331

DETERMINATIO ATTRACTIONIS

QUAM IN PUNCTUM QUODVIS POSITIONIS DATAE

EXERCERET PLANETA SI EIUS MASSA

PER TOTAM ORBITAM

RATIONE TEMPORIS QUO SINGULAE PARTES DESCRIBUNTUR

UNIFORMITER ESSET DISPERTITA

AUCTORE

CAROLO FRIDERICO GAUSS

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332

DETERMINATIO ATTRACTIONIS QUAM IN PUNCTUM QUODVIS POSITIONIS DATAE EXERCERET PLANETA SI EIUS MASSA PER TOTAM ORBITAM RATIONE TEMPORIS QUO SINGULAE PARTES DESCRIBUNTUR UNIFORMITER ESSET DISPERTITA.

1.

Variationes saeculares, quas elementa orbitae planetariae a perturbatione alius planetae patiuntur, ab huius positione in orbita sunt independentes, atque eaedem forent, sive planeta perturbans in orbita elliptica secundum Kepleri leges incedat, sive ipsius massa per orbitam eatenus aequabiliter dispertita concipiatur, ut orbitae partibus, alias aequali temporis intervallo descriptis, iam aequales massae partes tribuantur, siquidem tempora revolutionum planetae perturbati et perturbantis non sint commensurabilia. Theorema hoc elegans, si a nemine hucusque disertis verbis propositum est, saltem perfacile ex astronomiae physicae principiis demonstratur. Problema itaque se offert tum per se, tum propter plura artificia, quae eius solutio requirit, attentione perdignum: attractionem orbitae planetariae, aut si mavis, annuli elliptici, cuius crassities infinite parva, atque secundum legem modo explicatam variabilis, in punctum quodlibet positione datum exacte determinare.

2.

Denotando excentricitatem orbitae per e, atque puncti cuiusvis in ipsa anomaliam excentricam per E, huius elemento $\mathrm{d}E$ respondebit elementum anomaliae mediae $(1-e\cos E)\,\mathrm{d}E$; quamobrem elementum massae ei orbitae portiunculae, cui respondent illa elementa, tribuendum, erit ad massam integram, quam pro unitate accipiemus, ut $(1-e\cos E)\,\mathrm{d}E$ ad 2π, exprimente π semicircumferentiam circuli pro radio 1. Statuendo itaque distantiam puncti attracti a puncto orbitae $=\rho$, attractio ab orbitae elemento producta erit

$$=\frac{(1-e\cos E)\,\mathrm{d}E}{2\pi\rho\rho}$$

Designabimus semiaxem maiorem per a, semiaxem minorem per b, atque illum tamquam lineam abscissarum, centrumque ellipsis tamquam initium adoptabimus. Hinc erit $aa-bb=aaee$, abscissa puncti orbitae $=a\cos E$, ordinata $=b\sin E$. Denique distantiam puncti attracti a plano orbitae denotabimus per C, atque coordinatas reliquas axi maiori et minori parallelas per A et B. His ita praeparatis, attractio elementi orbitae decomponetur in duas axi maiori et minori parallelas atque tertiam plano orbitae normalem, puta

$$\frac{(A-a\cos E)(1-e\cos E)\,\mathrm{d}E}{2\pi\rho^3}=\mathrm{d}\xi$$

$$\frac{(B-b\sin E)(1-e\cos E)\,\mathrm{d}E}{2\pi\rho^3}=\mathrm{d}\eta$$

$$\frac{C(1-e\cos E)\,\mathrm{d}E}{2\pi\rho^3}=\mathrm{d}\zeta$$

ubi $\rho=\sqrt{((A-a\cos E)^2+(B-b\sin E)^2+CC)}$.

Integratis hisce differentialibus ab $E=0$ usque ad $E=360^0$, prodibunt attractiones partiales ξ, η, ζ secundum directiones, directionibus coordinatarum oppositas, e quibus attractio integra composita erit, et quas per methodum notam ad quaslibet alias directiones referre licebit.

3.

Rei summa iam in eo versatur, ut introducta loco ipsius E alia variabili, quantitas radicalis in formam simpliciorem redigatur. Ad hunc finem statuemus

$$\cos E=\frac{\alpha+\alpha'\cos T+\alpha''\sin T}{\gamma+\gamma'\cos T+\gamma''\sin T},\qquad \sin E=\frac{\beta+\beta'\cos T+\beta''\sin T}{\gamma+\gamma'\cos T+\gamma''\sin T}$$

ubi autem novem coëfficientes α, α', α'' etc. manifesto non sunt penitus arbitrarii, sed certis conditionibus satisfacere debent, quas ante omnia perscrutari oportet. Primo observamus, substitutionem eandem manere, si omnes coëfficientes per eundem factorem multiplicentur, ita ut absque generalitatis detrimento uni ex ipsis valorem determinatum tribuere, e. g. statuere liceret $\gamma=1$: attamen concinnitatis caussa omnes novem aliquantisper indefiniti maneant. Porro monemus, excludi debere valores tales, ubi $\alpha, \alpha', \alpha''$ vel β, β', β'' ipsis $\gamma, \gamma', \gamma''$ resp. proportionales essent: alioquin enim E haud amplius indeterminata maneret. Nequeunt igitur $\gamma'\alpha''-\gamma''\alpha'$, $\gamma''\alpha-\gamma\alpha''$, $\gamma\alpha'-\gamma'\alpha$ simul evanescere.

Manifesto coëfficientes $\alpha, \alpha', \alpha''$ etc. ita comparati esse debent, ut fiat indefinite

$$\left.\begin{aligned}&(\alpha+\alpha'\cos T+\alpha''\sin T)^2\\+&(\beta+\beta'\cos T+\beta''\sin T)^2\\-&(\gamma+\gamma'\cos T+\gamma''\sin T)^2\end{aligned}\right\}=0$$

unde necessario haec functio habere debet formam

$$k(\cos T^2+\sin T^2-1)$$

Hinc colligimus sex aequationes conditionales

$$\left.\begin{aligned}-\alpha\alpha-\beta\beta+\gamma\gamma&=k\\-\alpha'\alpha'-\beta'\beta'+\gamma'\gamma'&=-k\\-\alpha''\alpha''-\beta''\beta''+\gamma''\gamma''&=-k\\-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma''&=0\\-\alpha''\alpha-\beta''\beta+\gamma''\gamma&=0\\-\alpha\alpha'-\beta\beta'+\gamma\gamma'&=0\end{aligned}\right\}\quad\text{(I)}$$

Ab his aequationibus pendent plures aliae, quas evolvere operae pretium erit. Statuendo brevitatis caussa

$$\alpha\beta'\gamma''+\alpha'\beta''\gamma+\alpha''\beta\gamma'-\alpha\beta''\gamma'-\alpha'\beta\gamma''-\alpha''\beta'\gamma=\varepsilon\;\ldots\ldots\ldots\;\text{(II)}$$

e combinatione aequationum (I) facile derivantur novem sequentes:

$$\left.\begin{aligned}\varepsilon\alpha&=-k(\beta'\gamma''-\gamma'\beta'')\\\varepsilon\beta&=-k(\gamma'\alpha''-\alpha'\gamma'')\\\varepsilon\gamma&=+k(\alpha'\beta''-\beta'\alpha'')\\\varepsilon\alpha'&=+k(\beta''\gamma-\gamma''\beta)\\\varepsilon\beta'&=+k(\gamma''\alpha-\alpha''\gamma)\\\varepsilon\gamma'&=-k(\alpha''\beta-\beta''\alpha)\\\varepsilon\alpha''&=+k(\beta\gamma'-\gamma\beta')\\\varepsilon\beta''&=+k(\gamma\alpha'-\alpha\gamma')\\\varepsilon\gamma''&=-k(\alpha\beta'-\beta\alpha')\end{aligned}\right\}\quad\text{(III)}$$

E tribus primis harum aequationum rursus deducimus hanc:

$$\begin{aligned} &\varepsilon\alpha(\beta'\gamma''-\gamma'\beta'')+\varepsilon\beta(\gamma'\alpha''-\alpha'\gamma'')+\varepsilon\gamma(\alpha'\beta''-\beta'\alpha'') \\ &= -k(\beta'\gamma''-\gamma'\beta'')^2-k(\gamma'\alpha''-\alpha'\gamma'')^2+k(\alpha'\beta''-\beta'\alpha'')^2 \end{aligned}$$

cui aequivalens est haec:

$$\varepsilon\varepsilon = k(-\alpha'\alpha'-\beta'\beta'+\gamma'\gamma')(-\alpha''\alpha''-\beta''\beta''+\gamma''\gamma'')-k(-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma'')^2$$

quae adiumento aequationum 2, 3, 4 in (I) mutatur in hanc:

$$\varepsilon\varepsilon = k^3 \ldots\ldots\ldots\ldots (\text{IV})$$

Aeque facile ex aequationibus (I) derivantur hae:

$$\left.\begin{aligned} (\beta'\gamma''-\gamma'\beta'')^2 &= -k(k-\alpha'\alpha'-\alpha''\alpha'') \\ (\gamma'\alpha''-\alpha'\gamma'')^2 &= -k(k-\beta'\beta'-\beta''\beta'') \\ (\alpha'\beta''-\beta'\alpha'')^2 &= +k(k+\gamma'\gamma'+\gamma''\gamma'') \\ (\beta''\gamma-\gamma''\beta)^2 &= +k(k+\alpha\alpha-\alpha''\alpha'') \\ (\gamma''\alpha-\alpha''\gamma)^2 &= +k(k+\beta\beta-\beta''\beta'') \\ (\alpha''\beta-\beta''\alpha)^2 &= -k(k-\gamma\gamma+\gamma''\gamma'') \\ (\beta\gamma'-\gamma\beta')^2 &= +k(k+\alpha\alpha-\alpha'\alpha') \\ (\gamma\alpha'-\alpha\gamma')^2 &= +k(k+\beta\beta-\beta'\beta') \\ (\alpha\beta'-\beta\alpha')^2 &= -k(k-\gamma\gamma+\gamma'\gamma') \end{aligned}\right\} \quad (\text{V})$$

Exempli caussa evolutionem primae adscribimus, ad cuius instar reliquae facile formabuntur. Aequationes 4, 2, 3 in (I) scilicet suppeditant

$$(\gamma'\gamma''-\beta'\beta'')^2-(\gamma'\gamma'-\beta'\beta')(\gamma''\gamma''-\beta''\beta'') = \alpha'\alpha'\alpha''\alpha''-(\alpha'\alpha'-k)(\alpha''\alpha''-k)$$

quae aequatio evoluta protinus ipsam primam in (V) sistit.

Ex his aequationibus (V) concludimus, valorem $k=0$ in disquisitione nostra haud admissibilem esse; hinc enim omnes novem quantitates $\beta'\gamma''-\gamma'\beta''$ etc. necessario evanescerent, i. e. coëfficientes α, α', α'' tum ipsis β, β', β'', tum ipsis γ, γ', γ'' proportionales evaderent. Hinc etiam, propter aequationem IV, quantitas ε evanescere nequit; quamobrem k necessario debet esse quantitas positiva, siquidem omnes coëfficientes α, α', α'' etc. debent esse reales. Combinatis tribus aequationibus primis in (III) cum tribus primis in (V), hae novae prodeunt, quae manifesto a valore ipsius k non evanescente pendent:

$$\left.\begin{array}{l} \alpha\alpha - \alpha'\alpha' - \alpha''\alpha'' = -k \\ \beta\beta - \beta'\beta' - \beta''\beta'' = -k \\ \gamma\gamma - \gamma'\gamma' - \gamma''\gamma'' = +k \end{array}\right\} \text{(VI)}$$

Combinatio reliquarum easdem produceret. His denique adiungimus tres sequentes:

$$\left.\begin{array}{l} \beta\gamma - \beta'\gamma' - \beta''\gamma'' = 0 \\ \gamma\alpha - \gamma'\alpha' - \gamma''\alpha'' = 0 \\ \alpha\beta - \alpha'\beta' - \alpha''\beta'' = 0 \end{array}\right\} \text{(VII)}$$

quae facile ex aequationibus III derivantur; e. g. secunda, quinta et octava suppeditant:

$$\varepsilon\beta\gamma - \varepsilon\beta'\gamma' - \varepsilon\beta''\gamma'' = -k\gamma(\gamma'\alpha'' - \alpha'\gamma'') - k\gamma'(\gamma''\alpha - \alpha''\gamma) - k\gamma''(\gamma\alpha' - \alpha\gamma') = 0$$

Manifesto hae quoque aequationes ab exclusione valoris $k = 0$ sunt dependentes*).

Quoniam, ut iam supra monuimus, omnes coëfficientes α, α', α'' etc. per eundem factorem multiplicare licet, unde valor ipsius k per quadratum eiusdem factoris multiplicatus prodibit, abhinc semper supponemus

$$k = 1$$

quo pacto necessario quoque erit vel $\varepsilon = +1$ vel $\varepsilon = -1$. Patet itaque, novem coëfficientes α, α', α'' etc., inter quos sex aequationes conditionales adsunt, ad tres quantitates ab invicem independentes reducibiles esse debere, quod quidem commodissime per tres angulos sequenti modo efficitur:

$$\begin{array}{l} \alpha = \cos L \operatorname{tang} N \\ \beta = \sin L \operatorname{tang} N \\ \gamma = \sec N \\ \alpha' = \cos L \cos M \sec N \pm \sin L \sin M \\ \beta' = \sin L \cos M \sec N \mp \cos L \sin M \\ \gamma' = \cos M \operatorname{tang} N \\ \alpha'' = \cos L \sin M \sec N \mp \sin L \cos M \\ \beta'' = \sin L \sin M \sec N \pm \cos L \cos M \\ \gamma'' = \sin M \operatorname{tang} N \end{array}$$

*) Forsan haud superfluum erit monere, nos analysin praecedentem consulto elegisse atque alii derivationi relationum III—VII praetulisse, quae quamquam aliquantulum elegantior videretur, tamen, accurate examinata, quibusdam dubiis obnoxia inventa est, quae non sine ambagibus removere licuisset.

ubi signorum ambiguorum superiora referuntur ad casum $\varepsilon = +1$, inferiora ad casum $\varepsilon = -1$. Attamen tractatio analytica ad maximam partem elegantius sine usu horum angulorum absolvitur. Ceterum haud difficile foret, significationem geometricam tum horum angulorum, tum reliquarum quantitatum auxiliarium in hac disquisitione occurrentium assignare; hanc vero interpretationem ad institutum nostrum haud necessariam lectori perito explicandam linquimus.

4.

Si iam in expressione distantiae ρ pro $\cos E$ et $\sin E$ valores supra assumti substituuntur, illa in hanc formam transibit:

$$\rho = \frac{\sqrt{(G + G'\cos T^2 + G''\sin T^2 + 2H\cos T\sin T + 2H'\sin T + 2H''\cos T)}}{\gamma + \gamma'\cos T + \gamma''\sin T}$$

ubi coëfficientes α, α', α'' etc. ita determinabimus, ut salvis sex aequationibus conditionalibus

$$\left.\begin{aligned}
-\alpha\alpha - \beta\beta + \gamma\gamma &= 1 \\
-\alpha'\alpha' - \beta'\beta' + \gamma'\gamma' &= -1 \\
-\alpha''\alpha'' - \beta''\beta'' + \gamma''\gamma'' &= -1 \\
-\alpha'\alpha'' - \beta'\beta'' + \gamma'\gamma'' &= 0 \\
-\alpha''\alpha - \beta''\beta + \gamma''\gamma &= 0 \\
-\alpha\alpha' - \beta\beta' + \gamma\gamma' &= 0
\end{aligned}\right\} \quad [1]$$

adeoque etiam reliquis inde demanantibus, fiat

$$H = 0, \quad H' = 0, \quad H'' = 0$$

quo pacto problema generaliter loquendo erit determinatum. Quodsi itaque denominatorem ipsius ρ per t denotamus, transire debet functio trium quantitatum t, $t\cos E$, $t\sin E$ haec

$$(AA + BB + CC)tt + aa(t\cos E)^2 + bb(t\sin E)^2 - 2aAt.t\cos E - 2bBt.t\sin E$$

per substitutionem

$$\begin{aligned}
t\cos E &= \alpha + \alpha'\cos T + \alpha''\sin T \\
t\sin E &= \beta + \beta'\cos T + \beta''\sin T \\
t &= \gamma + \gamma'\cos T + \gamma''\sin T
\end{aligned}$$

in

$$G + G'\cos T^2 + G''\sin T^2$$

Manifesto hoc idem est, ac si dicas, functionem trium indeterminatarum x, y, z hanc (W)

$$aaxx + bbyy + (AA + BB + CC)zz - 2aAxz - 2bByz$$

per substitutionem

$$\begin{aligned} x &= \alpha u + \alpha' u' + \alpha'' u'' \\ y &= 6u + 6'u' + 6''u'' \\ z &= \gamma u + \gamma' u' + \gamma'' u'' \end{aligned}$$

in functionem indeterminatarum u, u', u'' hanc

$$Guu + G'u'u' + G''u''u''$$

transire debere. At quum ex his formulis, adiumento aequationum [1], facile sequatur

$$\begin{aligned} u &= -\alpha x - 6y + \gamma z \\ u' &= \quad \alpha' x + 6'y - \gamma' z \\ u'' &= \quad \alpha'' x + 6''y - \gamma'' z \end{aligned}$$

manifesto functio W identica esse debebit cum hac

$$G(-\alpha x - 6y + \gamma z)^2 + G'(\alpha' x + 6'y - \gamma' z)^2 + G''(\alpha'' x + 6''y - \gamma'' z)^2$$

unde habemus sex aequationes

$$\left.\begin{aligned} aa &= G\alpha\alpha + G'\alpha'\alpha' + G''\alpha''\alpha'' \\ bb &= G66 + G'6'6' + G''6''6'' \\ AA + BB + CC &= G\gamma\gamma + G'\gamma'\gamma' + G''\gamma''\gamma'' \\ bB &= G6\gamma + G'6'\gamma' + G''6''\gamma'' \\ aA &= G\gamma\alpha + G'\gamma'\alpha' + G''\gamma''\alpha'' \\ 0 &= G\alpha 6 + G'\alpha' 6' + G''\alpha'' 6'' \end{aligned}\right\} \quad [2]$$

Ex his duodecim aequationibus [1] et [2] incognitas nostras $G, G', G'', \alpha, \alpha', \alpha''$ etc., determinare oportebit.

5.

E combinatione aequationum [1] et [2] facile derivantur sequentes:

$$-\alpha aa+\gamma aA=\alpha G$$
$$-\beta bb+\gamma bB=\beta G$$
$$\gamma(AA+BB+CC)-\alpha aA-\beta bB=\gamma G$$

unde fit porro

$$\alpha=\frac{\gamma aA}{aa+G}\dots\dots\dots\dots\dots\dots[3]$$
$$\beta=\frac{\gamma bB}{bb+G}\dots\dots\dots\dots\dots\dots[4]$$
$$AA+BB+CC-\frac{aaAA}{aa+G}-\frac{bbBB}{bb+G}=G$$

Ultimam sic quoque exhibere possumus

$$\frac{AA}{aa+G}+\frac{BB}{bb+G}+\frac{CC}{G}=1\dots\dots[5]$$

Perinde e combinatione aequationum [1] et [2] deducimus

$$\alpha'aa-\gamma'aA=\alpha'G'$$
$$\beta'bb-\gamma'bB=\beta'G'$$
$$-\gamma'(AA+BB+CC)+\alpha'aA+\beta'bB=\gamma'G'$$

atque hinc

$$\alpha'=\frac{\gamma'aA}{aa-G'}\dots\dots\dots\dots\dots\dots[6]$$
$$\beta'=\frac{\gamma'bB}{bb-G'}\dots\dots\dots\dots\dots\dots[7]$$
$$\frac{AA}{aa-G'}+\frac{BB}{bb-G'}-\frac{CC}{G'}=1\dots\dots[8]$$

et prorsus simili modo

$$\alpha''=\frac{\gamma''aA}{aa-G''}\dots\dots\dots\dots\dots\dots[9]$$
$$\beta''=\frac{\gamma''bB}{bb-G''}\dots\dots\dots\dots\dots\dots[10]$$
$$\frac{AA}{aa-G''}+\frac{BB}{bb-G''}-\frac{CC}{G''}=1\dots\dots[11]$$

Patet itaque, G, $-G'$, $-G''$ esse radices aequationis

$$\frac{AA}{aa+x}+\frac{BB}{bb+x}+\frac{CC}{x}=1\dots\dots\dots[12]$$

quae rite evoluta ita se habet

$$x^3-(AA+BB+CC-aa-bb)xx+(aabb-aaBB-aaCC-bbAA-bbCC)x$$
$$-aabbCC=0\ \ldots\ldots\ldots\ldots\ [13]$$

6.

Iam de indole huius aequationis cubicae sequentia sunt notanda.

I. Ex aequationis termino ultimo $-aabbCC$ concluditur, eam certe habere radicem unam realem, et quidem vel positivam, vel, si $C=0$, cifrae aequalem. Denotemus hanc radicem realem non negativam per g.

II. Subtrahendo ab aequatione 12, ita exhibita

$$x=\frac{AAx}{aa+x}+\frac{BBx}{bb+x}+CC$$

hanc

$$g=\frac{AAg}{aa+g}+\frac{BBg}{bb+g}+CC$$

et dividendo per $x-g$, oritur nova, duas reliquas radices complectens

$$1=\frac{aaAA}{(aa+x)(aa+g)}+\frac{bbBB}{(bb+x)(bb+g)}$$

quae rite ordinata et soluta suppeditat [14]

$$2x=\frac{aaAA}{aa+g}+\frac{bbBB}{bb+g}-aa-bb\pm\surd\left(\left(aa-bb-\frac{aaAA}{aa+g}+\frac{bbBB}{bb+g}\right)^2+\frac{4aabbAABB}{(aa+g)(bb+g)}\right)$$

Haec expressio, quum quantitas sub signo radicali natura sua sit positiva, vel saltem non negativa, monstrat, etiam duas reliquas radices semper fieri reales.

III. Subtrahendo autem ab invicem aequationes istas sic exhibitas

$$gx=\frac{AAgx}{aa+x}+\frac{BBgx}{bb+x}+gCC$$
$$gx=\frac{AAgx}{aa+g}+\frac{BBgx}{bb+g}+xCC$$

et dividendo per $g-x$, prodit aequatio duas reliquas radices continens in hacce forma:

$$0=\frac{AAgx}{(aa+g)(aa+x)}+\frac{BBgx}{(bb+g)(bb+x)}+CC$$

cui manifesto, si g est quantitas positiva, per valorem positivum ipsius x satisfieri nequit. Unde concludimus, aequationem nostram cubicam radices positivas plures quam unam habere non posse.

IV. Quoties itaque 0 non est inter radices aequationis nostrae, aderunt necessario radix una positiva cum duabus negativis. Quoties vero $C = 0$, adeoque 0 una radicum, reliquas complectetur aequatio

$$xx-(AA+BB-aa-bb)x+aabb-aaBB-bbAA = 0$$

unde hae radices exprimentur per

$$\tfrac{1}{2}(AA+BB-aa-bb)\pm\tfrac{1}{2}\sqrt{((AA-BB-aa+bb)^2+4AABB)}$$

Tres casus hic iterum distinguere oportebit.

Primo si terminus ultimus $aabb-aaBB-bbAA$ est positivus (i. e. si punctum attractum in plano ellipsis attrahentis *intra* curvam iacet), ambae radices, quum reales esse debeant, eodem signo affectae erunt, adeoque quum simul positivae esse nequeant, necessario erunt negativae. Ceterum hoc etiam independenter ab iis, quae iam demonstrata sunt, inde concludi potest, quod coëfficiens medius, quem ita exhibere licet

$$(aabb-aaBB-bbAA)\left(\frac{1}{aa}+\frac{1}{bb}\right)+\frac{bbAA}{aa}+\frac{aaBB}{bb}$$

manifesto in hoc casu sit positivus.

Secundo, si terminus ultimus est negativus, sive punctum attractum in plano ellipsis *extra* curvam situm, necessario altera radix positiva erit, altera negativa.

Tertio autem, si terminus ultimus ipse evanesceret, sive punctum attractum in ipsa ellipsis circumferentia iaceret, etiam radix secunda fieret $= 0$, atque tertia

$$= -\frac{bbAA}{aa}-\frac{aaBB}{bb}$$

i. e. negativa. Ceterum hunc casum, physice impossibilem, et in quo attractio ipsa infinite magna evaderet, a disquisitione nostra, hocce saltem loco, excludemus.

7.

Ad determinandos coëfficientes $\gamma, \gamma', \gamma''$, ex aequationibus 1, 3, 4, 6, 7, 9, 10 invenimus

$$\left.\begin{aligned}\gamma &= \frac{1}{\sqrt{(1-(\frac{aA}{aa+G})^2-(\frac{bB}{bb+G})^2)}}\\ \gamma' &= \frac{1}{\sqrt{((\frac{aA}{aa-G'})^2+(\frac{bB}{bb-G'})^2-1)}}\\ \gamma'' &= \frac{1}{\sqrt{((\frac{aA}{aa-G''})^2+(\frac{bB}{bb-G''})^2-1)}}\end{aligned}\right\}\quad [15]$$

Ex his aequationibus rite cum 5, 8, 11 combinatis etiam sequitur:

$$\left.\begin{aligned}\gamma &= \sqrt{\frac{G}{(\frac{AG}{aa+G})^2+(\frac{BG}{bb+G})^2+CC}}\\ \gamma' &= \sqrt{\frac{G'}{(\frac{AG'}{aa-G'})^2+(\frac{BG'}{bb-G'})^2+CC}}\\ \gamma'' &= \sqrt{\frac{G''}{(\frac{AG''}{aa-G''})^2+(\frac{BG''}{bb-G''})^2+CC}}\end{aligned}\right\}\quad [16]$$

Hae posteriores expressiones ostendunt, nullam quantitatum G, G', G'' negativam esse posse, siquidem γ, γ', γ'' debent esse reales.

In casu itaque eo, ubi non est $C = 0$, necessario G aequalis statui debet radici positivae aequationis B, patetque adeo, $-G'$ aequalem esse debere alteri radici negativae, atque $-G''$ aequalem alteri*); utram vero radicem pro $-G'$, utram pro $-G''$ adoptemus, prorsus arbitrarium erit.

Quoties $C = 0$, punctumque attractum intra curvam situm, duas radices negativas aequationis 13 necessario pro $-G'$ et $-G''$ adoptare et proin $G = 0$ statuere oportet. Quoniam vero in hoc casu formula prima in 16 fit indeterminata, formulam primam in 15 eius loco retinebimus, quae suppeditat

$$\gamma = \frac{1}{\sqrt{(1-\frac{AA}{aa}-\frac{BB}{bb})}}$$

Quoties autem pro $C = 0$ punctum attractum extra ellipsin iacet. aequa-

*) Proprie quidem ex analysi praecedenti tantummodo sequitur, $-G'$ et $-G''$ satisfacere debere aequationi 13, unde dubium esse videtur, annon liceat, utramque $-G'$ et $-G''$ *eidem* radici negativae aequalem ponere, prorsus neglecta radice tertia. Sed facile perspicietur, siquidem aequationis radix secunda et tertia sint inaequales, ex $-G' = -G''$ sequi $\gamma' = \gamma''$, $\alpha' = \alpha''$, $\beta' = \beta''$, et proin $-\alpha'\alpha''-\beta'\beta''+\gamma'\gamma'' = -\alpha'\alpha'-\beta'\beta'+\gamma'\gamma' = 1$, quod aequationi quartae in [1] est contrarium. Conf. quae infra de casu duarum radicum aequalium aequationis 13 dicentur.

tionis 13 radix positiva statuenda est $= G$, atque vel negativa $= -G'$, et $G'' = 0$, vel radix negativa $= -G''$, et $G' = 0$; coëfficientem γ'' vel γ' vero inveniemus per formulam

$$\frac{1}{\sqrt{\left(\frac{AA}{aa} + \frac{BB}{bb} - 1\right)}}$$

Ceterum in casu iam excluso, ubi punctum attractum in ipsa circumferentia ellipsis situm supponeretur, coëfficientes γ et γ', vel γ et γ'' evaderent infiniti, quod indicat, transformationem nostram ad hunc casum omnino non esse applicabilem.

8.

Quamquam formulae 15, 16 ad determinationem coëfficientium $\gamma, \gamma', \gamma''$ sufficere possent, tamen etiam elegantiores assignare licet. Ad hunc finem multiplicabimus aequationem [5] per $aabb - GG$, unde prodit, levi reductione facta,

$$\frac{aaAA(bb+G)}{aa+G} - AAG + \frac{bbBB(aa+G)}{bb+G} - BBG + \frac{aabbCC}{G} - CCG = aabb - GG$$

Sed e natura aequationis cubicae fit

summa radicum $G - G' - G'' = AA + BB + CC - aa - bb$
productum radicum $GG'G'' = aabbCC$

Hinc aequatio praecedens transit in sequentem:

$$\frac{aaAA(bb+G)}{aa+G} + \frac{bbBB(aa+G)}{bb+G} + G'G'' - G(G - G' - G'' + aa + bb) = aabb - GG$$

quam etiam sic exhibere licet

$$\frac{aaAA(bb+G)}{aa+G} + \frac{bbBB(aa+G)}{bb+G} - (aa+G)(bb+G) + (G+G')(G+G'') = 0$$

Hinc valor coëfficientis γ e formula prima in [15] transmutatur in sequentem:

$$\gamma = \sqrt{\frac{(aa+G)(bb+G)}{(G+G')(G+G'')}} \quad \ldots\ldots\ldots [17]$$

Per analysin prorsus similem invenitur

$$\gamma' = \sqrt{\frac{(aa-G')(bb-G')}{(G+G')(G''-G')}} \quad \ldots\ldots\ldots [18]$$

$$\gamma'' = \sqrt{\frac{(aa-G'')(bb-G'')}{(G+G'')(G'-G'')}} \quad \ldots\ldots\ldots [19]$$

Postquam coëfficientes $\gamma, \gamma', \gamma''$ inventi sunt, reliqui $\alpha, \beta, \alpha', \beta', \alpha'', \beta''$ inde per formulas 3, 4, 6, 7, 9, 10 derivabuntur.

9.

Signa expressionum radicalium, per quas $\gamma, \gamma', \gamma''$ determinavimus, ad lubitum accipi posse facile perspicitur. Operae autem pretium est, inquirere, quomodo signum quantitatis ε cum signis istis nexum sit. Ad hunc finem consideremus aequationem tertiam in III art. 3.

$$\varepsilon\gamma = \alpha'\beta'' - \beta'\alpha''$$

quae per formulas 6, 7, 9, 10 transmutatur in hanc:

$$\begin{aligned}\varepsilon\gamma &= \frac{abAB\gamma'\gamma''}{(aa-G')(bb-G'')} - \frac{abAB\gamma'\gamma''}{(aa-G'')(bb-G')}\\ &= \frac{ab(aa-bb)AB(G''-G')\gamma'\gamma''}{(aa-G')(aa-G'')(bb-G')(bb-G'')}\end{aligned}$$

Sed e consideratione aequationis 13 facile deducimus

$$\begin{aligned}(aa+G)(aa-G')(aa-G'') &= aa(aa-bb)AA\\ (bb+G)(bb-G')(bb-G'') &= -bb(aa-bb)BB\end{aligned}$$

Hinc aequatio praecedens fit

$$\varepsilon\gamma = \frac{(aa+G)(bb+G)(G'-G'')\gamma'\gamma''}{ab(aa-bb)AB}$$

quae combinata cum aequatione 17 suppeditat

$$\gamma\gamma'\gamma'' = \frac{\varepsilon ab(aa-bb)AB}{(G+G')(G+G'')(G'-G'')}$$

Hinc patet, si pro $-G'$ electa sit aequationis cubicae radix negativa absolute maior, simulque coëfficientes $\gamma, \gamma', \gamma''$ omnes positive accepti sint, ε idem signum nancisci, quod habet AB, idemque evenire, si his quatuor conditionibus, vel omnibus vel duabus ex ipsis, contraria acta sint, oppositum vero, si uni vel tribus conditionibus adversatus fueris. Ceterum sequentes adhuc relationes notare convenit, e praecedentibus facile derivandas:

$$\alpha\alpha'\alpha'' = \frac{\varepsilon aabAAB}{(G+G')(G+G'')(G'-G'')}$$
$$\beta\beta'\beta'' = -\frac{\varepsilon abbABB}{(G+G')(G+G'')(G'-G'')}$$
$$\alpha\beta = \frac{abAB}{(G+G')(G+G'')}$$
$$\alpha'\beta' = -\frac{abAB}{(G+G')(G'-G'')}$$
$$\alpha''\beta'' = \frac{abAB}{(G+G'')(G'-G'')}$$

10.

Formulae nostrae quibusdam casibus indeterminatae fieri possunt, quos seorsim considerare oportet. Ac primo quidem discutiemus casum eum, ubi aequationis cubicae radices negativae $-G'$, $-G''$ aequales fiunt, unde, per formulas 18, 19, coëfficientes γ, γ'' valores infinitos nancisci videntur, qui autem revera sunt indeterminati.

Statuendo in formula 14, $g = G$, patet, ut duo valores ipsius x, i. e. ut $-G'$ et $-G''$ fiant aequales, necessario esse debere

$$AB = 0, \quad aa-bb-\frac{aaAA}{aa+G}+\frac{bbBB}{bb+G} = 0$$

Hinc facile intelligitur, quum $aa-bb$ natura sua sit vel quantitas positiva, vel $=0$, esse debere

$$B = 0$$
$$aa-bb = \frac{aaAA}{aa+G}, \quad \text{sive} \quad aa+G = \frac{aaAA}{aa-bb}$$

Substituendo hos valores in aequatione 14, fit

$$G' = G'' = bb$$

Substituendo porro valorem $x = -bb$ in aequatione cubica 13, prodit

$$(aa-bb)(CC+bb) = bbAA$$

Quoties haec aequatio conditionalis simul cum aequatione $B = 0$ locum habet, casus, quem hic tractamus, adducitur. Et quum fiat

$$G = \frac{aaAA}{aa-bb}-aa = \frac{aaCC}{bb}$$

formula 17 suppeditat

$$\gamma = \sqrt{\frac{aabbAA}{(aa-bb)(aaCC+b^4)}} = \sqrt{\frac{aaCC+aabb}{aaCC+b^4}}$$

ac dein formulae 3, 4

$$\alpha = \frac{\gamma(aa-bb)}{aA} = \frac{\gamma bbA}{a(CC+bb)} = \sqrt{\frac{bb(aa-bb)}{aaCC+b^4}} = \sqrt{\frac{b^4AA}{(CC+bb)(aaCC+b^4)}}$$

$$6 = 0$$

Valores coëfficientium γ', γ'' per formulas 18, 19 in hoc casu indeterminati manent, atque sic etiam valores coëfficientium reliquorum α', $6'$, α'', $6''$. Nihilominus per unum horum coëfficientium omnes quinque reliqui exprimi possunt, e. g. fit per formulam 6

$$\alpha' = \frac{\gamma' aA}{aa-bb}$$

ac dein

$$6' = \sqrt{(1-\alpha'\alpha'+\gamma'\gamma')}, \quad \gamma'' = \sqrt{(\gamma\gamma-1-\gamma'\gamma')}, \quad \alpha'' = \frac{\gamma'' aA}{aa-bb}, \quad 6'' = \sqrt{(1-\alpha''\alpha''+\gamma''\gamma'')}$$

Sed concinnius hoc ita perficitur. Ex

$$\gamma\gamma = 1+\alpha\alpha, \quad \alpha\alpha' = \gamma\gamma', \quad 1 = \alpha'\alpha'+6'6'-\gamma'\gamma'$$

sequitur

$$6'6'+\frac{\gamma'\gamma'}{\alpha\alpha} = 1-\alpha'\alpha'+\frac{\gamma\gamma\gamma'\gamma'}{\alpha\alpha} = 1$$

Quapropter statuere possumus

$$6' = \cos f, \quad \gamma' = \alpha\sin f, \quad \alpha' = \gamma\sin f$$

Dein vero e formulis

$$\varepsilon\alpha'' = 6\gamma'-\gamma 6', \quad \varepsilon 6'' = \gamma\alpha'-\alpha\gamma', \quad \varepsilon\gamma'' = 6\alpha'-\alpha 6', \quad \varepsilon\varepsilon = 1$$

invenimus

$$\alpha'' = -\varepsilon\gamma\cos f, \quad 6'' = \varepsilon\sin f, \quad \gamma'' = -\varepsilon\alpha\cos f$$

Valor anguli f hic arbitrarius est, nec non pro lubitu statui poterit vel $\varepsilon = +1$ vel $\varepsilon = -1$.

11.

Si G', G'' sunt inaequales, valores coëfficientium γ, γ', γ'' per formulas 17, 18, 19 indeterminati esse nequeunt, sed quoties aliqua quantitatum

$aa - G'$, $bb - G'$, $aa - G''$, $bb - G''$ evanescit, valor coëfficientis α', β', α'', γ'' per formulam 6, 7, 9, 10 resp. indeterminatus manere primo aspectu videtur, quod tamen secus se habere levis attentio docebit.

Supponumus e. g., esse $aa - G' = 0$, fietque, per aequationem 18, $\gamma' = 0$, nec non per aequationem 7, $\beta' = 0$ (siquidem non fuerit simul $aa = bb$) unde necessario esse debet $\alpha' = \pm 1$. Si vero simul $aa = bb$, formula, quae praecedit sextam in art. 5, suppeditat $\alpha' A + \beta' B = 0$, quae aequatio cum $\alpha'\alpha' + \beta'\beta' = 1$ iuncta, producit

$$\alpha' = \frac{B}{\sqrt{(AA+BB)}}, \quad \beta' = \frac{-A}{\sqrt{(AA+BB)}}$$

Hae expressiones manifesto indeterminatae esse nequeunt, nisi simul fuerit $A = 0$, $B = 0$; tunc vero ad casum in art. praec. iam consideratum delaberemur.

12.

Postquam duodecim quantitates G, G', G'', α, α', α'', β, β', β'', γ, γ', γ'' complete determinare docuimus, ad evolutionem differentialis $\mathrm{d}E$ progredimur. Statuamus

$$t = \gamma + \gamma' \cos T + \gamma'' \sin T \quad \ldots\ldots\ldots [20]$$

ita ut fiat

$$t \cos E = \alpha + \alpha' \cos T + \alpha'' \sin T \quad \ldots\ldots\ldots [21]$$

$$t \sin E = \beta + \beta' \cos T + \beta'' \sin T \quad \ldots\ldots\ldots [22]$$

Hinc deducimus

$$\begin{aligned} t\,\mathrm{d}E &= \cos E\,\mathrm{d}.\,t\sin E - \sin E\,\mathrm{d}.\,t\cos E \\ &= \cos E(\beta''\cos T - \beta'\sin T)\,\mathrm{d}T - \sin E(\alpha''\cos T - \alpha'\sin T)\,\mathrm{d}T \end{aligned}$$

adeoque

$$\begin{aligned} tt\,\mathrm{d}E &= (\alpha\beta'' - \alpha''\beta)\cos T\,\mathrm{d}T + (\alpha'\beta - \beta'\alpha)\sin T\,\mathrm{d}T + (\alpha'\beta'' - \beta'\alpha'')\,\mathrm{d}T \\ &= \varepsilon\gamma'\cos T\,\mathrm{d}T + \varepsilon\gamma''\sin T\,\mathrm{d}T + \varepsilon\gamma\,\mathrm{d}T = \varepsilon t\,\mathrm{d}T \end{aligned}$$

sive

$$t\,\mathrm{d}E = \varepsilon\,\mathrm{d}T \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots [23]$$

Observare convenit, quantitatem t natura sua semper positivam esse, si coëfficiens γ sit positivus, vel semper negativam, si γ sit negativus. Quum enim sit $(\gamma'\cos T + \gamma''\sin T)^2 + (\gamma''\cos T - \gamma'\sin T)^2 = \gamma'\gamma' + \gamma''\gamma'' = \gamma\gamma - 1$, erit semper

$\gamma'\cos T+\gamma''\sin T$, sine respectu signi, minor quam γ. Hinc concludimus, quoties $\varepsilon\gamma$ sit quantitas positiva, variabiles E et T semper simul crescere; quoties autem $\varepsilon\gamma$ sit quantitas negativa, necessario alteram variabilem semper decrescere, dum altera augeatur.

13.

Nexus inter variabiles E et T adhuc melius illustratur per ratiocinia sequentia. Statuendo $\sqrt{(\gamma\gamma-1)}=\delta$, ita ut fiat $\delta\delta=\alpha\alpha+6 6=\gamma'\gamma'+\gamma''\gamma''$, ex aequationibus 20, 21, 22 deducimus

$$\begin{aligned} t(\delta+\alpha\cos E+6\sin E)&\\ &=\gamma\delta+\alpha\alpha+66+(\gamma'\delta+\alpha\alpha'+66')\cos T+(\gamma''\delta+\alpha\alpha''+66'')\sin T\\ &=(\gamma+\delta)(\delta+\gamma'\cos T+\gamma''\sin T)\end{aligned}$$

Perinde ex aequationibus 21, 22 sequitur

$$t(\alpha\sin E-6\cos E)=\varepsilon(\gamma'\sin T-\gamma''\cos T)$$

Hae aequationes, statuendo

$$\frac{\alpha}{\delta}=\cos L,\quad \frac{6}{\delta}=\sin L,\quad \frac{\gamma'}{\delta}=\cos M,\quad \frac{\gamma''}{\delta}=\sin M$$

nanciscuntur formam sequentem:

$$\begin{aligned} t(1+\cos(E-L))&=(\gamma+\delta)(1+\cos(T-M))\\ t\sin(E-L)&=\varepsilon\sin(T-M)\end{aligned}$$

unde fit per divisionem, propter $(\gamma+\delta)(\gamma-\delta)=1$,

$$\begin{aligned} \operatorname{tang}\tfrac{1}{2}(E-L)&=\varepsilon(\gamma-\delta)\operatorname{tang}\tfrac{1}{2}(T-M)\\ \operatorname{tang}\tfrac{1}{2}(T-M)&=\varepsilon(\gamma+\delta)\operatorname{tang}\tfrac{1}{2}(E-L)\end{aligned}$$

Hinc non solum eadem conclusio derivatur, ad quam in fine art. praec. deducti sumus, sed insuper etiam patet, si valor ipsius E crescat 360 gradibus, valorem ipsius T tantundem vel crescere vel diminui, prout $\varepsilon\gamma$ sit vel quantitas positiva vel negativa. Ceterum statuendo $\delta=\operatorname{tang}N$, $\gamma=\sec N$, manifesto erit

$$\gamma-\delta=\operatorname{tang}(45^0-\tfrac{1}{2}N),\quad \gamma+\delta=\operatorname{tang}(45^0+\tfrac{1}{2}N)$$

14.

E combinatione aequationum 20, 21, 22 cum aequationibus art. 5 obtinemus:

$$at(A - a\cos E) = \alpha G - \alpha' G'\cos T - \alpha'' G''\sin T$$
$$bt(B - b\sin E) = \beta G - \beta' G'\cos T - \beta'' G''\sin T$$

Statuendo itaque brevitatis gratia

$$(\alpha G - \alpha' G'\cos T - \alpha'' G''\sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = aX$$
$$(\beta G - \beta' G'\cos T - \beta'' G''\sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = bY$$
$$C(\gamma + \gamma'\cos T + \gamma''\sin T)(\gamma - e\alpha + (\gamma' - e\alpha')\cos T + (\gamma'' - e\alpha'')\sin T) = Z$$

fit

$$\mathrm{d}\xi = \frac{\varepsilon X \mathrm{d}T}{2\pi t^3\rho^3}, \quad \mathrm{d}\eta = \frac{\varepsilon Y \mathrm{d}T}{2\pi t^3\rho^3}, \quad \mathrm{d}\zeta = \frac{\varepsilon Z \mathrm{d}T}{2\pi t^3\rho^3}$$

Sed habetur

$$t\rho = \pm\sqrt{(G + G'\cos T^2 + G''\sin T^2)}$$

signo superiore vel inferiore valente, prout t est quantitas positiva vel negativa (ρ enim natura sua semper positive accipitur), i. e. prout coëfficiens γ est positivus vel negativus. Hinc

$$\frac{\varepsilon \mathrm{d}T}{2\pi t^3\rho^3} = \pm\frac{\mathrm{d}T}{2\pi(G + G'\cos T^2 + G''\sin T^2)^{\frac{3}{2}}}$$

ubi signum ambiguum a signo quantitatis $\gamma\varepsilon$ pendet.

Ut iam valores ipsarum ξ, η, ζ obtineamus, integrationes differentialium exsequi oportet, a valore ipsius T, cui respondet $E = 0$, usque ad valorem, cui respondet $E = 360^0$, sive etiam (quod manifesto eodem redit) a valore ipsius T, cui respondet valor arbitrarius ipsius E, usque ad valorem, cui respondet valor ipsius E auctus 360^0; licebit itaque integrare a $T = 0$ usque ad $T = 360^0$, quoties $\varepsilon\gamma$ est quantitas positiva, vel a $T = 360^0$ usque ad $T = 0$, quoties $\varepsilon\gamma$ est negativa. Manifesto itaque, independenter a signo ipsius $\varepsilon\gamma$, erit:

$$\xi = \int\frac{X\mathrm{d}T}{2\pi(G + G'\cos T^2 + G''\sin T^2)^{\frac{3}{2}}}$$
$$\eta = \int\frac{Y\mathrm{d}T}{2\pi(G + G'\cos T^2 + G''\sin T^2)^{\frac{3}{2}}}$$
$$\zeta = \int\frac{Z\mathrm{d}T}{2\pi(G + G'\cos T^2 + G''\sin T^2)^{\frac{3}{2}}}$$

integrationibus a $T = 0$ usque ad $T = 360^0$ extensis.

15.

Nullo negotio perspicitur, integralia

$$\int \frac{\cos T \mathrm{d} T}{(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$
$$\int \frac{\sin T \mathrm{d} T}{(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$
$$\int \frac{\cos T \sin T \mathrm{d} T}{(G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$

a $T = 180^0$ usque ad $T = 360^0$ extensa obtinere valores aequales iis, quos nanciscantur, si a $T = 0$ usque ad $T = 180^0$ extendantur, sed signis oppositis affectos; quapropter ista integralia a $T = 0$ usque ad $T = 360^0$ extensa manifesto fiunt $= 0$. Hinc colligimus, esse

$$\xi = \int \frac{((\gamma - e\alpha)\alpha G - (\gamma' - e\alpha')\alpha' G' \cos T^2 - (\gamma'' - e\alpha'')\alpha'' G'' \sin T^2) \mathrm{d} T}{2\pi a (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$
$$\eta = \int \frac{((\gamma - e\alpha)\beta G - (\gamma' - e\alpha')\beta' G' \cos T^2 - (\gamma'' - e\alpha'')\beta'' G'' \sin T^2) \mathrm{d} T}{2\pi b (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$
$$\zeta = \int \frac{((\gamma - e\alpha)\gamma + (\gamma' - e\alpha')\gamma' \cos T^2 + (\gamma'' - e\alpha'')\gamma'' \sin T^2) C \mathrm{d} T}{2\pi (G + G' \cos T^2 + G'' \sin T^2)^{\frac{3}{2}}}$$

integralibus a $T = 0$ usque ad $T = 360^0$ extensis. Quodsi itaque valores integralium, eadem extensione acceptorum,

$$\int \frac{\cos T^2 \mathrm{d} T}{2\pi((G + G') \cos T^2 + (G + G'') \sin T^2)^{\frac{3}{2}}}$$
$$\int \frac{\sin T^2 \mathrm{d} T}{2\pi((G + G') \cos T^2 + (G + G'') \sin T^2)^{\frac{3}{2}}}$$

per P, Q denotamus, erit

$$a\xi = ((\gamma - e\alpha)\alpha G - (\gamma' - e\alpha')\alpha' G')P + ((\gamma - e\alpha)\alpha G - (\gamma'' - e\alpha'')\alpha'' G'')Q$$
$$b\eta = ((\gamma - e\alpha)\beta G - (\gamma' - e\alpha')\beta' G')P + ((\gamma - e\alpha)\beta G - (\gamma'' - e\alpha'')\beta'' G'')Q$$
$$\zeta = ((\gamma - e\alpha)\gamma + (\gamma' - e\alpha')\gamma') CP + ((\gamma - e\alpha)\gamma + (\gamma'' - e\alpha'')\gamma'') CQ$$

quo pacto problema nostrum complete solutum est.

16.

Quod attinet ad quantitates P, Q, manifesto quidem utraque fit

$$= \frac{1}{2(G+G')^{\frac{3}{2}}}$$

quoties $G' = G''$, in omnibus vero reliquis casibus ad transscendentes sunt referendae. Quas quomodo per series exprimere liceat, abunde constat. Lectoribus autem gratum fore speramus, si hacce occasione determinationem harum aliarumque transscendentium per algorithmum peculiarem expeditissimum explicemus, quo per multos iam abhinc annos frequenter usi sumus, et de quo alio loco copiosius agere propositum est.

Sint m, n duae quantitates positivae, statuamusque

$$m' = \tfrac{1}{2}(m+n), \quad n' = \sqrt{mn}$$

ita ut m', n' resp. sit medium arithmeticum et geometricum inter m et n. Medium geometricum semper positive accipi supponemus. Perinde fiat

$$m'' = \tfrac{1}{2}(m'+n'), \quad n'' = \sqrt{m'n'}$$
$$m''' = \tfrac{1}{2}(m''+n''), \quad n''' = \sqrt{m''n''}$$

et sic porro, quo pacto series m, m', m'', m''' etc., atque n, n', n'', n''' etc. versus *limitem communem* rapidissime convergent, quem per μ designabimus, atque simpliciter *medium arithmetico-geometricum* inter m et n vocabimus. Iam demonstrabimus, $\frac{1}{\mu}$ esse valorem integralis

$$\int \frac{\mathrm{d}T}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}}$$

a $T = 0$ usque ad $T = 360^0$ extensi.

Demonstr. Supponamus, variabilem T ita per aliam T' exprimi, ut fiat

$$\sin T = \frac{2m\sin T'}{(m+n)\cos T'^2 + 2m\sin T'^2}$$

perspicieturque facile, dum T' a valore 0 usque ad 90^0, 180^0, 270^0, 360^0 augeatur, etiam T (etsi inaequalibus intervallis) a 0 usque ad 90^0, 180^0, 270^0, 360^0 crescere. Evolutione autem rite facta, invenitur esse

$$\frac{\mathrm{d}T}{\sqrt{(mm\cos T^2 + nn\sin T^2)}} = \frac{\mathrm{d}T'}{\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)}}$$

adeoque valores integralium

$$\int \frac{\mathrm{d}\,T}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}}, \quad \int \frac{\mathrm{d}\,T'}{2\pi\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)}}$$

si utriusque variabilis a valore 0 usque ad valorem 360^0 extenditur, inter se aequales. Et quum perinde ulterius continuare liceat, patet, his valoribus etiam aequalem esse valorem integralis

$$\int \frac{\mathrm{d}\,\theta}{2\pi\sqrt{(\mu\mu\cos\theta^2 + \mu\mu\sin\theta^2)}}$$

a $\theta = 0$ usque ad $\theta = 360^0$, qui manifesto fit $= \frac{1}{\mu}$. Q. E. D.

17.

Ex aequatione, relationem inter T et T' exhibente,

$$(m-n)\sin T.\sin T'^2 = 2m\sin T' - (m+n)\sin T$$

facile deducitur

$$\sqrt{(mm\cos T^2 + nn\sin T^2)} = m - (m-n)\sin T.\sin T'$$
$$\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)} = m\,\text{cotang}\,T.\text{tang}\,T'$$

atque hinc, adiumento eiusdem aequationis,

$$\sin T.\sin T'.\sqrt{(mm\cos T^2 + nn\sin T^2)} + m'(\cos T^2 - \sin T^2)$$
$$= \cos T.\cos T'.\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)} - \tfrac{1}{2}(m-n)\sin T'^2$$

Multiplicata hac aequatione per

$$\frac{\mathrm{d}\,T}{\sqrt{(mm\cos T^2 + nn\sin T^2)}} = \frac{\mathrm{d}\,T'}{\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)}}$$

prodit

$$\frac{m'(\cos T^2 - \sin T^2)\,\mathrm{d}\,T}{\sqrt{(mm\cos T^2 + nn\sin T^2)}} = -\frac{\frac{1}{2}(m-n)\sin T'^2\,\mathrm{d}\,T'}{\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)}} + \mathrm{d}.\sin T'\cos T$$

Multiplicando hanc aequationem per $\frac{m-n}{\pi}$, substituendo $m'(m-n) = \frac{1}{2}(mm-nn)$, $(m-n)^2 = 4(m'm'-n'n')$, $\sin T'^2 = \frac{1}{2} - \frac{1}{2}(\cos T'^2 - \sin T'^2)$, et integrando, a valoribus T et $T' = 0$ usque ad 360^0, habemus:

$$(mm-nn)\int \frac{(\cos T^2 - \sin T^2).\,\mathrm{d}\,T}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}}$$
$$= -\frac{2(m'm'-n'n')}{\mu} + 2(m'm'-n'n')\int \frac{(\cos T'^2 - \sin T'^2)\,\mathrm{d}\,T'}{2\pi\sqrt{(m'm'\cos T'^2 + n'n'\sin T'^2)}}$$

Et quum integrale definitum ad dextram perinde transformare liceat, manifesto integrale

$$\int \frac{(\cos T^2 - \sin T^2)\,\mathrm{d}\,T}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}}$$

exprimetur per seriem infinitam citissime convergentem

$$-\frac{2(m'm'-n'n') + 4(m''m''-n''n'') + 8(m'''m'''-n'''n''') + \text{etc.}}{(mm-nn)\mu} = -\frac{\nu}{\mu}$$

Calculus numericus commodissime per logarithmos perficitur, si statuimus

$$\tfrac{1}{4}\sqrt{(mm-nn)} = \lambda,\quad \tfrac{1}{4}\sqrt{(m'm'-n'n')} = \lambda',\quad \tfrac{1}{4}\sqrt{(m''m''-n''n'')} = \lambda'' \text{ etc.}$$

unde erit

$$\lambda' = \frac{\lambda\lambda}{m'},\quad \lambda'' = \frac{\lambda'\lambda'}{m''},\quad \lambda''' = \frac{\lambda''\lambda''}{m'''} \text{ etc. atque}$$

$$\nu = \frac{2\lambda'\lambda' + 4\lambda''\lambda'' + 8\lambda'''\lambda''' + \text{etc.}}{\lambda\lambda}$$

18.

Per methodum hic explicatam etiam integralia *indefinita* (a valore variabilis $= 0$ inchoantia) maxima concinnitate assignare licet. Scilicet, si T'' perinde per m', n', T' determinari supponitur, uti T' per m, n, T, ac perinde rursus T''' per m'', n'', T'' et sic porro, etiam pro quovis valore determinato ipsius T, valores terminorum serie T, T', T'', T''' etc. ad limitem θ citissime convergent, eritque

$$\int \frac{\mathrm{d}\,T}{\sqrt{(mm\cos T^2 + nn\sin T^2)}} = \frac{\theta}{\mu}$$

$$\int \frac{(\cos T^2 - \sin T^2)\,\mathrm{d}\,T}{\sqrt{(mm\cos T^2 + nn\sin T^2)}} = -\frac{\nu\theta}{\mu} + \frac{\lambda'\cos T\sin T' + 2\lambda''\cos T'\sin T'' + 4\lambda'''\cos T''\sin T''' + \text{etc.}}{\lambda\lambda}$$

Sed haec obiter hic addigitavisse sufficiat, quum ad institutum nostrum non sint necessaria.

19.

Quodsi iam statuimus $m = \sqrt{(G+G')}$, $n = \sqrt{(G+G'')}$, valores quantitatum P, Q facile ad transscendentes μ, ν reducentur. Quum enim P, Q sint valores integralium

$$\int \frac{\cos T^2 \, \mathrm{d}T}{2\pi(mm\cos T^2 + nn\sin T^2)^{\frac{3}{2}}}, \quad \int \frac{\sin T^2 \, \mathrm{d}T}{2\pi(mm\cos T^2 + nn\sin T^2)^{\frac{3}{2}}}$$

a $T = 0$ usque ad $T = 360^0$ extensorum, primo statim obvium est, haberi

$$mmP + nnQ = \frac{1}{\mu} \; \ldots\ldots\ldots\ldots \; [24]$$

Porro fit

$$\frac{(\cos T^2 - \sin T^2)\,\mathrm{d}T}{2\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}} + \frac{(mm\cos T^2 - nn\sin T^2)\,\mathrm{d}T}{2\pi(mm\cos T^2 + nn\sin T^2)^{\frac{3}{2}}} = \frac{(mm\cos T^4 - nn\sin T^4)\,\mathrm{d}T}{\pi(mm\cos T^2 + nn\sin T^2)^{\frac{3}{2}}}$$

$$= \mathrm{d}.\frac{\cos T\sin T}{\pi\sqrt{(mm\cos T^2 + nn\sin T^2)}}$$

Integrando hanc aequationem a $T = 0$ usque ad $T = 360^0$, prodit

$$-\frac{\nu}{\mu} + mmP - nnQ = 0 \; \ldots\ldots\ldots \; [25]$$

E combinatione aequationum 24, 25 denique colligimus

$$P = \frac{1+\nu}{2mm\mu}, \quad Q = \frac{1-\nu}{2nn\mu}$$

356