3+\ldots$$
etc.
$$T = \alpha+\beta t+\gamma tt+\delta t^3+\ldots$$

Multiplicentur hae aequationes resp. per

$$\frac{1}{(a-b)(a-c)(a-d)\ldots.(a-t)}$$
$$\frac{1}{(b-a)(b-c)(b-d)\ldots.(b-t)}$$
$$\frac{1}{(c-a)(c-b)(c-d)\ldots.(c-t)}$$
$$\frac{1}{(d-a)(d-b)(d-c)\ldots.(d-t)}$$
etc.
$$\frac{1}{(t-a)(t-b)(t-c)\ldots.}$$

prodeatque inde per productorum additionem

$$\begin{aligned}&\frac{A}{(a-b)(a-c)(a-d)\ldots(a-t)}\\ &+\frac{B}{(b-a)(b-c)(b-d)\ldots(b-t)}\\ &+\frac{C}{(c-a)(c-b)(c-d)\ldots(c-t)}\\ &+\frac{D}{(d-a)(d-b)(d-c)\ldots(d-t)}\\ &+\text{etc.}\\ &+\frac{T}{(t-a)(t-b)(t-c)(t-d)\ldots} = W\end{aligned}$$

Tunc ex art. 1, ubi m idem denotabat, quod hic nobis est $m+1$, facile concludetur, fieri $W=0$; quamobrem multiplicando per $(t-a)(t-b)(t-c)\ldots$ prodit

$$\begin{aligned}T = &\frac{(t-b)(t-c)(t-d)\ldots}{(a-b)(a-c)(a-d)\ldots}A\\ &+\frac{(t-a)(t-c)(t-d)\ldots}{(b-a)(b-c)(b-d)\ldots}B\\ &+\frac{(t-a)(t-b)(t-d)\ldots}{(c-a)(c-b)(c-d)\ldots}C\\ &+\frac{(t-a)(t-b)(t-c)\ldots}{(d-a)(d-b)(d-c)\ldots}D\\ &+\text{etc.}\end{aligned}$$

4.

Formula in art. praec. inventa, ita comparata est, ut sponte sine omni calculo pateat, si pro t quantitatum $a, b, c, d\ldots$ aliqua in illa substituatur, valorem respondentem $A, B, C, D\ldots$ inde prodire. Neque hoc solo respectu sese commendat: certo enim ad usum practicum longe commodissima est, saltem quoties unicus tantum valor ipsius X e valoribus datis computandus est. Quando plures sunt eruendi, transformatio sequens formulae nostrae nonnunquam praeferri poterit

$$T = A + (t-a)\left(\frac{A}{a-b} + \frac{B}{b-a}\right)$$
$$+ (t-a)(t-b)\left(\frac{A}{(a-b)(a-c)} + \frac{B}{(b-a)(b-c)} + \frac{C}{(c-a)(c-b)}\right)$$
$$+ (t-a)(t-b)(t-c)\left(\frac{A}{(a-b)(a-c)(a-d)} + \frac{B}{(b-a)(b-c)(b-d)}\right.$$
$$\left. + \frac{C}{(c-a)(c-b)(c-d)} + \frac{D}{(d-a)(d-b)(d-c)}\right)$$
$$+ \text{etc.}$$

(quae ex formula art. 3 facillime derivatur, si in hac multitudo quantitatum $a, b, c, d\ldots$ ab una a ad duas a, b, inde ad tres a, b, c etc. successive increscere concipitur et quisque valor ipsius T hoc modo oriens a sequenti subtrahitur). Ponantur coëfficientes

$$\frac{A}{a-b} + \frac{B}{(b-a)}, \quad \frac{A}{(a-b)(a-c)} + \frac{B}{(b-a)(b-c)} + \frac{C}{(c-a)(c-b)} \text{ etc.} = A', \; A'' \text{ etc.}$$

porro designetur per B', B'' etc. id, quod fit ex A', A'' etc. si pro $a, b, c\ldots$ resp. scribitur $b, c, d\ldots$ atque pro $A, B, C\ldots$ resp. $B, C, D\ldots$; similiter designetur per C', C'' etc. id quod fit ex A', A'' etc., si pro $a, b, c\ldots$ resp. substituitur $c, d, e\ldots$ et pro $A, B, C\ldots$ resp. $C, D, E\ldots$ et sic porro. Tunc habebimus

$$A' = \frac{A-B}{a-b}, \quad B' = \frac{B-C}{b-c}, \quad C' = \frac{C-D}{c-d} \ldots$$
$$A'' = \frac{A'-B'}{a-c}, \quad B'' = \frac{B'-C'}{b-d}, \quad C'' = \frac{C'-D'}{c-e} \ldots$$
$$A''' = \frac{A''-B''}{a-d}, \quad B''' = \frac{B''-C''}{b-e}, \quad C''' = \frac{C''-D''}{c-f} \ldots$$
etc.

atque

$$T = A + A'(t-a) + A''(t-a)(t-b) + A'''(t-a)(t-b)(t-c) + \ldots$$

Multitudo quantitatum A', B', C' etc. hic computandarum erit $m-1$, multitudo quantitatum A'', B'', C'' etc. erit $m-2$ et sic porro.

5.

Sequens quoque mutatio formulae art. 3 non sine fructu in usum vocari poterit. Sit $\mathfrak{X}$ functio integra arbitraria indeterminatae x, neque tamen alterioris quam $m-1^{\text{ti}}$ ordinis; atque $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \ldots \mathfrak{T}$ eius valores pro $x = a, b, c, d \ldots t$ resp. Tunc per art. 1 erit

$$\begin{aligned} 0 = & \frac{\mathfrak{A}}{(a-b)(a-c)(a-d)\ldots(a-t)} \\ & + \frac{\mathfrak{B}}{(b-a)(b-c)(b-d)\ldots(b-t)} \\ & + \frac{\mathfrak{C}}{(c-a)(c-b)(c-d)\ldots(c-t)} \\ & + \frac{\mathfrak{D}}{(d-a)(d-b)(d-c)\ldots(d-t)} \\ & + \text{etc.} \\ & + \frac{\mathfrak{T}}{(t-a)(t-b)(t-c)(t-d)\ldots} \end{aligned}$$

Hinc facile deducitur

$$\begin{aligned} T = & \mathfrak{T} \\ & + \frac{(t-b)(t-c)(t-d)\ldots}{(a-b)(a-c)(a-d)\ldots}(A-\mathfrak{A}) \\ & + \frac{(t-a)(t-c)(t-d)\ldots}{(b-a)(b-c)(b-d)\ldots}(B-\mathfrak{B}) \\ & + \frac{(t-a)(t-b)(t-d)\ldots}{(c-a)(c-b)(c-d)\ldots}(C-\mathfrak{C}) \\ & + \frac{(t-a)(t-b)(t-c)\ldots}{(d-a)(d-b)(d-c)\ldots}(D-\mathfrak{D}) \\ & + \text{etc.} \end{aligned}$$

Quodsi functio $\mathfrak{X}$ ita eligitur, ut valores functionis X hi $A, B, C, D \ldots$ prope per illam repraesententur, hoc lucramur, ut coëfficientes $\frac{(t-b)(t-c)(t-d)\ldots}{(a-b)(a-c)(a-d)\ldots}$ etc. per quantitates parvas multiplicandi sint. Potest etiam pro $\mathfrak{X}$ quantitas constans assumi, e. g. una ex his $A, B, C, D \ldots$; in quo casu pars una e valore ipsius T excidet. Aut ita determinari potest $\mathfrak{X}$, ut duae pluresve harum quantitatum per $\mathfrak{X}$ repraesententur, in quo casu totidem partes ex T excident.

6.

Quae praecedunt, suppositioni innituntur, functionem X ultra potestatem x^{m-1} non egredi, sive differentiam m^{tam} una cum superioribus evanescere, in quo casu methodus interpolationis rigorose vera est. Si vero illa suppositio locum non habet, interpolatio eo tantummodo tendit, ut loco functionis X functio *simplicissima* eruatur, per quam valoribus propositis $A, B, C, D \ldots$ satisfiat. Iam ut errorem, qui a neglectis differentiis superioribus nascitur, diiudicare possimus, sint termini in X post potestatem x^{m-1} hi $\mu x^m + \nu x^{m+1} +$ etc. Erit itaque in art. 3 non $W = 0$, sed ut ex art. 1 sequitur, $W = \mu + \nu(a+b+c+d+\ldots+t) +$ etc. Hinc valori ipsius T illic tradito adhuc adiici debet

$$(t-a)(t-b)(t-c)(t-d)\ldots \times \{\mu + \nu(a+b+c+d+\ldots+t) + \text{etc.}\}$$

7.

Casus in praxi maxime frequens est, ubi $a, b, c, d\ldots$ progressionem arithmeticam constituunt. Ponendo intervallum $= 1$, ita ut sit $b = a+1$, $c = a+2$ etc., formula art. 4 fit

$$\begin{aligned} T = A & \\ & + (t-1)(B-A) \\ & + (t-1)(t-2)\left(\frac{C-2B+A}{2}\right) \\ & + (t-1)(t-2)(t-3)\left(\frac{D-3C+3B-A}{6}\right) \\ & + \text{etc.} \end{aligned}$$

sive

$$T = A + A'(t-1) + A''(t-1)(t-2) + A'''(t-1)(t-2)(t-3) + \ldots$$

ubi A', A'', A''' etc. computantur per algorithmum sequentem

$$\begin{array}{lll} A' = B-A, & B' = C-B, & C' = D-C \text{ etc.} \\ 2A'' = B'-A', & 2B'' = C'-B', & 2C'' = D'-C' \text{ etc.} \\ 3A''' = B''-A'', & 3B''' = C''-B'', & 3C''' = D''-C'' \text{ etc.} \end{array}$$

etc., quae formula cum vulgata interpolationis formula per differentias omnino convenit.

8.

Si pro satis multis valoribus ipsius x in serie arithmetica progredientibus $a, a+1, a+2\ldots$ valores respondentes functionis X cogniti sunt, ut seriem m valorum successivorum ipsius x ad lubitum eligere liceat ad computum ipsius T: quaestio oritur, *quosnam* valores ad hunc finem praeferre maxime praestet, siquidem plures quam m adhibere sive ultra differentiam $m-1^{\text{tam}}$ egredi nolimus?

Manifesto hoc ita est decidendum, ut error a differentia m^{ti} ordinis oriundus fiat quam minimus; hic vero error fit

$$= (t-a)(t-a-1)(t-a-2)\ldots(t-a-m+1)\frac{\Delta}{1.2.3\ldots m}$$

si adhibentur termini ad $x = a,\ a+1,\ a+2,\ldots a+m-1$ pertinentes, vel

$$= (t-a-1)(t-a-2)(t-a-3)\ldots(t-a-m)\frac{\Delta}{1.2.3\ldots m}$$

si adhibentur termini ad $x = a+1,\ a+2,\ a+3,\ldots a+m$ pertinentes, vel

$$= (t-a+1)(t-a)(t-a-1)\ldots(t-a-m+2)\frac{\Delta}{1.2.3\ldots m}$$

si adhibentur termini ad $x=a-1,\ a,\ a+1,\ldots a+m-2$ pertinentes, designando per Δ differentiam ordinis m^{ti}, quae siquidem ad differentias superiorum ordinum non respicitur, hic tamquam constans considerari potest. Quodsi igitur m termini ad $x = a,\ a+1,\ a+2,\ldots a+m-1$ pertinentes maxime idonei sunt, ex illis tribus expressionibus prima debet esse minima adeoque sine respectu signi $t-a <$ vel saltem non $> t-a-m$, necnon $t-a+1 >$ vel saltem non $< t-a-m+1$. Ex conditione priore facile deducitur, $t-a-m$ esse debere quantitatem negativam inter $-\frac{1}{2}m$ et $-\infty$ sitam, unde $t-a$ iacebit inter $+\frac{1}{2}m$ et $-\infty$; ex conditione posteriore autem erit $t-a+1$ quantitas positiva atque inter $\frac{1}{2}m$ et $+\infty$ sita; quare $t-a$ iacebit inter $\frac{1}{2}m-1$ et $\frac{1}{2}m$. Hinc sequitur, si m fuerit numerus par, valores ad interpolationem adhibendos ita eligendos esse, ut prior semissis ab una parte, posterior ab altera termini quaesiti iaceant; si autem m fuerit impar, termini ii sunt eligendi, quorum medius quaesito iaceat quam proximus. Quoties in hoc casu t est exacte medius inter duos valores consecutivos ipsius x, respectu erroris a neglecta differentia m^{ti} ordinis generaliter loquendo nihil intererit, sive terminus quaesito praecedens sive sequens pro medio adhibendorum adoptetur.

9.

Exempla. I. Invenire oporteat log. sin $24^0 30'$, si logarithmi sinuum per singulos gradus habentur, ita ut differentiae quartae et superiores negligantur. In hoc itaque casu secundum praecepta art. praec. adhibebimus logarithmos sinuum arcuum 23^0, 24^0, 25^0, 26^0, quibus per A, B, C, D designatis provenit per formulam art. 3 logarithmus quaesitus

$$= -\tfrac{1}{16}A + \tfrac{9}{16}B + \tfrac{9}{16}C - \tfrac{1}{16}D$$

cuius valor numericus ex

$$A = 9,5918780$$
$$B = 9,6093133$$
$$C = 9,6259483$$
$$D = 9,6418420$$

eruitur 9,6177271.5; tabulae dant 9,6177270.

Quodsi hic loco log. sin 23°, logarithmum sin 27° $= E =$ 9,6570468 adhibuissemus, formula

$$+\tfrac{5}{16}B + \tfrac{15}{16}C - \tfrac{5}{16}D + \tfrac{1}{16}E$$

dedisset 9,6177267.375.

Formula art. praec. pro hoc casu dat errorem ex neglecta differentia quarta Δ oriundum $= \tfrac{3}{128}\Delta$, si A, B, C, D, contra $= -\tfrac{5}{128}\Delta$, si B, C, D, E adhibentur; qui error valori calculato adiiciendus est. In exemplo nostro habetur $\Delta = -0,0000066$; correctio hinc calculata utrumque computum cum tabulis conciliat.

II. Sit tempus novilunii 16 Junii 1806 e longitudinibus solis et lunae pro singulis diebus computandum, ita ut differentiarum usque ad quartam incl. ratio habeatur. Habentur in ephemeridibus Parisiensibus pro meridie vero loci sequentes

	longit. solis	longit. lunae
1806 Junii 14	82°39′12″	53°19′15″
15	83 36 30	67 29 43
16	84 33 48	81 59 52
17	85 31 7	96 44 9
18	86 28 24	111 35 30

Sunt hic valores dati ipsius x differentiae inter longitudinem solis et lunae, valores respondentes ipsius X tempora, quae inde a meridie 16 Junii numerabuntur; quaeriturque valor ipsius X valori $x = 0$ respondens. Fit igitur

$a = -29^0 19' 57''$	$= -105596''$	$A = -2$
$b = -16\ \ 6\ 47$	$= -\ \ 58007$	$B = -1$
$c = -\ \ 2\ 33\ 56$	$= -\ \ 9236$	$C = \ \ 0$
$d = +11\ 13\ \ 2$	$= +\ \ 40382$	$D = +1$
$e = +25\ \ 7\ \ 6$	$= +\ \ 90426$	$E = +2$
$t = \ \ 0$		

Hinc computatur

$$\frac{(t-b)(t-c)(t-d)(t-e)}{(a-b)(a-c)(a-d)(a-e)} = +0{,}0149084$$
$$\frac{(t-a)(t-c)(t-d)(t-e)}{(b-a)(b-c)(b-d)(b-e)} = -0{,}1050660$$
$$\frac{(t-a)(t-b)(t-d)(t-e)}{(c-a)(c-b)(c-d)(c-e)} = +0{,}9624567$$
$$\frac{(t-a)(t-b)(t-c)(t-e)}{(d-a)(d-b)(d-c)(d-e)} = +0{,}1434435$$
$$\frac{(t-a)(t-b)(t-c)(t-d)}{(e-a)(e-b)(e-c)(e-d)} = -0{,}0157429$$

His coëfficientibus*) per $-2, -1, 0, 1, 2$ resp. multiplicatis, productorum summa fit $= +0{,}1872069$; quare novilunium erit 16 Junii, $4^h 29' 34'' 7$ temp. ver. Paris. sive $4^h 29' 41'' 4$ temp. med.; Connaissance des tems habet $4^h 27' 42''$.

10.

Sit X functio arcus indeterminati x huius formae

$$\alpha + \alpha' \cos x + \alpha'' \cos 2x + \alpha''' \cos 3x + \text{etc.}$$
$$+ \beta' \sin x + \beta'' \sin 2x + \beta''' \sin 3x + \text{etc.}$$

quae non excurrat in infinitum, sed cum $\cos mx$ et $\sin mx$ abrumpatur, ita ut multitudo coëfficientium (incognitorum) sit $2m+1$. Pro totidem valoribus diversis ipsius x, puta a, b, c, d etc. dati sint valores respondentes functionis X, puta $A, B, C, D \ldots$ (Ceterum valores ipsius x, quorum differentia est peripheria integra sive eius multiplum, manifesto hic pro diversis haberi nequeunt). Ex his datis quaeritur formula pro valore T, quem functio X pro quocunque alio valore ipsius x, puta t nanciscitur. Habentur itaque $2m+2$ aequationes

*) Confirmationi calculi inservit observatio ex art. 5 sine negotio derivanda, summam harum coëfficientium esse debere $= 1$.

$$
\begin{aligned}
A &= \alpha + \alpha' \cos a + \alpha'' \cos 2a + \alpha''' \cos 3a + \text{etc.} \\
&\quad + \beta' \sin a + \beta'' \sin 2a + \beta''' \sin 3a + \text{etc.} \\
B &= \alpha + \alpha' \cos b + \alpha'' \cos 2b + \alpha''' \cos 3b + \text{etc.} \\
&\quad + \beta' \sin b + \beta'' \sin 2b + \beta''' \sin 3b + \text{etc.} \\
C &= \alpha + \alpha' \cos c + \alpha'' \cos 2c + \alpha''' \cos 3c + \text{etc.} \\
&\quad + \beta' \sin c + \beta'' \sin 2c + \beta''' \sin 3c + \text{etc.} \\
D &= \alpha + \alpha' \cos d + \alpha'' \cos 2d + \alpha''' \cos 3d + \text{etc.} \\
&\quad + \beta' \sin d + \beta'' \sin 2d + \beta''' \sin 3d + \text{etc.} \\
&\text{etc.} \\
T &= \alpha + \alpha' \cos t + \alpha'' \cos 2t + \alpha''' \cos 3t + \text{etc.} \\
&\quad + \beta' \sin t + \beta'' \sin 2t + \beta''' \sin 3t + \text{etc.}
\end{aligned}
$$

Multiplicando has aequationes resp. per

$$
\begin{aligned}
&\frac{1}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots\sin\frac{1}{2}(a-t)} \\
&\frac{1}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots\sin\frac{1}{2}(b-t)} \\
&\frac{1}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots\sin\frac{1}{2}(c-t)} \\
&\frac{1}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots\sin\frac{1}{2}(d-t)} \\
&\text{etc.} \\
&\frac{1}{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots}
\end{aligned}
$$

addendoque producta prodeat

$$
\begin{aligned}
&\frac{A}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots\sin\frac{1}{2}(a-t)} \\
+&\frac{B}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots\sin\frac{1}{2}(b-t)} \\
+&\frac{C}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots\sin\frac{1}{2}(c-t)} \\
+&\frac{D}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots\sin\frac{1}{2}(d-t)} \\
+&\ \text{etc.} \\
+&\frac{T}{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots} = W
\end{aligned}
$$

Tunc ex art. 2, ubi m idem erat, quod hic est $2m+2$, adeoque casus secundus locum habet, fit $W=0$; quamobrem multiplicando per

$$\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots$$

prodit

$$\begin{aligned} T = &\frac{\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots.}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots.}A \\ &+\frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots.}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots.}B \\ &+\frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-d)\ldots.}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots.}C \\ &+\frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\ldots.}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots.}D \\ &+\text{etc.} \end{aligned}$$

Quum haec formula indefinite pro valore quocunque ipsius t locum habeat, manifestum est, si producta sinuum in numeratoribus in cosinus sinusque arcuum multiplicium evolvantur, id quod provenit cum

$$\begin{aligned} &\alpha+\alpha'\cos t+\alpha''\cos 2t+\alpha'''\cos 3t+\text{etc.} \\ &+\beta'\sin t+\beta''\sin 2t+\beta'''\sin 3t+\text{etc.} \end{aligned}$$

identicum esse debere, unde coëfficientes α, α', β', α'', β'' etc. innotescent. Ceterum formula pro T, ut hic exhibita est, ita est comparata, ut sponte et sine calculo pateat, substitutis pro t resp. a, b, c, d etc. valoribus propositis A, B, C, D etc. probe satisfieri.

11.

Si multitudo valorum datorum A, B, C, D etc. unitate minor esset, quam supposuimus, puta $=2m$, hi non sufficient ad determinationem $2m+1$ coëfficientium incognitorum, nisi inter eos relatio aliqua cognita fuerit. In hoc itaque casu expressio generalis pro T aliter exhiberi nequit, nisi ita, ut constantem incognitam implicet. Ratiocinia art. praec. hic adhiberi non possunt, quoniam summatio $W=0$, ex art. 2 petita, suppositioni innixa est, multitudinem quantitatum $A, B \ldots T$ esse parem. Formula quidem in art. praec. pro T inventa valoribus A, B etc. generaliter satisfacit, sive horum multitudo par sit sive impar: sed levi attentione perspicitur, ex evolutione productorum e sinubus prodire, in nostro casu, expressionem propositae non similem, sed huius formae

$$\mathfrak{A}\cos\tfrac{1}{2}t + \mathfrak{A}'\cos\tfrac{3}{2}t + \mathfrak{A}''\cos\tfrac{5}{2}t + \text{etc.}$$
$$+ \mathfrak{B}\sin\tfrac{1}{2}t + \mathfrak{B}'\sin\tfrac{3}{2}t + \mathfrak{B}''\sin\tfrac{5}{2}t + \text{etc.}$$

Quamobrem pro hoc casu viam aliam ingredi oportebit.

Multiplicetur X per $\cos(\frac{1}{2}x+k)$, proditque, si termini ultimi in X statuuntur $\alpha^m\cos mx + \beta^m\sin mx$

$$\begin{aligned}
&\alpha\cos(\tfrac{1}{2}x+k) + \tfrac{1}{2}\alpha'\cos(\tfrac{3}{2}x+k) + \tfrac{1}{2}\alpha''\cos(\tfrac{5}{2}x+k) + \ldots\\
&\qquad + \tfrac{1}{2}\alpha^m\cos((m+\tfrac{1}{2})x+k)\\
&+ \tfrac{1}{2}\alpha'\cos(\tfrac{1}{2}x-k) + \tfrac{1}{2}\alpha''\cos(\tfrac{3}{2}x-k) + \tfrac{1}{2}\alpha'''\cos(\tfrac{5}{2}x-k) + \ldots\\
&\qquad + \tfrac{1}{2}\alpha^m\cos((m-\tfrac{1}{2})x-k)\\
&+ \tfrac{1}{2}\beta'\sin(\tfrac{3}{2}x+k) + \tfrac{1}{2}\beta''\sin(\tfrac{5}{2}x+k) + \ldots\\
&\qquad + \tfrac{1}{2}\beta^m\sin((m+\tfrac{1}{2})x+k)\\
&+ \tfrac{1}{2}\beta'\sin(\tfrac{1}{2}x-k) + \tfrac{1}{2}\beta''\sin(\tfrac{3}{2}x-k) + \tfrac{1}{2}\beta'''\sin(\tfrac{5}{2}x-k) + \ldots\\
&\qquad + \tfrac{1}{2}\beta^m\sin((m-\tfrac{1}{2})x-k)
\end{aligned}$$

Iam si in hac expressione pro x successive substituitur $a, b, c, d \ldots . t$, quod inde provenit resp. per

$$\frac{1}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots\sin\frac{1}{2}(a-t)}$$
$$\frac{1}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots\sin\frac{1}{2}(b-t)}$$
$$\frac{1}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots\sin\frac{1}{2}(c-t)}$$
$$\frac{1}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots\sin\frac{1}{2}(d-t)}$$
etc.
$$\frac{1}{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots}$$

multiplicatur, productorumque aggregatum $= W$ ponitur: ex iis, quae in art. 2, (in casu priori, ubi m idem erat, quod hic est $2m+1$) tradidimus, facile sequitur, haberi

$$\begin{aligned}
W = &\ \tfrac{1}{2}(2i)^{2m}\alpha^m\cos(\tfrac{1}{2}(a+b+c+d+\ldots+t)+k)\\
&+ \tfrac{1}{2}(2i)^{2m}\beta^m\sin(\tfrac{1}{2}(a+b+c+d+\ldots+t)+k)
\end{aligned}$$

Ex altera vero parte erit

$$
\begin{aligned}
W = \; & \frac{A\cos(\frac{1}{2}a+k)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots\sin\frac{1}{2}(a-t)} \\
& + \frac{B\cos(\frac{1}{2}b+k)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots\sin\frac{1}{2}(b-t)} \\
& + \frac{C\cos(\frac{1}{2}c+k)}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots\sin\frac{1}{2}(c-t)} \\
& + \frac{D\cos(\frac{1}{2}d+k)}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots\sin\frac{1}{2}(d-t)} \\
& + \text{etc.} \\
& + \frac{T\cos(\frac{1}{2}t+k)}{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots}
\end{aligned}
$$

Quodsi iam quantitatem k, quae prorsus arbitraria est, $= -\frac{1}{2}t$ ponimus, summam autem $a+b+c+d+\ldots = s$ (exclusa t), prodit, multiplicando per $\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots$

$$
\begin{aligned}
T = \; & \frac{\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\ldots} A\cos\tfrac{1}{2}(t-a) \\
& + \frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-c)\sin\frac{1}{2}(t-d)\ldots}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\ldots} B\cos\tfrac{1}{2}(t-b) \\
& + \frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-d)\ldots}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\ldots} C\cos\tfrac{1}{2}(t-c) \\
& + \frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)\ldots}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\ldots} D\cos\tfrac{1}{2}(t-d) \\
& + \text{etc.} \\
& + 2^{2m-1}i^{2m}\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots \\
& \qquad \times\{\alpha^m\cos\tfrac{1}{2}s+\beta^m\sin\tfrac{1}{2}s\}
\end{aligned}
$$

Circa hanc formulam quasdam adhuc annotationes adiicimus

I. Est $i^{2m} = +1$, vel -1, prout m par est vel impar.

II. Quum formula indefinite pro quovis valore ipsius t valeat, necessario, evolutis sinuum productis in sinus et cosinus arcuum multiplicium, cum formula pro X, si pro x scribitur t, identica erit, i. e. cum

$$
\begin{aligned}
& \alpha+\alpha'\cos t+\alpha''\cos 2t+ \text{etc.} \\
& +\beta'\sin t+\beta''\sin 2t+ \text{etc.}
\end{aligned}
$$

III. Determinantur itaque omnes coëfficientes α, α', β', α'', β'' etc. per quantitates cognitas, atque unicam incognitam $\alpha^m\cos\frac{1}{2}s+\beta^m\sin\frac{1}{2}s$, quamobrem, si qua inter illos relatio insuper datur, omnes ex asse facile poterunt assignari.

IV. Quodsi autem tantummodo expressio eiusdem formae ut

$$\begin{aligned}&\alpha + \alpha' \cos t + \alpha'' \cos 2t + \text{etc.}\\ &+ \beta' \sin t + \beta'' \sin 2t + \text{etc.}\end{aligned}$$

quaeritur (sive cum hac identica sit, sive non), per quam valores propositi A, B, C, D etc. repraesententur, si pro t resp. a, b, c, d etc. substituitur: quantitas $\alpha^m \cos\frac{1}{2}s + \beta^m \sin\frac{1}{2}s$ ex incognita *arbitraria* evadit, quam si placet etiam $= 0$ statuere licebit; reveraque formula nostra pro T ita comparata est, ut absque calculo sponte pateat, hacce eam proprietate praeditam esse.

12.

Operae pretium erit, transformationem ei analogam, quam in art. 4 explicavimus, etiam ad formulas art. 10 et art. praec., si ibi $\alpha^m \cos\frac{1}{2}s + \beta^m \sin\frac{1}{2}s = 0$ ponitur, applicare, quo pacto hos duos casus diversos sub formula unica comprehendere licebit. Crescat itaque multitudo quantitatum $a, b, c, d \ldots$ successive ab una a, ad duas a, b, deinde ad tres a, b, c etc.; evolvatur alternis vicibus per artt. 10 et 11 valor, quem t pro singulis his suppositionibus nanciscitur, tandem quisque valor a proxime sequente subtrahatur, sequenti modo:

$$A$$

$$\frac{\sin\frac{1}{2}(t-b)}{\sin\frac{1}{2}(a-b)} A \cos\tfrac{1}{2}(t-a) + \frac{\sin\frac{1}{2}(t-a)}{\sin\frac{1}{2}(b-a)} B \cos\tfrac{1}{2}(t-b)$$

$$\frac{\sin\frac{1}{2}(t-b)\sin\frac{1}{2}(t-c)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)} A + \frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-c)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)} B + \frac{\sin\frac{1}{2}(t-a)\sin\frac{1}{2}(t-b)}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)} C$$

etc.

Hic differentia inter valorem primum et secundum fit

$$= \sin\tfrac{1}{2}(t-a)\cos\tfrac{1}{2}(t-b)\left\{\frac{A}{\sin\frac{1}{2}(a-b)} + \frac{B}{\sin\frac{1}{2}(b-a)}\right\}$$

Differentia inter valorem secundum et tertium

$$= \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\left\{\frac{A\cos\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)} + \frac{B\cos\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)} + \frac{C}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)}\right\}$$

Quae operatio si ultra continuatur, facile emergit lex generalis. Scilicet statuendo

$$A' = \frac{A}{\sin\frac{1}{2}(a-b)} + \frac{B}{\sin\frac{1}{2}(b-a)}$$

$$A'' = \frac{A\cos\frac{1}{2}(a-c)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)} + \frac{B\cos\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)} + \frac{C}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)}$$

$$A''' = \frac{A}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)} + \frac{B}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)}$$
$$+ \frac{C}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)} + \frac{D}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)}$$

$$A'''' = \frac{A\cos\frac{1}{2}(a-e)}{\sin\frac{1}{2}(a-b)\sin\frac{1}{2}(a-c)\sin\frac{1}{2}(a-d)\sin\frac{1}{2}(a-e)}$$
$$+ \frac{B\cos\frac{1}{2}(b-e)}{\sin\frac{1}{2}(b-a)\sin\frac{1}{2}(b-c)\sin\frac{1}{2}(b-d)\sin\frac{1}{2}(b-e)}$$
$$+ \frac{C\cos\frac{1}{2}(c-e)}{\sin\frac{1}{2}(c-a)\sin\frac{1}{2}(c-b)\sin\frac{1}{2}(c-d)\sin\frac{1}{2}(c-e)}$$
$$+ \frac{D\cos\frac{1}{2}(d-e)}{\sin\frac{1}{2}(d-a)\sin\frac{1}{2}(d-b)\sin\frac{1}{2}(d-c)\sin\frac{1}{2}(d-e)}$$
$$+ \frac{E}{\sin\frac{1}{2}(e-a)\sin\frac{1}{2}(e-b)\sin\frac{1}{2}(e-c)\sin\frac{1}{2}(e-d)}$$

etc. fiet

$$\begin{aligned} T = A &+ A' \sin\tfrac{1}{2}(t-a)\cos\tfrac{1}{2}(t-b) \\ &+ A'' \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b) \\ &+ A''' \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\cos\tfrac{1}{2}(t-d) \\ &+ A'''' \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d) \\ &+ A^{\mathrm{V}} \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\sin\tfrac{1}{2}(t-e)\cos\tfrac{1}{2}(t-f) \\ &+ A^{\mathrm{VI}} \sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\sin\tfrac{1}{2}(t-e)\sin\tfrac{1}{2}(t-f) \\ &+ \text{etc.} \end{aligned}$$

quae progressio ad totidem terminos continuanda est, quot valores functionis X dati sunt.

Coëfficientes A', A'', A''', A'''' etc. etiam per algorithmum sequentem computari possunt. Designetur per B', B'', B''', B'''' etc. id quod illi resp. fiunt, si

$$a,\ A,\ b,\ B,\ c,\ C,\ d,\ D \text{ etc.}$$

resp. mutantur in

$$b,\ B,\ c,\ C,\ d,\ D,\ e,\ E \text{ etc.}$$

Porro transeant

$$A',\ A'',\ A''',\ A'''' \text{ etc. in } C',\ C'',\ C''',\ C'''' \text{ etc. vel in } D',\ D'',\ D''',\ D'''' \text{ etc. etc.}$$

si

$$a,\ A,\ b,\ B,\ c,\ C,\ d,\ D \text{ etc.}$$

resp. mutantur in

$$c,\ C,\ d,\ D,\ e,\ E,\ f,\ F \text{ etc.}$$

vel in

$$d,\ D,\ e,\ E,\ f,\ F,\ g,\ G \text{ etc.}$$

etc.

Tunc erit

$$A' = \frac{A-B}{\sin\frac{1}{2}(a-b)},\quad B' = \frac{B-C}{\sin\frac{1}{2}(b-c)},\quad C' = \frac{C-D}{\sin\frac{1}{2}(c-d)} \text{ etc.}$$

$$A'' = \frac{A'\cos\frac{1}{2}(a-c)-B'}{\sin\frac{1}{2}(a-c)},\qquad B'' = \frac{B'\cos\frac{1}{2}(b-d)-C'}{\sin\frac{1}{2}(b-d)} \text{ etc.}$$

$$A''' = \frac{A''\cos\frac{1}{2}(a-c)+A'\sin\frac{1}{2}(a-c)-B''}{\sin\frac{1}{2}(a-d)},\qquad B''' = \frac{B''\cos\frac{1}{2}(b-d)+B'\sin\frac{1}{2}(b-d)-C''}{\sin\frac{1}{2}(b-e)} \text{ etc.}$$

$$A'''' = \frac{A'''\cos\frac{1}{2}(a-e)-B'''}{\sin\frac{1}{2}(a-e)},\qquad B'''' = \frac{B'''\cos\frac{1}{2}(b-f)-C'''}{\sin\frac{1}{2}(b-f)} \text{ etc.}$$

$$A^{\mathrm{v}} = \frac{A''''\cos\frac{1}{2}(a-e)+A'''\sin\frac{1}{2}(a-e)-B''''}{\sin\frac{1}{2}(a-f)},\qquad B^{\mathrm{v}} = \frac{B''''\cos\frac{1}{2}(b-f)+B'''\sin\frac{1}{2}(b-f)-C''''}{\sin\frac{1}{2}(b-g)} \text{ etc.}$$

$$A^{\mathrm{vi}} = \frac{A^{\mathrm{v}}\cos\frac{1}{2}(a-g)-B^{\mathrm{v}}}{\sin\frac{1}{2}(a-g)},\qquad B^{\mathrm{vi}} = \frac{B^{\mathrm{v}}\cos\frac{1}{2}(b-h)-C^{\mathrm{v}}}{\sin\frac{1}{2}(a-h)} \text{ etc.}$$

etc.

Lex formationis hic satis obvia est, si modo observetur, numeratores in valoribus pro A'', A''', A'''', A^{v}, A^{vi} etc. (valor pro A' ab hac regula excipiendus est) alternis vicibus e duabus vel tribus partibus constare.

13.

Theorema. *Si X est functio arcus x formae* (F)

$$\alpha+\alpha'\cos x+\alpha''\cos 2x+\alpha'''\cos 3x+ \text{ etc.}$$
$$+6'\sin x+6''\sin 2x+6'''\sin 3x+ \text{ etc.}$$

vel huius formae (G)

$$\gamma\cos\tfrac{1}{2}x+\gamma'\cos\tfrac{3}{2}x+\gamma''\cos\tfrac{5}{2}x+ \text{ etc.}$$
$$+\delta\sin\tfrac{1}{2}x+\delta'\sin\tfrac{3}{2}x+\delta''\sin\tfrac{5}{2}x+ \text{ etc.}$$

positoque $x=a$, valor functionis X fit $=0$: erit X divisibilis per $\sin\frac{1}{2}(x-a)$, quotiensque in casu priore formae G, in posteriore formae F.

Demonstratio. Casus prior. Si in functione X pro quavis parte $\cos nx$ substituitur $\cos nx-\cos na$, pro quavis parte $\sin nx$ autem $\sin nx-\sin na$, denique pro α, $\alpha - \alpha$ sive 0, manifestum est, partes determinatas hoc modo adiunctas, mutuo se tollere adeoque functionem X hoc modo non variari. Singulae vero partes ipsius X nunc per $\sin\frac{1}{2}(x-a)$ divisibiles erunt, puta

$$\begin{aligned}\frac{\cos nx - \cos na}{\sin\frac{1}{2}(x-a)} = & -2\sin((n-\tfrac{1}{2})x+\tfrac{1}{2}a) \\ & -2\sin((n-\tfrac{3}{2})x+\tfrac{3}{2}a) \\ & -2\sin((n-\tfrac{5}{2})x+\tfrac{5}{2}a) \\ & -\text{etc.} \\ & -2\sin((\tfrac{1}{2}x+(n-\tfrac{1}{2})a)\end{aligned}$$

$$\begin{aligned}\frac{\sin nx - \sin na}{\sin\frac{1}{2}(x-a)} = & +2\cos((n-\tfrac{1}{2})x+\tfrac{1}{2}a) \\ & +2\cos((n-\tfrac{3}{2})x+\tfrac{3}{2}a) \\ & +2\cos((n-\tfrac{5}{2})x-\tfrac{5}{2}a) \\ & +\text{etc.} \\ & +2\cos(\tfrac{1}{2}x+(n-\tfrac{1}{2})a)\end{aligned}$$

Uterque coëfficiens manifesto ad formam G reduci potest, quamobrem etiam tota X per $\sin\frac{1}{2}(x-a)$ divisibilis, quotiensque ad formam G reducibilis est. Q. E. D.

Casus posterior. Si in functione X pro quavis parte $\cos nx$ (designante iam n non ut ante integrum, sed integri imparis semissem) substituitur $\cos nx - \cos\frac{1}{2}(x-a)\cos na$, pro quavis parte $\sin nx$ autem $\sin nx - \cos\frac{1}{2}(x-a)\sin na$, manifestum est, partes, quae hoc modo functioni X accedunt, mutuo se tollere, adeoque X non variari. Singulae autem partes ipsius X nunc per $\sin\frac{1}{2}(x-a)$ divisibiles erunt, puta

$$\begin{aligned}\frac{\cos nx - \cos\frac{1}{2}(x-a)\cos na}{\sin\frac{1}{2}(x-a)} = & -2\sin((n-\tfrac{1}{2})x+\tfrac{1}{2}a) \\ & -2\sin((n-\tfrac{3}{2})x+\tfrac{3}{2}a) \\ & -2\sin((n-\tfrac{5}{2})x+\tfrac{5}{2}a) \\ & -\text{etc.} \\ & -2\sin(x+(n-1)a) \\ & -\sin na\end{aligned}$$

$$\begin{aligned}\frac{\sin nx - \cos\frac{1}{2}(x-a)\sin na}{\sin\frac{1}{2}(x-a)} = & +2\cos((n-\tfrac{1}{2})x+\tfrac{1}{2}a) \\ & +2\cos((n-\tfrac{3}{2})x+\tfrac{3}{2}a) \\ & +2\cos((n-\tfrac{5}{2})x+\tfrac{5}{2}a) \\ & +\text{etc.} \\ & +2\cos(x+(n-1)a) \\ & +\cos na\end{aligned}$$

ut per multiplicationem facile confirmatur. Uterque quotiens est formae F. Quamobrem functio tota X per $\sin\frac{1}{2}(x-a)$ divisibilis, quotiensque ad formam F reducibilis erit. Q. E. D.

14.

THEOREMA. *Si functio X formae F vel G pro pluribus valoribus diversis*) ipsius x, puta a, b, c, d etc. fit $=0$, per productum*

$$\sin\tfrac{1}{2}(x-a)\sin\tfrac{1}{2}(x-b)\sin\tfrac{1}{2}(x-c)\sin\tfrac{1}{2}(x-d)\,\ldots\,.$$

divisibilis, quotiensque vel eiusdem formae erit ut X vel diversae, prout multitudo valorum $a, b, c, d\ldots.$ par est vel impar.

Demonstr. Ponatur $X = X'\sin\frac{1}{2}(x-a)$, sitque M valor ipsius X' pro $x=b$. Hinc $M\sin\frac{1}{2}(b-a)$ erit valor ipsius X pro $x=b$, qui quum esse debeat $=0$, necessario erit $M=0$. Quare X' divisibilis erit per $\sin\frac{1}{2}(x-b)$ et simili ratione quotiens hinc oriundus per $\sin\frac{1}{2}(x-c)$, et sic porro. Quare X divisibilis erit per productum

$$\sin\tfrac{1}{2}(x-a)\sin\tfrac{1}{2}(x-b)\sin\tfrac{1}{2}(x-c)\sin\tfrac{1}{2}(x-d)\ldots \quad \text{Q. E. D.}$$

Altera theorematis pars tam obvia est, ut demonstratione opus non sit.

Ceterum aeque obvium est theorema inversum, si X per productum

$$\sin\tfrac{1}{2}(x-a)\sin\tfrac{1}{2}(x-b)\sin\tfrac{1}{2}(x-c)\sin\tfrac{1}{2}(x-d)\,\ldots\,.$$

divisibilis sit, ipsius valorem fieri $=0$, si ipsi x aliquis valorum a, b, c, d etc. tribuatur.

15.

Ex theoremate praec. sequitur, si functionis X, formae F, valores pro $x=a, b, c, d$ etc. resp. sint A, B, C, D etc., quamvis aliam similem functionem arcus x, quae iisdem valoribus satisfaciat, sub formula $X+PY$ contentam esse debere, ubi per P designamus productum

$$\sin\tfrac{1}{2}(x-a)\sin\tfrac{1}{2}(x-b)\sin\tfrac{1}{2}(x-c)\sin\tfrac{1}{2}(x-d)\ \text{etc.}$$

per Y autem functionem indefinitam arcus x, quae sit vel formae F vel formae

*) Adiecta eadem restrictione, quam in art. 10 exhibuimus.

G, prout multitudo valorum a, b, c, d etc., quam per μ designabimus, par est vel impar. Iam observamus:

I. In artt. 10, 11, 12 eruere docuimus functionem X ordinis m^{ti} (i. e. ultra $\cos mx$ et $\sin mx$ non progredientem), quae μ valoribus propositis A, B, C, D etc. satisfaciat, ita ut m sit $= \frac{1}{2}\mu - \frac{1}{2}$, vel $\frac{1}{2}\mu$, prout μ impar est vel par. Productum P manifesto est ordinis $\frac{1}{2}\mu^{ti}$. Quando itaque μ impar est, adeoque Y vel ordinis $\frac{1}{2}^{ti}$ vel altioris, erit PY ordinis ad minimum $\frac{1}{2}\mu + \frac{1}{2}^{ti}$ adeoque ordinis altioris quam X, unde colligitur, functiones alias quam X, quae iisdem valoribus propositis satisfaciant, non dari, nisi ordinis altioris. Quando vero μ est par, PY tunc tantummodo fit ordinis $\frac{1}{2}\mu^{ti}$, si pro Y accipitur quantitas definita; quare in hoc casu infinite multae quidem functiones similis formae et ordinis ut X dantur, quae iisdem valoribus satisfaciunt, omnes autem sub forma $X + hP$ comprehensae erunt, designante h quantitatem definitam. Eaedem conclusiones ex methodo, per quam in artt. 10, 11 functionem X derivavimus, sponte sequuntur.

II. Vice versa autem, si quae functio X' aliunde innotuisset, quae omnibus quidem valoribus propositis A, B, C, D etc. satisfacit, sed ad altiorem quam opus est ordinem ascendit, functionem Y ita determinare licebit, ut $X' + PY$ ad ordinem $\frac{1}{2}\mu - \frac{1}{2}^{tum}$ vel $\frac{1}{2}\mu^{tum}$ (prout μ impar est vel par) deprimatur. Sint termini summi in X', $K\cos nx + L\sin nx$; in P, $k\cos\frac{1}{2}\mu x + l\sin\frac{1}{2}\mu x$, accipianturque pro terminis summis in Y hi

$$\rho = -\frac{2(Kk + Ll)}{kk + ll}\cos(n - \tfrac{1}{2}\mu)x + \frac{2(Kl - Lk)}{kk + ll}\sin(n - \tfrac{1}{2}\mu)x$$

Hinc calculo facto invenietur, coëfficientes ipsorum $\cos nx$ et $\sin nx$ in $X' + PY$ fieri $= 0$, siquidem fuerit $n > \frac{1}{2}\mu$; similique modo ex terminis summis functionis evolutae $X' + P\rho$ (qui erunt ordinis $n - 1^{ti}$) definientur termini proxime inferiores in Y (ordinis $n - \frac{1}{2}\mu - 1^{ti}$). Haec operatio eousque repetenda erit, donec $X' + PY$ depressa sit ad ordinem non altiorem quam $\frac{1}{2}\mu$, adeoque ad ordinem $\frac{1}{2}\mu - \frac{1}{2}^{tum}$, quando μ impar est, ad ordinem $\frac{1}{2}\mu^{tum}$, quando μ par est. Ulterius hanc depressionem non patere, nullo negotio perspicitur.

III. In casu itaque priore (quando μ impar est) hoc modo necessario ad eandem functionem X perveniemus, quam per methodum supra traditam e valoribus datis A, B, C, D etc. eruissemus. In casu posteriore autem methodus modo exposita suppeditat functionem, quae quidem eiusdem ordinis erit, cuius est functio, per methodum art. 12, sive per methodum art. 11, si terminus ultimus

$= 0$ ponitur, oriunda, attamen ab hac diversa esse potest. Sit functio illa X'', haec X'''; adeoque $X''' = X'' + hP$, designante h quantitatem definitam. Ponamus terminos summos in X'' esse $K'\cos\frac{1}{2}\mu x + L'\sin\frac{1}{2}\mu x$; in X''' autem $K''\cos\frac{1}{2}\mu x + L''\sin\frac{1}{2}\mu x$, ita ut sit $K'' = K' + hk$, $L'' = L' + hl$. Iam non difficile est, ex art. 11 vel ex art. 12 demonstrare, si aggregatum $a+b+c+d+$ etc. designetur per s, esse $K''\cos\frac{1}{2}s + L''\sin\frac{1}{2}s = 0$, unde deducitur

$$h = -\frac{K'\cos\frac{1}{2}s + L'\sin\frac{1}{2}s}{k\cos\frac{1}{2}s + l\sin\frac{1}{2}s}$$
$$K'' = \frac{K'l - L'k}{k\cos\frac{1}{2}s + l\sin\frac{1}{2}s}\sin\frac{1}{2}s$$
$$L'' = -\frac{K'l - L'k}{k\cos\frac{1}{2}s + l\sin\frac{1}{2}s}\cos\frac{1}{2}s$$

Haud difficilius demonstratur, esse

$$k = \pm\frac{\cos\frac{1}{2}s}{2^{\mu-1}}, \qquad l = \pm\frac{\sin\frac{1}{2}s}{2^{\mu-1}}$$

ubi signum superius valet, quando $\frac{1}{2}\mu$ est par, inferius, quando est impar. Quare in valoribus modo traditis pro K'' et L'' denominator fit $= \pm\frac{1}{2^{\mu-1}}$ adeoque

$$K'' = (K'\sin\frac{1}{2}s - L'\cos\frac{1}{2}s)\sin\frac{1}{2}s$$
$$L'' = -(K'\sin\frac{1}{2}s - L'\cos\frac{1}{2}s)\cos\frac{1}{2}s$$

16.

Hactenus tales functiones consideravimus, in quibus tum cosinus tum sinus adsunt: saepissime vero aut hi aut illi absunt, quae functionum genera seorsim tractare conveniet. Utrumque quidem casum ad casum generalem hucusque consideratum reducere liceret; attamen etiam magis e re esse videtur, hanc disquisitionem ad problema art. 3 reducere.

Primo, si constat, omnes coëfficientes $\mathfrak{b}'$, $\mathfrak{b}''$, $\mathfrak{b}'''$ etc. in functione X (art. 10) esse $= 0$, multitudo incognitarum ad $m+1$ diminuitur, et proin pro tot valoribus diversis ipsius x valores respondentes functionis X novisse sufficit. Quoniam vero in hoc casu valor functionis X non mutatur, si pro x substituitur $-x$, manifestum est, non solum tales valores ipsius x, quorum differentia peripheriae vel multiplo peripheriae aequalis est, sed tales quoque, quorum summa est 0 vel peripheriae vel multiplo peripheriae aequalis, pro diversis non esse habendos. Aut generaliter, ut duos valores ipsius x pro diversis habere liceat, cosinus inaequales habere debent. Quibus ita intellectis sint $m+1$ valores dati functionis X hi A, B, C, D etc., valoribus ipsius x his a, b, c, d etc. respondentes, quaeriturque expressio generalis pro valore T, quem functio pro $x = 0$ adipiscitur. Quum functionem X in casu nostro etiam sub formam

$$\gamma + \gamma' \cos x + \gamma''(\cos x)^2 + \gamma'''(\cos x)^3 + \text{etc.} + \gamma^m(\cos x)^m$$

reducere liceat, habebimus per art. 3

$$\begin{aligned} T = &\frac{(\cos t - \cos b)(\cos t - \cos c)(\cos t - \cos d)\ldots}{(\cos a - \cos b)(\cos a - \cos c)(\cos a - \cos d)\ldots} A \\ &+ \frac{(\cos t - \cos a)(\cos t - \cos c)(\cos t - \cos d)\ldots}{(\cos b - \cos a)(\cos b - \cos c)(\cos b - \cos d)\ldots} B \\ &+ \frac{(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos d)\ldots}{(\cos c - \cos a)(\cos c - \cos b)(\cos c - \cos d)\ldots} C \\ &+ \frac{(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos c)\ldots}{(\cos d - \cos a)(\cos d - \cos b)(\cos d - \cos c)\ldots} D \\ &+ \text{etc.} \end{aligned}$$

Secundo, si omnes coëfficientes α, α', α'', α''' etc. evanescunt, multitudo incognitarum erit m, sufficitque adeo, valores functionis X pro totidem valoribus diversis ipsius x novisse. Quum valor functionis X pro $x = -t$ ex valore pro $x = +t$ sponte sequatur (per solam mutationem signi in oppositum), etiam hic tales valores pro diversis haberi nequeunt, unde facile colligitur, hic perinde ut in casu praec. eos tantummodo valores ipsius x pro diversis agnosci, quorum cosinus sunt inaequales. Praeterea quum valor functionis X pro $x = 0$, 180^0, 360^0 etc. et generaliter pro tali valore ipsius x, cuius sinus est $= 0$, sponte fiat $= 0$, adeoque iam ex natura problematis datus sit, talem valorem inter m datos non numerari supponimus. Iam quum constet, esse

$$\frac{\sin nx}{\sin x} = 2\cos(n-1)x + 2\cos(n-3)x + 2\cos(n-5)x + \ldots + 2\cos x$$

pro valore pari ipsius n, atque

$$= 2\cos(n-1)x + 2\cos(n-3)x + 2\cos(n-5)x + \ldots + 2\cos 2x + 1$$

pro valore impari ipsius n, facile concluditur $\frac{X}{\sin x}$ ad formam

$$\delta + \delta' \cos x + \delta''(\cos x)^2 + \delta'''(\cos x)^3 + \text{etc.} + \delta^{m-1}(\cos x)^{m-1}$$

reduci posse, adeoque, quum valores functionis $\frac{X}{\sin x}$ pro $x = a, b, c, d \ldots$ fiant $= \frac{A}{\sin a}, \frac{B}{\sin b}, \frac{C}{\sin c}, \frac{D}{\sin d} \ldots$, haberi per eundem art. 3

$$\begin{aligned} T = & \frac{\sin t(\cos t - \cos b)(\cos t - \cos c)(\cos t - \cos d)\ldots}{\sin a(\cos a - \cos b)(\cos a - \cos c)(\cos a - \cos d)\ldots} A \\ & + \frac{\sin t(\cos t - \cos a)(\cos t - \cos c)(\cos t - \cos d)\ldots}{\sin b(\cos b - \cos a)(\cos b - \cos c)(\cos b - \cos d)\ldots} B \\ & + \frac{\sin t(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos d)\ldots}{\sin c(\cos c - \cos a)(\cos c - \cos b)(\cos c - \cos d)\ldots} C \\ & + \frac{\sin t(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos c)\ldots}{\sin d(\cos d - \cos a)(\cos d - \cos b)(\cos d - \cos c)\ldots} D \\ & + \text{etc.} \end{aligned}$$

Ceterum in utroque casu formula pro T cum functione X identica erit, mutando t in x, quoniam illa indefinite pro valore quocunque ipsius t valet.

17.

E principiis algebraicis facile conclusio petitur, si, pro casu priore art. praec. productum

$$(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos c)(\cos t - \cos d) \text{ etc.}$$

pro posteriore autem productum

$$\sin t(\cos t - \cos a)(\cos t - \cos b)(\cos t - \cos c)(\cos t - \cos d) \text{ etc.}$$

per P designetur, quamvis functionem ipsi X similem, quae iisdem valoribus A, B, C, D etc. pro $x = a, b, c, d$ etc. satisfacit, necessario sub forma $X + PY$ contentam esse debere, ubi Y est functio indefinita arcus x formae

$$\gamma + \gamma' \cos x + \gamma'' \cos 2x + \gamma''' \cos 3x + \text{etc.}$$

Theorema inversum, scilicet quamlibet functionem sub forma $X + PY$ contentam illis valoribus satisfacere, sponte patet. Porro manifestum est, productum P esse ordinis $m+1^{\text{ti}}$, adeoque proxime altioris quam X; quamobrem alia functio ipsi X similis quae iisdem valoribus satisfaciat non datur, nisi ex ordine altiore quam X.

Vice versa, si quae functio X', ipsi X similis quidem, sed altioris ordinis, aliunde innotuit, quae valoribus datis A, B, C, D etc. satisfacit, functionem Y, eius quam tradidimus formae, ita determinare licet, ut $X' + PY$ ad ordinem

m^{tum} deprimatur, unde necessario functio X ipsa, quae e valoribus A, B, C, D etc. per methodum art. praec. eruitur, prodire debebit. Quum methodus terminum summum functionis Y atque inde deinceps inferiores evolvendi, tamquam casus specialis e methodo in art. 15, II tradita peti possit, non opus est, huic rei hic diutius inhaerere.

18.

Omnes disquisitiones praecedentes superstructae sunt conditioni, X esse progressionem cum cosinu sinuque arcus mx abruptam. Quae conditio si locum non habet, tot valores ipsius X, quot cognitos esse hactenus assumsimus, manifesto non sufficiunt ad determinationem completam, expressioque pro T in variis quos consideravimus casibus tradita, correctione opus habebit, a coëfficientibus sequentibus in X pendente et per summationes artt. 1 et 2 facile assignabili. Quodsi autem series X, sive in infinitum excurrat sive uspiam abrumpatur, tantopere convergit, ut partes post $\cos mx$ et $\sin mx$ sequentes pro evanescentibus haberi possint, etiam illam correctionem negligere licebit, adeoque valorem T saltem proxime verum nacti erimus. Quomodocunque vero haec se habeant, T certo functionem verae similem simplicissimam exhibet, per quam omnibus valoribus propositis satisfieri potest.

19.

Imprimis frequens, maximaque adeo attentione dignus est casus iste, ubi valores arcus x, pro quibus valores functionis X dati sunt, progressionem arithmeticam constituunt, in qua terminorum differentia aequalis est coëfficienti, ex divisione peripheriae integrae in totidem partes quot sunt termini orto, ita ut si progressio ultra terminum postremum continuaretur, termini primi peripheria integra aucti rursus deinceps prodituri essent. Postquam itaque disquisitionis generalis praecipua momenta in praecedentibus absolvimus, huncce casum, ubi termini dati quasi *periodum* completam constituunt, coronidis loco seorsim copiosius tractabimus. Antequam vero hanc disquisitionem ipsam aggrediamur, duo lemmata erunt praemittenda.

Lemma Primum. *Si arcus a, b, c, d etc., quorum multitudo est μ, constituunt progressionem arithmeticam, in qua terminorum differentia $b-a=c-b=d-c$ etc. est $=\frac{360^{\circ}}{\mu}$, productum ex μ factoribus $P=$*

$$\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d) \text{ etc.}$$

fit $= \mp\frac{\sin\frac{1}{2}\mu(t-a)}{2^{\mu-1}}$, *ubi signum superius vel inferius valet, prout* μ *par est vel impar.*

Demonstr. Potest quidem hoc theorema facile ex alio ab ill. Eulero in *Introd. in Anal. Infin.* I, §. 240 tradito derivari; ne quid vero hic desiderari videatur, demonstrationem aliam paucis hic explicabimus.

I. Si productum P primo ponitur sub formam

$$\cos\tfrac{1}{2}(t-a-180^0)\cos\tfrac{1}{2}(t-b-180^0)\cos\tfrac{1}{2}(t-c-180^0)\cos\tfrac{1}{2}(t-d-180^0)\ldots$$

atque tunc secundum algorithmum sinuum notum in aggregatum cosinuum evolvitur, facile perspicietur, inde prodire functionem vel formae F vel formae G (art. 13) ad ordinem $\frac{1}{2}\mu^{\text{tum}}$ ascendentem, terminumque *summum* fieri

$$\begin{aligned}
&= \tfrac{1}{2^{\mu-1}}\cos\tfrac{1}{2}(\mu t-a-b-c-d-\text{etc.}-\mu\times 180^0)\\
&= \tfrac{1}{2^{\mu-1}}\cos(\tfrac{1}{2}\mu(t-a-360^0)+90^0)\\
&= -\tfrac{1}{2^{\mu-1}}\sin\tfrac{1}{2}\mu(t-a-360^0)\\
&= -\tfrac{1}{2^{\mu-1}}\sin\tfrac{1}{2}\mu(t-a)\cos\mu\times 180^0\\
&= \mp\tfrac{1}{2^{\mu-1}}\sin\tfrac{1}{2}\mu(t-a)
\end{aligned}$$

signo superiore pro valore pari ipsius μ valente, inferiore pro impari.

II. Quum $\sin\frac{1}{2}\mu(t-a)$ fiat $=0$, si ipsi t aliquis valorum a. b, c, d etc. tribuitur, per theorema art. 14 erit $\sin\frac{1}{2}\mu(t-a)$ per productum P divisibilis; quotiens autem necessario erit quantitas definita $=h$, quoniam $\sin\frac{1}{2}\mu(t-a)$ et P eiusdem sunt ordinis. Manifesto autem (per I) ex aequatione $P=\frac{1}{h}\sin\frac{1}{2}\mu(t-a)$ sequitnr esse $h=\mp 2^{\mu-1}$, adeoque $P=\mp\frac{\sin\frac{1}{2}\mu(t-a)}{2^{\mu-1}}$ Q. E. D.

Lemma Secundum. *Si arcus* $a, b, c, d\ldots$ *eodem modo se habent, ut in lemmate primo, productum ex* μ *factoribus*

$$(\cos t-\cos a)(\cos t-\cos b)(\cos t-\cos c)(\cos t-\cos d)\ldots$$

fit $=\frac{\cos\mu t-\cos\mu a}{2^{\mu-1}}$

Demonstr. Quum fiat

$$\begin{aligned}
\cos t - \cos a &= 2\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(-t-a)\\
\cos t - \cos b &= 2\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(-t-b)\\
\cos t - \cos c &= 2\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(-t-c)\\
\cos t - \cos d &= 2\sin\tfrac{1}{2}(t-d)\sin\tfrac{1}{2}(-t-d)\\
&\text{etc.}
\end{aligned}$$

productum ex his factoribus fit, per lemma primum

$$= 2^\mu \times \mp\frac{\sin\frac{1}{2}\mu(t-a)}{2^{\mu-1}} \times \mp\frac{\sin\frac{1}{2}\mu(-t-a)}{2^{\mu-1}}$$

$$= \frac{\sin\frac{1}{2}\mu(t-a)\sin\frac{1}{2}\mu(-t-a)}{2^{\mu-2}} = \frac{\cos\mu t - \cos\mu a}{2^{\mu-1}} \quad \text{Q. E. D.}$$

Quum habeatur

$$\sin\tfrac{1}{2}\mu(t-a) = -\sin\tfrac{1}{2}\mu(t-b) = \sin\tfrac{1}{2}\mu(t-c) = -\sin\tfrac{1}{2}\mu(t-d) \text{ etc.}$$

nec non $\cos\mu a = \cos\mu b = \cos\mu c = \cos\mu d$ etc.: manifestum est, productum in lemmate primo fieri etiam

$$= \pm\frac{\sin\frac{1}{2}\mu(t-b)}{2^{\mu-1}} = \mp\frac{\sin\frac{1}{2}\mu(t-c)}{2^{\mu-1}} = \pm\frac{\sin\frac{1}{2}\mu(t-d)}{2^{\mu-1}} \text{ etc.}$$

nec non productum in lemmate secundo fieri etiam

$$= \frac{\cos\mu t - \cos\mu b}{2^{\mu-1}} = \frac{\cos\mu t - \cos\mu c}{2^{\mu-1}} = \frac{\cos\mu t - \cos\mu d}{2^{\mu-1}} \text{ etc.}$$

20.

Consideremus primo casum generalem art. 10, ubi X est formae

$$\alpha + \alpha'\cos x + \alpha''\cos 2x + \alpha'''\cos 3x + \ldots + \alpha^{(m)}\cos mx$$
$$+ \beta'\sin x + \beta''\sin 2x + \beta'''\sin 3x + \ldots + \beta^{(m)}\sin mx$$

atque $\mu = 2m+1$. Hic igitur erit

$$\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots = \frac{\sin\frac{1}{2}\mu(t-a)}{2^{\mu-1}\sin\frac{1}{2}(t-a)}$$

$$= \frac{1}{2^{\mu-1}}(1 + 2\cos(t-a) + 2\cos 2(t-a) + 2\cos 3(t-a) + \ldots + 2\cos m(t-a))$$

unde substituendo a pro t

$$\sin\tfrac{1}{2}(a-b)\sin\tfrac{1}{2}(a-c)\sin\tfrac{1}{2}(a-d)\ldots = \frac{\mu}{2^{\mu-1}}$$

Hinc in formula art. 10 pro T, fit coëfficiens ipsius A

$$= \frac{1}{\mu}(1+2\cos(t-a)+2\cos 2(t-a)+2\cos 3(t-a)+\ldots+2\cos m(t-a))$$

Simili modo fit

$$\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots = -\frac{\sin\frac{1}{2}\mu(t-b)}{2^{\mu-1}\sin\frac{1}{2}(t-b)}$$
$$= -\frac{1}{2^{\mu-1}}(1+2\cos(t-b)+2\cos 2(t-b)+2\cos 3(t-b)+\ldots+2\cos m(t-b))$$

atque

$$\sin\tfrac{1}{2}(b-a)\sin\tfrac{1}{2}(b-c)\sin\tfrac{1}{2}(b-d)\ldots = -\frac{\mu}{2^{\mu-1}}$$

Hinc coëfficiens ipsius B in formula pro T fit

$$= \frac{1}{\mu}(1+2\cos(t-b)+2\cos 2(t-b)+2\cos 3(t-b)+\ldots+2\cos m(t-b))$$

Prorsus similes erunt coëfficientes ipsorum C, D etc., unde tandem concluditur, fieri

$$\begin{aligned} T = & \ \frac{1}{\mu}(A+B+C+D+\ldots) \\ & +\frac{2}{\mu}(A\cos a+B\cos b+C\cos c+D\cos d+\ldots)\cos t \\ & +\frac{2}{\mu}(A\sin a+B\sin b+C\sin c+D\sin d+\ldots)\sin t \\ & +\frac{2}{\mu}(A\cos 2a+B\cos 2b+C\cos 2c+D\cos 2d+\ldots)\cos 2t \\ & +\frac{2}{\mu}(A\sin 2a+B\sin 2b+C\sin 2c+D\sin 2d+\ldots)\sin 2t \\ & + \text{etc.} \\ & +\frac{2}{\mu}(A\cos ma+B\cos mb+C\cos mc+D\cos md+\ldots)\cos mt \\ & +\frac{2}{\mu}(A\sin ma+B\sin mb+C\sin mc+D\sin md+\ldots)\sin mt \end{aligned}$$

Quum haec formula cum

$$\alpha+\alpha'\cos t+\beta'\sin t+\alpha''\cos 2t+\beta''\sin 2t+\text{ etc. }+\alpha^{(m)}\cos mt+\beta^{(m)}\sin mt$$

identica esse debeat, valores coëfficientium α, α', β', α'', β'' etc. hinc protinus habentur.

21.

Si progressio pro X cum terminis $\cos mx$ et $\sin mx$ non abrumpitur, sive in infinitum excurrat, sive finita sit, valor pro T in art. praec. inventus incompletus erit, itemque valores singulorum coëfficientium α, α', β', α'', β'' etc. ibi traditi correctione opus habebunt. Haec autem commodius per methodum sequentem quam per summationem art. 2 determinatur. Ante omnia observamus, esse

$$\cos na + \cos nb + \cos nc + \cos nd + \ldots = \mu \cos na$$
$$\sin na + \sin nb + \sin nc + \sin nd + \ldots = \mu \sin na$$

quoties n est integer per μ divisibilis; contra

$$\cos na + \cos nb + \cos nc + \cos nd + \ldots = 0$$
$$\sin na + \sin nb + \sin nc + \sin nd + \ldots = 0$$

quoties n est integer per μ non divisibilis. Pro casu priore res per se clara est; pro posteriore sit summa prima $= P$, secunda $= Q$, unde facile deducitur

$$P \cos n(b-a) - Q \sin n(b-a) = P$$
$$P \sin n(b-a) + Q \cos n(b-a) = Q$$

Multiplicando aequationem primam per $\cos n(b-a) - 1$, secundam per $\sin n(b-a)$, fit addendo delendoque quae mutuo se destruunt

$$2P(1 - \cos n(b-a)) = 0$$

Similiter multiplicando aequationem primam per $\sin n(b-a)$, secundam per $1 - \cos n(b-a)$, provenit addendo

$$2Q(1 - \cos n(b-a)) = 0$$

Iam pro casu quidem priore (ubi n per μ divisibilis est, et proin $\cos n(b-a) = 1$) hae aequationes identicae sunt, pro posteriore autem (ubi n per μ non est divisibilis, adeoque $\cos n(b-a) = \cos \frac{n}{\mu} \times 360^0$ non potest esse $= 1$) consistere nequeunt, nisi fuerit $P = 0$ et $Q = 0$.

Iam ponamus, post terminos $\alpha^m \cos mx + \beta^m \sin mx$ in expressione functionis X sequi

$$\alpha^{m+1} \cos(m+1)x + \alpha^{m+2} \cos(m+2)x + \text{etc}$$
$$+ \beta^{m+1} \sin(m+1)x + \beta^{m+2} \sin(m+2)x + \text{etc.}$$

valorem vero (incompletum), in suppositione, has partes non adesse, pro T in art. praec. inventum, esse

$$\gamma + \gamma' \cos t + \gamma'' \cos 2t + \gamma''' \cos 3t + \ldots + \gamma^m \cos mt$$
$$+ \delta' \sin t + \delta'' \sin 2t + \delta''' \sin 3t + \ldots + \delta^m \sin mt$$

ita ut habeatur $\gamma = \frac{1}{\mu}(A + B + C + D +$ etc.) etc. Substituantur pro A, B, C, D etc. valores sui, puta

$$A = \alpha + \alpha' \cos a + \alpha'' \cos 2a + \ldots + \alpha^m \cos ma + \alpha^{m+1} \cos(m+1)a \ldots$$
$$+ \beta' \sin a + \beta'' \sin 2a + \ldots + \beta^m \sin ma + \beta^{m+1} \sin(m+1)a \ldots$$
etc.

fietque (per summationem modo traditam)

$$\gamma = \alpha + \alpha^{\mu} \cos \mu a + \alpha^{2\mu} \cos 2\mu a + \alpha^{3\mu} \cos 3\mu a + \text{etc.}$$
$$+ \beta^{\mu} \sin \mu a + \beta^{2\mu} \sin 2\mu a + \beta^{3\mu} \sin 3\mu a + \text{etc.}$$

$$\gamma' = \alpha' + (\alpha^{\mu-1} + \alpha^{\mu+1}) \cos \mu a + (\alpha^{2\mu-1} + \alpha^{2\mu+1}) \cos 2\mu a + \text{etc.}$$
$$+ (\beta^{\mu-1} + \beta^{\mu+1}) \sin \mu a + (\beta^{2\mu-1} + \beta^{2\mu+1}) \sin 2\mu a + \text{etc.}$$

$$\delta' = \beta' - (\beta^{\mu-1} - \beta^{\mu+1}) \cos \mu a - (\beta^{2\mu-1} - \beta^{2\mu+1}) \cos 2\mu a - \text{etc.}$$
$$- (\alpha^{\mu-1} - \alpha^{\mu+1}) \sin \mu a - (\alpha^{2\mu-1} - \alpha^{2\mu+1}) \sin 2\mu a - \text{etc.}$$

$$\gamma'' = \alpha'' + (\alpha^{\mu-2} + \alpha^{\mu+2}) \cos \mu a + (\alpha^{2\mu-2} + \alpha^{2\mu+2}) \cos 2\mu \alpha + \text{etc.}$$
$$+ (\beta^{\mu-2} + \beta^{\mu+2}) \sin \mu a + (\beta^{2\mu-2} + \beta^{2\mu+2}) \sin 2\mu \alpha + \text{etc.}$$

$$\delta'' = \beta'' - (\beta^{\mu-2} - \beta^{\mu+2}) \cos \mu a - (\beta^{2\mu-2} - \beta^{2\mu+2}) \cos 2\mu a - \text{etc.}$$
$$- (\alpha^{\mu-2} - \alpha^{\mu+2}) \sin \mu a - (\alpha^{2\mu-2} - \alpha^{2\mu+2}) \sin 2\mu a - \text{etc.}$$

etc. usque ad

$$\gamma^m = \alpha^m + (\alpha^{\mu-m} + \alpha^{\mu+m}) \cos \mu a + (\alpha^{2\mu-m} + \alpha^{2\mu+m}) \cos 2\mu a + \text{etc.}$$
$$+ (\beta^{\mu-m} + \beta^{\mu+m}) \sin \mu a + (\beta^{2\mu-m} + \beta^{2\mu+m}) \sin 2\mu a + \text{etc.}$$

$$\delta^m = \beta^m - (\beta^{\mu-m} - \beta^{\mu+m}) \cos \mu a - (\beta^{2\mu-m} - \beta^{2\mu+m}) \cos 2\mu a - \text{etc.}$$
$$- (\alpha^{\mu-m} - \alpha^{\mu+m}) \sin \mu a - (\alpha^{2\mu-m} - \alpha^{2\mu+m}) \sin 2\mu a - \text{etc.}$$

Si itaque progressio tantopere convergit, ut α^{m+1}, β^{m+1}, α^{m+2}, β^{m+2} etc. negligi possint, valores γ, γ', δ' etc. pro veris α, α', β' etc. accipere licebit.

22.

Transimus ad casum alterum art. 11, ubi progressio X quidem eadem manet ut in art. 20, multitudo valorum datorum autem unitate minor est, quam multitudo coëfficientium incognitorum, puta $= 2m = \mu$. Hic habetur

$$\cos\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots$$
$$= -\frac{\sin\frac{1}{2}\mu(t-a)\cos\frac{1}{2}(t-a)}{2^{\mu-1}\sin\frac{1}{2}(t-a)}$$
$$= -\frac{\cos\frac{1}{2}(t-a)}{2^{\mu-2}}\left(\cos\tfrac{1}{2}(t-a)+\cos\tfrac{3}{2}(t-a)+\cos\tfrac{5}{2}(t-a)+\ldots+\cos\tfrac{\mu-1}{2}(t-a)\right)$$
$$= -\frac{1}{2^{\mu-1}}(1+2\cos(t-a)+2\cos 2(t-a)+2\cos 3(t-a)+\ldots+2\cos(m-1)(t-a)$$
$$+\cos m(t-a))$$

Quare in formula art. 11 pro T coëfficiens ipsius A fit

$$= \frac{1}{\mu}(1+2\cos(t-a)+2\cos 2(t-a)+2\cos 3(t-a)+\ldots$$
$$+2\cos(m-1)(t-a)+\cos m(t-a))$$

Et prorsus similes expressiones (mutato tantummodo a in b, c, d etc.) pro coëfficientibus ipsorum B, C, D etc. inveniuntur. Denique pars ultima ipsius T

$$2^{2m-1}i^{2m}\sin\tfrac{1}{2}(t-a)\sin\tfrac{1}{2}(t-b)\sin\tfrac{1}{2}(t-c)\sin\tfrac{1}{2}(t-d)\ldots\times\{\alpha^m\cos\tfrac{1}{2}s+\beta^m\sin\tfrac{1}{2}s\}$$

fit

$$= -i^{2m}\sin m(t-a)\{\alpha^m\cos(ma+(2m-1)90^0)+\beta^m\sin(ma+(2m-1)90^0)\}$$
$$= -\sin m(t-a)(\alpha^m\sin ma-\beta^m\cos ma)$$

Hinc tandem colligitur

$$\begin{aligned}
T = {} & \frac{1}{\mu}(A+B+C+D+\text{ etc.})\\
& +\frac{2}{\mu}(A\cos a+B\cos b+C\cos c+D\cos d+\text{ etc.})\cos t\\
& +\frac{2}{\mu}(A\sin a+B\sin b+C\sin c+D\sin d+\text{ etc.})\sin t\\
& +\frac{2}{\mu}(A\cos 2a+B\cos 2b+C\cos 2c+D\cos 2d+\text{ etc.})\cos 2t\\
& +\frac{2}{\mu}(A\sin 2a+B\sin 2b+C\sin 2c+D\sin 2d+\text{ etc.})\sin 2t\\
& +\text{ etc.}\\
& +\frac{1}{\mu}(A\cos ma+B\cos mb+C\cos mc+D\cos md+\text{ etc.})\cos mt\\
& +\frac{1}{\mu}(A\sin ma+B\sin mb+C\sin mc+D\sin md+\text{ etc.})\sin mt\\
& +(\alpha^m\sin ma-\beta^m\cos ma)(\sin ma\cos mt-\cos ma\sin mt)
\end{aligned}$$

Quae expressio, si, rescissa parte ultima, per

$$\gamma+\gamma'\cos t+\gamma''\cos 2t+\text{ etc. }+\gamma^m\cos mt$$
$$+\delta'\sin t+\delta''\sin 2t+\text{ etc. }+\delta^m\sin mt$$

exhibetur, omnes coëfficientes γ, γ', δ', γ'', δ'' etc. praecise eosdem valores habere patet, ut in art. praec., ubi erat $\mu = 2m+1$, exceptis duobus ultimis γ^m, δ^m, qui illic duplo erant maiores. Porro liquet, omnes coëfficientes α, α', $\alpha''\ldots\alpha^{m-1}$, β', $\beta''\ldots\beta^{m-1}$ illis γ, γ', $\gamma''\ldots\gamma^{m-1}$, δ', $\delta''\ldots\delta^{m-1}$ resp. aequales adeoque cognitos fieri, inter ultimos α^m, β^m autem haberi aequationes

$$\alpha^m = \gamma^m + (\alpha^m \sin ma - \beta^m \cos ma)\sin ma$$
$$\beta^m = \delta^m - (\alpha^m \sin ma - \beta^m \cos ma)\cos ma$$

quae tamen (ut ex natura problematis praevidere licebat) non sufficiunt ad determinationem completam coëfficientium α^m, β^m. Scilicet quum fiat

$$\cos ma = -\cos mb = \cos mc = -\cos md \text{ etc.}$$

atque

$$\sin ma = -\sin mb = \sin mc = -\sin md \text{ etc.}$$

adeoque

$$\gamma^m = \frac{\cos ma}{\mu}(A - B + C - D + \text{ etc.})$$
$$\delta^m = \frac{\sin ma}{\mu}(A - B + C - D + \text{ etc.})$$

facile perspicitur, utramque aequationem contentam esse sub unica

$$\alpha^m \cos ma + \beta^m \sin ma = \frac{1}{\mu}(A - B + C - D + \text{ etc.})$$

Aliam itaque insuper relationem inter α^m et β^m datam esse oportet, ut utrumque coëfficientem penitus assignare liceat.

23.

Si progressio X cum $\cos mx$ et $\sin mx$ non abrumpitur, valor ipsius T in art. praec. inventus incompletus erit, coëfficientesque γ, γ', δ' etc. a veris α, α', β' etc. diversi erunt. Et quidem facile perspicitur, si coëfficientes terminorum sequentium in X per eosdem characteres denotentur, quibus in art. 21 usi sumus, pro γ, γ', δ' etc. usque ad γ^{m-1}, δ^{m-1} prorsus eosdem valores prodire, quos illic eruimus; duos ultimos autem fieri

$$\gamma^m = \tfrac{1}{2}\alpha^m + \tfrac{1}{2}(\alpha^m + \alpha^{3m})\cos\mu a + \tfrac{1}{2}(\alpha^{3m} + \alpha^{5m})\cos 2\mu a + \text{ etc.}$$
$$+ \tfrac{1}{2}(\beta^m + \beta^{3m})\sin\mu a + \tfrac{1}{2}(\beta^{3m} + \beta^{5m})\sin 2\mu a + \text{ etc.}$$
$$\delta^m = \tfrac{1}{2}\beta^m - \tfrac{1}{2}(\beta^m - \beta^{3m})\cos\mu a - \tfrac{1}{2}(\beta^{3m} - \beta^{5m})\cos 2\mu a - \text{ etc.}$$
$$- \tfrac{1}{2}(\alpha^m - \alpha^{3m})\sin\mu a - \tfrac{1}{2}(\alpha^{3m} - \alpha^{5m})\sin 2\mu a - \text{ etc.}$$

In hoc itaque casu coëfficientes γ, γ', δ', γ'', δ'' ... γ^{m-1}, δ^{m-1} pro veris α, α', β', α'', β'' ... α^{m-1}, β^{m-1} adoptare licet, quatenus sequentes α^{m+1}, β^{m+1} etc. negligi possunt; ad ultimos vero γ^m, δ^m nisi ipsi pro negligendis haberi possunt, haec conclusio non est extendenda.

24.

Per disquisitionem inde ab art. 20 traditam eo pervenimus, ut propositis μ valoribus functionis X arcus indeterminati x, periodum completam formantibus, functionem $(X') =$

$$\gamma + \gamma' \cos x + \gamma'' \cos 2x + \ldots + \gamma^m \cos mx$$
$$+ \delta' \sin x + \delta'' \sin 2x + \ldots + \delta^m \sin mx$$

quae illis omnibus satisfaciat, assignare possimus, ita, ut numerus m, ordinem huius functionis exprimens sit vel $= \frac{1}{2}\mu - \frac{1}{2}$ vel $= \frac{1}{2}\mu$, prout μ impar est vel par. Ex artt. 15, 19 concluditur, quamvis aliam similem functionem, quae iisdem valoribus satisfacit, sub forma $X' + Y \sin \frac{1}{2}\mu(x-a)$ contentam esse debere, ubi Y functionem indefinitam arcus x exhibet formae G vel F (art. 13), prout μ impar est vel par. In casu priore praeter X' functio alia similis non datur, iisdem valoribus satisfaciens, nisi ex ordine altiore, unde X' cum X identica erit, si constat X ordinem m^{tum} non egredi. In casu posteriore autem infinitae quidem aliae similes functiones ordinis m^{ti} iisdem valoribus satisfaciunt, omnes tamen in eo convenient, quod sub forma $X' + h \sin m(x-a)$ contenti erunt, designante h quantitatem definitam: quamobrem, si constat X ordinem m^{tum} non egredi, saltem X et X' aliter quam in coëfficientibus ipsorum $\cos mx$ et $\sin mx$ non discrepabunt; in functione X' autem inter hos coëfficientes aequatio $\gamma^m \sin ma - \delta^m \cos ma = 0$ locum habebit. Porro patet, si qua functio X'' similis formae ut X', quae iisdem valoribus satisfaciat, aliunde constet, iisdem satisfieri per functionem quamcunque $X'' + Y \sin \frac{1}{2}\mu(x-a)$, atque Y ita determinari posse, ut haec functio ordinem m^{tum} non egrediatur. Ad hunc finem observamus, si qua pars ipsius Y exhibeatur per $K \cos(\varkappa x + k)$, designante K coëfficientem definitum, k arcum definitum, $\varkappa$ numerum vel integrum vel fractione $\frac{1}{2}$ affectum (prout μ par vel impar) — ad quam formam singulae partes reduci possunt, si cosinus et sinus eiusdem arcus contrahuntur — pars respondens producti $Y \sin \frac{1}{2}\mu(x-a)$ exhibebitur per

$$\tfrac{1}{2}K\sin((\varkappa+\tfrac{1}{2}\mu)x+k-\tfrac{1}{2}\mu a)-\tfrac{1}{2}K\sin((\varkappa-\tfrac{1}{2}\mu)x+k+\tfrac{1}{2}\mu a)$$

sive quod eodem redit per

$$L\sin(\lambda x+l)-L\sin((\lambda-\mu)x+l+\mu a)$$

designante λ integrum. Quocirca in X'' pro quavis parte $L\sin(\lambda x+l)$ substituere licebit hanc $L\sin((\lambda-\mu)x+l+\mu a)$ et simili ratione pro hac rursus $L\sin((\lambda-2\mu)x+l+2\mu a)$ porroque $L\sin((\lambda-3\mu)x+l+3\mu a)$ etc. Ut similis conclusio ad cosinus extendatur, sufficit observatio, cosinus pro sinubus prodire, si modo pro l scribatur $l+90^0$. Hinc colligitur, si in X'' occurrat terminus $L\cos\lambda x$, designante λ integrum maiorem quam $\tfrac{1}{2}\mu$, qui proin sub formam $\lambda=\nu\mu\pm\lambda'$ poni potest, ita ut ν sit integer atque λ' non maior quam $\tfrac{1}{2}\mu$, pro isto termino substitui posse

$$L\cos(\pm\lambda'x+\nu\mu a)=L\cos\nu\mu a\cos\lambda'x\mp L\sin\nu\mu a\sin\lambda'x$$

et similiter pro termino tali $L\sin\lambda x$ substitui poterit

$$L\sin(\pm\lambda'x+\nu\mu a)=L\sin\nu\mu a\cos\lambda'x\pm L\cos\nu\mu a\sin\lambda'x$$

Hoc modo manifesto functio X'' deprimetur ad aliam, ordinis certo non maioris quam $\tfrac{1}{2}\mu^{\text{ti}}$.

Per hanc itaque operationem X'' semper transibit in ipsam X', quoties μ impar est; in casu altero autem, ubi μ par est, certo in functionem talem, quae tantummodo in coëfficientibus ipsorum $\cos mx$ et $\sin mx$ a functione X' diversa esse potest. Ut ex illis coëfficientibus coëfficientes respondentes in X' deducantur, methodus generalis art. 15 adhiberi potest, ex qua sine negotio sequitur, si in illa functione termini ultimi sint $K\cos mx+L\sin mx$, pro his substitui debere, ut functio X' prodeat,

$$\begin{aligned}&(K\cos ma+L\sin ma)\cos ma\cos mx\\ &+(K\cos ma+L\sin ma)\sin ma\sin mx\end{aligned}$$

Hi termini fiunt $=K\cos mx+L\sin mx$, si inter K et L aequatio $K\sin ma=L\cos ma$ locum habet, ut per calculum facile confirmatur: tunc igitur illa functio cum X' omnino iam identica est.

25.

Si functio X cum terminis $\cos mx$, $\sin mx$ non abrumpitur, sed ulterius excurrit, coëfficientesque terminorum sequentium adhuc nimis considerabiles sunt, ita ut ipsos negligere non liceat, coëfficientes γ, γ', δ' etc. in functione X', quae μ valoribus functionis X periodum completam formantibus, pro $x = a, b, c, d$ etc. satisfacit, notabiliter a coëfficientibus respondentibus α, α', β' etc. discrepabunt. Quodsi itaque pro periodo $x = a, b, c, d$ etc. periodum aliam μ terminorum, puta pro $x = a', b', c', d'$ etc. simili modo tractamus, functionemque m^{ti} ordinis, his novis valoribus functionis X satisfacientem evolvimus, et perinde ut X' per expressionem

$$\begin{aligned}\gamma + \gamma'\cos x + \gamma''\cos 2x + \ldots + \gamma^m \cos mx \\ + \delta'\sin x + \delta''\sin 2x + \ldots + \delta^m \sin mx\end{aligned}$$

exhibemus: pro coëfficientibus γ, γ', δ' etc. iam valores nanciscemur, ab iis, quos ante invenimus, notabiliter diversos. Hinc intelligitur, quo pacto hos coëfficientes tamquam *variabiles* spectare liceat, et quidem, quum ex artt. 21, 23 singulos perinde per arcum $\mu a'$ determinari pateat, ut valores priores per arcum μa, singuli coëfficientes considerari poterunt tamquam functiones arcus indeterminati y, similiter ut X est functio arcus x. Iam supponamus, $\mu\nu = \pi$ valores functionis X periodum integram formantes, pro

$$x = a, a', a'', \ldots b, b', b'', \ldots c, c', c'', \ldots d, d', d'' \ldots \text{etc.}$$

cognitos esse, ita ut sit

$$\begin{aligned}
&a' = a + \tfrac{1}{\pi}360^0, && a'' = a + \tfrac{2}{\pi}360^0 \text{ etc.} \\
&b = a + \tfrac{\nu}{\pi}360^0 = a + \tfrac{1}{\mu}360^0, && b' = a + \tfrac{\nu+1}{\pi}360^0 = a' + \tfrac{1}{\mu}360^0 \text{ etc.} \\
&c = a + \tfrac{2\nu}{\pi}360^0 = a + \tfrac{2}{\mu}360^0 \text{ etc.}
\end{aligned}$$

patetque, hanc periodum π terminorum in ν periodos μ terminorum pro

$$\begin{aligned}
&x = a, \; b, \; c, \; d \text{ etc.} \\
&x = a', \; b', \; c', \; d' \text{ etc.} \\
&x = a'', \; b'', \; c'', \; d'' \text{ etc.} \\
&\text{etc.}
\end{aligned}$$

discerpi posse. Quodsi itaque eo quo docuimus modo pro singulis periodis functio

X' evolvitur, nanciscimur ν valores singulorum coëfficientium γ, γ', δ' etc., qui hisce deinceps valoribus ipsius y respondent

$$y = \mu a, \quad y = \mu a' = \mu a + \frac{1}{\nu} 360^0, \quad y = \mu a'' = \mu a + \frac{2}{\nu} 360^0 \text{ etc.}$$

adeoque periodos completas ν *terminorum formant.* Quocirca, per methodum nostram, singuli coëfficientes γ, γ', δ' etc. per expressionem talem

$$\begin{aligned} &\varepsilon + \varepsilon' \cos y + \varepsilon'' \cos 2y + \ldots + \varepsilon^n \cos ny \\ &+ \zeta' \sin y + \zeta'' \sin 2y + \ldots + \zeta^n \sin ny \end{aligned}$$

exhiberi poterunt, ita ut sit n vel $= \frac{1}{2}\nu - \frac{1}{2}$ vel $= \frac{1}{2}\nu$, prout ν impar est vel par. Manifestum est, talem expressionem eundem valorem ex substitutione $y = \mu b$, μc, μd etc. obtinere, quem obtinet ex $y = \mu a$; nec non eundem ex $y = \mu b'$, $\mu c'$, $\mu d'$ etc, quem obtinet ex $y = \mu a'$ et sic porro. Quamobrem si in

$$\begin{aligned} &\gamma + \gamma' \cos x + \gamma'' \cos 2x + \text{ etc.} \\ &+ \delta' \sin x + \delta'' \sin 2x + \text{ etc.} \end{aligned}$$

pro singulis coëfficientibus γ, γ', δ' etc. hae functiones arcus indeterminati y substituuntur, prodibit functio Z, duos arcus x, y involvens, transibitque Z in eam functionem X', quae e periodo prima deducta est, si pro y aliquis valorum μa, μb, μc, μd etc. substituitur; si vero pro y aliquis valorum $\mu a'$, $\mu b'$, $\mu c'$, $\mu d'$ etc. accipitur, transibit Z in eam functionem, quae eruta est e periodo secunda. Hinc tandem colligitur, si in Z pro y statim scribatur μx, ita ut functio solius x prodeat (X''), hanc ita comparatam fore, ut pro omnibus π valoribus ipsius x, valoribus propositis functionis X satisfaciat.

26.

Si methodus artt. 20, 22 ad omnes π valores propositos immediate sine praevia discerptione in ν periodos minores applicata esset, prodiisset functio (X''') ordinis $\frac{1}{2}\pi - \frac{1}{2}^{\text{ti}}$ vel $\frac{1}{2}\pi^{\text{ti}}$, prout π impar vel par, quam cum functione X'' iam comparabimus. Quum haec posterior, si rite reducta est, manifesto ad ordinem $\mu n + m^{\text{tum}}$ ascendat, tres casus distinguemus.

I. Quando tum μ tum ν est impar, erit $\mu n + m = \frac{1}{2}\mu\nu - \frac{1}{2} = \frac{1}{2}\pi - \frac{1}{2}$. unde patet, X'' cum X''' prorsus identicam esse debere.

II. Quando μ par est, ν impar, fit $\mu n + m = \frac{1}{2}\mu\nu = \frac{1}{2}\pi$, adeoque X'' eiusdem ordinis ut X'''. Sufficit itaque utriusque functionis terminos ultimos comparare. Haud difficile perspicitur, hos terminos ultimos in X'' prodire unice ex terminis $\gamma^m \cos mx + \delta^m \sin mx$, et quidem ex iis terminis coëfficientium γ^m, δ^m, qui continent $\cos ny$ et $\sin ny$ sive $\cos \mu n x$ et $\sin \mu n x$. Sint hi termini

$$\text{in } \gamma^m \ldots . \; k \cos \mu n x + l \sin \mu n x$$
$$\text{in } \delta^m \ldots . \; k' \cos \mu n x + l' \sin \mu n x$$

produceturque hinc in X''

$$(k \cos \mu n x + l \sin \mu n x) \cos m x + (k' \cos \mu n x + l' \sin \mu n x) \sin m x$$

unde prodeunt termini ordinis $\mu n + m^{\text{ti}}$ sequentes

$$\tfrac{1}{2}(k - l') \cos(\mu n + m) x + \tfrac{1}{2}(l + k') \sin(\mu n + m) x$$

Iam haud difficile quidem *ex formatione coëfficientium* demonstrari potest, esse

$$(k - l') \sin(\mu n + m) a = (l + k') \cos(\mu n + m) a$$

praeferimus tamen methodum sequentem, quae concinnior videtur. Ex art. 24 sequitur, esse

$$\gamma^m \sin \tfrac{1}{2} y - \delta^m \cos \tfrac{1}{2} y = 0$$

tum si pro y substituitur μa, atque pro γ^m, δ^m valores ex periodo prima deducti; *tum* si pro y substituitur $\mu a'$, atque pro γ^m, δ^m valores ex periodo secunda deducti etc. Quodsi itaque pro γ^m, δ^m expressiones, quae erutae sunt indefinitae substituuntur, ita ut $\gamma^m \sin \frac{1}{2} y - \delta^m \cos \frac{1}{2} y$ fiat functio arcus y formae G ad ordinem $n + \frac{1}{2} = \frac{1}{2}\nu$ ascendens, haec pro omnibus ν valoribus ipsius y his μa, $\mu a'$, $\mu a''$ etc. fiet $= 0$, adeoque per art. 14 per

$$\sin \tfrac{1}{2}(y - \mu a) \sin \tfrac{1}{2}(y - \mu a') \sin \tfrac{1}{2}(y - \mu a'') \text{ etc.}$$

divisibilis, hinc etiam per $\sin \frac{1}{2}\nu(y - \mu a)$ et proin formae $h \sin \frac{1}{2}\nu(y - \mu a)$, ita ut h sit quantitas definita. Quamobrem termini ultimi functionis $\gamma^m \sin \frac{1}{2} y - \delta^m \cos \frac{1}{2} y$ evolutae esse debebunt

$$- h \sin \tfrac{1}{2}\pi a \cos \tfrac{1}{2}\nu y + h \cos \tfrac{1}{2}\pi a \sin \tfrac{1}{2}\nu y$$

omnes vero antecedentes evanescere: illos vero fieri patet

$$= -\tfrac{1}{2}(l+k')\cos(n+\tfrac{1}{2})y + \tfrac{1}{2}(k-l')\sin(n+\tfrac{1}{2})y$$

Quocirca erit $\tfrac{1}{2}(k-l') = h\cos\tfrac{1}{2}\pi a = h\cos\tfrac{1}{2}(\mu n+m)a$

$$\tfrac{1}{2}(l+k') = h\sin\tfrac{1}{2}\pi a = h\sin\tfrac{1}{2}(\mu n+m)a \quad \text{Q. E. D.}$$

Haec vero aequatio ipsissimum erit criterium coëfficientium ultimorum functionis X''', ut in art. 24 ostentum est: quare in hoc quoque casu functio X'' cum X''' prorsus identica erit.

III. Quando ν par est, fit $\mu n+m = \tfrac{1}{2}\pi+m$, adeoque X'' ordinis altioris quam X'''. Facile autem perspicitur, omnes terminos in X'', qui ordinem $\tfrac{1}{2}\pi^{\text{tum}}$ transcendunt, provenire unice ex iis terminis singulorum coëfficientium γ', δ', γ'', δ'' etc., qui continent $\cos ny$ et $\sin ny$ sive $\cos\mu nx$ et $\sin\mu nx$. Sint hi termini, in expressione pro coëfficiente aliquo γ^λ vel δ^λ $k\cos\mu nx + l\sin\mu nx$, ex quibus producitur in X''

$$(k\cos\mu nx + l\sin\mu nx)\cos\lambda x \quad \text{in casu primo}$$
$$\text{vel } (k\cos\mu nx + l\sin\mu nx)\sin\lambda x \quad \text{in casu secundo}$$

unde

$$\tfrac{1}{2}k\cos(\mu n+\lambda)x + \tfrac{1}{2}l\sin(\mu n+\lambda)x$$
$$+\tfrac{1}{2}k\cos(\mu n-\lambda)x + \tfrac{1}{2}l\sin(\mu n-\lambda)x$$

in casu primo atque

$$-\tfrac{1}{2}l\cos(\mu n+\lambda)x + \tfrac{1}{2}k\sin(\mu n+\lambda)x$$
$$+\tfrac{1}{2}l\cos(\mu n-\lambda)x - \tfrac{1}{2}k\sin(\mu n-\lambda)x$$

in casu secundo. Per art. 24 autem substituere oportet pro $\cos(\mu n+\lambda)x$ sive $\cos(\tfrac{1}{2}\pi+\lambda)x$, $\cos\pi a\cos(\tfrac{1}{2}\pi-\lambda)x + \sin\pi a\sin(\tfrac{1}{2}\pi-\lambda)x$; atque pro $\sin(\mu n+\lambda)x$ sive $\sin(\tfrac{1}{2}\pi+\lambda)x$, $\sin\pi a\cos(\tfrac{1}{2}\pi-\lambda)x - \cos\pi a\sin(\tfrac{1}{2}\pi-\lambda)x$. Quare habebimus in casu primo

$$\tfrac{1}{2}(k+k\cos\pi a+l\sin\pi a)\cos(\tfrac{1}{2}\pi-\lambda)x$$
$$+\tfrac{1}{2}(l+k\sin\pi a-l\cos\pi a)\sin(\tfrac{1}{2}\pi-\lambda)x$$

in casu secundo autem

$$\tfrac{1}{2}(l-l\cos\pi a+k\sin\pi a)\cos(\tfrac{1}{2}\pi-\lambda)x$$
$$-\tfrac{1}{2}(k+l\sin\pi a+k\cos\pi a)\sin(\tfrac{1}{2}\pi-\lambda)x$$

Sed ex aequatione $k\sin\frac{1}{2}\pi a = l\cos\frac{1}{2}\pi a$, quae in utroque casu inter k et l locum habere debet (art. 24), nullo negotio deducitur

$$\tfrac{1}{2}(k + k\cos\pi a + l\sin\pi a) = k$$
$$\tfrac{1}{2}(l + k\sin\pi a - l\cos\pi a) = l$$

Hinc tandem colligitur, pro iis partibus in X'', quae ex terminis ultimis

$$k\cos ny + l\sin ny$$

cuiusvis coëfficientis γ^λ vel δ^λ producerentur, designante λ indicem maiorem quam 0, statim substitui posse

$$k\cos(\tfrac{1}{2}\pi - \lambda)x + l\sin(\tfrac{1}{2}\pi - \lambda)x \text{ in casu priore vel}$$
$$l\cos(\tfrac{1}{2}\pi - \lambda)x - k\sin(\tfrac{1}{2}\pi - \lambda)x \text{ in casu posteriore}$$

Hoc modo loco functionis X'', quae ad ordinem $\frac{1}{2}\pi + m^{\text{tum}}$ ascenderet, aliam X'''' nanciscimur, quae ordinem $\frac{1}{2}\pi^{\text{tum}}$ non egreditur. Nihil itaque superest, nisi ut coëfficientes cosinus et sinus arcus $\frac{1}{2}\pi x$, qui manifesto in hac functione iidem manserunt ut in X'', cum coëfficientibus respondentibus in X''' comparemus. Levi attentione autem perspicitur, hosce terminos unice produci ex coëfficiente primo γ, et quidem ex ultimis eius terminis; qui si per $k\cos ny + l\sin ny$ exhibentur, illi erunt $k\cos\frac{1}{2}\pi x + l\sin\frac{1}{2}\pi x$, adeoque necessario identici cum respondentibus in X'''. Hinc itaque colligitur, functionem X'''', quae per praecepta modo tradita loco functionis X'' obtinetur, cum functione X''' prorsus identicam fieri.

27.

Pro eo itaque casu, ubi multitudo valorum propositorum functionis X, periodum integram formantium, numerus compositus est $= \pi = \mu\nu$, per partitionem illius periodi in ν periodos μ terminorum eandem functionem cunctis valoribus datis satisfacientem eruere in artt. 25, 26 didicimus, quae per applicationem immediatam theoriae generalis ad periodum totam prodiret: illam vero methodum calculi mechanici taedium magis minuere, praxis tentantem docebit. Nulla iam amplius explicatione opus erit, quomodo illa partitio adhuc ulterius extendi et ad eum casum applicari possit, ubi multitudo omnium valorum propositorum numerus e tribus pluribusve factoribus compositus est, e. g. si numerus μ

rursus esset compositus, in quo casu manifesto quaevis periodus μ terminorum in plures periodos minores subdividi potest. Ceterum ex pluribus aliis annotationibus applicationi practicae inservientibus hic sequentes tantummodo attingemus. Valores talium aggregatorum

$$A\cos\lambda a + B\cos\lambda b + C\cos\lambda c + D\cos\lambda d + \text{etc.} = p$$
$$A\sin\lambda a + B\sin\lambda b + C\sin\lambda c + D\sin\lambda d + \text{etc.} = q$$

(ubi ut in praecedentibus, supponimus $b - a = c - b = d \quad c$ etc. $= \frac{1}{\mu}360^0 = \Delta$) saepius, siquidem nondum est $a = 0$, commodius determinantur per formulas

$$A + B\cos\lambda\Delta + C\cos 2\lambda\Delta + D\cos 3\lambda\Delta + \text{etc.} = P$$
$$B\sin\lambda\Delta + C\sin 2\lambda\Delta + D\sin 3\lambda\Delta + \text{etc.} = Q$$

$p = P\cos a - Q\sin a$, $q = P\sin a + Q\cos a$, sive etiam faciendo $\frac{Q}{P} = \operatorname{tang}\varphi$, atque $\frac{P}{\cos\varphi} = \frac{Q}{\sin\varphi} = R$, per has $p = R\cos(\varphi + a)$, $q = R\sin(\varphi + a)$.

28.

Exemplum. In commercio literario a clar. barone DE ZACH edito, vol. X p. 188 invenitur tabula, limitem borealem et australem zodiaci *Palladis* exhibens. Utriusque limites declinatio tamquam functio periodica ascensionis rectae spectatur, quae in tabula illa per singulos quinque gradus progreditur. Ad illustrationem disquisitionum praecedentium applicationem ad limitem borealem hic faciemus, cuius itaque declinatio nobis erit X, ascensio recta x. Excerpimus ex tabula illa periodum sequentem 12 terminorum

x	X		
0^0	$6^0 48'$ Bor.	$= +$	$408'$
30	1 29	+	89
60	1 6 Austr. ..	—	66
90	0 10 Bor.....	+	10
120	5 38	+	338
150	13 27	+	807
180	20 38	+	1238
210	25 11	+	1511
240	26 23	+	1583
270	24 22	+	1462
300	19 43	+	1183
330	13 24	+	804

Distribuamus hanc periodum primo in tres periodos quaternorum terminorum

$a = 0^0$	$A = +408$	$a' = 30^0$	$A' = +89$	$a'' = 60^0$	$A'' = -66$
$b = 90^0$	$B = +10$	$b' = 120^0$	$B' = +338$	$b'' = 150^0$	$B'' = +807$
$c = 180^0$	$C = +1238$	$c' = 210^0$	$C' = +1511$	$c'' = 240^0$	$C'' = +1583$
$d = 270^0$	$D = +1462$	$d' = 300^0$	$D' = +1183$	$d'' = 330^0$	$D'' = +804$

In formula

$$X' = \gamma + \gamma' \cos x + \gamma'' \cos 2x + \delta' \sin x + \delta'' \sin 2x$$

fit itaque

$$\gamma = \tfrac{1}{4}(A+B+C+D) = 779',5$$
$$\gamma' = \tfrac{1}{2}(A\cos a + B\cos b + C\cos c + D\cos d) = \tfrac{1}{2}(A-C) = -415',0$$
$$\delta' = \tfrac{1}{2}(A\sin a + B\sin b + C\sin c + D\sin d) = \tfrac{1}{2}(B-D) = -726',0$$
$$\gamma'' = \tfrac{1}{4}(A\cos 2a + B\cos 2b + C\cos 2c + D\cos 2d) = \tfrac{1}{4}(A-B+C-D) = +43',5$$
$$\delta'' = \tfrac{1}{4}(A\sin 2a + B\sin 2b + C\sin 2c + D\sin 2d) = 0$$

et similiter pro periodo secunda ac tertia. Hoc modo emergit

Pro periodo	ubi $y = 4x$	γ	γ'	δ'	γ''	δ''
prima	0^0	+779,5	−415,0	−726,0	+43,5	0
secunda	120^0	+780,2	−404,5	−721,4	+ 9,9	+17,1
tertia	240^0	+782,0	−413,5	−713,3	+11,7	−20,3

Hic porro exhibetur γ per formulam

$$\tfrac{1}{3}(779',5 + 780,2 + 782,0)$$
$$+\tfrac{2}{3}(779,5 + 780,2\cos 120^0 + 782,0\cos 240^0)\cos 4x$$
$$+\tfrac{2}{3}(780,2\sin 120^0 + 782,0\sin 240^0)\sin 4x$$

sive per

$$780,6 - 1,1\cos 4x - 1,0\sin 4x, \text{ et perinde}$$
$$\gamma' \text{ per } -411,0 - 4,0\cos 4x + 5,2\sin 4x$$
$$\delta' \text{ per } -720,2 - 5,8\cos 4x - 4,7\sin 4x$$
$$\gamma'' \text{ per } +21,7 + 21,8\cos 4x - 1,1\sin 4x$$
$$\delta'' \text{ per } -1,1 + 1,1\cos 4x + 21,6\sin 4x$$

Quibus valoribus in X' substitutis prodit formula 12 valores propositos exhibens

$$\begin{aligned}
&+780,6\\
&-411,0\cos x\ -720,2\sin x\\
&+\ \ 43,4\cos 2x-\ \ 2,2\sin 2x\\
&-\ \ \ 4,3\cos 3x+\ \ 5,5\sin 3x\\
&-\ \ \ 1,1\cos 4x-\ \ 1.0\sin 4x\\
&+\ \ \ 0,3\cos 5x-\ \ 0,3\sin 5x\\
&+\ \ \ 0,1\cos 6x
\end{aligned}$$

Distribuamus secundo eandem periodum in quatuor periodos ternorum terminorum, quibus singulis itaque per formulam primi ordinis

$$X' = \gamma + \gamma'\cos x + \delta'\sin x$$

satisfaciendum erit. Hic invenitur

Pro periodo	ubi $y = 3x$	γ	γ'	δ'
prima	0^0	$+776,3$	$-368,3$	$-718,8$
secunda	90^0	$+786,0$	$-414,5$	$-676,0$
tertia	180^0	$+785,0$	$-453,0$	$-721,1$
quarta	270^0	$+775,0$	$-408,2$	$-765,0$

unde deducuntur formulae, sub quibus γ, γ', δ' exhibentur sequentes:

$$\begin{aligned}
&\gamma \ldots\ldots +780,6-\ \ 4,3\cos 3x+\ \ 5,5\sin 3x+0,1\cos 6x\\
&\gamma' \ldots\ldots -411,0+42,3\cos 3x-\ \ 3,2\sin 3x+0,3\cos 6x\\
&\delta' \ldots\ldots -720,2+\ \ 1,2\cos 3x+44,5\sin 3x+0,3\cos 6x
\end{aligned}$$

His valoribus in $\gamma + \gamma'\cos x + \delta'\sin x$ substitutis, partibusque ultimis in γ, γ', δ', quae $\cos 6x$ continent, secundum praecepta art. 26 tractatis, prorsus eadem formula eruitur, ad quam supra pervenimus.

29.

Supersunt casus, ubi in functione X vel cosinus vel sinus soli adsunt, quos in artt. 16, 17 generaliter tractavimus. Supponemus, ut in praecc., datam esse periodum μ terminorum, puta $X = A, B, C, D$ etc. pro $x = a, b, c, d$ etc. Duo vero casus hic probe distinguendi sunt. Aliter enim tractari debet

1^{mo} casus, ubi a est complementum alicuius valorum sequentium b, c, d etc. ad 360^0 vel ad 0 vel ad multiplum peripheriae. Sit hic terminus $a + \frac{k}{\mu}360^0$, atque $a+\frac{k}{\mu}360^0+a=l\times 360^0$, sive $a=(\mu l-k)\frac{180}{\mu}$. Quare in hoc casu a per $\frac{180}{\mu}$ divisibilis erit, adeoque $\sin\mu a=0$; porro habetur $b=(\mu l-k+2)\frac{180^0}{\mu}$, atque $\sin\mu b=0$, et perinde de sequentibus c, d etc. Supponendo itaque, a esse minorem quam $\frac{360^0}{\mu}$, sed non negativum (quod permissum est, quum ab omnibus a, b, c, d etc. multiplum proxime minus totius peripheriae subtrahere et ex residuis minimum, quod proprietate illa praeditum erit pro periodi initio accipere liceat), erit vel $a=0$, atque $b=\frac{1}{\mu}360^0$, $c=\frac{2}{\mu}360^0$, $d=\frac{3}{\mu}360^0$ etc.; vel $a=\frac{1}{\mu}180^0$, $b=\frac{3}{\mu}180^0$, $c=\frac{5}{\mu}180^0$, $d=\frac{7}{\mu}180^0$ etc.

aliter 2do casus, ubi non est complementum ullius valorum sequentium b, c, d etc. ad 360^0 vel 0 vel multiplum peripheriae. Hic a non erit divisibilis per $\frac{180^0}{\mu}$, adeoque $\sin\mu a=\sin\mu b=\sin\mu c=\sin\mu d$ etc. non $=0$.

In casu posteriore methodus generalis art. 16 statim applicari potest. In priore autem ex valoribus a, b, c, d etc. (quos infra 360^0 reductos esse supponimus), omnes antea reiicere oportet, qui sunt majores quam 180^0, quippe quorum complementa ad 360^0 inter reliquos reperiuntur; insuperque, quando adsunt, valores 0 et 180^0, si in functione X sinus soli occurrunt: sed quoniam applicatio methodi generalis hac ratione minus concinna evaderet, hunc casum alio modo infra absolvemus. Initium iam ab illo casu faciemus, ubi $\sin\mu a$ non est $=0$.

30.

Sit primo X functio formae

$$\alpha+\alpha'\cos x+\alpha''\cos 2x+\alpha'''\cos 3x+\ldots+\alpha^n\cos nx$$

unde esse debebit $\mu=n+1$. Hic habetur per lemma secundum art. 19

$$(\cos t-\cos b)(\cos t-\cos c)(\cos t-\cos d)\text{ etc.}=\frac{1}{2^{\mu-1}}\times\frac{\cos\mu t-\cos\mu a}{\cos t-\cos a}$$
$$=\frac{1}{2^{\mu-1}\sin a}\{\sin\mu a+2\sin(\mu-1)a\cos t+2\sin(\mu-2)a\cos 2t+\text{etc.}+2\sin a\cos(\mu-1)t\}$$

Facile scilicet confirmatur, productum ex aggregato secundae partis huius aequationis per $\cos t-\cos a$ fieri $=\sin a(\cos\mu t-\cos\mu a)$. Quare fit quoque, scribendo a pro t

$$(\cos a-\cos b)(\cos a-\cos c)(\cos a-\cos d)\text{ etc.}=\frac{1}{2^{\mu-1}\sin a}\{\sin\mu a+2\sin(\mu-1)a\cos a$$
$$+2\sin(\mu-2)a\cos 2a+\text{ etc. }+2\sin a\cos(\mu-1)a\}$$

Aggregatum in hac expressione fit $=\mu\sin\mu a$, ut inde manifestum est, quod

$$\begin{aligned}\text{terminus secundus} &= \sin\mu a + \sin(\mu-2)a\\ \text{ultimus} &= \sin\mu a - \sin(\mu-2)a\\ \text{porro terminus tertius} &= \sin\mu a + \sin(\mu-4)a\\ \text{terminus penultimus} &= \sin\mu a - \sin(\mu-4)a\\ \text{etc.}\end{aligned}$$

Hinc colligitur, coëfficientem ipsius A in formula art. 16 pro T fieri

$$\frac{1}{\mu\sin\mu a}\{\sin\mu a + 2\sin(\mu-1)a\cos t + 2\sin(\mu-2)a\cos 2t + \text{etc.} + 2\sin a\sin(\mu-1)t\}$$

Prorsus similes expressiones proveniunt pro coëfficientibus ipsorum B, C, D etc., mutando tantummodo a in b, c, d etc. Quamobrem statuendo

$$\begin{aligned}\varepsilon &= \tfrac{1}{\mu}(A+B+C+D+\text{etc.})\\ \varepsilon' &= \frac{2}{\mu\sin\mu a}\{A\sin(\mu-1)a + B\sin(\mu-1)b + C\sin(\mu-1)c + D\sin(\mu-1)d + \text{etc.}\}\\ \varepsilon'' &= \frac{2}{\mu\sin\mu a}\{A\sin(\mu-2)a + B\sin(\mu-2)b + C\sin(\mu-2)c + D\sin(\mu-2)d + \text{etc.}\}\\ \varepsilon''' &= \frac{2}{\mu\sin\mu a}\{A\sin(\mu-3)a + B\sin(\mu-3)b + C\sin(\mu-3)c + D\sin(\mu-3)d + \text{etc.}\}\\ &\text{etc.}\\ \varepsilon^n &= \frac{2}{\mu\sin\mu a}(A\sin a + B\sin b + C\sin c + D\sin d + \text{etc.})\end{aligned}$$

erit

$$T = \varepsilon + \varepsilon'\cos t + \varepsilon''\cos 2t + \varepsilon'''\cos 3t + \text{etc.} + \varepsilon^n\cos nt$$

Quum haec formula generaliter pro valore quocunque ipsius t valere debeat, necessario cum X identica fiet, mutata t in x, adeoque coëfficientes $\varepsilon, \varepsilon', \varepsilon''$ etc. ipsis $\alpha', \alpha', \alpha''$ etc. resp. aequales.

31.

Quum per praecepta supra explicata functio X' formae

$$\begin{aligned}&\gamma + \gamma'\cos x + \gamma''\cos 2x + \ldots + \gamma^m\cos mx\\ &+ \delta'\sin x + \delta''\sin 2x + \ldots + \delta^m\sin mx\end{aligned}$$

quae omnibus μ valoribus propositis satisfaciat, et in qua sit $m = \frac{1}{2}\mu - \frac{1}{2}$ vel $\frac{1}{2}\mu$, prout μ impar est vel par: operae pretium est, hanc functionem cum functione modo inventa

$$\varepsilon + \varepsilon'\cos x + \varepsilon''\cos 2x + \varepsilon'''\cos 3x + \ldots + \varepsilon^{\mu-1}\cos(\mu-1)x$$

comparare, quam per X'' exprimemus. Comparando valores coëfficientium $\gamma, \gamma', \delta', \gamma'', \delta''$ etc. in artt. 20, 22 traditos cum valoribus coëfficientium $\varepsilon, \varepsilon', \varepsilon''$ etc. modo inventis, invenitur

$$\begin{aligned}
\varepsilon &= \gamma \\
\varepsilon' &= \gamma' - \delta' \operatorname{cotang} \mu a, & \varepsilon^{\mu-1} &= \delta' \operatorname{cosec} \mu a \\
\varepsilon'' &= \gamma'' - \delta'' \operatorname{cotang} \mu a, & \varepsilon^{\mu-2} &= \delta'' \operatorname{cosec} \mu a \\
\varepsilon''' &= \gamma''' - \delta''' \operatorname{cotang} \mu a, & \varepsilon^{\mu-3} &= \delta''' \operatorname{cosec} \mu a \\
&\text{etc. usque ad} \\
(\tfrac{1}{2})\varepsilon^m &= \gamma^m - \delta^m \operatorname{cotang} \mu a, & (\tfrac{1}{2})\varepsilon^{\mu-m} &= \delta^m \operatorname{cosec} \mu a
\end{aligned}$$

Factor $\frac{1}{2}$ ipsis ε^m et $\varepsilon^{\mu-m}$ praepositus tunc tantummodo valet, quando μ par est, omittique debet, quando μ est impar. Quo pacto, in casu posteriore, ubi $\mu - m = m + 1$ manifesto omnes μ coëfficientes $\varepsilon, \varepsilon', \varepsilon''$ etc. per γ, γ', δ' etc. determinati sunt; in priore vero, ubi $m = \mu - m$, pro coëfficiente $\varepsilon^{\frac{1}{2}\mu}$ valores duos $2\gamma^{\frac{1}{2}\mu} - 2\delta^{\frac{1}{2}\mu} \cot \mu a$, $2\delta^{\frac{1}{2}\mu} \operatorname{cosec} \mu a$ habemus, quorum aequalitatem ex aequatione $\gamma^{\frac{1}{2}\mu} \sin \frac{1}{2}\mu a = \delta^{\frac{1}{2}\mu} \cos \frac{1}{2}\mu a$ facile perspicere, et pro quibus igitur etiam hunc $\gamma^{\frac{1}{2}\mu} - \delta^{\frac{1}{2}\mu} \operatorname{cotg} \mu a + \delta^{\frac{1}{2}\mu} \operatorname{cosec} \mu a$ adoptare possumus.

Hinc facile deducitur, esse in utroque casu

$$\begin{aligned}
& X'' - (X' - \delta' \sin x - \delta'' \sin 2x - \delta''' \sin 3x - \text{etc.} - \delta^m \sin mx) \\
= & - \operatorname{cotg} \mu a (\delta' \cos x + \delta'' \cos 2x + \delta''' \cos 3x + \ldots + \delta^m \cos mx) \\
& + \operatorname{cosec} \mu a (\delta' \cos (\mu-1) x + \delta'' \cos (\mu-2) x + \ldots + \delta^m \cos (\mu - m) x)
\end{aligned}$$

Hinc patet, X'' ex X' deduci, si pro quovis termino $\delta^\lambda \sin \lambda x$ substituatur

$$- \operatorname{cotg} \mu a \, \delta^\lambda \cos \lambda x + \operatorname{cosec} \mu a \, \delta^\lambda \cos (\mu - \lambda) x$$

sive

$$\frac{- \delta^\lambda \cos \mu a \cos \lambda x + \delta^\lambda \cos (\mu - \lambda) x}{\sin \mu a}$$

cuius praecepti comparationem cum iis, quae in art. 24 explicata sunt, lectoribus linquimus.

32.

Si functio X cum $\cos nx$ non abrumpitur, sed ulterius excurrit, terminis sequentibus per $\alpha^\mu \cos \mu x + \alpha^{\mu+1} \cos (\mu + 1) x +$ etc. expressis, habebimus

$$\varepsilon = \alpha + \alpha^{\mu}\cos\mu a + \alpha^{2\mu}\cos 2\mu a + \text{etc.}$$

$$\varepsilon' = \frac{1}{\sin\mu a}\{\alpha'\sin\mu a + \alpha^{\mu+1}\sin 2\mu a + \alpha^{2\mu+1}\sin 3\mu a + \text{etc.}$$
$$- \alpha^{2\mu-1}\sin\mu a - \alpha^{3\mu-1}\sin 2\mu a - \alpha^{4\mu-1}\sin 3\mu a - \text{etc.}\}$$

$$\varepsilon'' = \frac{1}{\sin\mu a}\{\alpha''\sin\mu a + \alpha^{\mu+2}\sin 2\mu a + \alpha^{2\mu+2}\sin 3\mu + \text{etc.}$$
$$- \alpha^{2\mu-2}\sin\mu a - \alpha^{3\mu-2}\sin 2\mu a - \alpha^{4\mu-2}\sin 3\mu a - \text{etc.}\}$$

etc.

$$\varepsilon^{\mu-1} = \frac{1}{\sin\mu a}\{\alpha^{\mu-1}\sin\mu a + \alpha^{2\mu-1}\sin 2\mu a + \alpha^{3\mu-1}\sin 3\mu a + \text{etc.}$$
$$- \alpha^{\mu+1}\sin\mu a - \alpha^{2\mu+1}\sin 2\mu a - \alpha^{3\mu+1}\sin 3\mu a - \text{etc.}\}$$

ex quibus formulis iudicari potest, quatenus differentiam coëfficientium ε, ε', ε'' etc. a veris α, α', α'' etc. negligere liceat. In hoc itaque casu functio X'' non erit X ipsa, cum qua tamen in eo convenit, quod omnibus μ valoribus propositis satisfacit. Omnes autem similes functiones his valoribus satisfacientes sub forma $X'' + Y(\cos\mu x - \cos\mu a)$ contenti erunt, designante Y functionem indefinitam arcus x eiusdem formae ut X, scilicet a sinubus liberam. Functio X'' erit unica ordinis $\mu - 1^{\text{ti}}$: quaevis alia similis valoribus datis satisfaciens ad ordinem altiorem ascendet. Quodsi talis functio X''' aliunde constaret, omnes similes functiones, per quos valores dati repraesentantur, in hac quoque forma $X''' + Y(\cos\mu x - \cos\mu a)$ contenti erunt, poteritque Y ita determinari, ut functio ad ordinem $\mu - 1^{\text{tum}}$ depressa prodeat, quae erit ipsa X''. Methodus vero facillima ad hunc finem videtur esse, si primo ex X''' derivetur X' per art. 24 atque hinc X'' per art. 31. Sit itaque terminus quicunque in X''' $L\cos\lambda x$, ponaturque $\lambda = k\mu + \lambda'$, ita ut sit $\lambda' < \mu$ atque k integer. Tunc per art. 24 pro illo termino scribi debebit in X'

$L\cos k\mu a\cos\lambda' x - L\sin k\mu a\sin\lambda' x$, si $\lambda' < \frac{1}{2}\mu$, neque vero $\lambda' = 0$

$L\cos(k+1)\mu a\cos(\mu - \lambda')x + L\sin(k+1)\mu a\sin(\mu - \lambda')x$, si $\lambda' > \frac{1}{2}\mu$

$L\cos\frac{1}{2}\mu a\cos(k+\frac{1}{2})\mu a\cos\frac{1}{2}\mu x + L\sin\frac{1}{2}\mu a\cos(k+\frac{1}{2})\mu a\sin\frac{1}{2}\mu x$, si $\lambda' = \frac{1}{2}\mu$

Hinc autem prodit in X'', per art. praec. in casu primo

$$L\cos k\mu a\cos\lambda' x + L\sin k\mu a\,\text{cotang}\,\mu a\cos\lambda' x - L\sin k\mu a\,\text{cosec}\,\mu a\cos(\mu - \lambda')x$$

in casu secundo

$$L\cos(k+1)\mu a\cos(\mu-\lambda')x - L\sin(k+1)\mu a\,\text{cotg}\,\mu a\cos(\mu-\lambda')x$$
$$+ L\sin(k+1)\mu a\,\text{cosec}\,\mu a\cos\lambda' x$$

in casu tertio

$$L\cos\tfrac{1}{2}\mu a\cos(k+\tfrac{1}{2})\mu a\cos\tfrac{1}{2}\mu x - L\sin\tfrac{1}{2}\mu a\cos(k+\tfrac{1}{2})\mu a\,\mathrm{cotg}\,\mu a\cos\tfrac{1}{2}\mu x$$
$$+L\sin\tfrac{1}{2}\mu a\cos(k+\tfrac{1}{2})\mu a\,\mathrm{cosec}\,\mu a\cos\tfrac{1}{2}\mu x$$

quae expressio in omnibus tribus casibus reducitur ad

$$\frac{L\sin(k+1)\mu a\cos\lambda' x - L\sin k\mu a\cos(\mu-\lambda')x}{\sin\mu a}$$

Quoties autem $\lambda' = 0$, habemus in X' simpliciter $L\cos k\mu a$, qui terminus sine variatione in X'' retinetur.

33.

Sit secundo X functio formae

$$\beta'\sin x + \beta''\sin 2x + \beta'''\sin 3x + \text{etc.} + \beta^n\sin nx$$

unde esse debebit $\mu = n$. Hic fit, ex art. 30, coëfficiens ipsius A in formula secunda art. 16 pro T

$$= \frac{1}{\mu\sin\mu a}\times\frac{\sin t}{\sin a}(\sin\mu a + 2\sin(\mu-1)a\cos t + 2\sin(\mu-2)a\cos 2t + \text{etc.}$$
$$+ 2\sin a\cos(\mu-1)t)$$
$$= \frac{1}{\mu\sin\mu a}(2\cos(\mu-1)a\sin t + 2\cos(\mu-2)a\sin 2t + 2\cos(\mu-3)a\sin 3t + \text{etc.}$$
$$+ 2\cos a\sin(\mu-1)t + \sin\mu t)$$

Prorsus similes expressiones pro coëfficientibus ipsorum B, C, D etc. prodeunt, mutato tantummodo arcu a in b, c, d etc. Quamobrem faciendo

$$\zeta' = \frac{2}{\mu\sin\mu a}\{A\cos(\mu-1)a + B\cos(\mu-1)b + C\cos(\mu-1)c + D\cos(\mu-1)d + \text{etc.}\}$$
$$\zeta'' = \frac{2}{\mu\sin\mu a}\{A\cos(\mu-2)a + B\cos(\mu-2)b + C\cos(\mu-2)c + D\cos(\mu-2)d + \text{etc.}\}$$
$$\zeta''' = \frac{2}{\mu\sin\mu a}\{A\cos(\mu-3)a + B\cos(\mu-3)b + C\cos(\mu-3)c + D\cos(\mu-3)d + \text{etc.}\}$$

etc.

$$\zeta^{\mu-1} = \frac{2}{\mu\sin\mu a}\{A\cos a + B\cos b + C\cos c + D\cos d + \text{etc.}\}$$
$$\zeta^{\mu} = \frac{1}{\mu\sin\mu a}\{A + B + C + D + \text{etc.}\}$$

erit

$$T = \zeta'\sin t + \zeta''\sin 2t + \zeta'''\sin 3t + \text{etc.} + \zeta^{\mu}\sin\mu t$$

Quum haec formula generaliter pro valore quocunque ipsius t valeat, necessario cum X identica erit. mutando t in x, unde coëfficientes $\zeta', \zeta'', \zeta'''$ etc. ipsis $\beta', \beta'', \beta'''$ etc. resp. aequales erunt.

34.

Si ut in art. 31 omnibus valoribus propositis per functionem talem

$$\begin{aligned} X' = \gamma + \gamma'\cos x + \gamma''\cos 2x + \text{ etc. } + \gamma^m \cos mx \\ + \delta'\sin x + \delta''\sin 2x + \text{ etc. } + \delta^m \sin mx \end{aligned}$$

satisfactum est, existente $m = \frac{1}{2}\mu - \frac{1}{2}$ vel $= \frac{1}{2}\mu$, prout μ impar vel par est, comparatio huius functionis cum hac

$$X'' = \zeta'\sin x + \zeta''\sin 2x + \zeta'''\sin 3x + \text{ etc. } + \zeta^\mu \sin \mu x$$

quam in art. praec. eruimus, hasce aequationes suppeditant:

$$\begin{array}{ll} & \zeta^\mu = \gamma \operatorname{cosec} \mu a \\ \zeta' = \delta' + \gamma' \operatorname{cotg} \mu a, & \zeta^{\mu-1} = \gamma' \operatorname{cosec} \mu a \\ \zeta'' = \delta'' + \gamma'' \operatorname{cotg} \mu a, & \zeta^{\mu-2} = \gamma'' \operatorname{cosec} \mu a \\ \zeta''' = \delta''' + \gamma''' \operatorname{cotg} \mu a, & \zeta^{\mu-3} = \gamma''' \operatorname{cosec} \mu a \\ \text{etc. usque ad} & \\ (\tfrac{1}{2})\zeta^m = \delta^m + \gamma^m \operatorname{cotg} \mu a, & (\tfrac{1}{2})\zeta^{\mu-m} = \gamma^m \operatorname{cosec} \mu a \end{array}$$

ubi factor $\frac{1}{2}$ coëfficientibus $\zeta^m, \zeta^{\mu-m}$ praepositus pro eo tantum casu valet, ubi μ par est, pro altero vero, ubi μ impar est, omitti debet. In casu itaque posteriore ubi $\mu - m = m + 1$, pro quovis coëfficiente ζ', ζ'' etc. valorem unum per $\gamma, \gamma', \gamma''$ etc. et δ', δ'' etc. habemus; in priore vero, ubi $\mu - m = m$, pro coëfficiente $\zeta^{\frac{1}{2}\mu}$ duos aequales, pro quibus etiam valor $\delta^{\frac{1}{2}\mu} + \gamma^{\frac{1}{2}\mu} \operatorname{cotg} \mu a + \gamma^{\frac{1}{2}\mu} \operatorname{cosec} \mu a$ adoptari potest. Hinc colligitur

$$\begin{aligned} X'' = X' - \gamma - \gamma'\cos x - \gamma''\cos 2x - \text{ etc. } - \gamma^m \cos mx \\ + \operatorname{cotg} \mu a (\gamma' \sin x + \gamma'' \sin 2x + \gamma''' \sin 3x + \text{ etc. } + \gamma^m \sin mx) \\ + \operatorname{cosec} \mu a (\gamma \sin \mu x + \gamma' \sin(\mu-1)x + \gamma'' \sin(\mu-2)x + \text{ etc. } + \gamma^{\mu-m} \sin(\mu-m)x) . \end{aligned}$$

Quamobrem ex X' producitur X'', scribendo pro γ, $\gamma \operatorname{cosec} \mu a \sin \mu x = \frac{\gamma \sin \mu x}{\sin \mu a}$ et, pro quovis termino $\gamma^\lambda \cos \lambda x$

$$\gamma^\lambda \operatorname{cotg} \mu a \sin \lambda x + \gamma^\lambda \operatorname{cosec} \mu a \sin(\mu - \lambda) x \quad \text{sive} \quad \frac{\gamma^\lambda \cos \mu a \sin \lambda x + \gamma^\lambda \sin(\mu - \lambda) x}{\sin \mu a}$$

quod praeceptum lectores cum iis, quae in art. 24 tradidimus, ipsi comparent.

35.

Si functio X cum $\sin nx$ non abrumpitur, sed ulterius excurrit, terminis sequentibus per $\delta^{\mu+1}\sin(\mu+1)x + \delta^{\mu+2}\sin(\mu+2)x$ etc. expressis, habebimus:

$$\zeta' = \frac{1}{\sin \mu a}\{\delta' \sin \mu a + \delta^{\mu+1}\sin 2\mu a + \delta^{2\mu+1}\sin 3\mu a + \text{etc.}$$
$$+ \delta^{2\mu-1}\sin \mu a + \delta^{3\mu-1}\sin 2\mu a + \delta^{4\mu-1}\sin 3\mu a + \text{etc.}\}$$
$$\zeta'' = \frac{1}{\sin \mu a}\{\delta'' \sin \mu a + \delta^{\mu+2}\sin 2\mu a + \delta^{2\mu+2}\sin 3\mu a + \text{etc.}$$
$$+ \delta^{2\mu-2}\sin \mu a + \delta^{3\mu-2}\sin 2\mu a + \delta^{4\mu-2}\sin 3\mu a + \text{etc.}\}$$
$$\zeta''' = \frac{1}{\sin \mu a}\{\delta''' \sin \mu a + \delta^{\mu+3}\sin 2\mu a + \delta^{2\mu+3}\sin 3\mu a + \text{etc.}$$
$$+ \delta^{2\mu-3}\sin \mu a + \delta^{3\mu-3}\sin 2\mu a + \delta^{4\mu-3}\sin 3\mu a + \text{etc.}\}$$

etc. Pro coëfficiente ultimo autem

$$\zeta^\mu = \frac{1}{\sin \mu a}(\delta^\mu \sin \mu a + \delta^{2\mu}\sin 2\mu a + \delta^{3\mu}\sin 3\mu a + \text{etc.})$$

Hae formulae ostendunt, quatenus differentia inter X et X'' negligi possit. Haec posterior formula simplicissima erit inter omnes similes, per quas μ valoribus propositis satisfit; hae vero omnes in formula $X'' + Y\sin x(\cos \mu x - \cos \mu a)$ contentae erunt, designante Y, ut in art. 32 functionem indefinitam arcus x a sinubus liberam. Et generaliter, si X''' est functio quaecunque eiusdem formae ut X, i. e. solos sinus continens, per quam μ valoribus datis satisfit, formula $X''' + Y\sin x(\cos \mu x - \cos \mu a)$ omnes huiusmodi functiones continebit, quae si Y rite determinatur, ad ordinem μ^{tum} deprimi potest, quo pacto necessario functio X'' ipsa prodire debet. Prorsus simili modo ut in art. 32 regula generalis sequens ad hunc finem eruitur: Pro quovis termino in X''' tali $L\sin \lambda x$, ubi λ est maior quam μ, substituere oportet in X'', faciendo $\lambda = k\mu + \lambda'$, ita ut $k\mu$ sit multiplum ipsius μ proxime minus quam λ adeoque λ' inter limites 1 et μ incl. situs, terminos

$$L\frac{\sin(k+1)\mu a}{\sin \mu a}\sin \lambda' x + L\frac{\sin k\mu a}{\sin \mu a}\sin(\mu - \lambda')x$$

qui, quoties fit $\lambda' = \mu$, ad unum $L\frac{\sin \lambda a}{\sin \mu a}\sin \mu x$ reducuntur.

36.

Transformationes in art. praec. atque in art. 32 traditae concinnius ex theoremate quodam generali deduci possunt, quod quum per se quoque satis elegans sit, paucis hic adhuc attingemus.

Theorema. *Designantibus* λ, λ', λ'' *numeros integros quoscunque*, μ *numerum integrum, qui differentias inter illos,* $\lambda'-\lambda$, $\lambda''-\lambda'$, $\lambda-\lambda''$ *metitur (e. g. unitatem),* x *arcum indefinitum,* a *arcum definitum: functiones*

$$P = \sin(\lambda'-\lambda'')a\cos\lambda x + \sin(\lambda''-\lambda)a\cos\lambda' x + \sin(\lambda-\lambda')a\cos\lambda'' x$$
$$Q = \sin(\lambda'-\lambda'')a\sin\lambda x + \sin(\lambda''-\lambda)a\sin\lambda' x + \sin(\lambda-\lambda')a\cos\lambda'' x$$

per $\cos\mu x - \cos\mu a$ *erunt divisibiles.*

Demonstr. Quando λ per μ divisibilis est, ideoque etiam λ', λ'' per μ divisibiles erunt, facile confirmatur, valorem ipsarum P, Q, si substituatur $x = a$, esse identice $= 0$; quare P non mutabitur, si pro

$\cos\lambda x$	substituitur	$\cos\lambda x - \cos\lambda a$
$\cos\lambda' x$		$\cos\lambda' x - \cos\lambda' a$
$\cos\lambda'' x$		$\cos\lambda'' x - \cos\lambda'' a$

neque Q, si pro

$\sin\lambda x$	substituitur	$\sin\lambda x - \frac{\sin\lambda a\sin\mu x}{\sin\mu a}$
$\sin\lambda' x$		$\sin\lambda' x - \frac{\sin\lambda' a\sin\mu x}{\sin\mu a}$
$\sin\lambda'' x$		$\sin\lambda'' x - \frac{\sin\lambda'' a\sin\mu x}{\sin\mu a}$

Sed hae sex expressiones per $\cos\mu x - \cos\mu a$ divisibiles sunt, quod pro valore positivo ipsius λ de prima et quarta ostendisse sufficit. Scilicet facile per multiplicationem confirmatur, esse

$$\begin{aligned}&\sin\mu a(\cos\lambda x - \cos\lambda a)\\ &= (\cos\mu x - \cos\mu a)\{2\sin\mu a\cos(\lambda-\mu)x + 2\sin 2\mu a\cos(\lambda-2\mu)x\\ &\qquad + 2\sin 3\mu a\cos(\lambda-3\mu)x + \text{etc.} + 2\sin(\lambda-\mu)a\cos\mu x + \sin\lambda a\}\end{aligned}$$

$$\begin{aligned}&\sin\mu a\sin\lambda x - \sin\lambda a\sin\mu x\\ &= (\cos\mu x - \cos\mu a)\{2\sin\mu a\sin(\lambda-\mu)x + 2\sin 2\mu a\sin(\lambda-2\mu)x\\ &\qquad + 2\sin 3\mu a\sin(\lambda-3\mu)x + \text{etc.} + 2\sin(\lambda-\mu)a\sin\mu x\}\end{aligned}$$

Casus, ubi λ est negativus, ad hunc sponte reducitur. Hinc patet, functiones

P, Q ex partibus per $\cos\mu x - \cos\mu a$ divisibiles compositas, ideoque ipsas quoque per hunc divisorem divisibiles esse.

II. Quando λ per μ non est divisibilis, sit l numerus integer arbitrarius per μ divisibilis, ponaturque $\lambda = l + \theta$, $\lambda' = l' + \theta$, $\lambda'' = l'' + \theta$, unde etiam l', l'' per μ divisibiles erunt. Iam patet, si ponatur

$$\sin(l' - l'')a\cos lx + \sin(l'' - l)a\cos l'x + \sin(l - l')a\cos l''x = P'$$
$$\sin(l' - l'')a\sin lx + \sin(l'' - l)a\sin l'x + \sin(l - l')a\sin l''x = Q'$$

fieri

$$P = P'\cos\theta x - \theta'\sin\theta x, \quad Q = P'\sin\theta x + Q'\cos\theta x,$$

atque functiones P', Q', quippe quae sub casum primum iam absolutum pertinent, per $\cos\mu x - \cos\mu a$ divisibiles: hinc manifesto etiam P et Q per $\cos\mu x - \cos\mu a$ divisibiles erunt. Q. E. D. Ceterum demonstratio casus primi ita perfecta est, ut non sine quibusdam explicationibus applicari possit, quóties $\sin\mu a = 0$; tunc vero fit $P = 0$, $Q = 0$, ita ut demonstratione omnino non opus sit.

Quodsi itaque ponitur $\lambda - \lambda' = k\mu$, $\lambda - \lambda'' = (k+1)\mu$, patet, per $\cos\mu x - \cos\mu a$ divisibiles esse

$$\sin\mu a\cos\lambda x - \sin(k+1)\mu a\cos\lambda' x + \sin k\mu a\cos(\lambda' - \mu)x$$
$$\sin\mu a\sin\lambda x - \sin(k+1)\mu a\sin\lambda' x + \sin k\mu a\sin(\lambda' - \mu)x$$

adeoque etiam

$$\cos\lambda x - \frac{\sin(k+1)\mu a\cos\lambda' x - \sin k\mu a\cos(\mu - \lambda')x}{\sin\mu a}$$
$$\sin\lambda x - \frac{\sin(k+1)\mu a\sin\lambda' x + \sin k\mu a\sin(\mu - \lambda')x}{\sin\mu a}$$

unde ratio substitutionum in artt. 32, 35 statim elucet: quotiens enim ex divisione posteriore e solis sinubus constabit, adeoque manifesto denuo per $\sin x$ divisibilis erit.

37.

In artt. 30—36 supposuimus, $\sin\mu a$ non esse $= 0$: superest itaque, ut easdem disquisitiones pro eo casu resumamus, ubi $\sin\mu a = 0$. Hic statim supponemus, esse $a = 0$, vel $a = \frac{180^\circ}{\mu}$.

Sit primo X functio formae

$$\gamma + \gamma' \cos x + \gamma'' \cos 2x + \text{ etc. } + \gamma^n \cos nx$$

adeoque multitudo coëfficientium incognitorum $= 1 + n$. Iam scimus, ex omnibus valoribus propositis functionis X eos reiici debere, qui respondent valori ipsius maiori quam 180^0: quatuor itaque casus hic sunt distinguendi:

1) quando μ par, atque $a = 0$, erit 180^0 valor $\frac{1}{2}\mu + 1^{\text{tus}}$ ipsius x; quare quum sequentes reiici debeant, remanent valores $\frac{1}{2}\mu + 1$. Hinc esse debebit $n = \frac{1}{2}\mu$.

2) quando μ par, atque $a = \frac{180^0}{\mu}$, valor $\frac{1}{2}\mu^{\text{tus}}$ ipsius x erit $180^0 - \frac{180^0}{\mu}$; sequentes, qui fiunt maiores quam 180^0, reiiciendi sunt. Hinc esse debebit $n = \frac{1}{2}\mu - 1$.

3) quando μ impar est, atque $a = 0$, fit valor $\frac{1}{2}\mu + \frac{1}{2}^{\text{tus}} = 180^0 - \frac{180^0}{\mu}$, et

4) quando μ impar est, atque $a = \frac{180^0}{\mu}$, fit valor $\frac{1}{2}\mu + \frac{1}{2}^{\text{tus}} = 180^0$: sequentes in utroque casu reiici debent, adeoque erit $n = \frac{1}{2}\mu - \frac{1}{2}$.

Iam quoniam methodus in praecc. adhibita ad casum praesentem, ubi pars valorum datorum a periodo completa antea rescindenda esset, non sine quibusdam ambagibus applicari posset, methodum sequentem praeferimus.

Si per praecepta artt. 20, 22 functio formae

$$\gamma + \gamma' \cos x + \gamma'' \cos 2x + \text{ etc. } + \gamma^m \cos mx$$
$$+ \delta' \sin x + \delta'' \sin 2x + \text{ etc. } + \delta^m \sin mx$$

investigatur, per quam omnibus μ valoribus datis satisfit, et in qua $m = \frac{1}{2}\mu - \frac{1}{2}$ vel $= \frac{1}{2}\mu$, prout μ impar est vel par, coëfficientes δ', δ'', δ''' etc. sponte fient $= 0$. Nullo enim negotio patet, in expressione tali

$$A \sin \lambda a + B \sin \lambda b + C \sin \lambda c + D \sin \lambda d + \text{ etc.}$$

fieri vel partem primam $= 0$, atque ultimam $= -B \sin \lambda b$, penultimam $= -C \sin \lambda c$, antepenultimam $= -D \sin \lambda d$ etc. puta quando $a = 0$; vel ultimam $= -A \sin \lambda a$, penultimam $= -B \sin \lambda b$, antepenultimam $= -C \sin \lambda c$ etc., quando $a = \frac{180^0}{\mu}$, quum pro talibus valoribus ipsius x, quorum alter alterius complementum ad 360^0 est, valores functionis X aequales sint. Quamobrem functio

$$X' = \gamma + \gamma' \cos x + \gamma'' \cos 2x + \text{ etc. } + \gamma^m \cos mx$$

in qua coëfficientes γ, γ', γ'' etc. determinantur per formulas

$$\begin{aligned}
\gamma &= \tfrac{1}{\mu}(A+B+C+D+\text{ etc.})\\
\gamma' &= \tfrac{2}{\mu}(A\cos a\ +B\cos b\ +C\cos c\ +D\cos d\ +\text{ etc.})\\
\gamma'' &= \tfrac{2}{\mu}(A\cos 2a+B\cos 2b+C\cos 2c+D\cos 2d+\text{ etc.})
\end{aligned}$$

etc., ultimus autem, quando μ par est atque adeo $m=\frac{1}{2}\mu$, per hanc

$$\gamma^m = \tfrac{1}{\mu}(A\cos ma+B\cos mb+C\cos mc+D\cos md+\text{ etc.})$$

necessario cum functione X identica erit, siquidem haec non est gradus altioris quam supra definivimus. Namque in casibus 3 et 4 X est ordinis $\frac{1}{2}\mu-\frac{1}{2}^{\text{ti}}$, i.e. eiusdem ut X' et proin per art. 24 cum X' identica. In casu primo X est eiusdem ordinis ut X' et in casu secundo non maioris, quare tum hic tum illic omnes termini saltem usque ad ordinem $\frac{1}{2}\mu-1^{\text{tum}}$ in utraque functione convenient (art. 24). Terminos ordinis $\frac{1}{2}\mu^{\text{ti}}$ in his functionibus quoque convenire debere. inde per eundem art. 24 patet, quod in X aequatio conditionalis $K\sin ma = L\cos ma$ locum habet; scilicet fit $L=0$, atque in casu primo $\sin ma = 0$, in secundo, ubi X ad ordinem $\frac{1}{2}\mu-1$ tantummodo ascendit, $K=0$. Ceterum in casu secundo X' ordinis altioris esse videtur quam X, sed in hoc casu terminus ordinis $\frac{1}{2}\mu^{\text{ti}}$ in X' quoque evanescit, quum fiat

$$\gamma^{\frac{1}{2}\mu} = \tfrac{1}{\mu}(A\cos 90^0+B\cos 270^0+C\cos 450^0+D\cos 630^0+\text{ etc.}) = 0$$

ita ut in hoc quoque casu X' revera sit ordinis $m-1^{\text{ti}}$ sive $\frac{1}{2}\mu-1^{\text{ti}}$.

38.

Si functio X cum termino $\cos nx$ non abrumpitur, sed ulterius excurrit: denotatis terminis sequentibus per $\alpha^{n+1}\cos(n+1)x+\alpha^{n+2}\cos(n+2)x+$ etc. erit per artt. 21, 23

$$\begin{aligned}
\gamma &= \alpha\ \pm\alpha^{\mu}\ +\alpha^{2\mu}\ \pm\text{ etc.}\\
\gamma' &= \alpha'\ \pm\alpha^{\mu-1}\pm\alpha^{\mu+1}+\alpha^{2\mu-1}+\alpha^{2\mu+1}\pm\text{ etc.}\\
\gamma'' &= \alpha''\pm\alpha^{\mu-2}\pm\alpha^{\mu+2}+\alpha^{2\mu-2}+\alpha^{2\mu+2}\pm\text{ etc.}\\
\gamma''' &= \alpha'''\pm\alpha^{\mu-3}\pm\alpha^{\mu+3}+\alpha^{2\mu-3}+\alpha^{2\mu+3}\pm\text{ etc.}
\end{aligned}$$

et sic porro usque ad ultimum γ^m, quando μ impar est, vel ad penultimum γ^{m-1}, quando μ par est; signum inferius hic valet, quoties $a=\frac{180^0}{\mu}$, adeoque in casu 2 et 4, superius in casu 1 et 2; denique pro ultimo habebitur in casu primo

$$\gamma^m = \alpha^m + \alpha^{3m} + \alpha^{5m} + \text{etc.}$$

in casu secundo autem $\gamma^m = 0$. Hae aequationes ostendunt, quatenus differentiam inter functiones X et X' negligere permissum esse possit. Haec posterior functio inter omnes, quae μ valoribus propositis satisfaciunt, simplicissima erit, quae omnes sub forma $X' + Y\sin(\frac{1}{2}\mu x - \frac{1}{2}\mu a)$ contenti erunt, quae in casu 1 et 3 ad $X' + Y\sin\frac{1}{2}\mu x$, in casibus 2 et 4 vero ad $X' + Y\cos\frac{1}{2}\mu x$ reducitur. Manifesto autem, si haec expressio eiusdem formae esse debet ut X i. e. a sinubus libera, Y esse debet

in casu primo formae $g\sin x + g'\sin 2x + g''\sin 3x + \text{etc.}$
in casu secundo formae $g + g'\cos x + g''\cos 2x + \text{etc.}$
in casu tertio formae $g\sin\frac{1}{2}x + g'\sin\frac{3}{2}x + g''\sin\frac{5}{2}x + \text{etc.}$
in casu quarto formae $g\cos\frac{1}{2}x + g'\cos\frac{3}{2}x + g''\cos\frac{5}{2}x + \text{etc.}$

Et generalius, designante X'' functionem quamcunque ipsi X similem, quae μ valoribus propositis satisfacit, omnes huiusmodi formae sub formula $X'' + Y\sin\frac{1}{2}\mu x$ vel $X'' + Y\cos\frac{1}{2}\mu x$ contentae erunt, ubi Y functionem indefinitam eius, quam modo docuimus formae designat. Hoc ita perficere licet, ut sic functio ad ordinem $\frac{1}{2}\mu$, $\frac{1}{2}\mu - 1$, $\frac{1}{2}\mu - \frac{1}{2}$, $\frac{1}{2}\mu - \frac{1}{2}$ depressa prodeat, quae manifesto cum X' identica erit. Regula autem generalis pro reductione talis functionis X'' ad X' ex art. 24 facile deducitur. Pro quovis termino $L\cos\lambda x$ in X'' substitui debet in X', facto $\lambda = k\mu \pm \lambda'$, ita ut λ' non sit maior quam $\frac{1}{2}\mu$, terminus $\pm L\cos\lambda' x$, ubi signum inferius accipiendum est, quoties simul $a = \frac{180^\circ}{\mu}$ atque k par, superius in casibus reliquis; denique quoties in casu secundo, i. e. pro $a = \frac{180^\circ}{\mu}$ et valore pari ipsius μ, evadit $\lambda' = \frac{1}{2}\mu$, pro $L\cos\lambda x$ statim poni debet 0 in X', sive terminus ille omnino negligi.

39.

Si secundo functio X est formae

$$\beta'\sin x + \beta''\sin 2x + \beta'''\sin 3x + \text{etc.} + \beta^n\sin nx$$

adeoque multitudo coëfficientium incognitorum $= n$, etiam multitudo valorum datorum, subductis superfluis, esse debebit $= n$, ut ad coëfficientium determinationem completam sufficiant. Iam quum ut superflui in hoc casu reiiciendi sint valores functionis X ii, qui respondent valori ipsius x maiori quam 180^0 nec non valori 0 et 180^0, habebimus pro quatuor casibus supra distinctis:

1. quando μ par, $a = 0$, erit $n = \frac{1}{2}\mu - 1$
2. quando μ par, $a = \frac{180^0}{\mu}$, erit $n = \frac{1}{2}\mu$
3. quando μ impar, $a = 0$, erit $n = \frac{1}{2}\mu - \frac{1}{2}$
4. quando μ impar, $a = \frac{180^0}{\mu}$, erit $n = \frac{1}{2}\mu - \frac{1}{2}$

Iam prorsus simili modo ut in art. 37, in functione

$$\begin{aligned}&\gamma + \gamma'\cos x + \gamma''\cos 2x + \text{ etc. } + \gamma^m \cos mx\\ &+ \delta'\sin x + \delta''\sin 2x + \text{ etc. } + \delta^m \sin mx\end{aligned}$$

ad normam artt. 20, 22 eruta, quae omnibus μ valoribus satisfacit, et in qua m vel $= \frac{1}{2}\mu - \frac{1}{2}$, vel $= \frac{1}{2}\mu$, coëfficientes γ, γ', γ'' etc. sponte evanescent. Quum enim in serie A, B, C, D etc. vel termini ultimi ordine retrogrado in casu praesenti vel fiant $= -B, -C, -D$ etc. vel $= -A, -B, -C$ etc., prout $a = 0$, vel $= \frac{180^0}{\mu}$, insuperque pro illo casu $A = 0$, manifesto

$$A\cos\lambda a + B\cos\lambda b + C\cos\lambda c + D\cos\lambda d + \text{ etc.}$$

pro quovis valore ipsius x erit $= 0$. Quamobrem functio $X' =$

$$\delta'\sin x + \delta''\sin 2x + \delta'''\sin 3x + \text{ etc. } + \delta^m \sin mx$$

in qua coëfficientes δ', δ'', δ''' etc. determinantur per aequationes

$$\begin{aligned}\delta' &= \frac{2}{\mu}(A\sin a + B\sin b + C\sin c + D\sin d + \text{ etc.})\\ \delta'' &= \frac{2}{\mu}(A\sin 2a + B\sin 2b + C\sin 2c + D\sin 2d + \text{ etc.})\\ \delta''' &= \frac{2}{\mu}(A\sin 3a + B\sin 3b + C\sin 3c + D\sin 3d + \text{ etc.})\end{aligned}$$

etc., ultimus autem, quando μ par est, adeoque $m = \frac{1}{2}\mu$, per hanc

$$\delta^m = \frac{1}{\mu}(A\sin ma + B\sin mb + C\sin mc + D\sin md + \text{ etc.})$$

necessario cum X identica erit, quod eodem modo, ut in art. 37, facile demonstratur. Ceterum functio X' in casu primo, ubi $a = 0$, μ par, revera ad ordinem $\frac{1}{2}\mu-1$ tantum ascendit, quum fiat

$$\begin{aligned}\delta^m &= \frac{1}{\mu}(A\sin 0 + B\sin 180^0 + C\sin 360^0 + D\sin 540^0 + \text{etc.})\\ &= 0\end{aligned}$$

40.

Si functio X non est ordinis n^{ti}, ut supposuimus, sed ulterius excurrit, aequationes sequentes docebunt, quomodo differentia inter X' et X a coëfficientibus sequentibus pendeat (v. artt. 21, 23)

$$\begin{aligned}\delta' &= \mathfrak{b}' \mp \mathfrak{b}^{\mu-1} \pm \mathfrak{b}^{\mu+1} - \mathfrak{b}^{2\mu-1} + \mathfrak{b}^{2\mu+1} \mp \text{etc.}\\ \delta'' &= \mathfrak{b}'' \mp \mathfrak{b}^{\mu-2} \pm \mathfrak{b}^{\mu+2} - \mathfrak{b}^{2\mu-2} + \mathfrak{b}^{2\mu+2} \mp \text{etc.}\\ \delta''' &= \mathfrak{b}''' \mp \mathfrak{b}^{\mu-3} \pm \mathfrak{b}^{\mu+3} - \mathfrak{b}^{2\mu-3} + \mathfrak{b}^{2\mu+3} \mp \text{etc.}\end{aligned}$$

et sic porro usque ad ultimum δ^m vel penultimum δ^{m-1}, prout μ impar est vel par; signa superiora hic valent, quando $a=0$, inferiora, quando $a=\frac{180^0}{\mu}$: denique pro ultimo habetur in casu (1) $\delta^m=0$, in casu (2) vero $\delta^m = \mathfrak{b}^m - \mathfrak{b}^{3m} + \mathfrak{b}^{5m} - \mathfrak{b}^{7m} + \text{etc.}$ Omnes functiones periodicae, per quas μ valoribus propositis satisfit, et ex quibus X' est simplicissima, sub forma $X' + Y\sin(\frac{1}{2}\mu x - \frac{1}{2}\mu a)$ sive generalius sub forma $X'' + Y\sin(\frac{1}{2}\mu x - \frac{1}{2}\mu a)$ contentae erunt, designante X'' functionem talem quamcunque, quae formula pro casu 1 et 3 ad $X'' + Y\sin\frac{1}{2}\mu x$, pro casu 2 et 4 autem ad $X'' + Y\cos\frac{1}{2}\mu x$ reducitur; Y vero, siquidem alias functiones non consideramus, nisi quae ipsi X sunt similes, i. e. e solis sinubus compositae, necessario debet esse:

in casu 1 formae	$g + g'\cos x + g''\cos 2x + \text{etc.}$
in casu 2 formae	$g\sin x + g'\sin 2x + g''\sin 3x + \text{etc.}$
in casu 3 formae	$g\cos\frac{1}{2}x + g'\cos\frac{3}{2}x + g''\cos\frac{5}{2}x + \text{etc.}$
in casu 4 formae	$g\sin\frac{1}{2}x + g'\sin\frac{3}{2}x + g''\sin\frac{5}{2}x + \text{etc.}$

Functionem Y hic ita determinare licebit, ut prodeat functio ad ordinem $\frac{1}{2}\mu-1$, $\frac{1}{2}\mu$, $\frac{1}{2}\mu-\frac{1}{2}$, $\frac{1}{2}\mu-\frac{1}{2}$ depressa, quae cum X' necessario identica erit. Pro reductione functionis X'' ad X' regula generalis sequens habetur:

Quivis terminus in X'' talis $L\sin\lambda x = L\sin(k\mu\pm\lambda')x$, transmutetur aut in $\pm L\sin\lambda' x$ (quoties $a=0$, vel k par), aut in $\mp L\sin\lambda' x$ (quoties nec $a=0$, nec k par, i. e. quoties simul $a=\frac{180^0}{\mu}$ atque k impar): denique quoties in casu primo i. e. pro $a = 0$, et valore pari ipsius μ evadit $\lambda' = \frac{1}{2}\mu$, terminus $L\sin\lambda x$ omnino destruatur.

41.

Quum omnes casus speciales in artt. 29—40 considerati ad casum generalem in artt. 20—28 absolute reducti sint, omnia artificia, per quae in hoc casu calculus abbreviatur, qualia in artt. 25, 26, 27 explicavimus, etiam ad illos applicari poterunt. Quamobrem non opus erit, huic disquisitioni immorari, cui sequens exemplum ad artt. 39, 40 pertinens, finem imponet.

Aequatio centri pro novo planeta *Iunone*, adhibita excentricitate 0,254236, calculata est per methodum indirectam per singulos denos gradus, ut sequitur

Anomalia media $= x$		Aequatio Centri $= X$ $-$ $+$			
0^0	360^0	0			0
10	350	3^0	$50'$	$38''30$	$13838''30$
20	340	7	38	21,47	27501,47
30	330	11	20	8,79	40808,79
40	320	14	52	48,06	53568,06
50	310	18	12	49,21	65569,21
60	300	21	16	17,02	76577,02
70	290	23	58	42,92	86322,92
80	280	26	14	55,85	94495,85
90	270	27	58	52,36	100732,36
100	260	29	3	28,13	104608,13
110	250	29	20	33,68	105633,68
120	240	28	41	2,10	103262,10
130	230	26	55	22,77	96922,77
140	220	23	55	2,70	86102,70
150	210	19	35	0,79	70500,79
160	200	13	57	40,52	50260,52
170	190	7	16	58,33	26218,33
180	180	0			0

Discerpimus hanc periodum 36 terminorum in sex minores senorum terminorum; valores seni functionis X in singulis periodis contenti exhibebuntur per formulam talem

$$\gamma + \gamma'\cos x + \gamma''\cos 2x + \gamma'''\cos 3x$$
$$+ \delta'\sin x + \delta''\sin 2x + \delta'''\sin 3x$$

ubi pro coëfficientibus γ, γ', δ' etc. valores sequentes invenimus:

Periodus	$y=6x$	γ	γ'	γ''	γ'''	δ'	δ''	δ'''
prima	0°	0	0	0	0	−103830,165	+15406,638	0
secunda	60	+56″205	−217″757	+761″671	−1528,825	−103937,346	+15841,387	−882,667
tertia	120	+56,115	−217,467	+760,805	−1527,397	−104151,314	+16709,402	−2645,530
quarta	180	0	0	0	0	−104258,100	+17142,684	−3525,740
quinta	240	−56,115	−217,467	−760,805	+1527,397	−104151,314	+16709,402	−2645,530
sexta	300	−56,205	−217,757	−761,671	+1528,825	−103937,346	+15841,387	−882,667

Singuli coëfficientes γ, γ', γ'', γ''', δ', δ'', δ''' rursus sub formam talem

$$\varepsilon+\varepsilon'\cos 6x+\varepsilon''\cos 12x+\varepsilon'''\cos 18x$$
$$+\zeta'\sin 6x+\zeta''\sin 12x+\zeta'''\sin 18x$$

reducentur: nullo vero negotio perspicitur, pro quatuor prioribus evanescere debere ε, ε', ε'', ε'''; et pro tribus posterioribus, ζ', ζ'', ζ'''. Hoc modo invenitur

$$\gamma = +\ 64'',848\sin 6x+0'',052\sin 12x$$
$$\gamma' = -\ 251'',277\sin 6x-0'',167\sin 12x$$
$$\gamma'' = +\ 879'',002\sin 6x+0'',500\sin 12x$$
$$\gamma''' = -1764'',511\sin 6x-0'',824\sin 12x$$
$$\delta' = -104044'',264+213'',968\cos 6x+0'',132\cos 12x+0'',000\cos 18x$$
$$\delta'' = +\ 16275'',150-868'',020\cos 6x-0'',489\cos 12x-0'',003\cos 18x$$
$$\delta''' = -\ 1763'',689+1762'',868\cos 6x+0'',819\cos 12x+0'',002\cos 18x$$

His valoribus pro γ, γ' etc. substitutis, praeceptisque art. praecc. observatis, prodit functio sequens pro aequatione centri, in qua singuli coëfficientes intra centesimam minuti secundi partem exacti sunt.

$-104044''264\ \sin x$	$-1''643\ \sin 9x$
$+\ 16275,150\ \sin 2x$	$+0,494\ \sin 10x$
$-\ 3527,378\ \sin 3x$	$-0,149\ \sin 11x$
$+\ 873,511\ \sin 4x$	$+0,052\ \sin 12x$
$-\ 232,622\ \sin 5x$	$-0,017\ \sin 13x$
$+\ 64,848\ \sin 6x$	$+0,006\ \sin 14x$
$-\ 18,655\ \sin 7x$	$-0,004\ \sin 15x$
$+\ 5,491\ \sin 8x$	$+0,003\ \sin 16x$

Hoc modo manifesto calculus ita se habet, ac si pro arcu x alius $x' = x - a$ introduceretur. Saepenumero etiam calculus magnopere sublevari potest, si pro functione X alia $X - \mathfrak{X}$ adhibetur, ubi pro $\mathfrak{X}$ functio quaelibet ipsi X similis assumi potest, modo ordinem $\frac{1}{2}\mu - \frac{1}{2}^{\text{tum}}$ pro valore impari vel ordinem $\frac{1}{2}\mu^{\text{tum}}$ pro valore pari ipsius μ non egrediatur, et in casu posteriori coëfficientes terminorum $\cos\frac{1}{2}\mu x$ et $\sin\frac{1}{2}\mu x$ (siquidem eo ascendit) teneant debitam rationem ($\cos\frac{1}{2}\mu a : \sin\frac{1}{2}\mu a$). Manifestum est, si valores functionis $\mathfrak{X}$ pro $x = a, b, c, d$ etc. sint resp. $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., fore valores functionis $X - \mathfrak{X}$ pro iisdem valoribus resp. $A - \mathfrak{A}$, $B - \mathfrak{B}$, $C - \mathfrak{C}$, $D - \mathfrak{D}$ etc.; si porro functio Z hisce valoribus satisfaciens per praecepta praecedentia investigatur, functio $\mathfrak{X} + Z$ identica erit cum ea, quae ex applicatione directa horum praeceptorum ad valores ipsius functionis X prodiret. Semper autem functionem $\mathfrak{X}$ ita determinare oportet, ut valores propositi functionis X prope per illam exhibeantur, quo pacto quantitates $A - \mathfrak{A}$, $B - \mathfrak{B}$, $C - \mathfrak{C}$, $D - \mathfrak{D}$ etc. parvae et ad calculum magis tractabiles evadent. Ceterum pro $\mathfrak{X}$ etiam quantitas constans assumi potest, e. g. haec $\frac{1}{\mu}(A + B + C + D +$ etc.), quae functionis quaesitae partem invariabilem constituit.

BEMERKUNGEN.

Die vorliegende Abhandlung über Interpolation ist die wiederholte Ausarbeitung einer Untersuchung, von welcher ein früherer Entwurf im October 1805 begonnen zu sein scheint. Sie enthält die Methoden, die bei der Berechnung der Ausdrücke für die durch Störungskräfte hervorgebrachten Aenderungen der Elemente einer Planetenbahn vielfach angewandt worden sind. Dieser Ausmittelung der Störungen steht ein anderes Verfahren zur Seite, das direct die Werthe der Bahnelemente für einzelne Zeitpunkte ergibt und das, wie aus den im vorgefundenen handschriftlichen Nachlasse aufgezeichneten schliesslichen Formeln hervorgeht, auf folgenden Principien beruht.

Bilden die Werthe des Arguments, für welche die Werthe der Function gegeben sind, eine arithmetische Reihe, und hat man ein Argument x eingeführt, das bei jenen Werthen wie die ganzen Zahlen fortschreitet, bezeichnet die gegebenen Werthe mit fx, führt deren aufeinanderfolgende Differenzenreihen ein:

$$f(x+1)-fx=f^1(x+\tfrac{1}{2}),\quad f^1(x+\tfrac{1}{2})-f^1(x-\tfrac{1}{2})=f^2x,\quad f^2(x+1)-f^2x=f^3(x+\tfrac{1}{2})\ \text{u. s. f.}$$

ferner deren mit beliebigen Anfangsgliedern beginnende Summenreihen $f^{-1}(x+\frac{1}{2})$, $f^{-2}x$, $f^{-3}(x+\frac{1}{2})$ u. s. f. für welche

$$fx=f^{-1}(x+\tfrac{1}{2})-f^{-1}(x-\tfrac{1}{2}),\quad f^{-1}(x+\tfrac{1}{2})=f^{-2}(x+1)-f^{-2}x\ \text{ u. s. f.}$$

endlich die Mittelwerthe der aufeinander folgenden Glieder der so entstandenen Reihen nemlich

$$\ldots f^{-1}x=\tfrac{1}{2}f^{-1}(x+\tfrac{1}{2})+\tfrac{1}{2}f^{-1}(x-\tfrac{1}{2}),\quad f(x+\tfrac{1}{2})=\tfrac{1}{2}f(x+1)+\tfrac{1}{2}fx,\quad f^1x=\tfrac{1}{2}f^1(x+\tfrac{1}{2})+\tfrac{1}{2}f^1(x-\tfrac{1}{2})\ldots$$

also in übersichtlicher Anordnung zusammengestellt folgendes System von gegebenen und daraus zunächst berechneten Werthen

M. W.	II. Summe.	M. W.	I. Summe	gegeb. W.	M. W.	I. Differ.	M. W.	II. Differ.	M. W.
.	.	.	.	.	.	.	.	.	.
	$f^{-2}(x-1)$	$f^{-1}(x-1)$		$f(x-1)$			$f^{1}(x-1)$	$f^{2}(x-1)$	
. $f^{-2}(x-\frac{1}{2})$			$f^{-1}(x-\frac{1}{2})$		$f(x-\frac{1}{2})$	$f^{1}(x-\frac{1}{2})$			$f^{2}(x-\frac{1}{2})$.
	$f^{-2}x$	$f^{-1}x$		fx			$f^{1}x$	$f^{2}x$	
. $f^{-2}(x+\frac{1}{2})$			$f^{-1}(x+\frac{1}{2})$		$f(x+\frac{1}{2})$	$f^{1}(x+\frac{1}{2})$			$f^{2}(x+\frac{1}{2})$.
	$f^{-2}(x+1)$	$f^{-1}(x+1)$		$f(x+1)$			$f^{1}(x+1)$	$f^{2}(x+1)$	
.	.	.	.	.	.	.	.	.	.

so gibt Art. 4, wenn man 1[s] für $t, a, b, c, d, e\,..$ resp. $x+t,\ x,\ x+1,\ x-1,\ x+2,\ x-2$ 2[s] für $t, a, b, c, d, e\,..$ resp. $x+t,\ x,\ x-1,\ x+1,\ x-2,\ x+2\,...$ setzt, 3[s] aus den entsprechenden Seiten der beiden so erhaltenen Gleichungen die halbe Summe bildet, 4[s] aus denselben Gleichungen, nachdem zuvor in der zweiten $x+1$ und $t-1$ resp. für x und t gesetzt ist, die halbe Summe bildet und 5[s] in der zuletzt erhaltenen Gleichung $t=\frac{1}{2}$ macht, folgende Gleichungen für die durch Interpolatation zu bestimmenden Werthe $\varphi(x+t)$ und $\varphi(x+\frac{1}{2})$ derjenigen Function φx, die bei jedem ganzzahligen Werthe von x der gegebenen Grösse fx gleich wird:

$$\varphi(x+t) = fx + t.f^{1}(x+\tfrac{1}{2}) + \frac{t.t-1}{1.2}f^{2}x + \frac{t.t-1.t+1}{1.2.3}f^{3}(x+\tfrac{1}{2}) + \,..$$
$$+ \frac{\Pi(t+n-1)}{\Pi 2n.\Pi(t-n-1)}f^{2n}x + \frac{\Pi(t+n)}{\Pi(2n+1).\Pi(t-n-1)}f^{2n+1}(x+\tfrac{1}{2})\,..$$

$$\varphi(x+t) = fx + t.f^{1}(x-\tfrac{1}{2}) + \frac{t.t+1}{1.2}f^{2}x + \frac{t.t+1.t-1}{1.2.3}f^{3}(x-\tfrac{1}{2}) + \,..$$
$$+ \frac{\Pi(t+n)}{\Pi 2n.\Pi(t-n)}f^{2n}x + \frac{\Pi(t+n)}{\Pi(2n+1).\Pi(t-n-1)}f^{2n+1}(x-\tfrac{1}{2})\,...$$

$$\varphi(x+t) = fx + t.f^{1}x + \frac{t.t}{1.2}f^{2}x + \frac{t.t-1.t+1}{1.2.3}f^{3}x + \,..$$
$$+ \frac{\Pi(t+n-1)}{\Pi 2n.\Pi(t-n)}tf^{2n}x + \frac{\Pi(t+n)}{\Pi(2n+1)\Pi(t-n-1)}f^{2n+1}x + \,...$$

$$\varphi(x+t) = f(x+\tfrac{1}{2}) + (t-\tfrac{1}{2})f^{1}(x+\tfrac{1}{2}) + \frac{t.t-1}{1.2}f^{2}(x+\tfrac{1}{2}) + \frac{t.t-1}{1.2}(t-\tfrac{1}{2})f^{3}(x+\tfrac{1}{2}) + \,..$$
$$+ \frac{\Pi(t+n-1)}{\Pi 2n.\Pi(t-n-1)}f^{2n}(x+\tfrac{1}{2}) + \frac{\Pi(t+n-1)}{\Pi(2n+1).\Pi(t-n-1)}(t-\tfrac{1}{2})f^{2n+1}(x+\tfrac{1}{2}) + \,..$$

$$\varphi(x+\tfrac{1}{2}) = f(x+\tfrac{1}{2}) - \tfrac{1}{8}f^{2}(x+\tfrac{1}{2}) + \tfrac{3}{128}f^{4}(x+\tfrac{1}{2}) - \tfrac{5}{1024}f^{6}(x+\tfrac{1}{2}) + \tfrac{35}{32768}f^{8}(x+\tfrac{1}{2})\,...$$
$$+ (-1)^{n}\frac{\Pi 2n}{\Pi n.\Pi n}2^{-4n}.f^{2n}(x+\tfrac{1}{2}) + \,...$$

Die Coëfficienten von f^{m} in der dritten, vierten und fünften Gleichung sind gleich den Coëfficienten von $(2i\sin\omega)^{m}$ in den Reihenentwickelungen resp. für $\cos 2t\omega + i\frac{\sin 2t\omega}{\cos\omega}$, $\frac{\cos(2t-1)\omega}{\cos\omega} + i\sin(2t-1)\omega$ und $\frac{1}{\cos\omega}$ nach Potenzen der Grösse $2i\sin\omega$.

Setzt man nun

$$\Phi(x+\tfrac{1}{2}) = f^{-1}(x+\tfrac{1}{2}) + \tfrac{1}{24} f^{1}(x+\tfrac{1}{2}) - \tfrac{17}{5760} f^{3}(x+\tfrac{1}{2}) + \tfrac{367}{967680} f^{5}(x+\tfrac{1}{2}) - \tfrac{27859}{464486400} f^{7}(x+\tfrac{1}{2})$$
$$+ \tfrac{1295803}{122624409600} f^{9}(x+\tfrac{1}{2}) - \ldots$$

$$\Phi x = f^{-1}x - \tfrac{1}{12} f^{1}x + \tfrac{11}{720} f^{3}x - \tfrac{191}{60480} f^{5}x + \tfrac{2497}{3628800} f^{7}x$$
$$- \tfrac{14797}{95800320} f^{9}x + \tfrac{92427157}{2615348736000} f^{11}x - \ldots$$

$$\Psi x = f^{-2}x + \tfrac{1}{12} fx - \tfrac{1}{240} f^{2}x + \tfrac{31}{60480} f^{4}x - \tfrac{289}{3628800} f^{6}x$$
$$+ \tfrac{317}{22809600} f^{8}x - \ldots$$

$$\Psi(x+\tfrac{1}{2}) = f^{-2}(x+\tfrac{1}{2}) - \tfrac{1}{24} f(x+\tfrac{1}{2}) + \tfrac{17}{1920} f^{2}(x+\tfrac{1}{2}) - \tfrac{367}{193536} f^{4}(x+\tfrac{1}{2}) + \,.\, - \ldots$$

indem man als Coëfficienten von f^m in diesen vier Gleichungen die Coëfficienten nimmt, die bei den Entwicklungen resp. für $\frac{1}{2i\omega}$, $\frac{1}{2i\omega\cos\omega}$, $-\frac{1}{4\omega\omega}$, $-\frac{1}{4\omega\omega\cos\omega}$ noch Potenzen von $2i\sin\omega$ entstehen, so sind $\Phi(x+\frac{1}{2})$, Φx, Ψx, $\Psi(x+\frac{1}{2})$ die besonderen Werthe solcher Functionen $\Phi(x+t)$, $\Psi(x+t)$, für welche die Gleichungen

$$\Phi(x+t) - \Phi(x+t_0) = \int_{t_0}^{t} \varphi(x+t)\,\mathrm{d}t), \qquad \Psi(x+t) - \Psi(x+t_1) = \int_{t_1}^{t} \Phi(x+t)\,\mathrm{d}t$$

Statt haben, sie werden also bestimmten Integralen von $\varphi(x+t)\,\mathrm{d}t$ und $\varphi(x+t).\mathrm{d}t.\mathrm{d}t$ gleich, wenn man die Anfangswerthe der Summenreihen $f^{-1}(x-\frac{1}{2})$, $f^{-1}(x+\frac{1}{2})\ldots$, $f^{-2}x$, $f^{-2}(x+1)\ldots$ auf geeignete Weise ausgewählt.

Die Ableitung der Reihen fur Φ und Ψ aus der dritten und vierten der obigen Gleichungen fur $\varphi(x+t)$ erhält man unmittelbar, wenn man die Integrationen so ausführt, dass zunächst die Ausdrücke für $\Phi(x+\frac{1}{2}) - \Phi(x-\frac{1}{2})$, $\Phi(x+1) - \Phi x$, $\Psi(x+1) - 2\Psi x + \Psi(x-1)$ und $\Psi(x+\frac{3}{2}) - 2\Psi(x+\frac{1}{2}) + \Psi(x-\frac{1}{2})$ entstehen.

SCHERING.