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13.

Theoriae transcendentium Abelianarum primi ordinis adumbratio levis.

(Auctore Dr. *A. Göpel.*)

Quum theoria transcendentium *Abelianarum* jam inde a longo tempore geometras defatigaverit, cujus rei hice annales et praemium anno praeterito ab Academ. Paris. propositum testes sunt amplissimi, ita et ipse ante hos septem vel octo annos nonnulla in hac re excogitavi. Etenim quum functionibus ellipticis operam darem, statim deprehendi eas methodo ab *Abeliana* illa et *Jacobiana* omnino diversa tractari posse. Neque longum fuit quin animadverterem, hanc methodum quaestiones multo generaliores amplecti et fundamentum esse theoriae transcendentium *Abelianarum*, imo omnium quae ex integratione quantitatum algebraicarum oriuntur. Has meas lucubrationes cum geometris communicare in dies distuli, sperans ut aliquando tempus daretur, quo dissertationem accurate elaboratam et attentione doctorum dignam offerre possem; praesertim quum analystarum neminem talem tractationem ne suspicari quidem viderem. Verumtamen prodierunt nuperrime in lucem binae epistolae D^i^ *Hermite* tribus annis abhinc ad D^m^ *Jacobi* datae (cf. T. 33 hujus diarii) in quibus quum plura doctissime de transcendentibus altioribus disputavit, tum nonnulla e theoria functionum ellipticarum proposuit, quae meis tam similia erant, ut mihi eandem viam ingredi videretur, quam ipse absolveram. Quamobrem, ne mea obsolescerent, incomte quam omnino nihil dicere malui; quanquam doctissimus vir se idem in theoria transcendentium *Abelianarum* adhuc praestare non posse fassus est. Constitui itaque, editore hujus diarii humanissime admonente, theoriae novae specimen aliquod tradere, quantulumcunque est; in quo conscribendo quum magis quam optabam festinandum fuisset, si quid minus lucide et concinne dictum est, veniam lectoris pro sua benevolentia et illa rei conditione imprecor. Viam autem, quam hic geometris patefeci, aliis persequendam et excolendam relinquo, fortunatis, quibus „Deus haec otia fecit."

Praeliminaria.

Constat ex elementis, quamlibet expressionem quadraticam binarum variabilium $\mathfrak{a}$ et $\mathfrak{b}$, qualis est

$$(1.)\qquad \alpha\mathfrak{a}^2+2\beta\mathfrak{a}\mathfrak{b}+\gamma\mathfrak{b}^2+2\delta\mathfrak{a}+2\varepsilon\mathfrak{b}+\zeta,$$

adjecta idonea constante, transformari posse in summam duorum quadratorum:

$$(2.)\qquad (u+\mathfrak{a}K+\mathfrak{b}L)^2+(u'+\mathfrak{a}K'+\mathfrak{b}L')^2.$$

Quod in theoria sectionum conicarum est problema de reductione aequationis ad centrum et diametros. Excipiendus autem est casus, ubi $\beta^2=\alpha\gamma$ quae aequivalet conditioni $KL'-K'L=0$, qui talem transformationem non patitur. Nam quum in universum habeatur

$$X^2+Y^2=\left(\frac{pX+qY}{\sqrt{(p^2+q^2)}}\right)^2+\left(\frac{qX-pY}{\sqrt{(p^2+q^2)}}\right)^2,$$

expressio (2.) eo casu reduci potest ad formam

$$\left(\frac{K'u-Ku'}{\sqrt{(K^2+K'^2)}}\right)^2+\left(\frac{Ku+K'u'+\mathfrak{a}(K^2+K'^2)+\mathfrak{b}(KL+K'L')}{\sqrt{(K^2+K'^2)}}\right)^2$$

quae manifesto locum habere nequit, nisi est $\beta^2=\alpha\gamma$, et adjecta idonea constante revera unum tantum quadratum continet. Hunc itaque casum, ubi $KL'-K'L$ evanescit, etiam hoc loco excipimus atque e nostris disquisitionibus excludimus.

Constantibus, quae sunt in expressione (2.), speciem paulo diversam tribuere placet. Loco enim ipsorum u, K, L, u', K', L' ponemus $u\sqrt{r}$, $2K\sqrt{r}$, $2L\sqrt{r}$, $u'\sqrt{r'}$, $2K'\sqrt{r'}$, $2L'\sqrt{r'}$, et considerabimus formulam

$$(3.)\qquad r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+r'(u'+2\mathfrak{a}K'+2\mathfrak{b}L')^2.$$

Jam patet, quod omnium quae sequuntur fundamentum est, si variabilis $\mathfrak{a}$ unitate augeatur, idem esse ac si quantitas u capiat incrementum $2K$ *simul*que u' incrementum $2K'$. Item si variabilis $\mathfrak{b}$ unitate augeatur, idem esse ac si quantitates u, u' *simul* capiant incrementa $2L, 2L'$. Quare si formatur complexus terminorum, qui ex expressione (3.) pro singulis valoribus integris indicum $\mathfrak{a}$ et $\mathfrak{b}$ oriuntur, manifestum est, hunc complexum, positis $u+2K$, $u'+2K'$ pro u, u' (vel etiam $u+2L$, $u'+2L'$ pro u, u'), ipsum in se redire, quoniam quivis terminus in proximum abit. Idem omnino valet de complexu terminorum, qui ex qualibet functione expressionis (3.) pro singulis valoribus integris indicum $\mathfrak{a}$ et $\mathfrak{b}$ proveniunt.

Expositio problematis.

Fingatur itaque series bis infinita, cujus terminus generalis sit functio exponentialis ipsius expressionis (3.), quae denotetur per $e^{ru^2+r'u'^2}P'''(u,u')$,

$$e^{ru^2+r'u'^2}P'''(u,u') = S e^{r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+r'(u'+2\mathfrak{a}K'+2\mathfrak{b}L')^2}$$

sive explicite

$$e^{ru^2+r'u'^2}P'''(u,u') =$$

$$\left\{\begin{array}{llll}
\dots + \dots\dots\dots & & & \\
\dots + e^{r(u-2K-2L)^2+r'(u'-2K'-2L')^2} & + e^{r(u-2L)^2+r'(u'-2L')^2} & + e^{r(u+2K-2L)^2+r'(u'+2K'-2L')^2} & + \dots \\
\dots + e^{r(u-2K)^2+r'(u-2K')^2} & + e^{ru^2+r'u'^2} & + e^{r(u+2K)^2+r'(u'+2K')^2} & + \dots \\
\dots + e^{r(u-2K+2L)^2+r'(u'-2K'+2L')^2} & + e^{r(u+2L)^2+r'(u'+2L')^2} & + e^{r(u+2K+2L)^2+r'(u'+2K'+2L')^2} & + \dots \\
\dots + \dots\dots\dots & & &
\end{array}\right.$$

Ex antecedentibus constat hanc functionem ipsarum u et u' esse periodicam, ita ut pro valoribus simultaneis

$$\left\{\begin{array}{l}
u = u+2K,\quad u+4K,\quad \text{etc.},\quad u+2L,\quad u+4L,\quad \text{etc.},\\
u' = u'+2K',\quad u'+4K',\quad \text{etc.},\quad u'+2L',\quad u'+4L',\quad \text{etc.}
\end{array}\right.$$

in se ipsa redeat. Juvat autem valores, quos illa pro intermediis *argumentorum* u et u' valoribus accipit, peculiaribus notis insignire. Itidem seriebus, quae ex iisdem terminis, alternantibus tamen signis, compositae sunt, sua signa dabimus. Hoc modo oriuntur sedecim functiones ipsarum u et u', quas sequenti modo denotamus:

$$(4.)\quad \left\{\begin{array}{lll}
e^{ru^2+r'u'^2}P(u,u') & = & S(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+r'(u'+2\mathfrak{a}K'+2\mathfrak{b}L')^2}\\
e^{ru^2+r'u'^2}P'(u,u') & = & S(-1)^{\mathfrak{b}} \times \text{ in eundem factorem}\\
e^{ru^2+r'u'^2}P''(u,u') & = & S(-1)^{\mathfrak{a}} \times \text{ in eundem}\\
e^{ru^2+r'u'^2}P'''(u,u') & = & S\quad 1 \quad \times \text{ in eundem}\\
e^{ru^2+r'u'^2}iQ(u,u') & = & S(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{r(u+(2\mathfrak{a}+1)K+2\mathfrak{b}L)^2+r'(u'+(2\mathfrak{a}+1)K'+2\mathfrak{b}L')^2}\\
e^{ru^2+r'u'^2}Q'(u,u') & = & S(-1)^{\mathfrak{b}} \times \text{ in eundem}\\
e^{ru^2+r'u'^2}iQ''(u,u') & = & S(-1)^{\mathfrak{a}} \times \text{ in eundem}\\
e^{ru^2+r'u'^2}Q'''(u,u') & = & S\quad 1 \quad \times \text{ in eundem}\\
e^{ru^2+r'u'^2}iR(u,u') & = & S(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{r(u+2\mathfrak{a}K+(2\mathfrak{b}+1)L)^2+r'(u'+2\mathfrak{a}K'+(2\mathfrak{b}+1)L')^2}\\
e^{ru^2+r'u'^2}iR'(u,u') & = & S(-1)^{\mathfrak{b}} \times \text{ in eundem}\\
e^{ru^2+r'u'^2}R''(u,u') & = & S(-1)^{\mathfrak{a}} \times \text{ in eundem}\\
e^{ru^2+r'u'^2}R'''(u,u') & = & S\quad 1 \quad \times \text{ in eundem}
\end{array}\right.$$

$$(4.)\quad\begin{cases} e^{ru^2+r'u'^2}S(u,u') &= S(-1)^{\mathfrak{a}+\mathfrak{b}}e^{r(u+(2\mathfrak{a}+1)K+(2\mathfrak{b}+1)L)^2+r'(u'+(2\mathfrak{a}+1)K'+(2\mathfrak{b}+1)L')^2} \\ e^{ru^2+r'u'^2}iS'(u,u') &= S(-1)^{\mathfrak{b}}\times \text{ in eundem} \\ e^{ru^2+r'u'^2}iS''(u,u') &= S(-1)^{\mathfrak{a}}\times \text{ in eundem} \\ e^{ru^2+r'u'^2}S'''(u,u') &= S\quad 1\quad\times \text{ in eundem.} \end{cases}$$

Sed quum argumentorum et periodorum duplicitate major, quam commodum est, molestia scribentibus et legentibus creetur, abbreviationibus nonnunquam utemur. Ita pro $P(u,u')$ simplicius scribemus P vel $P(u)$. Similiter in quantitatibus exponentialibus eam partem, quae ad secundum argumentum u' spectat, saepius omittemus, ita ut $e^{r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+\dots}$ sit pro $e^{r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+r'(u'+2\mathfrak{a}K'+2\mathfrak{b}L')^2}$; $e^{4r\mathfrak{a}Ku+4r\mathfrak{a}^2K^2+\dots}$ pro $e^{4r\mathfrak{a}Ku+4r\mathfrak{a}^2K^2+4r'\mathfrak{a}K'u'+4r'\mathfrak{a}^2K'^2}$; $P(u+A)$ pro $P(u+A,u'+A')$ et cetera.

Hisce positis, oportet proprietates functionum P, Q, etc. investigare et relationes algebraicas vel differentiales inter illas invenire *).

Periodicitas functionum P, Q, etc. duplex; functionum $\frac{P}{Q}$, $\frac{P'}{Q}$, etc. quadruplex.

Quemadmodum in antecedentibus functionem $e^{ru^2+r'u'^2}P'''$ duplici periodo $2K$, $2L$ (cum correspondentibus $2K'$, $2L'$) gaudere vidimus, ita etiam functiones $e^{ru^2+r'u'^2}Q'''$, $e^{ru^2+r'u'^2}R'''$, $e^{ru^2+r'u'^2}S'''$ eadem gaudere in promptu est. Reliquae autem, quarum termini alternantia signa habent, applicatis illis periodis ex parte signum mutant, sicut tabula infra apposita demonstrat. Functio ex. gr. $e^{ru^2+r'u'^2}iQ$, positis $u+2K$, $u'+2K'$ pro u, u', in sequentem abit

$$e^{r(u+2K)^2+\dots}iQ(u+2K) = S(-1)^{\mathfrak{a}+\mathfrak{b}}e^{r(u+(2\mathfrak{a}+3)K+2\mathfrak{b}L)^2+\dots}$$

sive posito $\mathfrak{a}-1$ pro $\mathfrak{a}$, unde nihil mutatur, in

$$\begin{aligned} e^{r(u+2K)^2+\dots}iQ(u+2K) &= S(-1)^{\mathfrak{a}+\mathfrak{b}+1}e^{r(u+(2\mathfrak{a}+1)K+2\mathfrak{b}L)^2+\dots} \\ &= -e^{ru^2+\dots}iQ. \end{aligned}$$

Periodo autem iterum applicata, signum iterum mutatur et functio nostra in pristinum statum revertitur, ita ut sit

$$e^{r(u+4K)^2+\dots}Q(u+4K) = e^{ru^2+r'u'^2}Q.$$

Functiones itaque $e^{ru^2+r'u'^2}P$, etc. *mutatis* u, u' *in* $u+4K$, $u'+4K'$, *vel in* $u+4L$, $u'+4L'$ *nihil mutantur.* Quod primum est.

Verumtamen etiam functionibus ipsis P, Q, etc. suae sunt periodi, ab illis diversae. Divisis enim aequationibus (4.) utrimque per $e^{ru^2+r'u'^2}$, exponentes

*) Vix necessarium videtur peritos monere, simili modo functiones trium vel plurium argumentorum considerari posse; quae ad theoriam altiorum transcendentium perducunt.

singulorum terminorum fiunt expressiones ***lineares*** ipsarum u et u'. Erit ex. gr.

$$S = S(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{2(2\mathfrak{a}+1)(rKu+r'K'u')+2(2\mathfrak{b}+1)(rLu+r'L'u')+r((2\mathfrak{a}+1)K+(2\mathfrak{b}+1)L)^2+r'((2\mathfrak{a}+1)K'+(2\mathfrak{b}+1)L')^2}.$$

Quamobrem facile invenientur binae periodi $4A$ et $4B$ (cum correspondentibus $4A'$ et $4B'$), quibus applicatis quivis terminus in se ipse redeat. Debebit enim esse

$$(5.)\quad \begin{cases} 4rAK+4r'A'K' = \pi i, & 4rAL+4r'A'L' = 0, \\ 4rBK+4r'B'K' = 0, & 4rBL+4r'B'L' = -\pi i, \end{cases}$$

unde fit

$$(6.)\quad \begin{cases} A = \dfrac{\pi L' i}{4r(KL'-K'L)}, & A' = -\dfrac{\pi L i}{4r'(KL'-K'L)}, \\ B = \dfrac{\pi K' i}{4r(KL'-K'L)}, & B' = -\dfrac{\pi K i}{4r'(KL'-K'L)}. \end{cases}$$

Habentur itaque inter quadrantes A, B, K, L etc. relationes sequentes

$$AK' = BL', \qquad A'K = B'L.$$

Functiones itaque P, Q, etc. mutatis simul u, u' in $u+4A$, $u'+4A'$ vel in $u+4B$, $u'+4B'$ nihil mutantur. Quod secundum est *).

Quotientes $\frac{P}{Q}$, etc. manifesto habent periodos $4A$ et $4B$, quae singulis functionibus P, Q, etc. conveniunt. Sed est $\frac{P}{Q} = \frac{e^{ru^2+r'u'^2}P}{e^{ru^2+r'u'^2}Q}$, et similes. Unde quotiens $\frac{P}{Q}$ habet etiam periodos, quae singulis $e^{ru^2+r'u'^2}P$, $e^{ru^2+r'u'^2}Q$, etc. conveniunt; hasce dico $4K$ et $4L$.

Quotientes itaque binarum e functionibus P, Q, etc. quadrupliciter periodici sunt et mutatis simul u, u' in $u+4A$, $u'+4A'$, vel in $u+4B$, $u'+4B'$, vel in $u+4K$, $u'+4K'$ vel in $u+4L$, $u'+4L'$ nihil mutantur. Quod tertium est.

Quamvis autem in universum argumenta per integras periodos ($4A$ vel $4B$, vel etc.) incedere debeant, ut functiones P, Q, etc. vel $e^{ru^2+r'u'^2}P$, etc. eosdem valores accipiant, sunt tamen nonnullae, quae post dimidias jam periodos in se revertuntur; cujus rei supra vidisti exemplum. En tabulam conspectum mutationum offerentem, quas illae crescentibus argumentis per dimidias periodos subeunt.

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*) Videtur etiam tertia periodus $4b$ (et $4b'$) inveniri posse, siquidem ponitur

$$4rbK+4r'b'K' = \mu\pi i, \qquad 4rbL+4r'b'L' = \nu\pi i$$

existentibus μ et ν numeris integris. Sed invenitur $4b = 4\mu A - 4\nu B$, $4b' = 4\mu A' - 4\nu B'$; est itaque ex periodis $4A$, $4B$ conflata, itaque ab illis non diversa.

(7.)

	$2A$	$2B$	$2A+2B$	$2K$	$2L$	$2K+2L$
P	$+$	$+$	$+$	$-$	$-$	$+$
P'	$+$	$+$	$+$	$+$	$-$	$-$
P''	$+$	$+$	$+$	$-$	$+$	$-$
P'''	$+$	$+$	$+$	$+$	$+$	$+$
Q	$-$	$+$	$-$	$-$	$-$	$+$
Q'	$-$	$+$	$-$	$+$	$-$	$-$
Q''	$-$	$+$	$-$	$-$	$+$	$-$
Q'''	$-$	$+$	$-$	$+$	$+$	$+$
R	$+$	$-$	$-$	$-$	$-$	$+$
R'	$+$	$-$	$-$	$+$	$-$	$-$
R''	$+$	$-$	$-$	$-$	$+$	$-$
R'''	$+$	$-$	$-$	$+$	$+$	$+$
S	$-$	$-$	$+$	$-$	$-$	$+$
S'	$-$	$-$	$+$	$+$	$-$	$-$
S''	$-$	$-$	$+$	$-$	$+$	$-$
S'''	$-$	$-$	$+$	$+$	$+$	$+$.

Hinc ex. gr. facile desumitur esse $Q(u+2B)=Q$, $Q(u+2A+2B)=-Q$. Hoc tamen probe tenendum est, signa sub rubricis $2K$, $2L$, $2K+2L$ adnotata non referri ad ipsas functiones P, Q, etc., sed ad hasce $e^{ru^2+r'u'^2}P$, etc. Invenietur itaque ex. gr. non $Q(u+2L)=-Q(u)$, sed

$$e^{r(u+2L)^2+r'(u+2L')^2}Q(u+2L) = -e^{ru^2+r'u'^2}Q(u)$$

vel si mavis

$$e^{4rLu+4rL^2+\dots}Q(u+2L) = -Q(u).$$

Nexus inter sedecim functiones P, Q, etc. quoad argumenta.

Functiones $e^{ru^2+r'u'^2}P'''$, $e^{ru^2+r'u'^2}Q'''$, $e^{ru^2+r'u'^2}R'''$, $e^{ru^2+r'u'^2}S'''$ hoc modo alteras ex altera deduximus, ut argumenta u, u' quadrantibus periodorum (K, L, $K+L$ cum correspondentibus K', L', $K'+L'$) augeremus; de quo retrospicias ad aeqq. (4.). Idem valet de quaternis illis functionibus, quae vel duabus plagulis, vel una, vel nulla notatae sunt. Prorsus simili modo ex hisce P''', Q''', R''', S''' deduci possunt aliae, si argumenta quadrantibus A, B et $A+B$ (cum correspondentibus) augeatur. Quod si facis, in ipsas illas functiones P, P', P'', Q, etc. incidis, quas supra definivimus (vid. aeqq. (4.)); unde manifestum fit, non temere nos tales functiones introduxisse, quarum termini alternantibus signis affecti sint. Etenim quum habeatur ex. gr.

$$iQ = S(-1)^{\mathfrak{a}+\mathfrak{b}} e^{2(2\mathfrak{a}+1)(rKu+r'K'u')+4\mathfrak{b}(rLu+r'L'u')+a},$$

ubi a est constans, qua hoc loco non est opus, fit adhibitis aeqq. (5.)

$$iQ(u+A) = iS(-1)^{\mathfrak{b}} e^{2(2\mathfrak{a}+1)(rKu+r'K'u')+4\mathfrak{b}(rLu+r'L'u')+a},$$

$$iQ(u+B) = S(-1)^{\mathfrak{a}} e^{2(2\mathfrak{a}+1)(rKu+r'K'u')+4\mathfrak{b}(rLu+r'L'u')+a},$$

ergo e notatione nostra

$$Q(u+A) = Q', \qquad Q(u+B) = Q''.$$

Sequens tabula mutationes ante oculos ponit, quas functiones post quadrantes periodorum patiuntur:

(8.)

	A	B	$A+B$	K	L	$K+L$		A	B	$A+B$	K	L	$K+L$
P	P'	P''	P'''	iQ	iR	S	P''	P'''	P	P'	iQ''	R''	iS''
Q	Q'	Q''	Q'''	iP	$-iS$	$-R$	Q''	Q'''	Q	Q'	iP''	S''	iR''
R	R'	$-R''$	$-R'''$	$-iS$	iP	$-Q$	R''	R'''	R	R'	iS''	P''	iQ''
S	$-S'$	S''	S'''	$-iR$	$-iQ$	P	S''	S'''	$-S$	S'	iR''	Q''	iP''
P'	P	P'''	P''	Q'	iR'	iS'	P'''	P''	P'	P	Q'''	R'''	S'''
Q'	$-Q$	Q'''	$-Q''$	P'	iS'	iR'	Q'''	$-Q''$	Q'	$-Q$	P'''	S'''	R'''
R'	R	$-R'''$	$-R''$	S'	iP'	iQ'	R'''	R''	R'	R	S'''	P'''	Q'''
S'	S	$-S'''$	S''	R'	iQ'	iP'	S'''	$-S''$	S'	S	R'''	Q'''	P'''

De columnis, quae titulis K, L, $K+L$ inscriptae sunt, idem quod supra monendum est. Harum tabularum (7.) et (8.) maximus erit in sequentibus usus, quippe quarum ope ex data aequatione inter quantitates P, Q, etc. deducuntur quindecim aliae, quae inde auctis argumentis oriuntur. Ceterum in medio est, quomodo tabula (7.) ex (8.) derivari possit.

Valores argumentorum pro quibus sedecim functiones P, Q, etc. evanescunt.

Inter functiones P, Q, etc. decem sunt ordinis paris respectu variabilium simultanearum u, u'; videlicet P, P', P'', P'''; Q', Q'''; R'', R'''; S, S'''; reliquae sex imparis, nimirum Q, Q''; R, R'; S' S''. Ordinis autem paris ad similitudinem functionum algebraicarum eas dicimus, quae mutatis u, u' in $-u$, $-u'$ non mutantur; imparis quae signum mutant. Decem itaque illae evolvi poterunt in series hujusmodi

$$\alpha + (\beta u^2 + \beta_1 u u' + \beta_2 u'^2) + (\gamma u^4 + \gamma_1 u^3 u' + \gamma_2 u^2 u'^2 + \gamma_3 u u'^3 + \gamma_4 u'^4) + \text{ etc.};$$

reliquae sex in tales

$$(\alpha u + \alpha_1 u') + (\beta u^3 + \beta_1 u^2 u' + \beta_2 u u'^2 + \beta_3 u'^3) + \text{ etc.}$$

Quod ex ipsa expressionum (4.) forma cognoscere licet. Nam fit ex gr.

$$e^{ru^2+r'u'^2} i\, Q(-u,\,-u') = \mathfrak{S}(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{r(u-(2\mathfrak{a}+1)K-2\mathfrak{b}L)^2+\dots};$$

hinc positis $-\mathfrak{a}-1$, $-\mathfrak{b}$ pro $\mathfrak{a}$, $\mathfrak{b}$, quod salvo valore facere licet,

$$e^{ru^2+r'u'^2} i\, Q(-u,\,-u') = \mathfrak{S}(-1)^{\mathfrak{a}+\mathfrak{b}}\, e^{r(u+(2\mathfrak{a}+1)K+2\mathfrak{b}L)^2+\dots},$$

ergo $Q(-u,\,-u') = -Q(u,\,u')$. Similiter erit de functione Q''. Reliquae autem binae Q', Q''', quum singulis terminis signum $(-1)^{\mathfrak{a}}$ non sit praefixum, immutatae manent.

Hinc intelligitur, istas sex functiones imparis ordinis evanescere pro $u=0$, $u'=0$; bini enim termini se destruunt. Sed quum quaelibet harum functionum tabulae (8.) ope in reliquas quinque mutari possit, oportet eam etiam pro quinque aliis valoribus evanescere. Ita quum ex tabula (8.) habeatur

$$R'(u+A+B) = -R''(u),$$

$$e^{r(u+K+L)^2+\dots} R''(u+K+L) = e^{ru^2+\dots} i\, Q''(u)$$

fit propter $Q''(0)=0$, etiam

$$R''(K+L) = 0, \quad \text{nec non} \quad R'(A+B+K+L) = 0.$$

Hinc nascitur tabula exhibens valores, pro quibus sedecim functiones P, Q, etc. evanescunt.

(9.)

Q	Q''	R	R'	S'	S''	P	P'
0	0	0	0	0	0	K	L
B	B	A	A	K	L	L	$A+K$
$A+L$	L	$A+K$	K	$A+K$	$A+B$	$A+L$	$A+L$
$K+L$	$A+B+L$	$B+K$	$A+B+K$	$A+L$	$B+K$	$B+K$	$K+L$
$B+L$	$B+K+L$	$K+L$	$A+L+K$	$A+B$	$B+L$	$A+K+L$	$A+B+K$
$A+K+L$	$A+B+K+L$	$B+K+L$	$A+B+L+K$	$A+B+L$	$A+B+K$	$B+K+L$	$A+B+K+L$

P''	P'''	Q'	Q'''	R''	R'''	S	S'''
K	$A+K$	A	A	B	B	A	A
$B+K$	$B+L$	L	$A+B$	K	$A+B$	B	B
$B+L$	$A+B+K$	$A+B$	$A+L$	$A+B$	$A+K$	K	$A+L$
$K+L$	$A+B+L$	$K+L$	$B+L$	$K+L$	$B+K$	L	$B+K$
$A+B+L$	$A+K+L$	$A+B+L$	$B+K+L$	$A+B+K$	$A+K+L$	$A+B$	$A+B+K$
$A+B+K+L$	$B+K+L$	$A+K+L$	$A+B+K+L$	$B+K+L$	$A+B+K+L$	$K+L$	$A+B+L$

Monendum autem est primum, alteri argumento valores correspondentes 0, B', $A'+L'$, et sic porro tribuendos esse; secundum, functiones nostras propter periodicitatem etiam pro aliis et infinite multis valoribus evanescere;

eos dico, qui ex illis oriuntur additis semiperiodis $2A$, $2B$, $2K$ et $2L$. Quae consideratio etiam in construenda illa tabula adjumento fuit. Inventis enim sex valoribus argumentorum, qui ipsam Q nihilo aequalem reddant, quum sit $Q''(u+B) = Q(u)$, singulis addendus est quadrans B, ut inveniantur valores, pro quibus ipsa Q'' evanescit. Habentur itaque B, $2B$, $A+B+L$, $B+K+L$, $2B+L$, $A+B+K+L$, vel simplicius B, 0, $A+B+L$, $B+K+L$, L, $A+B+K+L$, quos in tabula legis. Et sic porro.

Animadvertisti in aequationibus (4.) notas functionales Q, Q'' etc., uno verbo eas, quae pro $u=0$, $u'=0$ annihilantur, factore imaginario i affectas esse. Quod quum arbitrarium esset, ideo factum est ut analogia functionum ellipticarum, quantum fieri posset, conservaretur. Consideratis enim quadrantibus K, L, sicut in illa theoria usuvenit, sub specie imaginaria Ki, Li, sedecim functiones nostrae sub specie reali exhiberi possunt, cujus ecce exempla:

$$P(u,u') = S(-1)^{\mathfrak{a}+\mathfrak{b}} q^{\mathfrak{a}^2} q_1^{2\mathfrak{a}\mathfrak{b}} q_2^{\mathfrak{b}^2} \cos(2\mathfrak{a}x+2\mathfrak{b}x'),$$
$$Q(u,u') = S(-1)^{\mathfrak{a}+\mathfrak{b}} q^{(\mathfrak{a}+\frac{1}{2})^2} q_1^{(2\mathfrak{a}+1)\mathfrak{b}} q_2^{\mathfrak{b}^2} \sin((2\mathfrak{a}+1)x+2\mathfrak{b}x'),$$

siquidem ponitur

$$2rKu+2r'K'u' = x, \qquad 2rLu+2r'L'u' = x',$$
$$e^{-4rK^2-4r'K'^2} = q, \qquad e^{-4rKL-4r'K'L'} = q_1, \qquad e^{-4rL^2-4r'L'^2} = q_2.$$

Relationes algebraicae inter functiones P, Q, etc.

Evolvantur quadrata ipsarum P, Q, etc. Quo facto invenietur, unumquodque lineariter per quatuor novas series exprimi posse. Datur itaque inter quina quadrata relatio linearis. Calculus autem hunc in modum se habet.

Ponatur

$$(10.)\qquad e^{ru^2+r'u'^2}X = S(-1)^{p\mathfrak{a}+q\mathfrak{b}} e^{r(u+(2\mathfrak{a}+\mu)K+(2\mathfrak{b}+\nu)L)^2+\dots},$$

quae forma omnes illas (4.) amplectitur; tribuendo enim literis p, q, μ, ν omnem ex ordine combinationem valorum 0 et 1 nascuntur sedecim illae aequationes. Ponatur iterum, quod idem est,

$$e^{ru^2+r'u'^2}X = S(-1)^{p\mathfrak{a}_1+q\mathfrak{b}_1} e^{r(u+(2\mathfrak{a}_1+\mu)K+(2\mathfrak{b}_1+\nu)L)^2+\dots}$$

et instituatur multiplicatio, quod fit multiplicando terminum generalem alterum per alterum. Hinc posito

$$(11.)\qquad \begin{cases} \mathfrak{a}+\mathfrak{a}_1 = \varepsilon, & \mathfrak{b}+\mathfrak{b}_1 = \vartheta, \\ \mathfrak{a}-\mathfrak{a}_1 = \varepsilon_1, & \mathfrak{b}-\mathfrak{b}_1 = \vartheta_1, \end{cases}$$

et advocata formula

$$x^2+y^2 = \tfrac{1}{2}(x-y)^2+\tfrac{1}{2}(x+y)^2,$$

prodit

$$(12.)\qquad e^{2ru^2+2r'u'^2}X^2 = S(-1)^{p\varepsilon+q\vartheta}\, e^{2r(\varepsilon_1 K+\vartheta_1 L)^2+\dots}\; e^{2r(u+(\varepsilon+\mu)K+(\vartheta+\nu)L)^2+\dots}.$$

Literis ε, ϑ, ε_1, ϑ_1 singuli deinceps valores integri tribuendi sunt, ea tamen conditione ut, quod aeqq. (11.) sibi postulant, ε et ε_1, item ϑ et ϑ_1, simul vel pares vel impares valores nanciscantur. Hinc patet, terminos seriei (12.) in quatuor genera distribui,

prout pares valores ipsius ε cum paribus ipsius ϑ combinantur
vel impares - - - ε cum paribus - -
vel pares - - - ε cum imparibus - -
vel impares - - - ε cum imparibus - -

Manifesto autem summa terminorum, qui ad idem genus pertinent, sub forma ducti e duabus seriebus hujusmodi exhiberi potest:

$$(13.)\qquad S e^{2r(\varepsilon_1 K+\vartheta_1 L)^2+\dots} \times S e^{2r(u+(\varepsilon+\mu)K+(\vartheta+\nu)L)^2+\dots},$$

namque factor $(-1)^{p\varepsilon+q\vartheta}$ pro terminis ejusdem generis idem manet.

Dispescitur itaque series (12.) in quatuor summandos ita:

$$(14.)\qquad e^{2ru^2+2r'u'^2}X^2 = a_{00}+(-1)^p a_{10}+(-1)^q a_{01}+(-1)^{p+q} a_{11},$$

ubi a_{00} denotat valorem expressionis (13.) pro paribus valoribus ipsius ε cum paribus ipsius ϑ, a_{10} valorem ejusdem expressionis pro imparibus ipsius ε cum paribus ipsius ϑ, et sic porro.

Jam descendimus ad valores particulares ipsarum p, q, μ, ν, et posito brevitatis causa

$$(15.)\qquad \begin{cases} e^{2ru^2+2r'u'^2}\,T = S e^{2r(u+2\varepsilon K+2\vartheta L)^2+\dots}, \\ e^{2ru^2+2r'u'^2}\,U = S e^{2r(u+(2\varepsilon+1)K+2\vartheta L)^2+\dots}, \\ e^{2ru^2+2r'u'^2}\,V = S e^{2r(u+2\varepsilon K+(2\vartheta+1)L)^2+\dots}, \\ e^{2ru^2+2r'u'^2}\,W = S e^{2r(u+(2\varepsilon+1)K+(2\vartheta+1)L)^2+\dots}, \end{cases}$$

itemque istarum valoribus particularibus pro $u=0$, $u'=0$,

$$(16.)\qquad \begin{cases} t = S e^{2r(2\varepsilon K+2\vartheta L)^2+\dots}, \\ u = S e^{2r((2\varepsilon+1)K+2\vartheta L)^2+\dots}, \\ v = S e^{2r(2\varepsilon K+(2\vartheta+1)L)^2+\dots}, \\ w = S e^{2r((2\varepsilon+1)K+(2\vartheta+1)L)^2+\dots}, \end{cases}$$

invenimus aequationes sequentes*),

*) Experti facile providebunt, hasce aequationes ad transformationem secundi ordinis nostrarum functionum pertinere. Nam si in nostras notationes $P(u, u')$ etc. quantitatem r introducere volumus, ita ut audiant $P(u, u'; r)$, functiones illae T, U, V, W exhibentur hoc modo $P'''(u, u'; 2r)$, $Q'''(u, u'; 2r)$ etc. De quo infra.

$$
(17.)\quad \left\{\begin{array}{ll}
P^2 = tT-uU-vV+wW, & R^2 = vT-wU-tU+uW,\\
P'^2 = tT+uU-vV-wW, & R'^2 = vT+wU-tV-uW,\\
P''^2 = tT-uU+vV-wW, & R''^2 = vT-wU+tV-uW,\\
P'''^2 = tT+uU+vV+wW, & R'''^2 = vT+wU+tV+uW,\\
Q^2 = uT-tU-wV+vW, & S^2 = wT-vU-uV+tW,\\
Q'^2 = uT+tU-wV-vW, & S'^2 = wT+vU-uV-tW,\\
Q''^2 = uT-tU+wV-vW, & S''^2 = wT-vU+uV-tW,\\
Q'''^2 = uT+tU+wV+vW, & S'''^2 = wT+vU+uV+tW.
\end{array}\right.
$$

Selectis itaque quatuor quibusvis e quadratis P^2, Q^2, etc., functiones T, U, V, W, ergo etiam reliqua duodecim quadrata poterunt per illa lineariter exprimi. Coëfficientes harum relationum pendebunt a valoribus constantibus t, u, v, w. Praestat autem valores particulares ipsarum P, Q, R, etc. introducere, quam alienas constantes. Quamobrem inveniendae sunt relationes inter t, u, v, w et inter valores, quos functiones P, Q, R, etc. pro $u=0$, $u'=0$ induunt, quos per correspondentes literas graecas easque minusculas significabimus. Id quod fit, ponendo in aeqq. (17.) $u=0$, $u'=0$.

Hinc habetur

$$
(18.)\quad \left\{\begin{array}{l}
\varpi^2 = t^2-u^2-v^2+w^2,\\
\varpi'^2 = t^2+u^2-v^2-w^2,\\
\varpi''^2 = t^2-u^2+v^2-w^2,\\
\varpi'''^2 = t^2+u^2+v^2+w^2,
\end{array}\right.
$$

$$
(19.)\quad \left\{\begin{array}{ll}
k'^2 = 2tu-2vw, & k'''^2 = 2tu+2vw,\\
\varrho''^2 = 2tv-2uw, & \varrho'''^2 = 2tv+2uw,\\
\sigma^2 = 2tw-2uv, & \sigma'''^2 = 2tw+2uv.
\end{array}\right.
$$

Valores k, k'', ϱ, ϱ', σ', σ'' evanescunt, sicuti fieri debere supra vidimus. Ex (18. et 19.) deducuntur sequentes:

$$
(20.)\quad \left\{\begin{array}{r}
\varpi^2+\varpi'^2+\varpi''^2+\varpi'''^2 = 4t^2,\\
-\varpi^2+\varpi'^2-\varpi''^2+\varpi'''^2 = 4u^2,\\
-\varpi^2-\varpi'^2+\varpi''^2+\varpi'''^2 = 4v^2,\\
\varpi^2-\varpi'^2-\varpi''^2+\varpi'''^2 = 4w^2,
\end{array}\right.
$$

$$
(21.)\quad \left\{\begin{array}{ll}
k'''^2+k'^2 = 4tu, & k'''^2-k'^2 = 4vw,\\
\varrho'''^2+\varrho''^2 = 4tv, & \varrho'''^2-\varrho''^2 = 4uw,\\
\sigma'''^2+\sigma^2 = 4tw, & \sigma'''^2-\sigma^2 = 4uv.
\end{array}\right.
$$

Eliminatis hinc ipsis t, u, v, w restant sex aequationes inter constantes ϖ, ϖ', etc., quarum itaque sex per reliquas quatuor inter se independentes exprimi poterunt. Verum hoc loco sufficiat, insigniores relationes inter ipsas apponere, quarum in posterum usus erit.

Ex aeqq. (21.) invenitur

$$(22.)\quad k'''^4 - k'^4 = \varrho'''^4 - \varrho''^4 = \sigma'''^4 - \sigma^4 \; (= 16\,tuvw).$$

Ex aeqq. (18.) et (19.) est

$$\varpi^2 + \sigma^2 = (t+w)^2 - (u+v)^2, \qquad \varpi^2 - \sigma^2 = (t-w)^2 - (u-v)^2,$$

ergo

$$(23.)\quad \begin{cases} \varpi^4 - \sigma^4 = \varpi'^4 - k'^4 = \varpi''^4 - \varrho''^4 \\ (= (t+u+v+w)(t+u-v-w)(t-u+v-w)(t-u-v+w)). \end{cases}$$

Similiter invenitur

$$\varpi'''^4 - \varpi^4 = k'''^4 + \varrho''^4 = k'^4 + \varrho'''^4 \; (= 4(t^2+w^2)(u^2+v^2)),$$

et hinc adhibitis aeqq. (22.) et (23.)

$$(24.)\quad \begin{cases} \varpi'''^4 - \varpi'^4 = \varrho'''^4 + \sigma^4 = \varrho''^4 + \sigma'''^4, \\ \varpi'''^4 - \varpi''^4 = \sigma'''^4 + k'^4 = \sigma^4 + k'''^4. \end{cases}$$

Porro est $\varpi^2\varpi'''^2 - \varpi'^2\varpi''^2 = 4t^2w^2 - 4u^2v^2$; dehinc et similiter:

$$(25.)\quad \begin{cases} \varpi^2\varpi'''^2 - \varpi'^2\varpi''^2 = \sigma^2\sigma'''^2, \\ \varpi'^2\varpi'''^2 - \varpi''^2\varpi^2 = k'^2k'''^2, \\ \varpi''^2\varpi'''^2 - \varpi^2\varpi'^2 = \varrho''^2\varrho'''^2, \end{cases}$$

$$(26.)\quad \begin{cases} \varpi^2k'''^2 - \varpi''^2k'^2 = \varrho'''^2\sigma^2, \\ \varpi'^2k'''^2 - \varpi'''^2k'^2 = \varrho''^2\sigma^2, \\ \varpi''^2k'''^2 - \varpi^2k'^2 = \varrho''^2\sigma'''^2, \\ \varpi'''^2k'''^2 - \varpi'^2k'^2 = \varrho'''^2\sigma'''^2, \end{cases}$$

$$(27.)\quad \begin{cases} \varpi^2\varrho'''^2 - \varpi'^2\varrho''^2 = k'''^2\sigma^2, \\ \varpi'^2\varrho'''^2 - \varpi^2\varrho''^2 = k'^2\sigma'''^2, \\ \varpi''^2\varrho'''^2 - \varpi'''^2\varrho''^2 = k'^2\sigma^2, \\ \varpi'''^2\varrho'''^2 - \varpi''^2\varrho''^2 = k'''^2\sigma'''^2, \end{cases}$$

$$(28.)^{*)}\quad \begin{cases} \varpi^2\sigma'''^2 - \varpi'''^2\sigma^2 = k'^2\varrho''^2, \\ \varpi'^2\sigma'''^2 - \varpi''^2\sigma^2 = k'^2\varrho'''^2, \\ \varpi''^2\sigma'''^2 - \varpi'^2\sigma^2 = k'''^2\varrho''^2, \\ \varpi'''^2\sigma'''^2 - \varpi^2\sigma^2 = k'''^2\varrho'''^2. \end{cases}$$

---

*) Aequationes (22—28.), si series dupliciter infinitas respicis, quae literis ϖ, $\varkappa$, ϱ, etc. denotantur, haud minus memorabiles sunt, quam illa una, quae in theoria functionum ellipti-

Aequationum (20. et 21.) ope quantitates t, u, v, w poterunt eliminari ex aeqq. (17.), quo facto eliminatis etiam ipsis T, U, V, W ex quinis, invenietur relatio cujus coëfficientes per constantes ϖ, ϖ', ... exprimuntur. Quae methodus nimium molestos calculos poscit. Semel autem cognita aequationum quaesitarum forma, ex. gr.

$$P^2 = aP'^2 + bS'^2 + cP''^2 + dS''^2,$$

res multo faciliori negotio per considerationem valorum particularium ipsorum u, u' absolvitur. Positis enim $u=0$, $u'=0$; vel $u=K$, $u'=K'$; vel similibus, mox habebitur tabulae (8.) ope sufficiens numerus aequationum, ex quibus coëfficientes a, b, c, d determinare liceat. Fit enim

pro $u = K$, $u' = K'$ *); $0 = ak'^2 - d\varrho''^2$,
pro $u = A+L$, $u' = A'+L'$; $0 = c\varrho'''^2 + dk'''^2$,
pro $u = B+K$, $u' = B'+K'$; $0 = ak'''^2 + b\varrho'''^2$,
pro $u = 0$, $u' = 0$; $\varpi^2 = a\varpi'^2 + c\varpi''^2$,
pro $u = A$, $u' = A'$; $\varpi'^2 = a\varpi^2 + b\sigma^2 + c\varpi'''^2 + d\sigma'''^2$.

Ex prioribus tribus statim concluditur esse $a = e\varrho''^2\varrho'''^2$, $b = -e\varrho''^2k'''^2$, $c = -ek'^2k'''^2$, $d = ek'^2\varrho'''^2$; quibus valoribus in quartam substitutis fit

$$\varpi^2 = e(\varpi'^2\varrho''^2\varrho'''^2 - \varpi''^2k'^2k'''^2),$$

sive propter aeqq. (25.)

$$\begin{aligned}\varpi^2 &= e(\varpi'^2(\varpi''^2\varpi'''^2 - \varpi^2\varpi'^2) - \varpi''^2(\varpi'^2\varpi'''^2 - \varpi''^2\varpi^2)) \\ &= -e\varpi^2(\varpi'^4 - \varpi''^4),\end{aligned}$$

unde $e = -\frac{1}{\varpi'^4 - \varpi''^4}$. Quodsi valores ipsarum a, b, c, d in aequationem quintam substituere mavis, habebis

$$\varpi'^2 = e(\varpi^2\varrho''^2\varrho'''^2 - \sigma^2\varrho''^2k'''^2 - \varpi'''^2k'^2k'''^2 + \sigma'''^2k'^2\varrho'''^2),$$

sive

$$\varpi'^2 = e(\varrho''^2(\varpi^2\varrho'''^2 - k'''^2\sigma^2) - k'^2(\varpi'''^2k'''^2 - \varrho'''^2\sigma'''^2)),$$

quae propter primam ipsarum (27.) et ultimam ipsarum (26.) abit in hanc:

$$\varpi'^2 = e(\varpi'^2\varrho''^4 - \varpi'^2k'^4) = e\varpi'^2(\varrho''^4 - k'^4),$$

---

carum earum locum tenet, videlicet

$$(1-2q+2q^4-2q^9+\ldots)^4 + (2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\ldots)^4 = (1+2q+2q^4+2q^9+\ldots)^4.$$

Cf. ***Jacobi***, Theoria funct. ellipt. p. 184.

*) Data aequatione *homogenea* gradus m^{ti} inter quantitates P, Q, etc., licet etiam quadrantes K, L, $K+L$ applicare. Quod manifestum fit multiplicando aequationem per $e^{mru^2 + mr'u'^2}$.

et haec aeqq. (23.) gratia in sequentem

$$1 = -e(\varpi'^4 - \varpi''^4),$$

unde $e = -\frac{1}{\varpi'^4 - \varpi''^4}$, sicut supra. Functio itaque P^2 hunc in modum exprimitur

$$(29.)\quad (\varpi'^4 - \varpi''^4)P^2 = -\varrho''^2\varrho'''^2 P'^2 + \varrho''^2 k'''^2 S'^2 + k'^2 k'''^2 P''^2 - k'^2\varrho'''^2 S''^2.$$

Ex hac deducuntur sequentes ope tabulae (8.) ponendo vicissim $u+K+L$, $u+A+B$, $u+A+B+K+L$ pro u:

$$(30.)\quad \begin{cases} (\varpi'^4 - \varpi''^4)S^2 = -\varrho''^2 k'''^2 P'^2 + \varrho''^2\varrho'''^2 S'^2 + k'^2\varrho'''^2 P''^2 - k'^2 k'''^2 S''^2, \\ (\varpi'^4 - \varpi''^4)P'''^2 = k'^2 k'''^2 P'^2 - k'^2\varrho'''^2 S'^2 - \varrho''^2\varrho'''^2 P''^2 + \varrho''^2 k'''^2 S''^2, \\ (\varpi'^4 - \varpi''^4)S'''^2 = k'^2\varrho'''^2 P'^2 - k'^2 k'''^2 S'^2 - \varrho''^2 k'''^2 P''^2 + \varrho''^2\varrho'''^2 S''^2. \end{cases}$$

Similiter similia:

$$(31.)\quad \begin{cases} (\varpi'^4 - \varpi''^4)Q'^2 = \varpi'^2 k'^2 P'^2 - \varpi''^2\varrho''^2 S'^2 - \varpi''^2 k'^2 P''^2 + \varpi'^2\varrho''^2 S''^2, \\ (\varpi'^4 - \varpi''^4)R'^2 = -\varpi''^2\varrho''^2 P'^2 + \varpi'^2 k'^2 S'^2 + \varpi'^2\varrho''^2 P''^2 - \varpi''^2 k'^2 S''^2, \\ \text{et cetera.} \end{cases}$$

Quoniam hoc modo functiones P, Q, etc. algebraice ad quaslibet quaternas (dic P', S', P'', S'') reductae sunt, restat ut relationes inter has quatuor investigentur, quarum manifesto nulla inter quadrata P'^2, S'^2, P''^2, S''^2 locum habere potest; tunc enim aliquis coëfficientium a, b, c, d, quos jamjam determinavimus, indeterminatus evadere debuisset. Oportet itaque producta binarum PQ, PR, etc. considerare. Quem ad finem itidem multiplicatio terminorum generalium institui potest. Attamen quum praevidere liceat, hoc modo ad relationes lineares inter producta binarum perveniri, calculus simplicissimus consideratione sequente perficitur.

Si datur relatio inter producta binarum, talis ex. gr.

$$PS = aP'Q + A,$$

ubi A summam reliquorum terminorum denotat, fit etiam ex tabula (7.)

$$PS = -aP'Q + A,$$

ergo addendo, $2PS = A + A$, ex qua productum $P'Q$ eliminatum est. Relatio itaque ea, quae ipsum PS continet, alios terminos continere nequit, nisi tales, qui adhibitis mutationibus tabulae (7.) immutati maneant. Qui sequentes esse inveniuntur: $P'S'$, $P''S''$, $P'''S'''$, QR, $Q'R'$, $Q''R''$, $Q'''R'''$; quapropter haberi poterit

$$aPS + bP'S' + cP''S'' + dP'''S''' + eQR + fQ'R' + gQ''R'' + hQ'''R''' = 0.$$

Hi tamen termini ope tabulae (8.) in dua systemata quaternorum discerpuntur. Mutato enim u in $u+K+L$, unde P in S et S in P mutatur, fit

$$aPS - bP'S' - cP''S'' + dP'''S''' + eQR - fQ'R' - gQ''R'' + hQ'''R''' = 0;$$

unde addendo et subtrahendo deducitur

$$aPS + dP'''S''' + eQR + hQ'''R''' = 0,$$
$$bP'S' + cP''S'' + fQ'R' + gQ''R'' = 0.$$

Et generaliter dato quolibet producto, verbi causa RS'', inveniri poterunt tria alia, e quibus cum illo relatio linearis formare liceat.

Ponatur itaque

$$aP'S' + bP''S'' + cQ'R' + dQ''R'' = 0.$$

Hinc tabulae (8.) beneficio deducuntur sequentes:

$$\begin{aligned} aPS + bP'''S''' - cQR + dQ'''R''' &= 0, \\ aP'''S''' + bPS + cQ'''R''' - dQR &= 0, \\ aQR + bQ'''R''' - cPS + dP'''S''' &= 0, \\ aQ'''R''' + bQR + cP'''S''' - dPS &= 0, \end{aligned}$$

unde, posito $u = 0$, $u' = 0$,

$$\begin{aligned} a\varpi\sigma + b\varpi'''\sigma''' + dk'''\varrho''' &= 0, \\ a\varpi'''\sigma''' + b\varpi\sigma + ck'''\varrho''' &= 0, \\ bk'''\varrho''' - c\varpi\sigma + d\varpi'''\sigma''' &= 0, \\ ak'''\varrho''' + c\varpi'''\sigma''' - d\varpi\sigma &= 0. \end{aligned}$$

Valoribus ipsarum a et b ex posterioribus in prioribus substitutis fit

$$dk'''^2\varrho'''^2 = d(\varpi'''^2\sigma'''^2 - \varpi^2\sigma^2),$$
$$ck'''^2\varrho'''^2 = c(\varpi'''^2\sigma'''^2 - \varpi^2\sigma^2),$$

quae propter aeqq. (28.) identicae sunt, ita ut c et d arbitrariae maneant. Posito itaque 1) $c = 1$, $d = 0$; 2) $c = 0$, $d = 1$, inveniuntur binae quae sequuntur:

$$(32.) \quad \begin{cases} k'''\varrho'''Q'R' = \varpi'''\sigma'''P'S' - \varpi\sigma P''S'', \\ k'''\varrho'''Q''R'' = -\varpi\sigma P'S' + \varpi'''\sigma'''P''S''. \end{cases}$$

Istae aequationes suppeditant novam, quam quaesivimus, relationem inter quatuor quantitates P', S', P'', S''. Nam quum ipsas Q', R' ex antecedentibus per illas P', S', P'', S'' exprimere liceat, obtinetur

$$k'''^2\varrho'''^2 Q'^2 R'^2 = (\varpi'''\sigma'''P'S' - \varpi\sigma P''S'')^2$$

sive e formulis (31.)

$$\begin{aligned} k'''^2\varrho'''^2 &(\varpi'^2k'^2P'^2 - \varpi''^2\varrho''^2S'^2 - \varpi''^2k'^2P''^2 + \varpi'^2\varrho''^2S''^2) \\ &\times(-\varpi''^2\varrho''^2P'^2 + \varpi'^2k'^2S'^2 + \varpi'^2\varrho''^2P''^2 - \varpi''^2k'^2S''^2) \\ &= (\varpi'^4 - \varpi''^4)^2(\varpi'''\sigma'''P'S' - \varpi\sigma P''S'')^2, \end{aligned}$$

unde evolvendo et reducendo ope formularum (22.) — (28.) fit

$$(33.)\quad \left\{\begin{array}{l} P'^4+S'^4-\dfrac{k'''^4+\varrho'''^4}{k'''^2\varrho'''^2}P'^2S'^2-\dfrac{\varpi'^4+\varpi''^4}{\varpi'^2\varpi''^2}(P'^2P''^2+S'^2S''^2) \\ +\dfrac{k'^4+\varrho''^4}{k'^2\varrho''^2}(P'^2S''^2+P''^2S'^2)-\dfrac{2\varpi\sigma\varpi'''\sigma'''(\varpi'^4-\varpi''^4)^2}{\varpi'^2\varpi^2k'^2\varrho''^2k'''^2\varrho'''^2}P'S'P''S'' \\ +P''^4+S''^4-\dfrac{k'''^4+\varrho'''^4}{k'''^2\varrho'''^2}P''^2S''^2 \end{array}\right\}=0.$$

Secunda aequationum (31.) ad alteram relationem inter quantitates P', S', P'', P'' perducere videtur, sed videtur tantum. Namque eliminatis inde litteris Q'', R'', ad eandem illam (33.) pervenitur. Quod vel a priori demonstrari potest. Manifesto enim relatio ex secunda illa aequatione proveniens eandem formam habebit, quam aeq. (33.), neque nisi respectu coëfficientum ab ea differre poterit. Quod si revera locum haberet, eliminari posset ex binis terminus $P'S'P''S''$, unde resultaret relatio secundi gradus inter *quadrata* P'^2, S'^2, P''^2, S''^2. Introductis itaque functionibus T, U, V, W ex aeqq. (17.) prodiret relatio rationalis secundi gradus inter ipsas T, U, V, W. At hae non differunt ab ipsis P''', Q''', R''', S''', nisi quod quantitas r duplicata est, sicut ibi adnotavimus. Mutato itaque $2r$ in r prodiret relatio secundi gradus inter ipsas P''', Q''', R''', S'''. Quod absurdum esse supra vidimus.

Eodem fere modo demonstratur, reliquas aequationes, quae inter producta binarum obtinent (quales sunt $\varrho''\sigma'''RS' = \varpi''k'''PQ'-\varpi k'P''Q'''$, $\varrho''\sigma RS'' = \varpi'k'''P'''Q'-\varpi'''k'P'Q'''$ et cetera), ad eandem illam relationem (33.) perducere.

Apponimus hic formulas sequentes, quas vel per methodum valorum particularium vel multis aliis modis ex formulis jam cognitis eruere licet:

$$(34.)\quad \left(\frac{\varpi'^4-\varpi''^4}{k'''^2\varrho'''^2}\right)(P'''^2S'''^2-P^2S^2)$$
$$=\left\{P'^4-\frac{k'''^4+\varrho'''^4}{k'''^2\varrho'''^2}P'^2S'^2+S'^4\right\}-\left\{P''^4-\frac{k'''^4+\varrho'''^4}{k'''^2\varrho'''^2}P''^2S''^2+S''^4\right\},$$

$$(35.)\quad (P'''S'''+PS)^2=\frac{(\varpi'''\sigma'''+\varpi\sigma)^2}{\varpi'^2\varpi''^2}(P'^2P''^2+S'^2S''^2)-\frac{k'''^2\varrho'''^2}{k'^2\varrho''^2}(P'^2S''^2+P''^2S'^2)$$
$$+2\left(\frac{\varpi\sigma\varpi'''\sigma'''(k'^4+\varrho''^4)}{\varpi'^2\varpi''^2k'^2\varrho''^2}-\frac{\varpi^2\varpi'''^2+\sigma^2\sigma'''^2}{\varpi'^2\varpi''^2}\right)P'P''S'S'',$$

$$(36.)\quad (P'''S'''-PS)^2=\frac{(\varpi'''\sigma'''-\varpi\sigma)^2}{\varpi'^2\varpi''^2}(P'^2P''^2+S'^2S''^2)-\frac{k'''^2\varrho'''^2}{k'^2\varrho''^2}(P'^2S''^2+P''^2S'^2)$$
$$+2\left(\frac{\varpi\sigma\varpi'''\sigma'''(k'^4+\varrho''^4)}{\varpi'^2\varpi''^2k'^2\varrho''^2}+\frac{\varpi^2\varpi'''^2+\sigma^2\sigma'''^2}{\varpi'^2\varpi''^2}\right)P'P''S'S''.$$

Quibus magis singularitatis quam utilitatis causa adjungimus hasce:

$$\begin{array}{ll}
P'''^2 S'''^2 - P^2 S^2 = Q'''^2 R'''^2 - Q^2 R^2, & P'^2 S'^2 - P'' S''^2 = Q'^2 R'^2 - Q''^2 R''^2, \\
P^2 S'''^2 - P'''^2 S^2 = Q'^2 R''^2 - Q''^2 R'^2, & P'^2 S''^2 - P'''^2 S'^2 = Q^2 R'''^2 - Q'''^2 R^2, \\
P''^2 S'''^2 - P'^2 S^2 = Q'''^2 R''^2 - Q^2 R'^2, & P'''^2 S''^2 - P^2 S'^2 = Q''^2 R'''^2 - Q'^2 R^2, \\
P'^2 S''^2 - P''^2 S^2 = Q'^2 R'''^2 - Q''^2 R^2, & P^2 S''^2 - P'''^2 S'^2 = Q^2 R''^2 - Q'''^2 R'^2, \\
\text{etc.} & \text{etc.}
\end{array}$$

quae facillime ex relationibus inter producta binarum deducuntur. Adnotetur etiam, positis $u=0$, $u'=0$, rursus in aeqq. (25.) — (28.) incidi.

Relationes differentiales inter functiones P, Q, etc.

Quum secunda relatio algebraica inter functiones P', S', P'', S'' inveniri nequeat, confugiendum erit ad differentialia. Consideremus itaque expressionem $P'\partial S' - S'\partial P'$. Et primum quidem manifestum est, illam habere periodos $4A$, $4B$, quum propter $P'(u+4A) = P'$ sit etiam $\partial P'(u+4A) = \partial P'$, etc. Secundo expressio $e^{2ru^2+2r'u'^2}(P'\partial S' - S'\partial P')$ habet periodos $4K$, $4L$, id quod hoc modo demonstratur. Est

$$\begin{aligned}
\partial(e^{ru^2+r'u'^2} S') &= e^{ru^2+r'u'^2}\partial S' + (2ru\,\partial u + 2r'u'\,\partial u')\,e^{ru^2+r'u'^2} S', \\
\partial(e^{ru^2+r'u'^2} P') &= e^{ru^2+r'u'^2}\partial P' + (2ru\,\partial u + 2r'u'\,\partial u')\,e^{ru^2+r'u'^2} P',
\end{aligned}$$

unde multiplicando per $e^{ru^2+r'u'^2}P'$ et $e^{ru^2+r'u'^2}S'$ et subtrahendo fit

$$\begin{aligned}
&(e^{ru^2+r'u'^2}P')\,\partial(e^{ru^2+r'u'^2}S') - (e^{ru^2+r'u'^2}S')\,\partial(e^{ru^2+r'u'^2}P') \\
&= e^{2ru^2+2r'u'^2}(P'\partial S' - S'\partial P').
\end{aligned}$$

Qua de re, quum alterum membrum aequationis periodos $4K$ et $4L$ habeat, habebit etiam alterum. Expressio $P'\partial S' - S'\partial P'$ poterit itaque per functiones P, Q, etc. exprimi; hoc est, lineariter per quadrata P^2, Q^2, etc. et producta binarum PQ, PR, etc. Hoc etiam, seriebus in usum vocatis, demonstrari potest.

Ponatur enim brevitatis causa

$$\begin{aligned}
M_1 &= e^{r(u+2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)^2+\dots}, \\
M &= e^{r(u+(2\mathfrak{a}+1)K+(2\mathfrak{b}+1)L)^2+\dots},
\end{aligned}$$

erit

$$\begin{aligned}
e^{ru^2+r'u'^2}P' &= S(-1)^{\mathfrak{b}_1} M_1, \\
e^{ru^2+r'u'^2}iS' &= S(-1)^{\mathfrak{b}} M, \\
\partial M_1 &= \{2r(u+2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)\partial u+\dots\}M_1, \\
\partial M &= \{2r(u+(2\mathfrak{a}+1)K+(2\mathfrak{b}+1)L)\partial u+\dots\}M,
\end{aligned}$$

$$\begin{aligned}
e^{ru^2+r'u'^2}\left[\partial P' + (2ru\,\partial u + 2r'u'\,\partial u')P'\right] &= S(-1)^{\mathfrak{b}_1}\{2r(2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)\partial u+\dots\}M_1 \\
&\quad + S(-1)^{\mathfrak{b}_1}\{2ru\,\partial u+\dots\}M_1,
\end{aligned}$$

ergo

$$e^{ru^2+r'u'^2}\partial P' = S(-1)^{\mathfrak{b}_1}\{2r(2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)\partial u+\ldots\}M_1$$

et similiter

$$e^{ru^2+r'u'^2}i\,\partial S' = S(-1)^{\mathfrak{b}}\{2r((2\mathfrak{a}+1)K+(2\mathfrak{b}+1)L)\partial u+\ldots\}M.$$

Dehinc

$$e^{2ru^2+2r'u'^2}i(P'\partial S'-S'\partial P')$$
$$= S(-1)^{\mathfrak{b}+\mathfrak{b}_1}\{2r((2\mathfrak{a}-2\mathfrak{a}_1+1)K+(2\mathfrak{b}-2\mathfrak{b}_1+1)L)\partial u+\ldots\}MM_1.$$

Posito autem

$$\mathfrak{a}+\mathfrak{a}_1 = \varepsilon,\qquad \mathfrak{b}+\mathfrak{b}_1 = \vartheta,$$
$$\mathfrak{a}-\mathfrak{a}_1 = \varepsilon_1,\qquad \mathfrak{b}-\mathfrak{b}_1 = \vartheta_1,$$

fit

$$MM_1 = e^{2r((\varepsilon_1+\frac{1}{2})K+(\vartheta_1+\frac{1}{2})L)^2+\ldots}\cdot e^{2r(u+(\varepsilon+\frac{1}{2})K+(\vartheta+\frac{1}{2})L)^2+\ldots}$$

unde

$$e^{2ru^2+2r'u'^2}i(P'\partial S'-S'\partial P')$$
$$= S\{2r((2\varepsilon_1+1)K+(2\vartheta_1+1)L)\partial u+\ldots\}\,e^{2r((\varepsilon_1+\frac{1}{2})K+(\vartheta_1+\frac{1}{2})L)^2+\ldots}$$
$$\times S(-1)^{\vartheta}e^{2r(u+(\varepsilon+\frac{1}{2})K+(\vartheta+\frac{1}{2})L)^2+\ldots},$$

qua in formula, quum ε et ε_1 simul vel pares vel impares numeri esse debeant itemque ϑ et ϑ_1, distinguendum erit inter valores pares et impares. Quare designata serie $Se^{2r(u+(\varepsilon+\frac{1}{2})K+(\vartheta+\frac{1}{2})L)^2+\ldots}$ per T_1, U_1, V_1, W_1, prout vel ε par, ϑ par, vel ε impar, ϑ par, etc. habebitur

$$(37.)\qquad e^{2ru^2+2r'u'^2}i(P'\partial S'-S'\partial P') = a_1T_1+b_1U_1+c_1V_1+d_1W_1.$$

Coëfficientes manifesto sunt valores differentialium functionum T_1, U_1, V_1, W_1 pro $u=0$, $u'=0$. Series autem T_1, U_1, V_1, W_1 eaedem illae sunt, in quas producta $P'S'$, $P''S''$, $Q'R'$, $Q''R''$, etc. evolvi possunt; id quod vel sine calculo ex formulis supra laudatis intelligitur. Praeter enim signa $(-1)^{\mathfrak{b}}$, $(-1)^{\mathfrak{b}_1}$ et factores illos differentiales res omnino eodemmodo se habet. Exprimantur itaque ipsa T_1, U_1, V_1, W_1 per producta PS, $P'''S'''$, QR, $Q'''R'''$, etc. et substituantur valores in aequat. (37.), quo facto habebitur aeq. hujusce formae

$$(38.)\qquad P'\partial S'-S'\partial P' = aPS+bP'''S'''+cQR+dQ'''R'''$$
$$+a_1P'S'+b_1P''S''+c_1Q'R'+d_1Q''R'',$$

existentibus a, b, c, d etc. differentialibus formae $f\partial u+f'\partial u'$, ubi f et f' constantes sunt. Vidimus autem singulas $Q'R'$, $Q''R''$ exprimi posse per $P'S'$ et $P''S''$, ergo (mutando u in $u+A$) etiam QR, $Q'''R'''$ per PS, $P'''S'''$, quadere aeq. (38.) mutatur in formam

$$(39.)\qquad P'\partial S'-S'\partial P' = aPS+bP'''S'''+a_1P'S'+b_1P''S''.$$

Accuratior determinatio earum facilius fit per valores particulares. Quem ob finem notandum est, differentialia functionum Q', Q''', R'', cet., scilicet earum, quae posito $u=0$, $u'=0$ valores finitos accipiunt, pro iisdem valoribus evanescere. Invenitur enim

$$\partial P' = S(-1)^{\mathfrak{b}_1}\{2r(2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)\partial u+\dots\}e^{r(2\mathfrak{a}_1 K+2\mathfrak{b}_1 L)^2+\dots};$$

mutatis itaque $\mathfrak{a}_1$, $\mathfrak{b}_1$ in $-\mathfrak{a}_1$, $-\mathfrak{b}_1$, id quod valorem ipsius $\partial P'$ non afficit, terminus generalis in oppositum mutatur; bini ergo termini se destruunt. Reliquarum autem sex, nempe ipsarum Q, Q'', R, R', S', S'' differentialia non evanescunt pro $u=0$, $u'=0$; quorum valores peculiaribus notis insigniantur ∂k, $\partial k''$, $\partial \varrho$, $\partial \varrho'$, $\partial \sigma'$, $\partial \sigma''$. Invenitur ex. gr.

$$\partial k = f\partial u+f'\partial u',$$

ubi positum est

$$if = S(-1)^{\mathfrak{a}+\mathfrak{b}}2r((2\mathfrak{a}+1)K+2\mathfrak{b}L)e^{r((2\mathfrak{a}+1)K+2\mathfrak{b}L)^2+\dots},$$

$$if' = S(-1)^{\mathfrak{a}+\mathfrak{b}}2r'((2\mathfrak{a}+1)K'+2\mathfrak{b}L')e^{r((2\mathfrak{a}+1)K+2\mathfrak{b}L)^2+\dots}$$

et sic porro. Quibus positis, determinatio coëfficientium hoc modo succedit. Habetur aequatio

$$(39.)\quad P'\partial S'-S'\partial P' = aPS+bP'''S'''+a_1P'S'+b_1P''S'',$$

unde mutando u in $u+K+L$, adjumento tabulae (8.) prodit

$$P'\partial S'-S'\partial P' = aPS+bP'''S'''-a_1P'S'-b_1P''S''.$$

Coëfficientes itaque a_1 et b_1 manifesto evanescunt, et restat

$$(40.)\quad P'\partial S'-S'\partial P' = aPS+bP'''S'''.$$

Hinc autem fit mutando u in $u+A+B$,

$$(41.)\quad P''\partial S''-S''\partial P'' = aP'''S'''+bPS,$$

unde ponendo $u=0$, $u'=0$ coëfficientes a et b determinantur per aeqq.

$$(42.)\quad \begin{cases} \varpi'\partial\sigma' = a\varpi\sigma+b\varpi'''\sigma''', \\ \varpi''\partial\sigma'' = a\varpi'''\sigma'''+b\varpi\sigma, \end{cases}$$

quae valores sequentes suppeditant:

$$(43.)\quad \begin{cases} k'''^2\varrho'''^2 a = \varpi''\varpi'''\sigma'''\partial\sigma''-\varpi\varpi'\sigma\partial\sigma', \\ k'''^2\varrho'''^2 b = \varpi'\varpi'''\sigma'''\partial\sigma'-\varpi\varpi''\sigma\partial\sigma''. \end{cases}$$

Inveniuntur autem alii valores sequenti modo. Mutando u in $u+K$ ex aeqq. (40.) et (41.) eruitur:

$$(44.)\quad Q'\partial R'-R'\partial Q' = aQR+bQ'''R''',$$

$$(45.)\quad Q''\partial R''-R''\partial Q'' = -aQ'''R'''-bQR,$$

unde ponendo $u = 0$, $u' = 0$ prodeunt hae:

$$(46.)\quad \begin{cases} k'\partial\varrho' = bk'''\varrho''', \\ \varrho''\partial k'' = ak'''\varrho''', \end{cases}$$

quae non solum multo simpliciores sunt, quam aeqq. (42.) vel (43.), sed etiam, eliminatis a et b, relationes inter differentialia $\partial\sigma'$, $\partial\sigma''$, $\partial\varrho'$, $\partial k''$ praebent, scilicet:

$$(47.)\quad \begin{cases} k'''\varrho'''\varrho''\partial k'' = \varpi''\varpi'''\sigma'''\partial\sigma'' - \varpi\varpi'\sigma\partial\sigma', \\ k'''\varrho'''k'\partial\varrho' = \varpi'\varpi'''\sigma'''\partial\sigma' - \varpi\varpi''\sigma\partial\sigma''. \end{cases}$$

Simili modo inveniuntur hae:

$$(48.)\quad \begin{cases} k'\varrho''\varrho'''\partial k = \varpi\varpi'\sigma'''\partial\sigma'' - \varpi''\varpi'''\sigma\partial\sigma', \\ k'\varrho''k'''\partial\varrho = \varpi\varpi''\sigma'''\partial\sigma' - \varpi'\varpi'''\sigma\partial\sigma''. \end{cases}$$

Plures autem relationes inter differentialia ∂k, $\partial\varrho$, etc. non dantur, quum propter independentia ∂u, $\partial u'$, bina arbitraria manere debeant. Adhibitis itaque aeqq. (46.) nacti summus relationes sequentes:

$$(49.)\quad \begin{cases} P'\partial S' - S'\partial P' = \dfrac{\varrho''\partial k''}{\varrho'''k'''}PS + \dfrac{k'\partial\varrho'}{k'''\varrho'''}P'''S''', \\ P''\partial S'' - S''\partial P'' = \dfrac{\varrho''\partial k''}{k'''\varrho'''}P'''S''' + \dfrac{k'\partial\varrho'}{k'''\varrho'''}PS, \end{cases}$$

ita ut tres jam cognitae sint aequationes inter quantitates P', S', P'', S'' (producta enim PS, $P'''S'''$ puta per ipsas P', S', P'', S'' expressa, quod ex antecedentibus licet), quarum altera (33.) est algebraica quarti gradus, alterae binae sunt differentiales primi ordinis. Reliquae, quas aeqq. (49.) similes invenire licet, algebraice ex illis tribus deduci possunt; quod vel a priori demonstrandum est.

Oportet itaque quartam aliquam eruere; quam ob rem ad differentialia secunda ascendendum erit. Dico autem formulam $P'\partial^2P' - \partial P'^2$ lineariter per quadrata ipsarum P', S', P'', S'' exprimi posse, *siquidem differentialia ∂u, $\partial u'$ constantia esse ponuntur.* Primo enim intelligitur propter $\partial P'(u+4A) = \partial P'$, $\partial^2P'(u+4A) = \partial^2P'$ habere periodos $4A$ et $4B$. Deinde quum, posito brevitatis causa $e^{ru^2+r'u'^2} = M$, sit

$$\partial(MP') = M\partial P' + (2ru\partial u + 2r'u'\partial u')MP',$$

$$\partial^2(MP') = M\partial^2P' + 2(2ru\partial u + 2r'u'\partial u')M\partial P' + (2ru\partial u + 2r'u'\partial u')^2MP' \\ + (2r\partial u^2 + 2r'\partial u'^2)MP' + (2ru\partial^2u + 2r'u'\partial^2u')MP',$$

prodit

$$(MP')\partial^2(MP')-\partial(MP')^2 = M^2(P'\partial^2P'-\partial P'^2)+(2r\partial u^2+2r'\partial u'^2)(MP')^2$$
$$+(2ru\partial^2u+2r'u'\partial^2u')(MP')^2,$$

unde elicitur, formulam $e^{2ru^2+2r'u'^2}(P'\partial^2P'-\partial P'^2)$ habere periodos $4K$ et $4L$, siquidem ponitur $\partial^2u=0$, $\partial^2u'=0$. Potest itaque illud $P'\partial^2P'-\partial P'^2$ expressioni hujus formae aequari:

$$aP^2+bQ^2+cR^2+\ldots.+\alpha PQ+\beta PR+\gamma PS+\ldots.$$

Applicatis autem argumentorum incrementis, quae in tabula (7.) leguntur, ipsum $P'\partial^2P'-\partial P'^2$ non mutatur. Nec magis quadrata P^2, Q^2, etc. Productorum autem PQ, PR, etc., quodvis unum aliquando oppositum valorem induit; quare per se evanescere debent. Quadrata denique P^2, Q^2, etc. aeqq. (31.) ope ad quaterna P'^2, S'^2, P''^2, S''^2 reduci possunt. Restat itaque aequatio hujus formae:

$$P'\partial^2P'-\partial P'^2 = aP'^2+bS'^2+cP''^2+dS''^2.$$

Coëfficientes more solito determinantur. Quem ad finem animadvertendum est, sex differentialia secunda, nimirum ∂^2Q, ∂^2Q'', ∂^2R, ∂^2R', ∂^2S', ∂^2S'', evanescere pro valoribus $u=0$, $u'=0$. Reliqua finitos valores accipiunt et denotantur per $\partial^2\varpi$, $\partial^2\varpi'$, etc. Habent autem formam sequentem:

$$\partial^2\varpi = a\partial u^2+2b\partial u\partial u'+c\partial u'^2,$$

ubi a, b, c constantes. Ceterum haec differentialia omnia per unum quodlibet exprimi possunt. Nam quum habeatur ex. gr. $P\partial P'-P'\partial P =$ summae ex productis binarum, differentiando et ponendo $u=0$, $u'=0$ prodit $\varpi\partial^2\varpi'-\varpi'\partial^2\varpi$ $=$ quantitati a constantibus et a quantitatibus ∂k, $\partial\varrho$, etc. pendentem.

His positis invenitur aequatio

$$(50.)\quad P'\partial^2P'-\partial P'^2 = \frac{\varpi'^3\partial^2\varpi'-\varpi''^3\partial^2\varpi''}{\varpi'^4-\varpi''^4}P'^2+\frac{\varrho''^2\partial k''^2-k'^2\partial\varrho'^2}{\varpi'^4-\varpi''^4}S'^2$$
$$+\frac{k'^2\partial k''^2-\varrho''^2\partial\varrho'^2}{\varpi'^4-\varpi''^4}P''^2+\frac{\varpi''^2\partial\sigma''^2-\varpi'^2\partial\sigma''^2}{\varpi'^4-\varpi''^4}S''^2,$$

nec non similes pro expressionibus $P\partial^2P-\partial P^2$, $Q\partial^2Q-\partial Q^2$, etc.; quas ad secundam speciem nostrarum functionum facere infra videbis. Imprimis autem notamus formulam

$$(51.)\quad (P'P''+S'S'')\partial^2(P'P''+S'S'')-\partial(P'P''+S'S'')^2$$
$$=a(P'P''+S'S'')^2+b(P'^2S''^2+P''^2S'^2)+cP'P''S'S''+d(P'^2S'^2+P''^2S''^2),$$

ubi constantes a, b, c, d determinatae sunt per aeqq.

$$(52.)\quad \begin{cases} a\varpi'\varpi'' = \varpi'\partial^2\varpi''+\varpi''\partial^2\varpi'+2\partial\sigma\partial\sigma'', \\ bk'^2\varrho''^2 = -(k'\partial k''+\varrho''\partial\varrho')^2, \\ dk'''^2\varrho'''^2 = (k'''\partial k-\varrho'''\partial\varrho)^2, \\ a(\varpi\varpi'''+\sigma\sigma''')^2+b(\varpi^2\sigma'''^2+\varpi'''^2\sigma^2)+d(\varpi^2\sigma^2+\varpi'''^2\sigma'''^2)+c\varpi\varpi'''\sigma\sigma''' \\ \quad = (\varpi\varpi'''+\sigma\sigma''')(\varpi\partial^2\varpi'''+\varpi'''\partial^2\varpi+\sigma\partial^2\sigma'''+\sigma'''\partial^2\sigma). \end{cases}$$

Adnotetur, quod gravissimum est, differentialia $\partial^2\varpi$, $\partial^2\varpi'$, etc. eliminari posse ex valore ipsius c advocatis aequationibus

$$(53.)\quad \begin{cases} \varrho''\varrho'''(\varpi'\partial^2\varpi-\varpi\partial^2\varpi') = k'\sigma'''\partial k\partial\sigma''-k'''\sigma\partial k''\partial\varrho', \\ \varrho''\varrho'''(\varpi''\partial^2\varpi'''-\varpi'''\partial^2\varpi'') = k'\sigma\partial k\partial\sigma'-k'''\sigma'''\partial k''\partial\sigma'', \\ \varrho''k'''(\sigma'''\partial^2\varpi''-\varpi''\partial^2\sigma''') = \varpi'''\varrho'''\partial\sigma''\partial k''+\varpi k'\partial\sigma'\partial\varrho, \\ \varrho''k'''(\sigma\partial^2\varpi'-\varpi'\partial\sigma) = \varpi'''k'\partial\sigma''\partial\varrho+\varpi\varrho'''\partial\sigma''\partial k''. \end{cases}$$

Quamvis hoc ad propositum nostrum satis sit aequationum differentialium, juvat tamen adnotare, posse etiam omnium ordinum inveniri aequationes hujuscemodi $X\partial^n Y-n\partial X\partial^{n-1}Y+n_2\partial^2X\partial^{n-2}Y-n_3\partial^3X\partial^{n-3}Y+$ etc. $=$ functioni rationali integrae secundi gradus ipsarum P', S', P'', S'', ubi X et Y locum tenent binarum quarumlibet e functionibus P, Q, R, etc. Quod utraque methodo, qua initio hujus capitis usi sumus, demonstrari potest. Immo non datur aequatio differentialis, quin ex talibus composita sit. Eadem illa aequatio valet etiam, si pro X et Y quaelibet functiones integrae homogeneae ipsarum P, Q, R, etc. ejusdem dimensionis substituuntur. Valet denique etiam, si argumenta alterius (X) quibuslibet constantibus augeantur, alterius (Y) *iisdem* constantibus minuantur; ita ut hoc modo audiat

$X(u+\alpha, u'+\alpha')\partial^n Y(u-\alpha, u'-\alpha')-n\partial X(u+\alpha, u'+\alpha')\partial^{n-1}Y(u-\alpha, u'-\alpha')$ $+$ etc. $=$ functioni rationali ipsarum P', S', P'', S'', etc. Quam infra ad tertiam speciem functionum quadrupliciter periodicarum facere videbis.

Transformatio aequationum differentialium.

Superest nunc, ut aequationes et finitas et differentiales, quas vidimus inter quatuor quantitates locum habere, paullo accuratius examinemus, ut demonstretur, *functiones nostras ad transcendentes Abelianas primi ordinis pertinere.*

Et quidem prae caeteris hoc unum et memorabile observandum est, quod priores tres aequationes, nimirum (33.) et (49.) quamvis ad determinationem quatuor quantitatem P', S', P'', S'' non sufficiant, tamen quotientes e binis omni rigore definiunt, scilicet tres quantitates $\frac{S'}{P'}$, $\frac{P''}{P'}$, $\frac{S''}{P'}$, quas functiones quadrupliciter periodicas esse supra comperimus. Qua propter in hoc loco consistimus et reliquam aeqq. (50.) impraesentiarum mittimus.

Aequationes (49.)

$$P'\partial S' - S'\partial P' = \frac{\varrho''\partial k''}{k'''\varrho'''}PS + \frac{k'\partial\varrho'}{k'''\varrho'''}P'''S''',$$

$$P''\partial S'' - S''\partial P'' = \frac{k'\partial\varrho'}{k'''\varrho'''}PS + \frac{\varrho''\partial k''}{k'''\varrho'''}P'''S'''$$

addimus et subtrahimus, unde fit

$$(54.)\quad \begin{cases} \dfrac{(P'\partial S'-S'\partial P')+(P''\partial S''-S''\partial P'')}{P'''S'''+PS} = \dfrac{k'\partial\varrho'+\varrho''\partial k''}{k'''\varrho'''} = \partial\mu, \\[2ex] \dfrac{(P'\partial S'-S'\partial P')-(P''\partial S''-S''\partial P'')}{P'''S'''-PS} = \dfrac{k'\partial\varrho'-\varrho''\partial k''}{k'''\varrho'''} = \partial\nu, \end{cases}$$

quae posito brevitatis causa

$$(55.)\quad \frac{S'}{P'} = p,\quad \frac{S''}{P''} = q,\quad \frac{P'}{P''} = s,\quad \frac{P'''S'''+PS}{P'P''} = \varphi,\quad \frac{P'''S'''-PS}{P'P''} = \psi,$$

transformantur in hasce:

$$(56.)\qquad \frac{s\partial p + \frac{1}{s}\partial q}{\varphi} = \partial\mu,\qquad \frac{s\partial p - \frac{1}{s}\partial q}{\psi} = \partial\nu.$$

Valores ipsarum s, φ, ψ, per quantitates P', S', P'', S'' vel potius p, q expressi, ex aeqq. (33.), (35.), (36.) repetendi sunt. Introductis enim denotationibus sequentibus

$$(57.)\quad \begin{cases} 2E = \dfrac{k'''^4+\varrho'''^4}{k'''^2\varrho'''^2},\quad 2F = \dfrac{\varpi'^4+\varpi''^4}{\varpi'^2\varpi''^2},\quad 2C = \dfrac{k'^4+\varrho''^4}{k'^2\varrho''^2}, \\[2ex] 2D = \dfrac{\varpi\sigma\varpi'''\sigma'''(\varpi'^4-\varpi''^4)^2}{\varpi'^2\varpi''^2k'^2\varrho''^2k'''^2\varrho'''^2}, \\[2ex] \quad b = \dfrac{\varpi'''^2\sigma'''^2+\varpi^2\sigma^2}{\varpi'^2\varpi''^2},\quad c = \dfrac{\varpi\sigma\varpi'''\sigma'''(k'^4+\varrho''^4)}{\varpi'^2\varpi''^2k'^2\varrho''^2},\quad a = \dfrac{k'''^2\varrho'''^2}{k'^2\varrho''^2}, \\[2ex] \quad b_1 = \dfrac{2\varpi\varpi'''\sigma\sigma'''}{\varpi'^2\varpi''^2},\quad c_1 = \dfrac{\varpi^2\varpi'''^2+\sigma^2\sigma'''^2}{\varpi'^2\varpi''^2},\ \text{unde} \\[2ex] \qquad\qquad E^2-1 = \dfrac{(k'''^4-\varrho'''^4)^2}{4k'''^4\varrho'''^4} = \dfrac{(\varpi'^4-\varpi''^4)^2}{4k'''^4\varrho'''^4}, \end{cases}$$

aeqq. (33.) — (36.) mutantur in sequentes:

$$(58.)\quad (1-2Ep^2+p^4)s^4-2(F(1+p^2q^2)-C(p^2+q^2)+2Dpq)s^2+(1-2Eq^2+q^4)=0,$$

$$(59.)\quad 2\sqrt{(E^2-1)}\left(\frac{P'''^2S'''^2+P^2S^2}{P'^2P''^2}\right) = (1-2Ep^2+p^4)s^2-(1-2Eq^2+q^4)\frac{1}{s^2},$$

$$(60.)\quad \left(\frac{P'''S'''+PS}{P'P''}\right)^2 = \varphi^2 = (b+b_1)(1+p^2q^2)-a(p^2+q^2)+2(c-c_1)pq,$$

$$(61.)\quad \left(\frac{P'''S'''-PS}{P'P''}\right)^2 = \psi^2 = (b-b_1)(1+p^2q^2)-a(p^2+q^2)+2(c+c_1)pq.$$

Ad solutionem aequationis biquadraticae (58.) observo fieri ex aeqq. (59.) — (61.):

$$(1-2Ep^2+p^4)s^2-(1-2Eq^2+q^4)\frac{1}{s^2} = 2\sqrt{(E^2-1)}\cdot\varphi\psi.$$

At ex ipsa (58.) fit

$$(1-2Ep^2+p^4)s^2+(1-2Eq^2+q^4)\frac{1}{s^2} = 2(F(1+p^2q^2)-C(p^2+q^2)+2Dpq),$$

unde

$$(62.)\quad s^2 = \frac{F(1+p^2q^2)-C(p^2+q^2)+2Dpq+\sqrt{(E^2-1)}\cdot\varphi\psi}{1-2Ep^2+p^4}.$$

$$(63.)\quad \frac{1}{s^2} = \frac{F(1+p^2q^2)-C(p^2+q^2)+2Dpq-\sqrt{(E^2-1)}\cdot\varphi\psi}{1-2Eq^2+q^4}.$$

His itaque valoribus ex aeqq. (60.) — (64.) in aeqq. (56.) substitutis, *determinatio ipsarum p, q reducta est ad integrationem quantitatum algebraicarum irrationalium, a duabus aequationibus secundi gradus pendentium,* earumque mixtarum quidem, utpote quum duae variabiles independentes p et q ingrediantur, tamen *algebraicarum.* Integratis enim aeqq. (56.) prodeunt sequentes:

$$(64.)\quad \int\frac{s\partial p+\frac{1}{s}\partial q}{\varphi} = \mu, \qquad \int\frac{s\partial p-\frac{1}{s}\partial q}{\psi} = \nu,$$

quarum differentialia $\dfrac{s\partial p+\frac{1}{s}\partial q}{\varphi}$ et $\dfrac{s\partial p-\frac{1}{s}\partial q}{\psi}$ manifesto sunt differentialia exacta.

Opera nunc danda est, ut variabiles separentur; hoc est, ut loco ipsarum p et q aliae binae variabiles introducantur, quae integrationem ad simplices quadraturas secundum singulas variabiles perducant. Arduum sane incoeptum, si complicationem formularum, valores ipsarum φ, ψ, s, $\frac{1}{s}$ exprimentium, respicis. Quod tamen feliciter succedit per substitutiones

$$(65.)\quad p = \frac{y\Delta z+z\Delta y}{1-y^2z^2}, \qquad q = \frac{y\Delta z-z\Delta y}{1-y^2z^2},$$

ubi positum est $\Delta y = \sqrt{(1-2Ey^2+y^4)}$, $\Delta z = \sqrt{(1-2Ez^2+z^4)}$. Etenim instituto calculo fit

$$(66.)\quad\begin{cases}\dfrac{\partial p}{\Delta p}=\dfrac{\partial y}{\Delta y}+\dfrac{\partial z}{\Delta z},\quad \dfrac{\partial q}{\Delta q}=\dfrac{\partial y}{\Delta y}-\dfrac{\partial z}{\Delta z},\\ (1-y^2z^2)^2\varphi^2\\ =(b+b_1)\left\{1-2\dfrac{C-E-D}{F-1}y^2+y^4\right\}\left\{1-2\dfrac{C+E+D}{F+1}z^2+z^4\right\},\\ (1-y^2z^2)^2\psi^2\\ =(b-b_1)\left\{1-2\dfrac{C+E-D}{F+1}y^2+y^4\right\}\left\{1-2\dfrac{C-E+D}{F-1}z^2+z^4\right\},\\ \tfrac{1}{2}(1-y^2z^2)^2\left(s\Delta p+\dfrac{\Delta q}{s}\right)^2\\ =(F+1)\left\{1-2\dfrac{C+E-D}{F+1}y^2+y^4\right\}\left\{1-2\dfrac{C+E+D}{F+1}z^2+z^4\right\},\\ \tfrac{1}{2}(1-y^2z^2)^2\left(s\Delta p-\dfrac{\Delta q}{s}\right)^2\\ =(F-1)\left\{1-2\dfrac{C-E-D}{F-1}y^2+y^4\right\}\left\{1-2\dfrac{C-E+D}{F-1}z^2+z^4\right\},\end{cases}$$

aequationes (56.) autem sequenti modo repraesentari possunt

$$(67.)\quad\begin{cases}\dfrac{\left(s\Delta p+\dfrac{\Delta q}{s}\right)}{\varphi}\left(\dfrac{\partial p}{2\Delta p}+\dfrac{\partial q}{2\Delta q}\right)+\dfrac{\left(s\Delta p-\dfrac{\Delta q}{s}\right)}{\varphi}\left(\dfrac{\partial p}{2\Delta p}-\dfrac{\partial q}{2\Delta q}\right)=\partial\mu,\\ \dfrac{\left(s\Delta p-\dfrac{\Delta q}{s}\right)}{\psi}\left(\dfrac{\partial p}{2\Delta p}+\dfrac{\partial q}{2\Delta q}\right)+\dfrac{\left(s\Delta p+\dfrac{\Delta q}{s}\right)}{\psi}\left(\dfrac{\partial p}{2\Delta p}-\dfrac{\partial q}{2\Delta q}\right)=\partial\nu,\end{cases}$$

unde substitutis valoribus ex aeqq. (66.) et sublatis factoribus communibus in nominatoribus et denominatoribus et posito brevitatis causa

$$\frac{C-E-D}{F-1}=E_1,\quad \frac{C+E-D}{F+1}=E_2,$$
$$\frac{C+E+D}{F+1}=E_3,\quad \frac{C-E+D}{F-1}=E_4,$$

oriuntur aequationes separatae

$$(68.)\quad\begin{cases}\sqrt{(2(F+1))}\dfrac{\partial y}{\sqrt{(1-2Ey^2+y^4)}}\sqrt{\dfrac{1-2E_2y^2+y^4}{1-2E_1y^2+y^4}}\\ \quad+\sqrt{(2(F-1))}\dfrac{\partial z}{\sqrt{(1-2Ez^2+z^4)}}\sqrt{\dfrac{1-2E_4z^2+z^4}{1-2E_3z^2+z^4}}=\sqrt{(b+b_1)}\,\partial\mu,\\ \sqrt{(2(F-1))}\dfrac{\partial y}{\sqrt{(1-2Ey^2+y^4)}}\sqrt{\dfrac{1-2E_1y^2+y^4}{1-2E_2y^2+y^4}}\\ \quad+\sqrt{(2(F+1))}\dfrac{\partial z}{\sqrt{(1-2Ez^2+z^4)}}\sqrt{\dfrac{1-2E_3z^2+z^4}{1-2E_4z^2+z^4}}=\sqrt{(b-b_1)}\,\partial\nu.\end{cases}$$

Attamen quum illa substitutio non ita primo intuitu liqueat, haud inscitum mihi videtur ejus analysin hic apponere; praesertim quum in rebus insolitis, quas nemo hucusque vel e longinquo addigitavit, operae pretium sit, methodos cognovisse, e quibus aliquid utilitatis capi possit.

Separatio variabilium.

Respectis et *periodicitate quadruplice* functionum p et q et reductione ad *bina* argumenta independentia et forma *algebraica* differentialium, facile conjicitur adhibita idonea substitutione factum iri

$$\frac{s\,\partial p+\frac{1}{s}\partial q}{\varphi} = \frac{(\alpha+\beta x)\partial x}{\sqrt{X}} + \frac{(\alpha+\beta x')\partial x'}{\sqrt{X'}} = \partial\mu,$$

$$\frac{s\,\partial p-\frac{1}{s}\partial q}{\psi} = \frac{(\alpha_1+\beta_1 x)\partial x}{\sqrt{X}} + \frac{(\alpha_1+\beta_1 x')\partial x'}{\sqrt{X'}} = \partial\nu,$$

ubi X et X' functiones rationales integras quinti vel sexti gradus repraesentant, ut est in transcendentibus Abelianis primi ordinis *). E contrario itaque oportet etiam a transcendentibus Abelianis, facta idonea substitutione pro variabilibus x et x', ad nostras aeqq. differentiales perveniri posse. Ipsae autem x et x' spectari possunt, ut radices aequationis quadraticae $Ux^2-2U'x+U''=0$, quarum coëfficientes sunt functiones rationales vel si mavis „uniformes" (transcendentes) binorum argumentorum μ et ν, sive u et u'. Quamobrem

*) Quae D. *Jacobi* de functionibus plurium argumentorum multipliciter periodicis, quas in theoriam transcendentium Abelianarum introducere convenit, acutissime disseruit (cf. commentt. hujus diarii 32[dam] T. 9 et 2[dam] T. 13) pro satis et abunde notis habenda sunt. Omittimus itaque hoc loco et ad aliam occasionem, si qua dabitur, relegamus ea, quae nosmet ipsi olim de hisce rebus meditati sumus; quamvis multo latius pateant, quippe quae ex consideratione integralium in universum, neque vero ex singulari casu, ubi radix quadrata e polynomio in factores *reales* resolubili tractatur, depromta sint. — Unum tamen jam nunc adnotare juvat de argumento, quod clarissimus vir ex absurditate periodicitatis triplicis et a fortiori quadruplicis contra considerationem functionum unius solius variabilis petivit. Quod enim omnem functionem tripliciter periodicam dicit habere periodum omni data quantitate minorem idque absurdum esse, priori quidem parti toto animo assentiendum esse existimamus, verum quid huic rei absurditatis insit (cum pace tanti viri dicatur) plane non intelligimus. Quonam deinde loco hujus absurditatis causam positam esse censeamus? Num in elementis, an in notione integralis vel functionis? Quibus omnibus *) bene perpensis, hoc sane loco functio $u=\int\frac{(\alpha+\beta x)\partial x}{\sqrt{X}}$ rejicienda esse videtur, non tamen propter putatam absurditatem, sed quatenus de relationibus *algebraicis* inter ternas $x(u)$, $x(v)$ et $x(u+v)$ disquiritur, talis autem relatio non datur. Quin immo, si quis relationes *transcendentes* inter illas investigare velit (quod vix nostrae aetatis esse videtur), eum amplissimum laboribus campum inventurum esse arbitramur.

*) !? *J.*

licet etiam, eas sicut functiones rationales algebraicas ipsarum p et q et reliquorum quotientium e binis $\left(\frac{Q}{S}, \frac{P}{R}, \text{etc.}\right)$ considerare, quippe qui sunt ipsi functiones uniformes argumentorum u et u'. Huc accedit, quod talis substitutio $Ux^2 - 2U'x + U'' = 0$ bellissime cum nostris aequationibus quadrare videtur, nam quum habeatur $x = \frac{U' + \sqrt{(U'^2 - UU'')}}{U}$ radices $\sqrt{X}$ et $\sqrt{X'}$ manifesto talem formam induunt $\sqrt{\frac{M + N\sqrt{(U'^2 - UU'')}}{L}}$, qualem habet ipsum s (cf. aeq. (62.), ubi productum $\varphi\psi$ radicem quadratam e functione rationali repraesentat). Verumtamen huic rei unum obstat, quod gravissimum est. Etenim duo radicalia $\sqrt{(U'^2 - UU'')}$ et $\varphi\psi$, quae algebraice commensurabilia esse oportet (scilicet $\frac{\sqrt{(U'^2 - UU'')}}{\varphi\psi}$ = functioni rationali ipsarum p, q, et reliquarum quotientium e binis) naturae valde diversae sunt. Alterum $\varphi\psi$, quod posuimus aequale quantitati $\frac{P'''^2 S'''^2 - P^2 S^2}{P'^2 P''^2}$, functionem rationalem esse vides, alterum autem $\sqrt{(U'^2 - UU'')}$ sicut functio irrationalis spectandum est, siquidem singulas variabiles x et x' per argumenta u et u' rationaliter exprimi non posse censemus.

Transformandum itaque erit illud s ope formulae

$$\sqrt{(a + \sqrt{b})} = \sqrt{\frac{a + \sqrt{(a^2 - b)}}{2}} + \sqrt{\frac{a - \sqrt{(a^2 - b)}}{2}}$$

in aliam, quae sibi propria est, formam. Quod facillime perficitur, respectis formulis (62.), (63.), unde multiplicando elicitur

$$(69.)\quad \{F(1 + p^2 q^2) - C(p^2 + q^2) + 2Dpq\}^2 - (E^2 - 1)\varphi^2\psi^2 = \Delta p^2 \Delta q^2.$$

Erit itaque

$$(70.)\quad \begin{cases} s = \dfrac{\sqrt{(2G + 2\Delta p \Delta q)}}{2\Delta p} + \dfrac{\sqrt{(2G - 2\Delta p \Delta q)}}{2\Delta p}, \\[2ex] \dfrac{1}{s} = \dfrac{\sqrt{(2G + 2\Delta p \Delta q)}}{2\Delta q} - \dfrac{\sqrt{(2G - 2\Delta p \Delta q)}}{2\Delta q}, \end{cases}$$

ubi brevitatis causa positum est

$$G = F(1 + p^2 q^2) - C(p^2 + q^2) + 2Dpq;$$

quamobrem aequationes differentiales (56.) mutantur in eas quas sub numero (67.) legis:

$$(71.)\quad \begin{cases} \dfrac{\sqrt{(2G + 2\Delta p \Delta q)}}{\varphi}\left(\dfrac{\partial p}{2\Delta p} + \dfrac{\partial q}{2\Delta q}\right) + \dfrac{\sqrt{(2G - 2\Delta p \Delta q)}}{\varphi}\left(\dfrac{\partial p}{2\Delta p} - \dfrac{\partial q}{2\Delta q}\right) = \partial\mu, \\[2ex] \dfrac{\sqrt{(2G - 2\Delta p \Delta q)}}{\psi}\left(\dfrac{\partial p}{2\Delta p} + \dfrac{\partial q}{2\Delta q}\right) + \dfrac{\sqrt{(2G + 2\Delta p \Delta q)}}{\psi}\left(\dfrac{\partial p}{2\Delta p} - \dfrac{\partial q}{2\Delta q}\right) = \partial\nu, \end{cases}$$

quarum forma pulcherrime ad considerationes supra memoratas accommodatur. Quantitas enim $\Delta p \Delta q$, quae sub radicalibus comparuit, revera est irrationalis, id quod ex evolutionibus, a quibus initio profecti sumus, satis liquet. Nunc vero illas considerationes mittimus, quum reliqua sua sponte sese offerant.

Semel enim erutis aeqq. (71.) nihil simplicius, quam differentialibus ellipticis, quae illic conspicienda sunt, hasce substitutiones applicare:

$$(72.)\qquad \frac{\partial p}{2\Delta p}+\frac{\partial q}{2\Delta q}=\frac{\partial y}{\Delta y},\qquad \frac{\partial p}{2\Delta p}-\frac{\partial q}{2\Delta q}=\frac{\partial z}{\Delta z},$$

quae e noto de additione theoremate integralia offerunt, quae sequuntur

$$(73.)\qquad p=\frac{y\Delta z+z\Delta y}{1-y^2z^2},\qquad q=\frac{y\Delta z-z\Delta y}{1-y^2z^2}.$$

Hinc prodit

$$(74.)\qquad pq=\frac{y^2-z^2}{1-y^2z^2},\qquad p^2+q^2=2\,\frac{(y^2+z^2)(1+y^2z^2)-4Ey^2z^2}{(1-y^2z^2)^2},$$

$$(75.)\qquad \Delta p\Delta q=\frac{(1+y^2z^2)^2-2E(1+y^2z^2)(y^2+z^2)+(y^2+z^2)^2}{(1-y^2z^2)^2},$$

quibus valoribus in expressionibus ipsarum s, $\frac{1}{s}$, φ, ψ (aeqq. 69., 60., 61.) substitutis, facillime pervenitur ad aeqq., quas jam sub numero (66.) dedimus, neque ad resolutionem in factores quicquam aliud opus erit, nisi relationes inter constantes, quas vocavimus C, D, E, F, a, b, c, b_1, c_1 cognovisse. Quem ob finem aut ad aequationes elementares (25.) — (28.) redeundum, aut quod multo simplicius est, aequationis (68.) utrumque membrum evolvendum erit. Instituta comparatione prodeunt aeqq. seqq.

$$(76.)\qquad E^2-1=\frac{F^2-1}{b^2-b_1^2}=\frac{D^2-E^2+1}{c^2-c_1^2}=\frac{CF-E}{ab}=\frac{DF}{bc+b_1c_1}=\frac{CD}{ac},$$

nec non ex ipsis (57.)

$$\frac{D}{ab_1}=E^2-1,$$

unde sine labore eliciuntur sequentes:

$$(77.)\qquad \begin{cases} a=\dfrac{\sqrt{(C^2-1)}}{\sqrt{(E^2-1)}},\quad b=\dfrac{CF-E}{\sqrt{(E^2-1)}\sqrt{(C^2-1)}},\quad c=\dfrac{CD}{\sqrt{(E^2-1)}\sqrt{(C^2-1)}},\\ b_1=\dfrac{D}{\sqrt{(E^2-1)}\sqrt{(C^2-1)}},\quad c_1=\dfrac{CE-F}{\sqrt{(E^2-1)}\sqrt{(C^2-1)}},\\ D^2+1=C^2+E^2+F^2-2CEF. \end{cases}$$

Neque vero mirandum est, tres coëfficientes indeterminatas manere, quum inter ipsas quantitates ϖ, $\varkappa$, ϱ, etc. quatuor algebraice indeterminatae restiterint; namque hae nostrae C, D, etc. sunt functiones quotientium e binis et aeqq. inter ipsas $\varpi, \varkappa, \varrho$, etc. homogeneae erant. Haec insuper naturae transcendentium Abelianarum omnino consentanea sunt, quippe quae tres modulos habeant. Ceterum ipsum calculum, utpote elementarem, lectoribus relinquimus et ad separationem variabilium, ut est in aeqq. (68.), perventum esse putamus.

Reductio differentialium ad formam symmetricam et consuetam.

Variabiles jam quidem separatae sunt, neque tamen, quod exspectandum erat, locum symmetricum in aequationibus (68.) occupant. Hujus rei causa in eo quaerenda est, quod quum novas variabiles per aequationes differentiales (72.) introduxerimus, constantem integrationis male elegimus; talem nimirum ut evanescentibus y et z etiam ipsa p et q evanescerent. Introducenda erit itaque nova variabilis pro altera illarum ex. gr. pro ipsa z, per aequationem

$$\frac{\partial z}{\Delta z} = \frac{\partial y'}{\Delta y'}$$

unde fit $z = \frac{y' \Delta e + e \Delta y'}{1 - e^2 y'^2}$, ubi e est constans arbitraria. Hanc autem omnino ita determinare oportet, ut facta substitutione differentialia (68.) in similem sui formam redeant; id quod aperto fieri nequit, nisi ipsum z^2 per y'^2 rationaliter expressum sit. Facimus itaque $\Delta e = 0$, unde posito

$$\Delta e^2 = 1 - 2Ee^2 + e^4 = (1 - \alpha^2 e^2)(1 - \beta^2 e^2),$$

fit $e = \alpha$ (sive β). Hinc nanciscimur substitutionem sequentem

$$(78.)\quad \begin{cases} z^2 = \dfrac{\alpha^2 - y'^2}{1 - \alpha^2 y'^2}, \quad \Delta z = \dfrac{\beta(\alpha^4 - 1) y'}{1 - \alpha^2 y'^2}, \quad \dfrac{\partial z}{\Delta z} = \dfrac{\partial y'}{\Delta y'}, \\ \tfrac{1}{2}\left(z^2 + \dfrac{1}{z^2}\right) = \dfrac{E - 2y'^2 + Ey'^4}{1 - 2Ey'^2 + y'^4}, \end{cases}$$

quae ad aeqq. (68.) applicata fit

$$\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right) - E_4 = \frac{(E - F_4) - 2(1 - EE_4)y'^2 + (E - E_4)y'^4}{1 - 2Ey'^2 + y'^4},$$

$$\tfrac{1}{2}\left(z^2 + \frac{1}{z^2}\right) - E_3 = \frac{(E - E_3) - 2(1 - EE_3)y'^2 + (E - E_3)y'^4}{1 - 2Ey'^2 + y'^4}.$$

Sed quum introductis ipsarum E_1, E_2, E_3, E_4 valoribus et considerata aequatione (77.) habeatur

$$1 - E(E_1 + E_3) + E_1 E_3 = 0,$$

$$1 - E(E_2 + E_4) + E_2 E_4 = 0,$$

$$\frac{E - E_4}{E - E_3} = \frac{F + 1}{F - 1},$$

erit

$$\frac{1-2E_4z^2+z^4}{1-2E_3z^2+z^4} = \frac{F+1}{F-1}\cdot\frac{1-2E_2y'^2+y'^4}{1-2E_1y'^2+y'^4},$$

ergo etiam

$$\sqrt{(2(F-1))}\cdot\frac{\partial z}{\varDelta z}\sqrt{\frac{1-2E_4z^2+z^4}{1-2E_3z^2+z^4}} = \sqrt{(2(F+1))}\cdot\frac{\partial y'}{\varDelta y'}\sqrt{\frac{1-2E_2y'^2+y'^4}{1-2E_1y'^2+y'^4}},$$

quo facto reductio aequationum differentialium ad formam symmetricam respectu variabilium perfecta est, sicut desiderabatur.

Introductis itaque valoribus (78.) in aeqq. (73.) — (75.) nanciscimur substitutiones sequentes, quae nostras aequationes differentiales (56.) ad separationem variabilium symmetricam transformant:

$$(79.)\quad\begin{cases} p = \dfrac{\beta(\alpha^4-1)yy'+\alpha\varDelta y\varDelta y'}{1+y^2y'^2-\alpha^2(y^2+y'^2)}, \qquad q = \dfrac{\beta(\alpha^4-1)yy'-\alpha\varDelta y\varDelta y'}{1+y^2y'^2-\alpha^2(y^2+y'^2)}, \\ pq = \dfrac{y^2+y'^2-\alpha^2(1+y^2y'^2)}{1+y^2y'^2-\alpha^2(y^2+y'^2)}, \\ p^2+q^2 = 2\alpha^2\dfrac{(1+y^2y'^2)^2+(y^2+y'^2)^2-2E(y^2+y'^2)(1+y^2y'^2)+8(E^2-1)y^2y'^2}{\{1+y^2y'^2-\alpha^2(y^2+y'^2)\}^2}, \\ \varDelta p\varDelta q = -\dfrac{\beta^2(\alpha^4-1)^2(y^2-y'^2)(1-y^2y'^2)}{\{1+y^2y'^2-\alpha^2(y^2+y'^2)\}^2}, \end{cases}$$

quarum beneficio habebitur

$$(80.)\quad\begin{cases} \dfrac{\partial y}{\varDelta y}\sqrt{\dfrac{1-2E_2y^2+y^4}{1-2E_1y^2+y^4}}+\dfrac{\partial y'}{\varDelta y'}\sqrt{\dfrac{1-2E_2y'^2+y'^4}{1-2E_1y'^2+y'^4}} = \dfrac{\sqrt{(b+b_1)}}{\sqrt{(2(F+1))}}\partial\mu, \\ \dfrac{\partial y}{\varDelta y}\sqrt{\dfrac{1-2E_1y^2+y^4}{2-2E_2y^2+y^4}}+\dfrac{\partial y'}{\varDelta y'}\sqrt{\dfrac{1-2E_1y'^2+y'^4}{1-2E_2y'^2+y'^4}} = \dfrac{\sqrt{(b-b_1)}}{\sqrt{(2(F-1))}}\partial\nu. \end{cases}$$

Variabiles y et y' hoc modo per p et q exprimuntur. Ex aeqq. (79.) fit

$$p+q = \frac{2\beta(\alpha^4-1)yy'}{N}, \qquad 1+\alpha^2pq = \frac{(1-\alpha^4)(1+y^2y'^2)}{N},$$

$$1+\beta^2pq = \frac{(\beta^2-\alpha^2)(y^2+y'^2)}{N},$$

atque hinc

$$\frac{p+q}{1+\alpha^2pq} = \frac{-2\beta yy'}{1+y^2y'^2}, \qquad\qquad \frac{p+q}{1+\beta^2pq} = \frac{-2\alpha yy'}{y^2+y'^2},$$

unde

$$\frac{(1+\alpha p)(1+\alpha q)}{(1-\alpha p)(1-\alpha q)} = \left(\frac{1-yy'}{1+yy'}\right)^2 = f, \qquad \frac{(1+\beta p)(1+\beta q)}{(1-\beta p)(1-\beta q)} = \left(\frac{y-y'}{y+y'}\right)^2 = g.$$

Extractis itaque radicibus invenitur

$$yy' = \frac{1-\sqrt{f}}{1+\sqrt{f}}, \qquad \frac{y}{y'} = \frac{1+\sqrt{g}}{1-\sqrt{g}},$$

unde

$$(81.)\quad y^2 = \frac{1-\sqrt{f}}{1+\sqrt{f}}\cdot\frac{1+\sqrt{g}}{1-\sqrt{g}},\qquad y'^2 = \frac{1-\sqrt{f}}{1+\sqrt{f}}\cdot\frac{1-\sqrt{g}}{1+\sqrt{g}}.$$

Hinc elucet, aequationem, cujus radices sunt y et y', ad octavum gradum ascendere. Id quod vel per se jam satis indicat, functiones aliquas illarum sicut novas variabiles introducendas esse, quae ab aequatione secundi tantum gradus pendeant. Ipsae etiam aeqq. (80.) talem substitutionem suppeditant, quippe quae positis $y^2+\frac{1}{y^2}=x$, $y'^2+\frac{1}{y'^2}=x'$ valde simplificantur. Eadem fere ex aeqq. (81.) petitur, namque invenimus

$$\frac{1-y^2}{1+y^2} = \frac{\sqrt{f}-\sqrt{g}}{1-\sqrt{(fg)}},\qquad \frac{1-y'^2}{1+y'^2} = \frac{\sqrt{f}+\sqrt{g}}{1+\sqrt{(fg)}},$$

quarum quadrata unam tantum radicem $\sqrt{(fg)}$ continent:

$$\left(\frac{1-y^2}{1+y^2}\right)^2 = \frac{f+g-2\sqrt{(fg)}}{1+fg-2\sqrt{(fg)}},\qquad \left(\frac{1-y'^2}{1+y'^2}\right)^2 = \frac{f+g+2\sqrt{(fg)}}{1+fg+2\sqrt{(fg)}}.$$

Introductis itaque variabilibus x et x' per substitutiones

$$(82.)\quad x = \left(\frac{1-y^2}{1+y^2}\right)^2,\qquad x' = \left(\frac{1-y'^2}{1+y'^2}\right)^2,$$

et advocatis valoribus ipsarum f et g, inveniuntur formulae sequentes:

$$(83.)\quad x = \frac{E(1+p^2q^2)-(p^2+q^2)+\Delta p\Delta q}{(E+1)(1+pq)^2},\qquad x' = \frac{E(1+p^2q^2)-(p^2+q^2)-\Delta p\Delta q}{(E+1)(1+pq)^2}$$

et propter

$$\frac{\partial y}{\Delta y} = \frac{1}{2\sqrt{(2(1-E))}}\cdot\frac{\partial x}{\sqrt{x}\sqrt{(1-x)}\sqrt{\left(1-\frac{E+1}{E-1}x\right)}},$$

aequationes differentiales:

$$(84.)\quad \left\{\begin{aligned} &\frac{\partial x}{\sqrt{x}\sqrt{(1-x)}\sqrt{(1-mx)}}\cdot\frac{\sqrt{(1-m_2x)}}{\sqrt{(1-m_1x)}}+\frac{\partial x'}{\sqrt{x'}\sqrt{(1-x')}\sqrt{(1-mx')}}\cdot\frac{\sqrt{(1-m_2x')}}{\sqrt{(1-m_1x')}}\\ &\qquad = 2\sqrt{(b+b_1)}\sqrt{\frac{(1-E)(1-E_1)}{(1+F)(1-E_2)}}\,\partial\mu,\\ &\frac{\partial x}{\sqrt{x}\sqrt{(1-x)}\sqrt{(1-mx)}}\cdot\frac{\sqrt{(1-m_1x)}}{\sqrt{(1-m_2x)}}+\frac{\partial x'}{\sqrt{x'}\sqrt{(1-x')}\sqrt{(1-mx')}}\cdot\frac{\sqrt{(1-m_1x')}}{\sqrt{(1-m_2x')}}\\ &\qquad = 2\sqrt{(b-b_1)}\sqrt{\left(\frac{(1-E)(1-E_2)}{(F-1)(1-E_1)}\right.}\,\partial\nu,\end{aligned}\right.$$

siquidem ponitur

$$(85.)\quad m=\frac{E+1}{E-1},\quad m_1=\frac{E_1+1}{E_1-1}=\frac{EC-F+D}{C+1\cdot E-1},\quad m_2=\frac{E_2+1}{E_2-1}=\frac{EC-F+D}{C-1\cdot E-1}.$$

Praeter aeqq. (83), quae nexum inter variabiles primitivas p, q et derivatas x, x' exprimunt, notentur etiam sequentes:

$$(86.)\quad \begin{cases} \sqrt{x}+\sqrt{x'} = \frac{\alpha\sqrt{2}}{\sqrt{(E+1)}}\cdot\frac{\sqrt{((1-\beta^2p^2)(1-\beta^2q^2))}}{1+pq}, \\ \sqrt{x}-\sqrt{x'} = \frac{\beta\sqrt{2}}{\sqrt{(E+1)}}\cdot\frac{\sqrt{((1-\alpha^2p^2)(1-\alpha^2q^2))}}{1+pq}, \\ \sqrt{(1-x)}+\sqrt{(1-x')} = \frac{\sqrt{2}}{\sqrt{(E+1)}}\cdot\frac{\sqrt{((1+\alpha p)(1+\alpha q)(1+\beta p)(1+\beta q))}}{1+pq}, \\ \sqrt{(1-x)}-\sqrt{(1-x')} = -\frac{\sqrt{2}}{\sqrt{(E+1)}}\cdot\frac{\sqrt{((1-\alpha p)(1-\alpha q)(1-\beta p)(1-\beta q))}}{1+pq}, \\ \sqrt{(1-mx)}+\sqrt{(1-mx')} = \frac{\sqrt{2}}{\sqrt{(E-1)}}\cdot\frac{\sqrt{(-(1-\alpha p)(1+\alpha q)(1+\beta p)(1-\beta q))}}{1+pq}, \\ \sqrt{(1-mx)}-\sqrt{(1-mx')} = \frac{\sqrt{2}}{\sqrt{(E-1)}}\cdot\frac{\sqrt{(-(1+\alpha p)(1-\alpha q)(1-\beta p)(1+\beta q))}}{1+pq}, \\ \sqrt{(x(1-x)(1-mx))}+\sqrt{(x'(1-x')(1-mx'))} \\ \qquad = 2\sqrt{2}\frac{q^3-p-Eq(1-pq)}{(E+1)\sqrt{(1-E)}(1+pq)^3}\Delta p, \\ \sqrt{(x(1-x)(1-mx))}-\sqrt{(x'(1-x')(1-mx'))} \\ \qquad = 2\sqrt{2}\frac{p^3-q-Ep(1-pq)}{(E+1)\sqrt{(1-E)}(1+pq)^3}\Delta q, \\ \sqrt{\frac{1-m_2x}{1-m_1x}}+\sqrt{\frac{1-m_2x'}{1-m_1x'}} = \sqrt{2}\sqrt{\frac{(b+b_1)(E_1-1)}{(F+1)(E_2-1)}}\cdot\frac{s\Delta p}{\varphi}, \\ \sqrt{\frac{1-m_2x}{1-m_1x}}-\sqrt{\frac{1-m_2x'}{1-m_1x'}} = \sqrt{2}\sqrt{\frac{(b+b_1)(E_1-1)}{(F+1)(E_2-1)}}\cdot\frac{\Delta q}{s\varphi}, \\ \sqrt{\frac{1-m_1x}{1-m_2x}}+\sqrt{\frac{1-m_1x'}{1-m_2x'}} = \sqrt{2}\sqrt{\frac{(b-b_1)(E_2-1)}{(F-1)(E_1-1)}}\cdot\frac{s\Delta p}{\psi}, \\ \sqrt{\frac{1-m_1x}{1-m_2x}}+\sqrt{\frac{1-m_1x'}{1-m_2x'}} = -\sqrt{2}\sqrt{\frac{(b-b_1)(E_2-1)}{(F-1)(E_1-1)}}\cdot\frac{\Delta q}{s\psi}. \end{cases}$$

Limites integrationis.

Reducto problemate de functionibus quadrupliciter periodicis ad integrationem radicis quadraticae e quantitate algebraica quinti gradus, plenior determinatio considerationem limitum postulat.

Quem ad finem observo primum, mutato u in $u+A+B$ vel in $u+K+L$, quantitates p et q (quas ipsis $\frac{S'}{P'}$, $\frac{S''}{P''}$ aequales posuimus) vel in se invicem mutari vel in valores reciprocos $\frac{1}{p}$ et $\frac{1}{q}$; unde etiam elucet, mutato u in

$u+A+B+K+L$, mutari p in $\frac{1}{q}$, q in $\frac{1}{p}$. Hinc fit, ut variabiles x et x', quae per aeqq. (83.) exprimuntur, factis illis mutationibus sive immutatae maneant, sive in se invicem abeant. Secundum est, quod applicatis ad argumentum u semiperiodis $2A$, $2B$, $2(A+B)$, etc. quantitates p et q aut non mutantur aut oppositos valores $-p$, $-q$ induunt, quod e tabula (7.) cognoscitur. Unde quum ipsis x et x' nulla mutatio ingruat, sequitur in universum, variabiles x et x' mutato u in

$$u+n_1(A+B)+n_2(A-B)+n_3(K+L)+n_4(K-L),$$

vel in se ipsas vel invicem abire, siquidem literis n_1, n_2, n_3, n_4 numeri integri denotantur. Si itaque argumento u valores A, B, K, L, etc. ex ordine tribuuntur, quaterni ad eosdem valores ipsarum x et x' perducunt, ita ut in quatuor classes distribui possint, quarum prima amplectitur valores 0, $A+B$, $K+L$, $A+B+K+L$; secunda quadrantes A, B, $A+K+L$, $B+K+L$; tertia quadrantes K, L, $A+B+K$, $A+B+L$; quarta quadrantes $A+K$, $B+K$, $A+L$, $B+L$. Et in universum, si valor multiplex $n_1(A+B)+n_2(A-B)+n_3(K+L)+n_4(K-L)$ brevitatis gratia per H designatur, omnis summa quotlibet quadrantium ad unam aliquam formarum H, $A+H$, $K+H$, $A+K+H$ revocari potest; quarum singulae ad eosdem valores variabilium x, x' perducunt.

Determinatio valorum ipsarum x et x', qui ad singulas classes pertinent, commodissime ope tabulae (8.) perficitur. Namque invenitur

1) pro $u=0$, $p=0$, $q=0$;

2) pro $u=A$, $p=\frac{\sigma}{\varpi}$, $q=\frac{\sigma'''}{\varpi'''}$;

3) pro $u=K$, $p=0$, $q=\infty$;

4) pro $u=A+K$, p indeterminatum, $q=\frac{\mathfrak{S}'''}{k'''}$, unde $\varDelta q=0$,

quibus valoribus in aeqq. (83.) substitutis et respectis formulis (25.) — (28.), (57.) et (85.), nanciscimur valores sequentes:

1) pro $u=H$, $x=1$, $x'=\frac{1}{m}$;

2) pro $u=A+H$, $x=\frac{1}{m_1}$, $x'=\frac{1}{m_2}$;

3) pro $u=K+H$, $x=0$, $x'=\infty$;

4) pro $u=A+K+H$, $x=x'$ arbitrariam;

quorum utroslibet pro limitibus integrationis correspondentibus eligere licet.

Sumantur itaque ex. causa valores $u=0$, $x=1$, $x'=\frac{1}{m}$ pro limitibus inferioribus. Unde, quum differentialia $\partial\varrho'$, $\partial k''$, ideoque etiam $\partial\mu$ et $\partial\nu$ habeant formam $g\,\partial u+g'\,\partial u'$ et $h\,\partial u+h'\,\partial u'$, sicut supra adnotavimus, fit

$$(87.)\quad \begin{cases} \displaystyle\int_{x\div 1}\frac{(1-m_2 x)\partial x}{\sqrt{X}}+\int_{x'\div\frac{1}{m}}\frac{(1-m_2 x')\partial x'}{\sqrt{X'}} = gu+g'u', \\ \displaystyle\int_{x\div 1}\frac{(1-m_1 x)\partial x}{\sqrt{X}}+\int_{x'\div\frac{1}{m}}\frac{(1-m_1 x')\partial x'}{\sqrt{X'}} = hu+h'u', \end{cases}$$

ubi positum est

$$X = x(1-x)(1-mx)(1-m_1x)(1-m_2x),$$
$$X' = x'(1-x')(1-mx')(1-m_1x')(1-m_2x');$$

ita ut variabiles x, x', ergo etiam p et q, ab omni parte determinatae sint.

Priusquam ad sequentia pergitur, haud abs re esse videtur, objectionem aliquam praecidere, quae ex applicatione limitum ad quartam classem pertinentium oriri potest. Sumto enim valore $u=A+K$ pro limite inferiori, et designata per f quantitate arbitraria, inveniuntur aequationes

$$(88.)\quad \begin{cases} \displaystyle\int_{x\div f}\frac{(1-m_2 x)\partial x}{\sqrt{X}}+\int_{x'\div f}\frac{(1-m_2 x')\partial x'}{\sqrt{X'}} = g(u-A-K)+g'(u'-A'-K'), \\ \displaystyle\int_{x\div f}\frac{(1-m_1 x)\partial x}{\sqrt{X}}+\int_{x'\div f}\frac{(1-m_1 x')\partial x'}{\sqrt{X'}} = h(u-A-K)+h'(u'-A'-K'), \end{cases}$$

quae profecto propter limitem arbitrarium aliquid absurdi habere videntur. Animadvertendum est autem, pro $u=A+K$ fieri quidem $x=x'$, non tamen $\sqrt{X}=\sqrt{X'}$, sed $\sqrt{X}=-\sqrt{X'}$. Namque ex aeqq. (86.) sequitur, propter $\Delta q=0$, primum

$$\sqrt{(x(1-x)(1-mx))}-\sqrt{(x'(1-x')(1-mx'))} = 0$$

deinde propter $s=\frac{P'}{P''}=0$,

$$\sqrt{\frac{1-m_1 x}{1-m_2 x}}+\sqrt{\frac{1-m_1 x'}{1-m_2 x'}} = 0$$

unde $\sqrt{X}=-\sqrt{X'}$. Hinc manifesto habetur

$$\int_{f\div g}\frac{(1-m_2 x)\,dx}{\sqrt{X}}+\int_{f\div g}\frac{(1-m_2 x')\,dx'}{\sqrt{X'}} = 0,$$

qua aequatione ad aeqq. (88.) addita, manifestum fit mutato f in g summam duorum integralium, quae sunt in aeq. (88.), omnino non mutari. Quae consideratio solutionem paradoxi praebet.

Solutio problematis inversi de transcendentibus Abelianis.

Venimus nunc ad problema inversum, quale vulgo proponi solebat: Datis aequationibus differentialibus

$$\int\frac{(\alpha+\beta y)\partial y}{\sqrt{Y}}+\int\frac{(\alpha+\beta y')\partial y'}{\sqrt{Y'}} = v,$$

$$\int\frac{(\alpha'+\beta' y)\partial y}{\sqrt{Y}}+\int\frac{(\alpha'+\beta' y')\partial y'}{\sqrt{Y'}} = v',$$

ubi Y *et* Y' significant expressiones integras variabilium y et y' quinti vel sexti gradus, invenire ipsas y et y'. Cujus solutio ex antecedentibus facile derivatur.

Redigantur enim primum differentialia ope substitutionis idoneae ad formam (87.), quod cum elementare sit hoc loco fusius non exsequendum est, et ponatur

$$\int\frac{1-m_2 x}{\sqrt{X}}\partial x+\int\frac{1-m_2 x'}{\sqrt{X'}}\partial x' = gu+g'u',$$

$$\int\frac{1-m_1 x}{\sqrt{X}}\partial x+\int\frac{1-m_1 x'}{\sqrt{X'}}\partial x' = hu+h'u'.$$

E quibus aequationibus postquam constantes g, h, g', h', et quadrantes $K, L, K', L', A, B, A', B'$ inventae fuerint, tota res peracta erit.

Hunc ad finem instituatur integratio inter limites in capite praecedente traditos et invenietur

$$\int\limits_{0\div 1}\frac{1-m_2 x}{\sqrt{X}}\partial x+\int\limits_{\infty\div\frac{1}{m}}\frac{1-m_2 x'}{\sqrt{X'}}\partial x' = (gK+g'K')+(gH+g'H'),$$

$$\int\limits_{0\div 1}\frac{1-m_1 x}{\sqrt{X}}\partial x+\int\limits_{\infty\div\frac{1}{m}}\frac{1-m_1 x'}{\sqrt{X'}}\partial x' = (hK+h'K')+(hH+h'H'),$$

$$\int\limits_{\frac{1}{m_1}\div 1}\frac{1-m_2 x}{\sqrt{X}}\partial x+\int\limits_{\frac{1}{m_2}\div\frac{1}{m}}\frac{1-m_2 x'}{\sqrt{X'}}\partial x' = (gA+g'A')+(gH+g'H'),$$

$$\int\limits_{\frac{1}{m_1}\div 1}\frac{1-m_1 x}{\sqrt{X}}\partial x+\int\limits_{\frac{1}{m_2}\div\frac{1}{m}}\frac{1-m_1 x'}{\sqrt{X'}}\partial x' = (hA+h'A')+(hH+h'H'),$$

unde elucet, integralia nostra habere valores multiplices, quales per H designavimus (id quod pro casu speciali jam notum erat *)). Hos itaque valores a calculo integrali postulamus, et facta comparatione valores expressionum $gK+g'K'$,

*) ! J.

$hK+h'K'$, $gA+g'A'$, $hA+h'A'$, nec non reliquarum, e quibus $gH+g'H'$ et $hH+h'H'$ conflantur, videlicet $gL+g'L'$, $hL+h'L'$, $gB+g'B'$, $hB+h'B'$, inventos esse putamus. Sed quum propter $AK'-A'K=BL'-B'L$ (cf. aeqq. (6.)) habeatur identice

$$\begin{aligned}&(gA+g'A')(hK+h'K')-(hA+h'A')(gK+g'K')\\ &=(gB+g'B')(hL+h'L')-(hB+h'B')(gL+g'L'),\end{aligned}$$

alteruter e septem reliquis sponte sequitur; quapropter ad determinationem duodecim quantitatum g, h, g', h', A, B, K, L, etc. septem tantum aequationes relinquuntur, quibus accedunt binae illae ex aeqq. (6.) derivatae $AK'=BL'$ et $A'K=B'L$. Tres itaque qnantitatum illarum indeterminatae manent, quod *mirum* videri posset, nisi e sequente consideratione eluceret eas omnino arbitrarias esse.

Nam quum ab expressionibus hujusmodi profecti simus

$$e^{ru^2+r'u'^2}P''' = S e^{r(u+2\mathfrak{a}K+2\mathfrak{b}L)^2+r'(u'+2\mathfrak{a}K'+2\mathfrak{b}L')^2},$$

quae sex constantes continent, facile demonstratur, tres earum salvo valore ipsius P''' ad arbitrium assumi posse. Fit enim

$$P''' = S e^{4\mathfrak{a}(rKu+r'K'u')+4\mathfrak{b}(rLu+r'L'u')+4\mathfrak{a}^2(rK^2+r'K'^2)+8\mathfrak{a}\mathfrak{b}(rKL+r'K'L')+4\mathfrak{b}^2(rL^2+r'L'^2)}$$

quare selectis sex aliis constantibus r_1, K_1, L_1, r_1', K_1', L_1' talibus ut sit

$$\begin{aligned}r_1K_1^2+r_1'K_1'^2&=rK^2+r'K'^2,\\ r_1K_1L_1+r_1'K_1'L_1'&=rKL+r'K'L',\\ r_1L_1^2+r_1'L_1'^2&=rL^2+r'L^2,\end{aligned}$$

et introductis insuper variabilibus u_1 et u_1', per aequationes

$$\begin{aligned}r_1K_1u_1+r_1'K_1'u_1'&=rKu+r'K'u',\\ r_1L_1u_1+r_1'L_1'u_1'&=rLu+r'L'u',\end{aligned}$$

expressio ipsius P''' in se ipsa redit. Inter sex quantitates r_1, K_1, L_1, r_1', K_1', L_1' autem, quum tres tantum aequationes habeantur, tres arbitrariae sunt.

Rite jam determinatis quadrantibus A, B, K, etc., formentur quantitates P, P', P'', etc. sicut praescriptum est in aeqq. (4.); quo facto aequationes (83.) solutionem completam problematis praebent.