

## Werk

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## Übergeordnetes Werk

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ELEGANTIORES INTEGRALIS  $\int \frac{dx}{\sqrt{(1-x^2)}}$  PROPRIETATES.

---

[1.]

Valorem huius integralis ab  $x = 0$  usque ad  $x = 1$  semper per  $\frac{1}{2}\pi$  designamus. Variabilem  $x$  respectu integralis per signum sin lemn denotamus, respectu vero complementi integralis ad  $\frac{1}{2}\pi$  per cos lemn. Ita ut

$$\sin \text{lemn} \int \frac{dx}{\sqrt{(1-x^2)}} = x, \quad \cos \text{lemn} \left( \frac{1}{2}\pi - \int \frac{dx}{\sqrt{(1-x^2)}} \right) = x$$

Variabilis  $x$  tamquam radius vector curvae, integrale autem tamquam curvae arcus respondens considerari potest, curva vero erit ea quam Lemniscatam dixerunt. Haec sufficiunt ad intelligenda quae sequuntur.

[2.]

$$1 = ss + cc + sscc \quad \text{sive} \quad 2 = (1+ss)(1+cc) = \left(\frac{1}{ss} - 1\right)\left(\frac{1}{cc} - 1\right)$$

$$s = \sqrt{\frac{1-cc}{1+cc}}, \quad c = \sqrt{\frac{1-ss}{1+ss}}$$

$$\sin \text{lemn}(a \pm b) = \frac{s'c' \pm s'c}{1 \mp ss'c'}$$

$$\cos \text{lemn}(a \pm b) = \frac{cc' \mp ss'}{1 \pm ss'c'}$$

$$\sin \text{lemn}(-a) = -\sin \text{lemn} a, \quad \cos \text{lemn}(-a) = \cos \text{lemn} a$$

$$\sin \text{lemn} k\pi = 0 \quad \sin \text{lemn} \left(k + \frac{1}{2}\right)\pi = \pm 1$$

$$\cos \text{lemn} k\pi = \pm 1 \quad \cos \text{lemn} \left(k + \frac{1}{2}\right)\pi = 0$$

$k$  denotante numerum integrum quemcunque positivum seu negativum, signum superius sumendum quoties  $k$  est par, inferius quoties est impar.

[3.]

$$\sin \text{lemn } \varphi = s$$

$$\sin \text{lemn } 2\varphi = sc(1+ss)\frac{2}{1+s^4} = sc(1+cc)\frac{2}{1+c^4}$$

$$\sin \text{lemn } 3\varphi = s \frac{3-6s^4-s^8}{1+6s^4-3s^8}$$

$$\sin \text{lemn } 4\varphi = 4sc(1+ss)\frac{1-5s^4-5s^8+s^{12}}{1+20s^4-26s^8+20s^{12}+s^{16}}$$

$$\sin \text{lemn } 5\varphi = s \cdot \frac{5-2s^4+s^8}{1-2s^4+5s^8} \cdot \frac{1-12s^4-26s^8+52s^{12}+s^{16}}{1+52s^4-26s^8+12s^{12}+s^{16}}$$

$$\sin \text{lemn } n\varphi = s \cdot \frac{n - \frac{n \cdot n \cdot n - 1 \cdot n \cdot n + 6}{60} s^4 - \frac{n^6 - 13n^4 + 36nn + 420 \cdot n \cdot n \cdot n + 1}{10080} s^8 \dots}{1 + \frac{n \cdot n \cdot n - 1}{12} s^4 - \frac{n \cdot n \cdot n - 1 \cdot n \cdot n - 4 \cdot n \cdot n + 75}{10080} s^8 \dots}$$

$$\cos \text{lemn } \varphi = c$$

$$\cos \text{lemn } 2\varphi = -\frac{1-2cc-c^4}{1+2cc-c^4} = \frac{1-2ss-s^4}{1+2ss+s^4}$$

$$\cos \text{lemn } 3\varphi = c \frac{1-4ss-6s^4-4s^8+s^{12}}{1+4ss-6s^4+4s^8+s^{12}}$$

$$\cos \text{lemn } 4\varphi = \frac{1-8ss-12s^4+8s^8-38s^{12}-8s^{16}-12s^{20}+8s^{24}+s^{28}}{1+8ss-12s^4-8s^8-38s^{12}+8s^{16}-12s^{20}-8s^{24}+s^{28}}$$

[4.]

$$\arcsin \text{lemn } x = x + \frac{1}{2} \cdot \frac{1}{5} x^5 + \frac{1 \cdot 3 \cdot 1}{2 \cdot 4 \cdot 9} x^9 + \frac{1 \cdot 3 \cdot 5 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 13} x^{13} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 17} x^{17} + \dots$$

$$\sin \text{lemn } \varphi = \varphi - \frac{1}{10} \varphi^5 + \frac{1}{120} \varphi^9 - \frac{11}{15600} \varphi^{13} + \frac{211}{3536000} \varphi^{17} - \frac{1607}{318240000} \varphi^{21} + \dots$$

$$P\varphi = \varphi - \frac{1}{60} \varphi^5 - \frac{1}{10080} \varphi^9 + \frac{23}{259459200} \varphi^{13} + \frac{107}{207484333056000} \varphi^{17} + \dots$$

$$= \varphi - \frac{2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \varphi^5 - \frac{36}{1 \cdot \dots \cdot 9} \varphi^9 + \frac{552}{1 \cdot \dots \cdot 13} \varphi^{13} + \frac{5136}{1 \cdot \dots \cdot 17} \varphi^{17} + \frac{5146848}{1 \cdot \dots \cdot 21} \varphi^{21} \dots$$

$$Q\varphi = 1 + \frac{1}{12} \varphi^4 - \frac{1}{10080} \varphi^8 + \frac{17}{19958400} \varphi^{12} + \frac{283}{435891456000} \varphi^{16} \dots$$

$$= 1 + \frac{2}{1 \cdot 2 \cdot 3 \cdot 4} \varphi^4 - \frac{4}{1 \cdot \dots \cdot 8} \varphi^8 + \frac{408}{1 \cdot \dots \cdot 12} \varphi^{12} + \frac{13584}{1 \cdot \dots \cdot 16} \varphi^{16} \dots$$

$$\cos \text{lemn } \varphi = 1 - \varphi\varphi + \frac{1}{2}\varphi^4 - \frac{3}{10}\varphi^6 + \frac{7}{40}\varphi^8 - \frac{61}{600}\varphi^{10} + \frac{71}{1200}\varphi^{12} \dots$$

$$\left. \begin{matrix} p \\ q \end{matrix} \varphi \right\} = 1 \mp \frac{1}{2}\varphi\varphi - \frac{1}{24}\varphi^4 \mp \frac{1}{240}\varphi^6 + \frac{17}{40320}\varphi^8 \mp \frac{1}{403200}\varphi^{10} + \frac{37}{159667200}\varphi^{12} \\ \pm \frac{113}{4151347200}\varphi^{14} + \frac{4171}{6974263296000}\varphi^{16} \dots$$

Formulae pro  $P, Q, p, q$  in infinitum continuatae quavis convergentia data citius convergunt, formula autem pro  $\sin \text{lemn } \varphi$  diverget, si  $\varphi$  ponetur  $> \frac{\pi}{\sqrt{2}}$  sive  $\varphi^4 > \frac{\pi^4}{4}$ , formula autem pro  $\cos \text{lemn } \varphi$  diverget, si  $\varphi > \pi$

*Einige neue Formeln die Lemniscatischen Functionen betreffend.*

$$\text{Es sei } \int \frac{dx}{\sqrt{(1-x^4)}} = \varphi, \text{ oder } \varphi = x + \frac{1}{2} \cdot \frac{1}{5} x^5 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{9} x^9 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{13} x^{13} \dots$$

$$\text{Man hat dann} \quad \varphi\varphi = xx + \frac{3}{5} \cdot \frac{1}{3} x^6 + \frac{3 \cdot 7}{5 \cdot 9} \cdot \frac{1}{5} x^{10} + \frac{3 \cdot 7 \cdot 11}{5 \cdot 9 \cdot 13} \cdot \frac{1}{7} x^{14}.$$

$$\text{Es sei} \quad x = \sin \text{lemn } \varphi, \quad y = \sin \text{lemn } \psi, \quad z = \sin \text{lemn } (\varphi + \psi)$$

so hat man

$$z = \frac{x\sqrt{(1-y^4)} + y\sqrt{(1-x^4)}}{1+xyy} = \frac{xx-yy}{x\sqrt{(1-y^4)} - y\sqrt{(1-x^4)}}$$

$$\sqrt{\frac{1-zz}{1+zz}} = \frac{-2xy + \sqrt{(1-x^4)}\sqrt{(1-y^4)}}{1+xx+yy-xyy} = \frac{1-xx-yy-xyy}{2xy + \sqrt{(1-x^4)}\sqrt{(1-y^4)}}$$

$$\sqrt{\frac{1-z^4}{zz}} = \frac{x(1+y^4)\sqrt{(1-x^4)} - y(1+x^4)\sqrt{(1-y^4)}}{(1+xyy)(xx-yy)}$$

$$\sqrt{(1-z^4)} = \frac{(1-xyy)\sqrt{(1-x^4)}\sqrt{(1-y^4)} - 2xy(xx+yy)}{(1+xyy)^2}$$

$$\sqrt{(1-zz)} = \frac{\sqrt{(1-xx)}\sqrt{(1-yy)} - xy\sqrt{(1+xx)}\sqrt{(1+yy)}}{1+xyy}$$

$$\sqrt{(1+zz)} = \frac{\sqrt{(1+xx)}\sqrt{(1+yy)} + xy\sqrt{(1-xx)}\sqrt{(1-yy)}}{1+xyy}$$

Die einfachste Manier  $\sin \operatorname{lemn} \varphi$  in eine Reihe nach Potenzen von  $\varphi$  zu entwickeln scheint folgende zu sein: man hat

$$\sin \operatorname{lemn} (1+i) \varphi = \frac{(1+i) \sin \operatorname{lemn} \varphi}{\sqrt{1-\sin \operatorname{lemn} \varphi^4}}$$

setzt man also

$$(\sin \operatorname{lemn} \varphi)^{-2} = \varphi^{-2} (1 + \alpha \varphi^4 + \beta \varphi^8 + \gamma \varphi^{12} \dots)$$

und folglich

$$(\sin \operatorname{lemn} (1+i) \varphi)^{-2} = -\frac{1}{2} i \varphi^{-2} (1 - 4 \alpha \varphi^4 + 16 \beta \varphi^8 - 64 \gamma \varphi^{12} \dots)$$

also

$$\begin{aligned} \sin \operatorname{lemn} \varphi^2 &= -2i (\sin \operatorname{lemn} (1+i) \varphi)^{-2} + (\sin \operatorname{lemn} \varphi)^{-2} \\ &= 5 \alpha \varphi^2 - 15 \beta \varphi^6 + 65 \gamma \varphi^{10} - 255 \delta \varphi^{14} \dots \end{aligned}$$

man hat also

$$(1 + \alpha t + \beta t^2 + \gamma t^3 + \delta t^4 + \dots) (5 \alpha - 15 \beta t + 65 \gamma t^2 - 255 \delta t^3 + 1025 \epsilon t^4 \dots) = 1$$

hieraus

|                                                                                                        |                                |                                      |                           |
|--------------------------------------------------------------------------------------------------------|--------------------------------|--------------------------------------|---------------------------|
| $5 \alpha = 1$                                                                                         | $\alpha = \frac{1}{5}$         | $= \frac{1}{5}$                      |                           |
| $3 \beta = \alpha \alpha$                                                                              | $\beta = \frac{1}{75}$         | $= \frac{1}{3.25}$                   | $= \frac{1}{15} \alpha$   |
| $13 \gamma = 2 \alpha \beta$                                                                           | $\gamma = \frac{2}{4875}$      | $= \frac{2}{3.5^2.13}$               | $= \frac{2}{65} \beta$    |
| $51 \delta = 14 \alpha \gamma - 3 \beta \beta$                                                         | $\delta = \frac{1}{82875}$     | $= \frac{1}{3.5^3.13.17}$            | $= \frac{1}{34} \gamma$   |
| $205 \epsilon = 50 \alpha \delta - 10 \beta \gamma$                                                    | $\epsilon = \frac{2}{6215625}$ | $= \frac{2}{9.5^3.3.17}$             | $= \frac{2}{75} \delta$   |
| $819 \zeta = 206 \alpha \epsilon - 54 \beta \delta + 13 \gamma \gamma$                                 | $\zeta = \frac{2}{242409375}$  | $= \frac{2}{3^2.5^3.13^2.17}$        | $= \frac{1}{39} \epsilon$ |
| $3277 \eta = 818 \alpha \zeta - 202 \beta \epsilon + 38 \gamma \delta$                                 | $\eta = \frac{4}{19527421875}$ | $= \frac{4}{3.5^7.13^3.17.29}$       | $= \frac{18}{725} \zeta$  |
| $13107 \theta = 3278 \alpha \eta - 822 \beta \zeta + 218 \gamma \epsilon - 51 \delta \delta, \theta =$ | $\frac{223}{44815433203125}$   | $= \frac{223}{3^4.5^8.13^3.17^2.29}$ | $= \frac{223}{9180} \eta$ |

Die Grenze des Verhältnisses zweier aufeinander folgender Glieder ist

$$(2,6220 \dots)^4 : 1 = 47,27 : 1$$

Sehr nahe ist

$$\zeta = \frac{92}{(2\pi)^{24}}, \quad \eta = \frac{108}{(2\pi)^{28}}, \quad \theta = \frac{124}{(2\pi)^{32}}$$

$$\begin{aligned} \log P &= \log \varphi - \frac{1}{12} \alpha \varphi^4 - \frac{1}{56} \beta \varphi^8 - \frac{1}{132} \gamma \varphi^{12} - \frac{1}{240} \delta \varphi^{16} - \frac{1}{380} \epsilon \varphi^{20} - \frac{1}{552} \zeta \varphi^{24} \dots \\ &= \log \varphi - \frac{1}{60} \varphi^4 - \frac{1}{4200} \varphi^8 - \frac{1}{321750} \varphi^{12} - \frac{1}{19890000} \varphi^{16} - \frac{1}{1180968750} \varphi^{20} \\ &\quad - \frac{22}{1756255921875} \varphi^{24} - \dots \end{aligned}$$

$$\begin{aligned} \log \sin \text{lemn} \varphi &= \log \varphi - \frac{3}{6} \alpha \varphi^4 + \frac{7}{28} \beta \varphi^8 - \frac{33}{66} \gamma \varphi^{12} + \frac{127}{120} \delta \varphi^{16} - \frac{513}{190} \epsilon \varphi^{20} + \frac{2047}{276} \zeta \varphi^{24} \dots \\ &= \log \varphi - \frac{1}{10} \varphi^4 + \frac{1}{300} \varphi^8 - \frac{1}{4875} \varphi^{12} + \frac{127}{9945000} \varphi^{16} - \frac{3}{3453125} \varphi^{20} \\ &\quad + \frac{89}{1454456250} \varphi^{24} - \dots \end{aligned}$$

Die Coëfficienten  $\alpha$ ,  $\beta$ ,  $\gamma$  u. s. f. lassen sich auch mittelst folgender Gleichung bestimmen

$$1 + \alpha t + \beta t t + \gamma t^3 + \dots = (1 + \frac{3}{2.3} \alpha t + \frac{7}{8.7} \beta t t + \frac{33}{32.11} \gamma t^3 + \frac{127}{128.15} \delta t^4 \dots)^2$$

Die bequemste Art  $\log \cos \text{lemn} \varphi$  in eine Reihe zu entwickeln ist folgende. Es ist

$$\frac{d \log \cos \text{lemn} \varphi}{d \varphi} = - \frac{2 d \varphi'}{d \log \sin \text{lemn} \varphi} = -(1-i) \sin \text{lemn} (1+i) \varphi$$

hieraus ergibt sich

$$\begin{aligned} \log \cos \text{lemn} \varphi &= -\varphi \varphi - \frac{2}{15} \varphi^6 - \frac{2}{75} \varphi^{10} - \frac{44}{6825} \varphi^{14} - \frac{422}{248625} \varphi^{18} - \frac{6428}{13673375} \varphi^{22} \\ &\quad - \frac{6044}{44890625} \varphi^{26} - \frac{20824792}{527240390625} \varphi^{30} - \dots \end{aligned}$$

$$P \frac{d^4 P}{d \varphi^4} - 4 \frac{d P}{d \varphi} \cdot \frac{d^2 P}{d \varphi^2} + 3 \left( \frac{d d P}{d \varphi^2} \right)^2 = 2 P P$$

$$\begin{aligned} P \varphi &= \varphi - \frac{2}{1 \dots 5} \varphi^5 - \frac{36}{1 \dots 9} \varphi^9 + \frac{552}{1 \dots 13} \varphi^{13} + \frac{5136}{1 \dots 17} \varphi^{17} + \frac{5146848}{1 \dots 21} \varphi^{21} - \dots \\ &= \varphi - \frac{1}{4.3.5} \varphi^5 - \frac{1}{32.9.5.7} \varphi^9 + \frac{23}{129.81.25.7.11.13} \varphi^{13} + \frac{107}{2048.243.125.49.11.13.17} \varphi^{17} \\ &\quad + \frac{23.37}{8192.729.625.49.11.13.17.19} \varphi^{21} - \dots \end{aligned}$$

$$\begin{aligned}
 P\psi\omega &= +2.6220575\psi \\
 &\quad -2.0656648\psi^5 \\
 &\quad -0.5811918\psi^9 \\
 &\quad +0.0245475\psi^{13} \\
 &\quad +0.0001890\psi^{17} \\
 &\quad +0.0000620\psi^{21}
 \end{aligned}$$

$$P\frac{1}{2}\omega = +1.2453446 = \sqrt[4]{\frac{1}{2}} \cdot e^{\frac{1}{8}\pi}$$

$$\frac{d \log Q}{d \varphi^2} = \frac{P P}{Q Q}$$

Die Halbierung geschieht sehr bequem so

$$\begin{aligned}
 \cos \text{lemn } \varphi &= \tan u \\
 \sin \text{lemn } \varphi &= \sqrt{\cos 2u} \\
 \cos \text{lemn } \varphi &= \sqrt{\tan \frac{1}{2}(45^\circ + u)} \\
 \sin \text{lemn } \varphi &= \sqrt{\tan \frac{1}{2}(45^\circ - u)}
 \end{aligned}$$

$$\begin{aligned}
 \sin \text{lemn } (\varphi + i\psi) &= r(\cos v + i \sin v) \\
 \sin \text{lemn } \varphi^2 &= \tan \Phi \\
 \sin \text{lemn } \psi^2 &= \tan \Psi \\
 r &= \sqrt{\tan(\Phi + \Psi)} \\
 \tan v &= \sqrt{\frac{\tan 2\Psi}{\tan 2\Phi}}
 \end{aligned}$$

$$\begin{aligned}
 \sin \text{lemn } (\varphi + \psi) &= \frac{\sqrt{(\cos 2\Phi \sin 2\Psi) + (\sin 2\Phi \cos 2\Psi)}}{\cos(\Phi - \Psi) \cdot \sqrt{2}} \\
 &= \frac{\sin(\Psi - \Phi) \cdot \sqrt{2}}{\sqrt{(\cos 2\Phi \sin 2\Psi) - (\sin 2\Phi \cos 2\Psi)}}
 \end{aligned}$$

Es sei

$$\sin \text{lemn } A = x, \quad \sin \text{lemn } B = y, \quad \sin \text{lemn } \frac{A+B}{1+i} = z$$

so ist

$$\begin{aligned}
 (1+i)z &= \frac{\sqrt{(1+xx)(1-yy)} - \sqrt{(1-xx)(1+yy)}}{x-y} \\
 \frac{1-i}{z} &= \frac{\sqrt{1+xx}(1-yy) + \sqrt{(1-xx)(1+yy)}}{x+y}
 \end{aligned}$$

$$\frac{\sin \text{lemn} \frac{A-B}{1+i}}{\sin \text{lemn} \frac{A+B}{1+i}} = \frac{\sin \text{lemn} A - \sin \text{lemn} B}{\sin \text{lemn} A + \sin \text{lemn} B}$$

$$\begin{aligned} pp &= QQ - PP, & qq &= QQ + PP \\ P(a-b) \cdot Q(a+b) &= Pa \cdot Qa \cdot \sqrt{(Qb^4 - Pb^4)} - Pb \cdot Qb \cdot \sqrt{(Qa^4 - Pa^4)} \\ P(a-b) \cdot P(a+b) &= Pa^2 \cdot Qb^2 - Qa^2 \cdot Pb^2 \\ Q(a-b) \cdot Q(a+b) &= Pa^2 \cdot Pb^2 + Qa^2 \cdot Qb^2 \end{aligned}$$

$$\begin{aligned} q(a-b) \cdot q(a+b) &= pa^2 \cdot Pb^2 + qa^2 \cdot Qb^2 \\ &= pb^2 \cdot Pa^2 + qb^2 \cdot Qa^2 \end{aligned}$$

$$\begin{aligned} q(a-b) \cdot Q(a+b) &= qa \cdot qb \cdot Qa \cdot Qb - pa \cdot pb \cdot Pa \cdot Pb \\ q(a-b) \cdot P(a+b) &= qa \cdot pb \cdot Pa \cdot Qb + pa \cdot qb \cdot Qa \cdot Pb \end{aligned}$$

$$P\varphi = P$$

$$Q\varphi = Q$$

$$p\varphi = \sqrt{(QQ - PP)}$$

$$q\varphi = \sqrt{(QQ + PP)}$$

$$P2\varphi = 2PQ\sqrt{(Q^4 - P^4)}$$

$$Q2\varphi = Q^4 + P^4$$

$$p2\varphi = Q^4 - 2QQPP - P^4$$

$$q2\varphi = Q^4 + 2QQPP + P^4$$

$$P3\varphi = 3Q^8P - 6Q^4P^5 - P^9$$

$$Q3\varphi = Q^9 + 6Q^5P^4 - 3QP^8$$

$$p3\varphi = \sqrt{(QQ - PP)} \cdot (Q^8 - 4Q^6PP - 6Q^4P^4 - 4QQP^6 + P^8)$$

$$q3\varphi = \sqrt{(QQ + PP)} \cdot (Q^4 + 4Q^6PP - 6Q^4P^4 + 4QQP^6 + P^8)$$

$$P4\varphi = 4PQ\sqrt{(Q^4 - P^4)} \cdot (Q^{12} - 5Q^8P^4 - 5Q^4P^8 + P^{12})$$

$$Q4\varphi = Q^{16} + 20Q^{12}P^4 - 26Q^8P^8 + 20Q^4P^{12} + P^{16}$$

$$\begin{aligned} p4\varphi &= Q^{16} - 8Q^{14}PP - 12Q^{12}P^4 - 8Q^{10}P^6 + 38Q^8P^8 + 8Q^6P^{10} - 12Q^4P^{12} \\ &\quad + 8QQP^4 + P^{16} \end{aligned}$$

$$\begin{aligned} q4\varphi &= Q^{16} + 8Q^{14}PP - 12Q^{12}P^4 + 8Q^{10}P^6 + 38Q^8P^8 - 8Q^6P^{10} - 12Q^4P^{12} \\ &\quad - 8QQP^4 + P^{16} \end{aligned}$$



$$\begin{aligned}
 \frac{Qn\varphi}{Q\varphi^{nn}} &= 1 + \frac{1}{12}(n^4 - nn)\left(\frac{P\varphi}{Q\varphi}\right)^4 - \frac{1}{10080}(n^8 + 70n^6 - 371n^4 + 300nn)\left(\frac{P\varphi}{Q\varphi}\right)^8 \\
 &\quad + \frac{1}{19958400}(17n^{12} + 165n^{10} + 4191n^8 - 106865n^6 + 426492n^4 \\
 &\quad \quad - 324000nn) \cdot \left(\frac{P\varphi}{Q\varphi}\right)^{12} \\
 &= 1 + \frac{1}{12}nn(n-1)\left(\frac{P\varphi}{Q\varphi}\right)^4 - \frac{1}{10080}nn(nn-1)(nn-4)(nn+75)\left(\frac{P\varphi}{Q\varphi}\right)^8 \\
 &\quad + \frac{1}{19958400}nn(nn-1)(nn-4)(nn-9)(17n^4 + 403nn + 9000) \cdot \left(\frac{P\varphi}{Q\varphi}\right)^{12}
 \end{aligned}$$

$$Pi\varphi = iP\varphi$$

$$Qi\varphi = Q\varphi$$

$$pi\varphi = q\varphi$$

$$qi\varphi = p\varphi$$

$$P(1+i)\varphi = (1+i)PQ$$

$$Q(1+i)\varphi = \sqrt{(Q^4 - P^4)} = pq$$

$$p(1+i)\varphi = QQ - iPP$$

$$q(1+i)\varphi = QQ + iPP$$

$$P(2+i)\varphi = (2+i)PQ^4 - iP^5$$

$$Q(2+i)\varphi = Q^5 - (1-2i)QP^4$$

$$p(2+i)\varphi = \sqrt{(QQ + PP)} \cdot \{Q^4 - (2+2i)QQPP + P^4\}$$

$$q(2+i)\varphi = \sqrt{(QQ - PP)} \cdot \{Q^4 + (2+2i)QQPP + P^4\}$$

$$P(3+i)\varphi = (3+i)PQ^3 - (2+6i)P^5Q^5 + (3+i)P^9Q$$

$$Q(3+i)\varphi = \sqrt{(Q^4 - P^4)} \cdot \{Q^8 + (2+8i)P^4Q^4 + P^8\}$$

$$p(3+i)\varphi = Q^{10} - (4-3i)Q^8PP - (2-4i)Q^6P^4 + (4+2i)Q^4P^6 - (3+4i)QQP^8 - iP^{10}$$

$$q(3+i)\varphi = Q^{10} + (4-3i)Q^8PP - (2-4i)Q^6P^4 - (4+2i)Q^4P^6 - (3+4i)QQP^8 + iP^{10}$$

$$P(1+2i)\varphi = (1+2i)PQ^4 - P^5$$

$$Q(1+2i)\varphi = Q^5 - (1+2i)QP^4$$

$$P(1+3i)\varphi = (1+3i)PQ^9 - (6+2i)P^5Q^5 + (1+3i)P^9Q$$

$$Q(1+3i)\varphi = Q^{10} \dots$$

$$P(1+4i)\varphi = (1+4i)PQ^{16} - (20+12i)P^5Q^{12} - (10-28i)P^9Q^8$$

$$+ (12-20i)P^{13}Q^4 + P^{17}$$

$$Q(1+4i)\varphi =$$

$$P(1+ni)\varphi = (1+in)PQ^{nn} - \left(\frac{nn \cdot nn-1}{12} + \frac{n \cdot nn-1 \cdot nn-4}{60}i\right)P^5Q^{nn-4} + \\ = (1+in)PQ^{nn} - \frac{n \cdot nn-1 \cdot 1+in \cdot nn-4i}{60}P^5Q^{nn-4}$$

Unter den Zahlen  $y = x, 2x+1, 2x+i, 2x-1, 2x-i$  ist immer wenigstens eine (oder drei oder alle), die die Auflösung der Congruenz  $1-y^4 \equiv zz$  nach irgend einem Modulus möglich macht.

$$\sin \text{lemn } X = x, \quad \sin \text{lemn } Y = y, \quad \sin \text{lemn } Z = z$$

$$\sin \text{lemn } (X+Y+Z) = \frac{-2xyz(xx+yy+zz-xyyz) + x\sqrt{(1-y^4)} \cdot \sqrt{(1-z^4)} \cdot (1-yzz+xxz+xyy) + y\sqrt{(1-x^4)} \cdot \sqrt{(1-z^4)} \cdot (1+yyz-xxz+xyy) + z\sqrt{(1-x^4)} \cdot \sqrt{(1-y^4)} \cdot (1+yyz+xxz-xyy)}{1+2yyz+2xxz+2xyy+y^4z^4+x^4z^4+x^4y^4-2x^4yyz-2xyy^4z-2xyyz^4}$$

$$\text{Ist} \quad \varphi + \varphi' + \varphi'' = 0, \quad \begin{aligned} \sin \text{lemn } \varphi &= x & \cos \text{lemn } \varphi &= y \\ \sin \text{lemn } \varphi' &= x' & \cos \text{lemn } \varphi' &= y' \\ \sin \text{lemn } \varphi'' &= x'' & \cos \text{lemn } \varphi'' &= y'' \end{aligned}$$

so ist

$$x(y-y'y'') = x'(y'-yy'') = x''(y''-yy'), \quad y'' = \frac{xy-x'y'}{xy'-x'y}$$

Q

|         |    |                                                                 |        |
|---------|----|-----------------------------------------------------------------|--------|
| $-1+2i$ | 5  | $1+(-1+2i)x^4$                                                  |        |
| $-3$    | 9  | $1+6x^4$                                                        | $3x^8$ |
| $+3+2i$ | 13 | $1+(-11+10i)x^4 + (7-4i)x^8 + (3+2i)x^{12}$                     |        |
| $+1+4i$ | 17 | $1+(+12-20i)x^4 + (-10+28i)x^8 - (20+12i)x^{12} + (1+4i)x^{16}$ |        |
| 5       | 25 | $1+50x^4 - 125x^8 + 300x^{12} - 105x^{16} - 62x^{20} + 5x^{24}$ |        |

$$\text{für} \quad x^4 = +1 \quad \text{und} \quad x^4 = -1$$

werden diese Functionen respective den Quadraten und Würfeln

$$\begin{array}{cccccc} \text{von} & 1-i & -2 & -2-2i & -4i & +8 \\ \text{gleich für } M = & 5 & 9 & 13 & 17 & 25 \end{array}$$

$$Q(a+bi)\varphi = Q^{aa+bb} + \frac{1}{12}\{(a+bi)^4 - (aa+bb)\}Q^{aa+bb-4}P^4 \\ - \frac{1}{1680}\{(a+bi)^8 + 70(a+bi)^4(aa+bb) - 35(aa+bb)^2 - 336(a+bi)^4 + 300(aa+bb)\} \cdot Q^{aa+bb-8}P^8 \dots$$