ELEGANTIORES INTEGRALIS $\int \frac{\mathrm{d}x}{\sqrt{(1-x^4)}}$ PROPRIETATES.

[1.]

Valorem huius integralis ab $x = 0$ usque ad $x = 1$ semper per $\frac{1}{2}\varpi$ designamus. Variabilem x respectu integralis per signum sin lemn denotamus, respectu vero complementi integralis ad $\frac{1}{2}\varpi$ per cos lemn. Ita ut

$$\sin \operatorname{lemn} \int \frac{\mathrm{d}x}{\sqrt{(1-x^4)}} = x, \qquad \cos \operatorname{lemn} \left(\tfrac{1}{2}\varpi - \int \frac{\mathrm{d}x}{\sqrt{(1-x^4)}}\right) = x$$

Variabilis x tamquam radius vector curvae, integrale autem tamquam curvae arcus respondens considerari potest, curva vero erit ea quam Lemniscatam dixerunt. Haec sufficiunt ad intelligenda quae sequuntur.

[2.]

$$1 = ss + cc + sscc \quad \text{sive} \quad 2 = (1+ss)(1+cc) = \left(\frac{1}{ss} - 1\right)\left(\frac{1}{cc} - 1\right)$$

$$s = \sqrt{\frac{1-cc}{1+cc}}, \qquad c = \sqrt{\frac{1-ss}{1+ss}}$$

$$\sin \operatorname{lemn} (a \pm b) = \frac{sc' \pm s'c}{1 \mp scs'c'}$$

$$\cos \operatorname{lemn} (a \pm b) = \frac{cc' \mp ss'}{1 \pm ss'cc'}$$

$$\sin \operatorname{lemn} (-a) = -\sin \operatorname{lemn} a, \qquad \cos \operatorname{lemn} (-a) = \cos \operatorname{lemn} a$$

$$\sin \operatorname{lemn} k\varpi = 0 \qquad \sin \operatorname{lemn} (k + \tfrac{1}{2})\varpi = \pm 1$$

$$\cos \operatorname{lemn} k\varpi = \pm 1 \qquad \cos \operatorname{lemn} (k + \tfrac{1}{2})\varpi = 0$$

k denotante numerum integrum quemcunque positivum seu negativum, signum superius sumendum quoties k est par, inferius quoties est impar.

[3.]

$$\sin\text{lemn}\,\varphi = s$$

$$\sin\text{lemn}\,2\varphi = sc(1+ss)\frac{2}{1+s^4} = sc(1+cc)\frac{2}{1+c^4}$$

$$\sin\text{lemn}\,3\varphi = s\,\frac{3-6s^4-s^8}{1+6s^4-3s^8}$$

$$\sin\text{lemn}\,4\varphi = 4sc(1+ss)\frac{1-5s^4-5s^8+s^{12}}{1+20s^4-26s^8+20s^{12}+s^{16}}$$

$$\sin\text{lemn}\,5\varphi = s\,.\,\frac{5-2s^4+s^8}{1-2s^4+5s^8}\,.\,\frac{1-12s^4-26s^8+52s^{12}+s^{16}}{1+52s^4-26s^8+12s^{12}+s^{16}}$$

$$\sin\text{lemn}\,n\varphi = s\,.\,\frac{n-\dfrac{n\,.\,nn-1\,.\,nn+6}{60}s^4-\dfrac{n^6-13n^4+36nn+420\,.\,n\,.\,nn+1}{10080}s^8\ldots}{1+\dfrac{n\,.\,n\,.\,nn-1}{12}s^4-\dfrac{nn\,.\,nn-1\,.\,nn-4\,.\,nn+75}{10080}s^8\ldots}$$

$$\cos\text{lemn}\,\varphi = c$$

$$\cos\text{lemn}\,2\varphi = -\frac{1-2cc-c^4}{1+2cc-c^4} = \frac{1-2ss-s^4}{1+2ss+s^4}$$

$$\cos\text{lemn}\,3\varphi = c\,\frac{1-4ss-6s^4-4s^6+s^8}{1+4ss-6s^4+4s^6+s^8}$$

$$\cos\text{lemn}\,4\varphi = \frac{1-8ss-12s^4+8s^6-38s^8-8s^{10}-12s^{12}+8s^{14}+s^{16}}{1+8ss-12s^4-8s^6-38s^8+8s^{10}-12s^{12}-8s^{14}+s^{16}}$$

[4.]

$$\text{arc}\sin\text{lemn}\,x = x+\frac{1}{2}\,.\,\frac{1}{5}x^5+\frac{1\,.\,3}{2\,.\,4}\frac{1}{9}x^9+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{1}{13}x^{13}+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}\frac{1}{17}x^{17}+\ldots$$

$$\sin\text{lemn}\,\varphi = \varphi-\frac{1}{10}\varphi^5+\frac{1}{120}\varphi^9-\frac{11}{15600}\varphi^{13}+\frac{211}{3536000}\varphi^{17}-\frac{1607}{318240000}\varphi^{21}+\ldots$$

$$P\varphi = \varphi-\frac{1}{60}\varphi^5-\frac{1}{10080}\varphi^9+\frac{23}{259459200}\varphi^{13}+\frac{107}{207484333056000}\varphi^{17}+\ldots$$

$$= \varphi-\frac{2}{1.2.3.4.5}\varphi^5-\frac{36}{1\ldots.9}\varphi^9+\frac{552}{1\ldots.13}\varphi^{13}+\frac{5136}{1\ldots.17}\varphi^{17}+\frac{5146848}{1\ldots.21}\varphi^{21}\ldots$$

$$Q\varphi = 1+\frac{1}{12}\varphi^4-\frac{1}{10080}\varphi^8+\frac{17}{19958400}\varphi^{12}+\frac{283}{435891456000}\varphi^{16}\ldots$$

$$= 1+\frac{2}{1.2.3.4}\varphi^4-\frac{4}{1\ldots.8}\varphi^8+\frac{408}{1\ldots.12}\varphi^{12}+\frac{13584}{1\ldots.16}\varphi^{16}\ldots$$

$$\cos \text{lemn}\, \varphi = 1 - \varphi\varphi + \tfrac{1}{2}\varphi^4 - \tfrac{3}{10}\varphi^6 + \tfrac{7}{40}\varphi^8 - \tfrac{61}{600}\varphi^{10} + \tfrac{71}{1200}\varphi^{12} \dots$$

$$\left.\begin{matrix} p\varphi \\ q\varphi \end{matrix}\right\} = 1 \mp \tfrac{1}{2}\varphi\varphi - \tfrac{1}{24}\varphi^4 \mp \tfrac{1}{240}\varphi^6 + \tfrac{17}{40320}\varphi^8 \mp \tfrac{1}{403200}\varphi^{10} + \tfrac{37}{159667200}\varphi^{12}$$
$$\pm \tfrac{113}{4151347200}\varphi^{14} + \tfrac{4171}{6974263296000}\varphi^{16} \dots$$

Formulae pro P, Q, p, q in infinitum continuatae quavis convergentia data citius convergunt, formula autem pro $\sin \text{lemn}\, \varphi$ diverget, si φ ponetur $> \frac{\varpi}{\sqrt{2}}$ sive $\varphi^4 > \frac{\varpi^4}{4}$, formula autem pro $\cos \text{lemn}\, \varphi$ diverget, si $\varphi > \varpi$

Einige neue Formeln die Lemniscatischen Functionen betreffend.

Es sei $\int \frac{dx}{\sqrt{(1-x^4)}} = \varphi$, oder $\varphi = x + \frac{1}{2} \cdot \frac{1}{5} x^5 + \frac{1.3}{2.4} \cdot \frac{1}{9} x^9 + \frac{1.3.5}{2.4.6} \frac{1}{13} x^{13} \dots$

Man hat dann $\varphi\varphi = xx + \frac{3}{5} \cdot \frac{1}{3} x^6 + \frac{3.7}{5.9} \cdot \frac{1}{5} x^{10} + \frac{3.7.11}{5.9.13} \cdot \frac{1}{7} x^{14}.$

Es sei $x = \sin \text{lemn}\, \varphi$, $y = \sin \text{lemn}\, \psi$, $z = \sin \text{lemn}\, (\varphi + \psi)$

so hat man

$$z = \frac{x\sqrt{(1-y^4)} + y\sqrt{(1-x^4)}}{1+xxyy} = \frac{xx-yy}{x\sqrt{(1-y^4)} - y\sqrt{(1-x^4)}}$$

$$\sqrt{\frac{1-zz}{1+zz}} = \frac{-2xy + \sqrt{(1-x^4)}\sqrt{(1-y^4)}}{1+xx+yy-xxyy} = \frac{1-xx-yy-xxyy}{2xy + \sqrt{(1-x^4)}\sqrt{(1-y^4)}}$$

$$\sqrt{\frac{1-z^4}{zz}} = \frac{x(1+y^4)\sqrt{(1-x^4)} - y(1+x^4)\sqrt{(1-y^4)}}{(1+xxyy)(xx-yy)}$$

$$\sqrt{(1-z^4)} = \frac{(1-xxyy)\sqrt{(1-x^4)}\sqrt{(1-y^4)} - 2xy(xx+yy)}{(1+xxyy)^2}$$

$$\sqrt{(1-zz)} = \frac{\sqrt{(1-xx)}\sqrt{(1-yy)} - xy\sqrt{(1+xx)}\sqrt{(1+yy)}}{1+xxyy}$$

$$\sqrt{(1+zz)} = \frac{\sqrt{1+xx)}\sqrt{(1+yy)} + xy\sqrt{(1-xx)}\sqrt{(1-yy)}}{1+xxyy}$$

Die einfachste Manier $\sin \operatorname{lemn} \varphi$ in eine Reihe nach Potenzen von φ zu entwickeln scheint folgende zu sein: man hat

$$\sin \operatorname{lemn}(1+i)\varphi = \frac{(1+i)\sin \operatorname{lemn}\varphi}{\sqrt{(1-\sin \operatorname{lemn}\varphi^4)}}$$

setzt man also

$$(\sin \operatorname{lemn}\varphi)^{-2} = \varphi^{-2}(1+\alpha\varphi^4+\beta\varphi^8+\gamma\varphi^{12}\ldots.)$$

und folglich

$$(\sin \operatorname{lemn}(1+i)\varphi)^{-2} = -\tfrac{1}{2}i\varphi^{-2}(1-4\alpha\varphi^4+16\beta\varphi^8-64\gamma\varphi^{12}\ldots)$$

also

$$\begin{aligned}\sin \operatorname{lemn}\varphi^2 &= -2i(\sin \operatorname{lemn}(1+i)\varphi)^{-2}+(\sin \operatorname{lemn}\varphi)^{-2}\\ &= 5\alpha\varphi\varphi-15\beta\varphi^6+65\gamma\varphi^{10}-255\delta\varphi^{14}\ldots\end{aligned}$$

man hat also

$$(1+\alpha t+\beta t^2+\gamma t^3+\delta t^4+\ldots)(5\alpha-15\beta t+65\gamma t^2-255\delta t^3+1025\varepsilon t^4\ldots)=1$$

hieraus

$$\begin{array}{llll}
5\alpha = 1 & \alpha = \frac{1}{5} & = \frac{1}{5} & \\
3\beta = \alpha\alpha & \beta = \frac{1}{75} & = \frac{1}{3.25} & = \frac{1}{15}\alpha \\
13\gamma = 2\alpha\beta & \gamma = \frac{2}{4875} & = \frac{2}{3.5^3.13} & = \frac{2}{65}\beta \\
51\delta = 14\alpha\gamma - 3\beta\beta & \delta = \frac{1}{82875} & = \frac{1}{3.5^3.13.17} & = \frac{1}{34}\gamma \\
205\varepsilon = 50\alpha\delta - 10\beta\gamma & \varepsilon = \frac{2}{6215625} & = \frac{2}{9.5^5.13.17} & = \frac{2}{75}\delta \\
819\zeta = 206\alpha\varepsilon - 54\beta\delta + 13\gamma\gamma & \zeta = \frac{2}{242409375} & = \frac{2}{3^3.5^5.13^2.17} & = \frac{1}{39}\varepsilon \\
3277\eta = 818\alpha\zeta - 202\beta\varepsilon + 38\gamma\delta & \eta = \frac{4}{19527421875} & = \frac{4}{3.5^7.13^2.17.29} & = \frac{18}{725}\zeta \\
13107\theta = 3278\alpha\eta - 822\beta\zeta + 218\gamma\varepsilon - 51\delta\delta, & \theta = \frac{223}{44815433203125} & = \frac{223}{3^4.5^8.13^2.17^2.29} & = \frac{223}{9180}\eta
\end{array}$$

Die Grenze des Verhältnisses zweier aufeinander folgender Glieder ist

$$(2{,}6220\ldots)^4 : 1 = 47{,}27 : 1$$

Sehr nahe ist

$$\zeta = \frac{92}{(2\varpi)^{24}}, \qquad \eta = \frac{108}{(2\varpi)^{28}}, \qquad \theta = \frac{124}{(2\varpi)^{32}}$$

$$\log P = \log\varphi - \tfrac{1}{12}\alpha\varphi^4 - \tfrac{1}{56}\beta\varphi^8 - \tfrac{1}{132}\gamma\varphi^{12} - \tfrac{1}{240}\delta\varphi^{16} - \tfrac{1}{380}\varepsilon\varphi^{20} - \tfrac{1}{552}\zeta\varphi^{24}\ldots$$

$$= \log\varphi - \tfrac{1}{60}\varphi^4 - \tfrac{1}{4200}\varphi^8 - \tfrac{1}{321750}\varphi^{12} - \tfrac{1}{19890000}\varphi^{16} - \tfrac{1}{1180968750}\varphi^{20}$$

$$- \tfrac{22}{1756255921875}\varphi^{24} - \ldots$$

$$\log\sin\operatorname{lemn}\varphi = \log\varphi - \tfrac{3}{6}\alpha\varphi^4 + \tfrac{7}{28}\beta\varphi^8 - \tfrac{33}{66}\gamma\varphi^{12} + \tfrac{127}{120}\delta\varphi^{16} - \tfrac{513}{190}\varepsilon\varphi^{20} + \tfrac{2047}{276}\zeta\varphi^{24}\ldots$$

$$= \log\varphi - \tfrac{1}{10}\varphi^4 + \tfrac{1}{300}\varphi^8 - \tfrac{1}{4875}\varphi^{12} + \tfrac{127}{9945000}\varphi^{16} - \tfrac{3}{3453125}\varphi^{20}$$

$$+ \tfrac{89}{1454456250}\varphi^{24} - \ldots$$

Die Coëfficienten α, β, γ u. s. f. lassen sich auch vermittelst folgender Gleichung bestimmen

$$1 + \alpha t + \beta tt + \gamma t^3 + \ldots = \left(1 + \tfrac{3}{2.3}\alpha t + \tfrac{7}{8.7}\beta tt + \tfrac{33}{32.11}\gamma t^3 + \tfrac{127}{128.15}\delta t^4\ldots\right)^2$$

Die bequemste Art $\log\cos\operatorname{lemn}\varphi$ in eine Reihe zu entwickeln ist folgende. Es ist

$$\frac{d\log\cos\operatorname{lemn}\varphi}{d\varphi} = -\frac{2\,d\varphi}{d\log\sin\operatorname{lemn}\varphi} = -(1-i)\sin\operatorname{lemn}(1+i)\varphi$$

hieraus ergibt sich

$$\log\cos\operatorname{lemn}\varphi = -\varphi\varphi - \tfrac{2}{15}\varphi^6 - \tfrac{2}{75}\varphi^{10} - \tfrac{44}{6825}\varphi^{14} - \tfrac{422}{248625}\varphi^{18} - \tfrac{6428}{13673375}\varphi^{22}$$

$$- \tfrac{6044}{44890625}\varphi^{26} - \tfrac{20824792}{527240390625}\varphi^{30} - \ldots$$

$$P\frac{d^4P}{d\varphi^4} - 4\frac{dP}{d\varphi}\cdot\frac{d^3P}{d\varphi^3} + 3\left(\frac{ddP}{d\varphi^2}\right)^2 = 2PP$$

$$P\varphi = \varphi - \tfrac{2}{1\ldots.5}\varphi^5 - \tfrac{36}{1\ldots.9}\varphi^9 + \tfrac{552}{1\ldots.13}\varphi^{13} + \tfrac{5136}{1\ldots.17}\varphi^{17} + \tfrac{5146848}{1\ldots.21}\varphi^{21} - \ldots$$

$$= \varphi - \tfrac{1}{4.3.5}\varphi^5 - \tfrac{1}{32.9.5.7}\varphi^9 + \tfrac{23}{128.81.25.7.11.13}\varphi^{13} + \tfrac{107}{2048.243.125.49.11.13.17}\varphi^{17}$$

$$+ \tfrac{23.37}{8192.729.625.49.11.13.17.19}\varphi^{21} - \ldots$$

$$
\begin{aligned}
P\psi\varpi = & + 2.6220575\,\psi \\
& - 2.0656648\,\psi^5 \\
& - 0.5811918\,\psi^9 \\
& + 0.0245475\,\psi^{13} \\
& + 0.0001890\,\psi^{17} \\
& + 0.0000620\,\psi^{21}
\end{aligned}
$$

$$P\tfrac{1}{2}\varpi = +1.2453446 = \sqrt[4]{\tfrac{1}{2}}\,.\,e^{\frac{1}{8}\pi}$$

$$\frac{\mathrm{d}\,\mathrm{d}\log Q}{\mathrm{d}\varphi^2} = \frac{PP}{QQ}$$

Die Halbirung geschieht sehr bequem so

$$
\begin{aligned}
\cos\operatorname{lemn}\varphi &= \operatorname{tang} u \\
\sin\operatorname{lemn}\varphi &= \sqrt{\cos 2u} \\
\cos\operatorname{lemn}\varphi &= \sqrt{\operatorname{tang}\tfrac{1}{2}(45^0+u)} \\
\sin\operatorname{lemn}\varphi &= \sqrt{\operatorname{tang}\tfrac{1}{2}(45^0-u)}
\end{aligned}
$$

$$
\begin{aligned}
\sin\operatorname{lemn}(\varphi+i\psi) &= r(\cos v + i\sin v) \\
\sin\operatorname{lemn}\varphi^2 &= \operatorname{tang}\Phi \\
\sin\operatorname{lemn}\psi^2 &= \operatorname{tang}\Psi \\
r &= \sqrt{\operatorname{tang}(\Phi+\Psi)} \\
\operatorname{tang} v &= \sqrt{\frac{\operatorname{tang}2\Psi}{\operatorname{tang}2\Phi}}
\end{aligned}
$$

$$
\begin{aligned}
\sin\operatorname{lemn}(\varphi+\psi) &= \frac{\sqrt{(\cos 2\Phi\sin 2\Psi)}+\sqrt{(\sin 2\Phi\cos 2\Psi)}}{\cos(\Phi-\Psi)\,.\sqrt{2}} \\
&= \frac{\sin(\Psi-\Phi)\,.\sqrt{2}}{\sqrt{(\cos 2\Phi\sin 2\Psi)}-\sqrt{(\sin 2\Phi\cos 2\Psi)}}
\end{aligned}
$$

Es sei

$$\sin\operatorname{lemn} A = x, \qquad \sin\operatorname{lemn} B = y, \qquad \sin\operatorname{lemn}\frac{A+B}{1+i} = z$$

so ist

$$
\begin{aligned}
(1+i)z &= \frac{\sqrt{(1+xx)(1-yy)}-\sqrt{(1-xx)(1+yy)}}{x-y} \\
\frac{1-i}{z} &= \frac{\sqrt{1+xx)(1-yy)}+\sqrt{(1-xx)(1+yy)}}{x+y}
\end{aligned}
$$

$$\frac{\sin \operatorname{lemn} \frac{A-B}{1+i}}{\sin \operatorname{lemn} \frac{A+B}{1+i}} = \frac{\sin \operatorname{lemn} A - \sin \operatorname{lemn} B}{\sin \operatorname{lemn} A + \sin \operatorname{lemn} B}$$

$$pp = QQ - PP, \qquad qq = QQ + PP$$

$$P(a-b).Q(a+b) = Pa.Qa.\sqrt{(Qb^4 - Pb^4)} - Pb.Qb.\sqrt{(Qa^4 - Pa^4)}$$
$$P(a-b).P(a+b) = Pa^2.Qb^2 - Qa^2.Pb^2$$
$$Q(a-b).Q(a+b) = Pa^2.Pb^2 + Qa^2.Qb^2$$

$$q(a-b).q(a+b) = pa^2.Pb^2 + qa^2.Qb^2$$
$$= pb^2.Pa^2 + qb^2.Qa^2$$

$$q(a-b).Q(a+b) = qa.qb.Qa.Qb - pa.pb:Pa.Pb$$
$$q(a-b).P(a+b) = qa.pb.Pa.Qb + pa.qb.Qa.Pb$$

$$P\varphi = P$$
$$Q\varphi = Q$$
$$p\varphi = \sqrt{(QQ - PP)}$$
$$q\varphi = \sqrt{(QQ + PP)}$$

$$P2\varphi = 2PQ\sqrt{(Q^4 - P^4)}$$
$$Q2\varphi = Q^4 + P^4$$
$$p2\varphi = Q^4 - 2QQPP - P^4$$
$$q2\varphi = Q^4 + 2QQPP + P^4$$

$$P3\varphi = 3Q^8P - 6Q^4P^5 - P^9$$
$$Q3\varphi = Q^9 + 6Q^5P^4 - 3QP^8$$
$$p3\varphi = \sqrt{(QQ - PP)}.(Q^8 - 4Q^6PP - 6Q^4P^4 - 4QQP^6 + P^8)$$
$$q3\varphi = \sqrt{(QQ + PP)}.(Q^4 + 4Q^6PP - 6Q^4P^4 + 4QQP^6 + P^8)$$

$$P4\varphi = 4PQ\sqrt{(Q^4 - P^4)}.(Q^{12} - 5Q^8P^4 - 5Q^4P^8 + P^{12})$$
$$Q4\varphi = Q^{16} + 20Q^{12}P^4 - 26Q^8P^8 + 20Q^4P^{12} + P^{16}$$
$$p4\varphi = Q^{16} - 8Q^{14}PP - 12Q^{12}P^4 - 8Q^{10}P^6 + 38Q^8P^8 + 8Q^6P^{10} - 12Q^4P^{12} + 8QQP^4 + P^{16}$$
$$q4\varphi = Q^{16} + 8Q^{14}PP - 12Q^{12}P^4 + 8Q^{10}P^6 + 38Q^8P^8 - 8Q^6P^{10} - 12Q^4P^{12} - 8QQP^4 + P^{16}$$

$$\frac{Qn\varphi}{Q\varphi^{nn}} = 1 + \tfrac{1}{12}(n^4 - nn)\left(\frac{P\varphi}{Q\varphi}\right)^4 - \tfrac{1}{10080}(n^8 + 70n^6 - 371n^4 + 300nn)\left(\frac{P\varphi}{Q\varphi}\right)^8$$
$$+ \tfrac{1}{19958400}(17n^{12} + 165n^{10} + 4191n^8 - 106865n^6 + 426492n^4 - 324000nn).\left(\frac{P\varphi}{Q\varphi}\right)^{12}$$
$$= 1 + \tfrac{1}{12}nn(n-1)\left(\frac{P\varphi}{Q\varphi}\right)^4 - \tfrac{1}{10080}nn(nn-1)(nn-4)(nn+75)\left(\frac{P\varphi}{Q\varphi}\right)^8$$
$$+ \tfrac{1}{19958400}nn(nn-1)(nn-4)(nn-9)(17n^4 + 403nn + 9000).\left(\frac{P\varphi}{Q\varphi}\right)^{12}$$

$$Pi\varphi = iP\varphi$$
$$Qi\varphi = Q\varphi$$
$$pi\varphi = q\varphi$$
$$qi\varphi = p\varphi$$

$$P(1+i)\varphi = (1+i)PQ$$
$$Q(1+i)\varphi = \sqrt{(Q^4 - P^4)} = pq$$
$$p(1+i)\varphi = QQ - iPP$$
$$q(1+i)\varphi = QQ + iPP$$

$$P(2+i)\varphi = (2+i)PQ^4 - iP^5$$
$$Q(2+i)\varphi = Q^5 - (1-2i)QP^4$$
$$p(2+i)\varphi = \sqrt{(QQ+PP)}.\{Q^4 - (2+2i)QQPP + P^4\}$$
$$q(2+i)\varphi = \sqrt{(QQ-PP)}.\{Q^4 + (2+2i)QQPP + P^4\}$$

$$P(3+i)\varphi = (3+i)PQ^3 - (2+6i)P^5Q^5 + (3+i)P^9Q$$
$$Q(3+i)\varphi = \sqrt{(Q^4 - P^4)}.\{Q^8 + (2+8i)P^4Q^4 + P^8\}$$
$$p(3+i)\varphi = Q^{10} - (4-3i)Q^8PP - (2-4i)Q^6P^4 + (4+2i)Q^4P^6 - (3+4i)QQP^8 - iP^{10}$$
$$q(3+i)\varphi = Q^{10} + (4-3i)Q^8PP - (2-4i)Q^6P^4 - (4+2i)Q^4P^6 - (3+4i)QQP^8 + iP^{10}$$

$$P(1+2i)\varphi = (1+2i)PQ^4 - P^5$$
$$Q(1+2i)\varphi = Q^5 - (1+2i)QP^4$$
$$P(1+3i)\varphi = (1+3i)PQ^9 - (6+2i)P^5Q^5 + (1+3i)P^9Q$$
$$Q(1+3i)\varphi = Q^{10} \;..$$
$$P(1+4i)\varphi = (1+4i)PQ^{16} - (20+12i)P^5Q^{12} - (10-28i)P^9Q^8 + (12-20i)P^{13}Q^4 + P^{17}$$

$$Q(1+4i)\varphi =$$

$$P(1+ni)\varphi = (1+in)PQ^{nn} - \left(\tfrac{nn.nn-1}{12} + \tfrac{n.nn-1.nn-4}{60}\,i\right)P^5 Q^{nn-4} +$$
$$= (1+in)PQ^{nn} - \frac{n.nn-1.1+in.n-4i}{60}P^5 Q^{nn-4}$$

Unter den Zahlen $y = x$, $2x+1$, $2x+i$, $2x-1$, $2x-i$ ist immer wenigstens eine (oder drei oder alle), die die Auflösung der Congruenz $1-y^4 \equiv zz$ nach irgend einem Modulus möglich macht.

$$\sin\text{lemn}\, X = x, \qquad \sin\text{lemn}\, Y = y, \qquad \sin\text{lemn}\, Z = z$$

$$\sin\text{lemn}(X+Y+Z) = \frac{\begin{array}{r}-2xyz(xx+yy+zz-xxyyzz) + x\sqrt{(1-y^4)}.\sqrt{(1-z^4)}.(1-yyzz+xxzz+xxyy)\\ +y\sqrt{(1-x^4)}.\sqrt{(1-z^4)}.(1+yyzz-xxzz+xxyy)\\ +z\sqrt{(1-x^4)}.\sqrt{(1-y^4)}.(1+yyzz+xxzz-xxyy)\end{array}}{1+2yyzz+2xxzz+2xxyy+y^4z^4+x^4z^4+x^4y^4-2x^4yyzz-2xxy^4zz-2xxyyz^4}$$

Ist $\quad \varphi+\varphi'+\varphi'' = 0, \quad \sin\text{lemn}\,\varphi = x \quad \cos\text{lemn}\,\varphi = y$

$\sin\text{lemn}\,\varphi' = x' \quad \cos\text{lemn}\,\varphi' = y'$

$\sin\text{lemn}\,\varphi'' = x'' \quad \cos\text{lemn}\,\varphi' = y''$

so ist

$$x(y-y'y'') = x'(y'-yy'') = x''(y''-yy'), \qquad y'' = \frac{xy-x'y'}{xy'-x'y}$$

Q		
$-1+2i$	5	$1+(-1+2i)x^4$
-3	9	$1+6x^4-3x^8$
$+3+2i$	13	$1+(-11+10i)x^4+(7-4i)x^8+(3+2i)x^{12}$
$+1+4i$	17	$1+(+12-20i)x^4+(-10+28i)x^8-(20+12i)x^{12}+(1+4i)x^{16}$
5	25	$1+50x^4-125x^8+300x^{12}-105x^{16}$ $-62x^{20}+5x^{24}$

für $\quad x^4 = +1 \quad$ und $\quad x^4 = -1$

werden diese Functionen respective den Quadraten und Würfeln

von	$1-i$	-2	$-2-2i$	$-4i$	$+8$
gleich für $M =$	5	9	13	17	25

$$Q(a+bi)\varphi = Q^{aa+bb} + \tfrac{1}{12}\{(a+bi)^4 - (aa+bb)\}Q^{aa+bb-4}P^4$$
$$-\tfrac{1}{10080}\{(a+bi)^8 + 70(a+bi)^4(aa+bb) - 35(aa+bb)^2 - 336(a+bi)^4 + 300(aa+bb)\}.$$
$$.\,Q^{aa+bb-8}P^8 \ldots$$