[III.]

[ZUR THEORIE DER NEUEN TRANSSCENDENTEN.]

Die Theoreme in Beziehung auf diejenigen Reihen und unendlichen Producte, welche zu der Theorie der Arithmetisch Geometrischen Mittel gehören, ordnen wir so:

1. $$1-x.1-xx.1-x^3.1-x^4\ldots=[x]$$

2. $$1+x.1+xx.1+x^3.1+x^4\ldots=\frac{[xx]}{[x]}$$

3. $$1-x.1-x^3.1-x^5.1-x^7\ldots=\frac{[x]}{[xx]}$$

4. $$1+x.1+x^3.1+x^5.1+x^7\ldots=\frac{[xx]^2}{[x][x^4]}$$

5. $$[-x]=\frac{[xx]^3}{[x][x^4]}$$

6. $$1+xy.1+x^3y.1+x^5y\ldots1+xy^{-1}.1+x^3y^{-1}.1+x^5y^{-1}\ldots$$

evolvitur in seriem

$$Fx.\{1+(y+y^{-1})x+(y^2+y^{-2})x^4+(y^3+y^{-3})x^9+\ldots\}$$

7. also $$Fx=\frac{(1-x)^2(1-x^3)^2(1-x^5)^2\ldots}{1-2x+2x^4-2x^9+\ldots}=\frac{[x]^2}{[xx]^2}\cdot\frac{1}{1-2x+2x^4-2x^9+\ldots}$$

8. $$Fx = \frac{1+xx\,.\,1+x^6\,.\,1+x^{10}\,\ldots}{1-2x^4+2x^{16}-2x^{36}+\ldots} = \frac{[x^4]^2}{[xx][x^8]}\cdot\frac{1}{1-2x^4+2x^{16}-\ldots}$$

9. $$[xx]Fx = [x^8]Fx^4 = [x^{32}]Fx^{16} = [x^{128}]Fx^{64} = \text{etc.} = 1$$

10. $$1-2x+2x^4-\,.\,. = \frac{[x]^2}{[xx]} = \frac{1-x\,.\,1-xx\,.\,1-x^3\,.\,.}{1+x\,.\,1+xx\,.\,1+x^3\,.\,.}$$

11. $$1+2x+2x^4+\,.\,. = \frac{[xx]^5}{[x]^2[x^4]^2} = \frac{1+x\,.\,1-xx\,.\,1+x^3\,.\,1-x^4\,.\,.}{1-x\,.\,1+xx\,.\,1-x^3\,.\,1+x^4\,.\,.}$$

Andere Beweise dieser Sätze.

Wenn man in 6 statt y, xy schreibt, so wird

$$1+\frac{1}{yy}\,.\,1+xxyy\,.\,1+x^4yy\,.\,1+x^6yy\,\ldots\,1+xxy^{-2}\,.\,1+x^4y^{-2}\,.\,1+x^6y^{-2}\,.\,.$$
$$= \frac{1}{[xx]}\{(1+y^{-2})+(y^2+y^{-4})xx+(y^4+y^{-6})x^6+\,.\,.\}$$

oder

12. $$y+\frac{1}{y}\,.\,1+xxyy\,.\,1+x^4yy\,.\,1+x^6yy\,\ldots\,1+xxy^{-2}\,.\,1+x^4y^{-2}\,.\,1+x^6y^{-2}\,.\,.$$
$$= \frac{1}{[xx]x^{\frac{1}{4}}}\{(y+y^{-1})x^{\frac{1}{4}}+(y^3+y^{-3})x^{\frac{9}{4}}+\,.\,.\}$$

Anderer Beweis

13. $$x^{\frac{1}{4}}\frac{[x^4]^2}{[xx]} = x^{\frac{1}{4}}+x^{\frac{9}{4}}+\ldots$$

oder

14. $$\frac{[xx]^2}{[x]} = 1+x+x^3+x^6+x^{10}+\,.\,. = \frac{1-xx\,.\,1-x^4\,.\,.}{1-x\,.\,1-x^3\,.\,.}$$

Anderer Beweis.

15. $$(1-2x+2x^4-\,.\,.)\;(1+2x+2x^4+\,.\,.) = (1-2xx+2x^8-\,.\,.)^2$$

16. $$(1-2x+2x^4-\,.\,.)^2+(1+2x+2x^4+\,.\,.) = 2(1+2xx+2x^8+\,.\,.)^2$$

17. $$(1+2x+2x^4+\,.\,.)^2\;(x^{\frac{1}{4}}+x^{\frac{9}{4}}+\,.\,.) = (x^{\frac{1}{8}}+x^{\frac{9}{8}}+\,.\,.)^2$$

18. $$(1+2x+2x^4+\,.\,.)^2+(2x^{\frac{1}{4}}+2x^{\frac{9}{4}}+\,.\,.)^2 = (1+2x^{\frac{1}{2}}+2x^2+\,.\,.)^2$$

19. $$(1+2x+2x^4+\,.\,.)^4 = (1-2x+2x^4-\,.\,.)^4+(2x^{\frac{1}{4}}+2x^{\frac{9}{4}}+\,.\,.)^4$$

Anwendung auf arithm. geom. Mittel.

20. $$\begin{array}{ll} a\ = h(1+2x+2x^4+..)^2 & b\ = h(1-2x+2x^4-..)^2 \\ a'\ = h(1+2xx+2x^8+..)^2 & b'\ = h(1-2xx+2x^8-..)^2 \\ a''\ = h(1+2x^4+2x^{16}+..)^2 & b''\ = h(1-2x^4+2x^{16}-..)^2 \\ a'''= h(1+2x^8+2x^{32}+..)^2 & b'''= h(1-2x^8+2x^{32}-..)^2 \\ \text{etc.} & \text{etc.} \end{array}$$

$$c\ = \sqrt{(aa-bb)}$$

$$c'\ = \sqrt{(a'a'-b'b')} = \tfrac{1}{2}(a-b) = \frac{cc}{4a'} = \frac{cc}{4a'}, \qquad \sqrt{\frac{c'}{4h}} = \frac{c}{4}\frac{1}{\sqrt{a'h}}$$

$$c'' = \sqrt{(a''a''-b''b'')} = \tfrac{1}{2}(a'-b') = \frac{c'c'}{4a''} = \frac{c^4}{64a'a'a''}, \qquad \sqrt[4]{\frac{c''}{4h}} = \frac{c}{4a'^{\frac{1}{2}}a''^{\frac{1}{4}}h^{\frac{1}{4}}}$$

$$c'''= \sqrt{(a'''a'''-b'''b''')} = \tfrac{1}{2}(a''-b'') = \frac{c''c''}{4a'''} = \frac{c^8}{2^{14}a'^4a''^2a'''}, \qquad \sqrt[8]{\frac{c'''}{4h}} = \frac{c}{4a'^{\frac{1}{2}}a''^{\frac{1}{4}}a'''^{\frac{1}{8}}h^{\frac{1}{8}}}$$

etc. etc.

21. $$x^{\frac{1}{2}} = \frac{c}{4a'^{\frac{1}{2}}a''^{\frac{1}{4}}a'''^{\frac{1}{8}}..} = \frac{c}{4a'}\left[\frac{a'}{a''}\right]^{\frac{1}{2}}\left[\frac{a''}{a'''}\right]^{\frac{1}{4}}\left[\frac{a'''}{a''''}\right]^{\frac{1}{8}} \text{ etc.}$$

22. $$x = \frac{a-b}{8a''}\left[\frac{a''}{a'''}\right]^{\frac{1}{2}}\left[\frac{a'''}{a''''}\right]^{\frac{1}{4}} \text{ etc.}$$

Setzt man in 6 statt x, x^3 und statt y, $-x$, so wird

$$1-x^4 . 1-x^{10} . 1-x^{16} \ldots 1-x^2 . 1-x^8 . 1-x^{14} \ldots$$
$$= \frac{1}{[x^6]}\{1-xx-x^4+x^{10}+x^{14}-x^{24}-x^{30}+..\}$$

oder

23. $$[x] = 1-x-xx+x^5+x^7-x^{12}-x^{15}+ \text{ etc.}$$

Anderer Beweis.

Man setze in 6 statt x, x^3 und statt y, $+x$, so wird

24. $1+x+xx+x^5+x^7+x^{12}+x^{15}+\text{etc.} = [x^3].1+x.1+xx.1+x^4.1+x^5 \ldots$
$$= \frac{[xx][x^3]^2}{[x][x^6]}$$

Man setze in 6 statt x, x^3 und statt y, xxy, so wird

$$1+\frac{x}{y}+x^5y+\frac{x^8}{yy}+x^{16}yy+\text{etc.}$$
$$= 1+\frac{x}{y}.1+\frac{x^7}{y}.1+\frac{x^{13}}{y}\ldots 1+x^5y.1+x^{11}y.1+x^{17}y\ldots[x^6]$$

oder statt x, x^3 gesetzt

$$x+\frac{x^4}{y}+x^{16}y+\frac{x^{25}}{yy}+x^{49}yy+\text{etc.}$$
$$= x.1+\frac{x^3}{y}.1+\frac{x^{21}}{y}.1+\frac{x^{39}}{y}\ldots 1+x^{15}y.1+x^{33}y.1+x^{51}y\ldots[x^{18}]$$

also

25. $x+x^4+x^{16}+x^{25}+x^{49}+\ldots$
$$= x.1+x^3.1+x^{15}.1+x^{21}.1+x^{33}.1+x^{39}\ldots[x^{18}] = x\frac{[x^6]^2[x^9][x^{36}]}{[x^3][x^{12}][x^{18}]}$$

26. $x-x^4-x^{16}+x^{25}+x^{49}-\text{etc.} = x.1-x^3.1-x^{15}.1-x^{21}.1-x^{33}\ldots[x^{18}]$
$$= x\frac{[x^3][x^{18}]^2}{[x^6][x^9]}$$

Da nun

$$\frac{x.1-x^3.1-x^{15}.1-x^{21}.1-x^{33}\ldots[x^{18}]}{x.1-x^3.1-x^9.1-x^{15}.1-x^{21}\ldots} = \frac{[x^{18}]^2}{[x^9]} = 1+x^9+x^{27}+x^{54}+\ldots$$

so ist

27. $x-x^4-x^{16}+x^{25}+x^{29}-\ldots$
$$= 1-x^3.1-x^9.1-x^{15}\ldots\frac{1}{x^{\frac{1}{8}}}\{x^{\frac{9}{8}}+x^{\frac{81}{8}}+x^{\frac{225}{8}}+x^{\frac{441}{8}}+\text{etc.}\}$$

ferner folgt aus 24, wenn man statt x, x^3 setzt, weil

$$\frac{1-2x^9+2x^{36}-2x^{81}+\ldots}{1-x^3.1-x^9.1-x^{15}\ldots} = \frac{[x^6][x^9]^2}{[x^3][x^{18}]} = 1+x^3+x^6+x^{15}+x^{21}+\ldots$$

28. $1-2x^9+2x^{36}-2x^{81}+\ldots = 1-x^3.1-x^9.1-x^{15}\ldots\frac{1}{x^{\frac{1}{4}}}\{x^{\frac{1}{4}}+x^{\frac{25}{4}}+x^{\frac{49}{4}}+\text{etc.}\}$

also aus der Verbindung von 27 und 28

29. $1-2x+2x^4-2x^9+2x^{16}-\,..$

$$= 1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots\frac{1}{x^{\frac{1}{8}}}\{x^{\frac{1}{8}}-2x^{\frac{9}{8}}+x^{\frac{25}{8}}+x^{\frac{49}{8}}-2x^{\frac{81}{8}}-\text{etc.}\}$$

ferner folgt aus 23

$$x^{\frac{1}{8}}-x^{\frac{25}{8}}-x^{\frac{49}{8}}+x^{\frac{121}{8}}+\,..=x^{\frac{1}{8}}[x^3]$$

also

30. $1-2x^3+2x^{12}-2x^{27}+2x^{48}-\,..$

$$= 1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots\frac{1}{x^{\frac{1}{8}}}\{x^{\frac{1}{8}}-x^{\frac{25}{8}}-x^{\frac{49}{8}}+x^{\frac{121}{8}}+x^{\frac{169}{8}}-\,..\}$$

Aus der Summation von 29 und 30

31. $\{1-2x+2x^4-2x^9+\,..\}+\{1-2x^3+2x^{12}-2x^{27}+\,..\}$

$$= 1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots\frac{2}{x^{\frac{1}{8}}}\{x^{\frac{1}{8}}-x^{\frac{9}{8}}-x^{\frac{81}{8}}+x^{\frac{121}{8}}+x^{\frac{169}{8}}-\,..\}$$

Aus der Summation von 28 und 30

32. $\{1-2x^3+2x^{12}-2x^{27}+\,..\}+\{1-2x^9+2x^{36}-2x^{81}+\,..\}$

$$= 1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots\frac{2}{x^{\frac{1}{8}}}\{x^{\frac{1}{8}}+x^{\frac{121}{8}}+x^{\frac{169}{8}}+x^{\frac{529}{8}}+\,..\}$$

Setzt man in 6 statt x, x^6 und statt y, xy, so wird

$$1+x^7y\,.\,1+x^{19}y\,.\,1+x^{31}y\ldots 1+\frac{x^5}{y}\,.\,1+\frac{x^{17}}{y}\ldots[x^{12}]$$
$$= 1+\frac{x^5}{y}+x^7y+\frac{x^{22}}{yy}+x^{26}yy+\,..$$

Man hat demnach die Zerlegungen in Factoren

33. $(1-2x^3+2x^{12}-2x^{27}+\,..)+(1-2x^9+2x^{36}-2x^{81}+\,..)$

$$= 2[x^{36}]\,.\,1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots 1+x^{15}\,.\,1+x^{21}\,.\,1+x^{51}\,.\,1+x^{57}\ldots$$

34. $(1-2x+2x^4-2x^9+\,..)+(1-2x^3+2x^{12}-2x^{27}+\,..)$

$$= 2[x^{12}]\,.\,1-x\,.\,1-x^3\,.\,1-x^5\,.\,1-x^7\ldots 1+x^5\,.\,1+x^7\,.\,1+x^{17}\,.\,1+x^{19}\ldots$$

35. $(1+2x^3+2x^{12}+2x^{27}+\,..)+(1+2x^9+2x^{36}+2x^{81}+\,..)$

$$= 2[x^{36}]\,.\,1+x^3\,.\,1+x^9\,.\,1+x^{15}\ldots 1-x^{15}\,.\,1-x^{21}\,.\,1-x^{51}\ldots$$

36. $(1+2x+2x^4+2x^9+\,..)+(1+2x^3+2x^{12}+2x^{27}+\,..)$

$$= 2[x^{12}]\,.\,1+x\,.\,1+x^3\,.\,1+x^5\,.\,1+x^7\ldots 1-x^5\,.\,1-x^7\,.\,1-x^{17}\,.\,1-x^{19}\ldots$$

Hieraus ergeben sich zugleich die Factoren des letzten Theils in 31.

Aus der Subtraction von 28 und 30

37. $$(1-2x^9+2x^{36}-2x^{81}+\,.\,.)-(1-2x^3+2x^{12}-\,.\,.)$$
$$=2\,.\,1-x^3\,.\,1-x^9\,.\,1-x^{15}\,.\,.\,\frac{1}{x^{\frac{1}{4}}}(x^{\frac{25}{8}}+x^{\frac{49}{8}}+x^{\frac{289}{8}}+x^{\frac{361}{8}}+\,.\,.)$$

Setzt man in 6 statt x, x^6 und statt y, x^5y, so wird

$$1+x^{11}y\,.\,1+x^{23}y\,.\,1+x^{35}y\,\ldots\,1+\frac{x}{y}\,.\,1+\frac{x^{13}}{y}\,.\,1+\frac{x^{25}}{y}\,\ldots\,.\,[x^{12}]$$
$$=1+\frac{x}{y}+x^{11}y+\frac{x^{14}}{yy}+x^{34}yy+\ldots$$

Also die Zerlegung in Factoren

38. $$(1-2x^5+2x^{36}-2x^{81}+\,.\,.)-(1-2x^3+2x^{12}-\,.\,.)$$
$$=2x^3[x^{36}]\,.\,1+x^3\,.\,1+x^{33}\,.\,1+x^{39}\,.\,1+x^{69}\ldots\,1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots$$

39. $$(1-2x^3+2x^{12}-2x^{27}+\,.\,.)-(1-2x+2x^4-\,.\,.)$$
$$=2x[x^{12}]\,.\,1+x\,.\,1+x^{11}\,.\,1+x^{13}\,.\,1+x^{23}\ldots\,1-x\,.\,1-x^3\,.\,1-x^5\ldots$$

40. $$(1+2x^3+2x^{12}+2x^{27}+\,.\,.)-(1+2x^9+2x^{36}+\,.\,.)$$
$$=2x^3[x^{36}]\,.\,1-x^3\,.\,1-x^{33}\,.\,1-x^{39}\,.\,1-x^{69}\ldots\,1+x^3\,.\,1+x^9\,.\,1+x^{15}\ldots$$

41. $$(1+2x+2x^4+2x^9+\,.\,.)-(1+2x^3+2x^{12}+\,.\,.)$$
$$=2x[x^{12}]\,.\,1-x\,.\,1-x^{11}\,.\,1-x^{13}\,.\,1-x^{23}\ldots\,1+x\,.\,1+x^3\,.\,1+x^5\ldots$$

Aus der Subtraction von 29 und 30 folgt

42. $$(1-2x^3+2x^{12}-2x^{27}+\,.\,.)-(1-2x+2x^4-\,.\,.)$$
$$=2\,.\,1-x^3\,.\,1-x^9\,.\,1-x^{15}\ldots\,\frac{1}{x^{\frac{1}{4}}}(x^{\frac{9}{8}}-x^{\frac{25}{8}}-x^{\frac{49}{8}}+x^{\frac{81}{8}}+x^{\frac{225}{8}}-\,.\,.)$$

Woraus die Zerlegung des letzten Gliedes dieser Gleichung in Factoren folgt.

Aus der Multiplication von 34 und 39 folgt

43. $$(1-2x^3+2x^{12}-2x^{27}+\,.\,.)^2-(1-2x+2x^4-\,.\,.)^2$$
$$=4x[x^{12}]^2\,.\,(1-x)^2(1-x^3)^2(1-x^5)^2\ldots\,1+x\,.\,1+x^5\,.\,1+x^7\,.\,1+x^{11}\ldots$$
$$=4x\frac{[x^{12}]^3[x]^2}{[xx]^2}\cdot\frac{[xx]^2[x^3][x^{12}]}{[x][x^4][x^6]^2}=4x\frac{[x][x^3][x^{12}]^3}{[x^4][x^6]^2}$$

Ebenso aus der Multiplication von 36 und 41

44. $$(1+2x+2x^4+\,.\,.)^2-(1+2x^3+2x^{12}+2x^{27}+\,.\,.)^2$$
$$=4x[x^{12}]^2.(1+x)^2(1+x^3)^2(1+x^5)^2\ldots 1-x.\,1-x^5.\,1-x^7.\,1-x^{11}\ldots$$
$$=4x\frac{[x^{12}]^2[xx]^4}{[x]^2[x^4]^2}\cdot\frac{[x][x^6]}{[xx][x^3]}=4x\frac{[xx]^3[x^6][x^{12}]^2}{[x][x^8][x^4]^2}$$

Also der Quotient

45. $$\frac{(1-2x^8+2x^{12}-2x^{27}+\,.\,.)^2-(1-2x+2x^4-\,.\,.)^2}{(1+2x^8+2x^{12}+2x^{27}+\,.\,.)^2-(1+2x+2x^4+\,.\,.)^2}$$
$$=-\left(\frac{1-x}{1+x}\right)^3\left(\frac{1-x^3}{1+x^3}\right)\left(\frac{1-x^5}{1+x^5}\right)^3\left(\frac{1-x^7}{1+x^7}\right)^3\left(\frac{1-x^9}{1+x^9}\right)\ldots$$
$$=\frac{[x]^2[x^8]^2[x^4][x^{12}]}{[xx]^3[x^6]^3}$$

und das Product

45b. $$\{(1-2x^3+2x^{12}-\,.\,.)^2-(1-2x+2x^4-\,.\,.)^2\}$$
$$\times\{(1+2x^3+2x^{12}+\,.\,.)^2-(1+2x+2x^4+\,.\,.)^2\}$$
$$=-16xx\frac{[xx]^3[x^{12}]^5}{[x^4]^3[x^6]}$$

Aus $28+i30$ folgt

46. $$(1-2x^9+2x^{36}-\,.\,.)-i(1-2x^3+2x^{12}-\,.\,.)$$
$$=1-x^3.\,1-x^9.\,1-x^{15}\ldots\frac{1-i}{x^{\frac{1}{8}}}(x^{\frac{1}{8}}+ix^{\frac{25}{8}}+ix^{\frac{49}{8}}+x^{\frac{121}{8}}+\,.\,.)$$

Nun findet man aus 6 nach dem, was zwischen 22 und 23 gezeigt ist

$$1+x^4y.\,1+x^{10}y.\,1+x^{16}y\ldots 1+\frac{xx}{y}.\,1+\frac{x^8}{y}.\,1+\frac{x^{14}}{y}\ldots 1-x^6.\,1-x^{12}\ldots$$
$$=1+\frac{xx}{y}+x^4y+\frac{x^{10}}{yy}+x^{14}yy+\ldots$$

Also

$$1+x^4i.\,1-x^{10}i.\,1+x^{16}i\ldots 1-\frac{xx}{i}.\,1+\frac{x^8}{i}.\,1-\frac{x^{14}}{i}\ldots 1+x^6.\,1-x^{12}.\,1+x^{18}\ldots$$
$$=1+ixx+ix^4+x^{10}+x^{14}+\ldots$$

Daher die Zerlegung in Factoren

47. $$(1-2x^3+2x^{12}-\,.\,.)-i(1-2x+2x^4-\,.\,.)$$
$$=\;1-x.\,1-x^3.\,1-x^5\ldots 1-i.\,1+x^3.\,1-x^6.\,1+x^9\ldots$$
$$\times 1+ix.\,1+ixx.\,1-ix^4.\,1-ix^5.\,1+ix^7.\,1+ix^8\ldots$$

und ferner

48. $$(1-2x^3+2x^{12}-\,.\,.)+i(1-2x+2x^4-\,.\,.)$$
$$= 1-x\,.\,1-x^3\,.\,1-x^5\ldots\ 1+i\,.\,1+x^3\,.\,1-x^6\,.\,1+x^9\ldots$$
$$\times 1-ix\,.\,1-ixx\,.\,1+ix^4\,.\,1+ix^5\,.\,1-ix^7\,.\,1-ix^8\ldots$$

49. $$(1+2x^3+2x^{12}+\,.\,.)-i(1+2x+2x^4+\,.\,.)$$
$$= 1+x\,.\,1+x^3\,.\,1+x^5\ldots\ 1-i\,.\,1-x^3\,.\,1-x^6\,.\,1-x^9\ldots$$
$$\times 1-ix\,.\,1+ixx\,.\,1-ix^4\,.\,1+ix^5\ldots$$

50. $$(1+2x^3+2x^{12}+\,.\,.)+i(1+2x+2x^4+\,.\,.)$$
$$= 1+x\,.\,1+x^3\,.\,1+x^5\ldots\ 1+i\,.\,1-x^3\,.\,1-x^6\,.\,1-x^9\ldots$$
$$\times 1+ix\,.\,1-ixx\,.\,1+ix^4\,.\,1-ix^5\ldots$$

Also aus der Multiplication von 47 und 48

51. $$(1-2x^3+2x^{12}-\ldots)^2+(1-2x+2x^4-\ldots)^2$$
$$= 2(1-x)^2(1-x^3)^2(1-x^5)^2\ldots(1+x^3)^2(1-x^6)^2(1+x^9)^2\ldots$$
$$\times 1+xx\,.\,1+x^4\,.\,1+x^8\,.\,1+x^{10}\ldots$$
$$= 2(1-x)^2(1-x^3)^2(1-x^5)^2\ldots(1+x^3)^2\,1+x^9)^2\ldots$$
$$\times(1-x^6)^3(1-x^{12})^2\ldots\ 1+xx\,.\,1+x^4\,.\,1+x^6\,.\,1+x^8\ldots$$
$$= 2\,\frac{[x]^2[x^6]^4}{[xx]^2[x^3]^2[x^{12}]^2}\cdot\frac{[x^6]^3}{[x^{12}]}\cdot\frac{[x^4]}{[xx]} = \frac{2[x]^2[x^4][x^6]^7}{[x^2]^3[x^3]^2[x^{12}]^3}$$

und aus der Multiplication von 49 und 50

52. $$(1+2x^3+2x^{12}+\,.\,.)^2+(1+2x+2x^4+\ldots)^2$$
$$= 2(1+x)^2(1+x^3)^2(1+x^5)^2\ldots(1-x^3)^2(1-x^6)^2(1-x^9)^2\ldots$$
$$\times 1+xx\,.\,1+x^4\,.\,1+x^8\,.\,1+x^{10}\ldots$$
$$= 2\,\frac{[xx]^4}{[x]^2[x^4]^2}\,[x^3]^2\,\frac{[x^4][x^6]}{[xx][x^{12}]} = \frac{2[xx]^3[x^3]^2[x^6]}{[x]^2[x^4][x^{12}]}$$

und der Quotient

53. $$\frac{(1-2x^3+2x^{12}-\,.\,.)^2+(1-2x+2x^4-\,.\,.)^2}{(1+2x^3+2x^{12}+\,.\,.)^2+(1+2x+2x^4+\,.\,.)^2}$$
$$= \left(\frac{1-x}{1+x}\right)^2\left(\frac{1-x^3}{1+x^3}\right)^2\left(\frac{1-x^5}{1+x^5}\right)^2\ldots\left(\frac{1+x^3}{1-x^3}\right)^2\left(\frac{1+x^9}{1-x^9}\right)^2\ldots$$
$$= \left(\frac{1-x}{1+x}\right)^2\left(\frac{1-x^5}{1+x^5}\right)^2\left(\frac{1-x^7}{1+x^7}\right)^2\ldots = \frac{[x]^4[x^4]^2[x^6]^6}{[x^2]^6[x^3]^4[x^{12}]^2}$$

und das Product

54. $$\{(1-2x^3+..)^2+(1-2x+..)^2\}\{(1+2x^3+..)^2-(1+2x+..)^2\}$$
$$= 4\frac{[x^6]^8}{[x^{12}]^4} = 4(1-2x^6+2x^{24}-..)^2$$

Aus Formel 23 folgt

55. $$x+x^9+x^{25}+.. = x.1+x^8.1+x^{16}.1+x^{24}...(1-x^{16}-x^{32}+x^{80}+x^{112}-..)$$

$$\tfrac{3}{2}\text{ Exponent} = \square - 1$$

Aus Formel 26 folgt

56. $$x^3+x^{27}+x^{75}+.. = x.1+x^8.1+x^{16}.1+x^{24}...(x^2-x^{10}-x^{42}+x^{66}+x^{130}-..)$$

oder $x^{\frac{1}{3}}.1+x^8.1+x^{16}.1+x^{24}... = A$, $x^2 = t^3$ gesetzt

55. $$x+x^9+x^{25}+.. = A(t-t^{25}-t^{49}+t^{121}+t^{169}-...)$$

56. $$x^3+x^{27}+x^{75}+.. = A(t^4-t^{16}-t^{64}+t^{100}+t^{196}-...)$$

Nun folgt aus der Factorenzerlegung in 24 sehr leicht, wenn man statt x, it statt y, i setzt

$$it-it^4+it^{16}-it^{25}-it^{49}+..$$
$$= it.1-t^3.1+t^{15}.1+t^{21}.1-t^{33}...1+t^{18}.1-t^{36}.1+t^{54}...$$

Also aus 55 — 56

57. $$(x+x^9+x^{25}+..)-(x^3+x^{27}+x^{75}+..)$$
$$= x.1-xx.1+x^{10}.1+x^{14}.1-x^{22}.1-x^{26}...$$
$$\times 1+x^{12}.1-x^{24}.1+x^{36}...1+x^8.1+x^{16}.1+x^{24}...$$

Und so sehr leicht

58. $$(x+x^9+x^{25}+..)+(x^3+x^{27}+x^{75}+..)$$
$$= x.1+xx.1-x^{10}.1-x^{14}.1+x^{22}.1+x^{26}...$$
$$\times 1+x^{12}.1-x^{24}.1+x^{36}...1+x^8.1+x^{16}.1+x^{24}...$$

Also durch Multiplication

59. $$(x+x^9+x^{25}+\,.\,.)^2-(x^3+x^{27}+x^{75}+\,.\,.)^2$$
$$= xx.1-x^4.1-x^{20}.1-x^{28}.1-x^{44}.1-x^{52}\ldots$$
$$\times(1+x^{12})^2(1-x^{24})^2(1+x^{36})^2\ldots(1+x^8)^2(1+x^{16})^2\ldots$$
$$= xx\frac{[x^4][x^{16}]^2[x^{24}]^7}{[x^8]^3[x^{12}]^3[x^{48}]^2}$$

Eben so folgt aus der Factorenzerlegung in 24, wenn man statt x, t und statt y, $-i$ setzt

$$t+it^4-it^{16}-t^{25}-t^{49}\ldots = t.1+it^3.1-it^{15}.1+it^{21}.1-it^{33}\ldots[t^{18}]$$

also

60. $$(x+x^9+x^{25}+\ldots)+i(x^3+x^{27}+x^{75}+\ldots)$$
$$= x.1+ixx.1-ix^{10}.1+ix^{14}.1-ix^{22}\ldots 1+x^8.1+x^{16}\ldots 1-x^{12}.1-x^{24}\ldots$$

und eben so

61. $$(x+x^9+x^{25}+\ldots)-i(x^3+x^{27}+x^{75}+\ldots)$$
$$= x.1-ixx.1+ix^{10}.1-ix^{14}.1+ix^{22}\ldots 1+x^8.1+x^{16}\ldots 1-x^{12}.1-x^{24}\ldots$$

Also durch Multiplication

62. $$(x+x^9+x^{25}+\ldots)^2+(x^3+x^{27}+\,.\,.)^2$$
$$= xx\frac{[x^8]^2}{[x^4][x^{16}]}\cdot\frac{[x^{12}][x^{48}]}{[x^{24}]^2}\cdot\frac{[x^{16}]^2}{[x^8]^2}\cdot[x^{12}]^2 = xx\frac{[x^{12}]^3[x^{16}][x^{48}]}{[x^4][x^{24}]^2}$$

Also Product von 59 und 62

63. $$(x+x^9+x^{25}+\ldots)^4-(x^3+x^{27}+x^{75}+\,.\,.)^4 = x^4\frac{[x^{16}]^3[x^{24}]^5}{[x^8]^3[x^{48}]}$$

Durch Multiplication von 62 und 44 folgt

64. $$\left\{(x^{\frac{1}{2}}+x^{\frac{9}{2}}+x^{\frac{25}{2}}+\,.\,.)^2+(x^{\frac{3}{2}}+x^{\frac{27}{2}}+\,.\,.)^2\right\}$$
$$\times\left\{(1+2x^2+2x^8+\,.\,.)^2-(1+2x^6+2x^{24}+\,.\,.)^2\right\}$$
$$= 4xx\frac{[x^4]^3[x^{12}][x^{24}]^2}{[x^2][x^6][x^8]^2}\cdot x\frac{[x^6]^3[x^8][x^{24}]}{[x^2][x^{12}]^2} = 4x^3\frac{[x^4]^3[x^6]^2[x^{24}]^3}{[x^2]^2[x^{12}][x^8]}$$

Durch Multiplication von 59 und 44

65. $$\left\{\left(x^{\frac{1}{2}}+x^{\frac{9}{2}}+x^{\frac{25}{2}}+\,.\,.\right)^2-\left(x^{\frac{3}{2}}+x^{\frac{27}{2}}+\,.\,.\right)^2\right\}$$
$$\times\left\{\left(1+2x^2+2x^8+\,.\,.\right)^2-\left(1+2x^6+2x^{24}+\,.\,.\right)^2\right\}$$
$$=4x^3\frac{[xx][x^8]^2[x^{12}]^7}{[x^4]^3[x^6]^3[x^{24}]^2}\cdot\frac{[x^4]^3[x^{12}][x^{24}]^2}{[x^2][x^6][x^8]^2}$$
$$=4x^3\frac{[x^{12}]^8}{[x^6]^4}=4\left(x^{\frac{3}{4}}+x^{\frac{27}{4}}+x^{\frac{75}{4}}+\,.\,.\right)^4$$

Man hat ferner

66. $$[x]=\frac{a^{\frac{1}{12}}b^{\frac{1}{3}}(aa-bb)^{\frac{1}{24}}}{2^{\frac{1}{3}}x^{\frac{1}{24}}\sqrt{h}}\qquad\frac{a}{h}=\frac{[x^2]^{10}}{[x]^4[x^4]^4}$$

67. $$[xx]=\frac{a^{\frac{1}{6}}b^{\frac{1}{6}}(aa-bb)^{\frac{1}{12}}}{2^{\frac{2}{3}}x^{\frac{1}{12}}\sqrt{h}}\qquad\frac{b}{h}=\frac{[x]^4}{[xx]^2}$$

68. $$[x^4]=\frac{a^{\frac{1}{12}}b^{\frac{1}{12}}(aa-bb)^{\frac{1}{6}}}{2^{\frac{4}{3}}x^{\frac{1}{6}}\sqrt{h}}\qquad\frac{aa-bb}{hh}=16x\frac{[x^4]^8}{[xx]^4}$$

Für die Fünf-Theilung $\left\{\begin{matrix}a & b\\ A & B\end{matrix}\right\}$

$$\left(\frac{a-A}{5A-a}\right)^4=\frac{AA-BB}{aa-bb}\cdot\frac{BB}{bb}$$
$$\left(\frac{B-b}{5B-b}\right)^4=\frac{AA-BB}{aa-bb}\cdot\frac{AA}{aa}$$
$$\frac{a-A}{B-b}=\frac{5B-b}{5A-a}=\sqrt[4]{\frac{aB}{bA}}$$

$$(a-A)(5A-a)^5=256\,Aabb(aa-bb)$$
$$(B-b)(5B-b)^5=256\,Baab(aa-bb)$$

$$(a-A)^5(5A-a)=256\,AaBB(AA-BB)$$
$$(B-b)^5(5B-b)=256\,AAbB(AA-BB)$$

$$a-A=\frac{4x[xx]^2[x^5][x^{20}]}{[x][x^4]}=2^{\frac{4}{3}}\frac{a^{\frac{1}{6}}A^{\frac{1}{6}}B^{\frac{5}{12}}(AA-BB)^{\frac{5}{24}}}{b^{\frac{1}{12}}(aa-bb)^{\frac{1}{24}}}$$

$$5A-a=4\frac{[x][x^4][x^{10}]^2}{[x^5][x^{20}]}=2^{\frac{4}{3}}\frac{a^{\frac{1}{6}}A^{\frac{1}{6}}b^{\frac{5}{12}}(aa-bb)^{\frac{5}{24}}}{B^{\frac{1}{12}}(AA-BB)^{\frac{1}{24}}}$$

Zu der Theorie der Fünftheilung gehören folgende Theoreme. Wir bezeichnen $1+xy.1+x^3y.1+x^5y\ldots 1+\frac{x}{y}.1+\frac{x^3}{y}.1+\frac{x^5}{y}\ldots$ durch (x,y), so ist

[69] $$(x,\alpha y).(x,\tfrac{y}{\alpha}) = P\{1+xx(yy+\tfrac{1}{yy})+x^8(y^4+\tfrac{1}{y^4})+\ldots\}$$
$$+Q\{y+\tfrac{1}{y}+x^4(y^3+\tfrac{1}{y^3})+x^{12}(y^5+\tfrac{1}{y^5})+\ldots\}$$

wo P, Q von y unabhängig.

Also

$$(x,\alpha i)(x,\tfrac{\alpha}{i}) = (xx,\alpha\alpha) = P(1-2xx+2x^8-\ldots) = \frac{[xx]^2}{[x^4]}P$$

oder

$$P = (xx,\alpha\alpha)\frac{[x^4]}{[xx]^2}$$

Ferner für $y=-\alpha x$

$$(x,-\alpha\alpha x).(x,-x) = 0$$
$$= P(1+\tfrac{1}{\alpha\alpha}+\alpha\alpha x^4+\tfrac{x^4}{\alpha^4}+\ldots)-Q(\alpha x+\tfrac{1}{\alpha x}+\tfrac{x}{\alpha^3}+\alpha^3x^7+\tfrac{x^7}{\alpha^5}+\ldots)$$

d. i.

$$P\{\alpha+\tfrac{1}{\alpha}+x^4(\alpha^3+\tfrac{1}{\alpha^3})+x^{12}(\alpha^5+\tfrac{1}{\alpha^5})+\ldots\}$$
$$=\frac{Q}{x}\{1+xx(\alpha\alpha+\tfrac{1}{\alpha\alpha})+x^8(\alpha^4+\tfrac{1}{\alpha^4})+\ldots\}$$

Nun ist

$$P = \frac{1}{[xx]^2}\{1+xx(\alpha\alpha+\tfrac{1}{\alpha\alpha})+\ldots\}$$

Also

$$Q = \frac{x}{[xx]}\{(\alpha+\tfrac{1}{\alpha})+x^4(\alpha^3+\tfrac{1}{\alpha^3})+\ldots\}$$

und unser Theorem

70. $$(x,\alpha y).(x,\tfrac{y}{\alpha}) = \frac{1}{[xx]^2}\left\{\begin{array}{l}\{1+xx(\alpha\alpha+\tfrac{1}{\alpha\alpha})+x^8(\alpha^4+\tfrac{1}{\alpha^4})+\ldots\}\\ \times\{1+xx(yy+\tfrac{1}{yy})+x^8(y^4+\tfrac{1}{y^4})+\ldots\}\\ +x\{\alpha+\tfrac{1}{\alpha}+x^4(\alpha^3+\tfrac{1}{\alpha^3})+x^{12}(\alpha^5+\tfrac{1}{\alpha^5})+\ldots\}\\ \times\{y+\tfrac{1}{y}+x^4(y^3+\tfrac{1}{y^3})+x^{12}(y^5+\tfrac{1}{y^5})+\ldots\}\end{array}\right\}$$

oder

$$(x,\alpha y).(x,\tfrac{y}{\alpha}) = \frac{[x^4]^2}{[xx]^2}\{(xx,\alpha\alpha).(xx,yy)+x\alpha y\,(xx,\alpha\alpha xx).(xx,xxyy)\}$$

(Man kann auch leicht die Reihe, wodurch P multiplicirt ist, $=0$ machen, durch $y = ix$)

71. $$(x,y)+(x,\tfrac{x}{y})\sqrt[4]{\tfrac{x}{yy}} = (x^{\frac{1}{4}},y^{\frac{1}{2}})\frac{[x^{\frac{1}{2}}]}{[xx]}$$

$$(x,\alpha x) = \tfrac{1}{\alpha}(x,\tfrac{x}{\alpha}), \qquad (x,\alpha xx) = \tfrac{1}{\alpha x}(x,\alpha)$$

Den Satz 70 kann man auch so enonciren

72. $$(x,\alpha).(x,\beta) = \frac{[x^4]^2}{[xx]^2}\{(xx,\alpha\beta).(xx,\tfrac{\alpha}{\beta})+x\alpha(xx,\alpha\beta xx).(xx,\tfrac{\alpha xx}{\beta})\}$$

Hieraus folgt

$$(x,\tfrac{x}{\alpha}).(x,\tfrac{x}{\beta}) = \frac{[x^4]^2}{[xx]^2}\{(xx,\tfrac{xx}{\alpha\beta}).(xx,\tfrac{\alpha}{\beta})+\alpha(xx,\alpha\beta).(xx,\tfrac{\alpha xx}{\beta})\}$$

hieraus ferner

73. $$(x,\alpha).(x,\beta)+(x,\tfrac{x}{\alpha}).(x,\tfrac{x}{\beta})\sqrt{\tfrac{x}{\alpha\beta}} = \frac{[x]^2}{[xx]^2}.(x^{\frac{1}{2}},\sqrt{\alpha\beta}).(x^{\frac{1}{2}},\sqrt{\tfrac{\beta}{\alpha}})$$

Nun ist

$$\frac{1+2x+2x^4+\ldots}{1+2x^5+2x^{20}+\ldots} = \frac{(ix^{\frac{5}{2}},\, ix^{\frac{3}{2}}).(ix^{\frac{5}{2}},\, -ix^{\frac{1}{2}})}{(ix^{\frac{5}{2}},\, -ix^{\frac{3}{2}}).(ix^{\frac{5}{2}},\, ix^{\frac{1}{2}})}$$
$$= \frac{(-x^5,xx)(-x^5,-x)+x(-x^5,-x^3)(-x^5,x^4)}{(-x^5,xx)(-x^5,-x)-x(-x^5,-x^3)(-x^5,x^4)}$$

Woraus der erste zu beweisende Satz von selbst folgt. Ebenso ist

$$\frac{1+2x+2x^4+\ldots}{1+2x^5+2x^{20}+\ldots} = \frac{1-\varepsilon x.1-\varepsilon\varepsilon x.1-\varepsilon^3 x.1-\varepsilon^4 x.1+\varepsilon xx.1+\varepsilon\varepsilon xx.1+\varepsilon^3 xx.1+\varepsilon^4 xx\ldots}{1+\varepsilon x.1+\varepsilon\varepsilon x.1+\varepsilon^3 x.1+\varepsilon^4 x.1-\varepsilon xx.1-\varepsilon\varepsilon xx.1-\varepsilon^3 xx.1-\varepsilon^4 xx\ldots}$$

$$= \frac{1-\varepsilon^4.1-\varepsilon^3}{1+\varepsilon^4.1+\varepsilon}.\frac{(ix^{\frac{1}{2}},\, i\varepsilon x^{\frac{1}{2}}).(ix^{\frac{1}{2}},\, i\varepsilon\varepsilon x^{\frac{1}{2}})}{(ix^{\frac{1}{2}},\, -i\varepsilon x^{\frac{1}{2}}).(ix^{\frac{1}{2}},\, i\varepsilon\varepsilon x^{\frac{1}{2}})}$$

$$= \frac{\varepsilon\varepsilon-\varepsilon^3.\varepsilon-\varepsilon^4}{\varepsilon\varepsilon+\varepsilon^3.\varepsilon+\varepsilon^4}.\frac{(-x,-\varepsilon^3 x).(-x,\varepsilon)-\varepsilon x(-x,\varepsilon^3 xx)(-x,-\varepsilon x)}{(-x,-\varepsilon^3 x).(-x,\varepsilon)+\varepsilon x(-x,\varepsilon^3 xx)(-x,-\varepsilon x)}$$

$$= \frac{-\varepsilon+\varepsilon\varepsilon+\varepsilon^3-\varepsilon^4}{+\varepsilon+\varepsilon\varepsilon+\varepsilon^3+\varepsilon^4}.\frac{(x,\varepsilon^3 x).(x,-\varepsilon)+\varepsilon^3(x,-\varepsilon^3)(x,\varepsilon x)}{(x,\varepsilon^3 x).(x,-\varepsilon)-\varepsilon^3(x,-\varepsilon^3)(x,\varepsilon x)}$$

Woraus der zweite zu beweisende Satz von selbst folgt.

$$(x,x) = 2(1+xx)^2(1+x^4)^2 \ldots = 2\frac{[x^4]^2}{[xx]^2}$$

$$(x,1) = (1+x)^2\ (1+x^3)^2 \ldots = \frac{[xx]^4}{[x]^2[x^4]^2}$$

Man hat also

$$(x,\alpha)^2 = \frac{[x^4]^2}{[xx]^2}\left\{(xx,\alpha\alpha)\frac{[x^4]^4}{[xx]^2[x^8]^2} + 2x\alpha(xx,\alpha\alpha xx)\frac{[x^8]^2}{[x^4]^2}\right\}$$

Durch die Entwickelung von $(x,y)^3$ erhält man

$$P\left\{1+x^3\left(y^3+\frac{1}{y^3}\right)+\ldots\right\}+Q\left\{x^{\frac{1}{3}}\left(y+\frac{1}{y}\right)+x^{\frac{4}{3}}\left(yy+\frac{1}{yy}\right)+x^{\frac{16}{3}}\left(y^4+\frac{1}{y^4}\right)+\ldots\right\}$$

also für $y = -\varepsilon x$

$$(1-\varepsilon\varepsilon)^3\frac{[x^6]^3}{[x^2]^3} = Qx^{-\frac{2}{3}}\{\varepsilon-\varepsilon\varepsilon-(\varepsilon-\varepsilon\varepsilon)xx-\ldots\}$$

oder

$$3x^{\frac{2}{3}}.\frac{[x^6]^3}{[x^2]^3} = Q(1-xx-x^4+x^{10}+x^{14}-\ldots) = [xx]Q$$

also

$$Q = 3x^{\frac{2}{3}}.\frac{[x^6]^3}{[xx]^4}$$

Durch Entwickelung von $(x,y).(x,-y)^2$ erhält man

$$P'\left\{1+x^3\left(y^3+\frac{1}{y^3}\right)+\ldots\right\}+Q'\left\{x^{\frac{1}{3}}\left(y+\frac{1}{y}\right)+x^{\frac{4}{3}}\left(yy+\frac{1}{yy}\right)+\ldots\right\}$$

$$(1-\varepsilon\varepsilon)(1+\varepsilon\varepsilon)^2\frac{[x^6][x^8][x^{12}]^3}{[xx][x^4]^3[x^{24}]} = Q'x^{-\frac{2}{3}}(\varepsilon-\varepsilon\varepsilon)[xx]$$

$$Q' = -x^{\frac{2}{3}}\frac{[x^6][x^8][x^{12}]^3}{[xx]^2[x^4]^3[x^{24}]}$$

Man folgert hieraus leicht, dass man setzen darf

$$T(x,y)^3 + U(x,y)(x,-y)^2 = (x^3,y^3)$$

so dass T und U Functionen von x. Um sie zu bestimmen setzen wir

1) $y = x$ so wird $T(x, x)^3 = (x^3, x^3)$ oder $T = \frac{1}{4}\frac{[x^{12}]^2[xx]^6}{[x^6]^2[x^4]^6}$

2) $y = -\varepsilon x$ so wird $T(x, -\varepsilon x)^2 = U(x, +\varepsilon x)^2$

oder

$$T\frac{(1-\varepsilon\varepsilon)^2[x^6]^2}{[xx]^2} = \frac{[x^{12}]^2[xx]^2}{[x^6]^2[x^4]^2}U(1+\varepsilon\varepsilon)^2$$

oder

$$T = -\frac{1}{3}U\frac{[xx]^4[x^{12}]^2}{[x^6]^4[x^4]^2}$$

oder

$$U = -\frac{3}{4}\frac{[x^6]^2[xx]^2}{[x^4]^4}$$

folglich

$$\frac{T\left[\frac{(x,-y)}{(x,y)}\right]^3 + U\frac{(x,-y)}{(x,y)}}{T + U\left[\frac{(x,-y)}{(x,y)}\right]^2} = \frac{(x^3,-y^3)}{(x^3,y^3)}$$

also

$$\frac{\frac{[xx]^4}{[x^4]^2}\left[\frac{(x,-y)}{(x,y)}\right]^3 - 3\frac{[x^6]^4}{[x^{12}]^2}\frac{(x,-y)}{(x,y)}}{\frac{[xx]^4}{[x^4]^2} - 3\frac{[x^6]^4}{[x^{12}]^2}\left[\frac{(x,-y)}{(x,y)}\right]^2} = \frac{(x^3,-y^3)}{(x^3,y^3)}$$