7.

De aequationibus quarti et sexti gradus quae in theoria linearum et superficierum secundi gradus occurrunt.

(Auctore *F. Joachimsthal,* Prof. ord.)

Quae olim de normalibus linearum et superficierum secundi gradus exposui theoremata, ope duarum aequationum, quarum una quarti altera autem sexti gradus, demonstrari possunt. Cum autem in utraque, quam ea de re conscripsi, commentatione geometrica, harum aequationum proprietates algebraicas consulte praetermiserim, haud inutile fore credo, formulas, quibus antea usus sum, cum Geometris nunc communicare. In fine commentationis nonnulla de aequatione generaliore $2n$ti gradus adjiciam, quae illas tanquam casus speciales continet.

I.

Problema de inveniendis normalibus a puncto (ξ, η) ad conicam ad coordinatas rectangulares relatam

$$(1.)\qquad \frac{x^2}{a}+\frac{y^2}{b} = 1$$

ductis, a resolutione aequationis (1.) atque

$$(2.)\qquad a\frac{\xi-x}{x} = b\frac{\eta-y}{y}$$

pendet, ubi x et y tanquam incognitae spectandae sunt. Introducendo aliam incognitam u, valoribus fractionum in (2.) contentarum aequalem, habemus

$$(2^*.)\qquad x = \frac{a\xi}{a+u}\qquad y = \frac{b\eta}{b+u},$$

atque hinc, per aequationem (1.)

$$(3.)\qquad \frac{a\xi^2}{(a+u)^2}+\frac{b\eta^2}{(b+u)^2} = 1,$$

sive, si brevitatis causa loco ipsarum $a\xi^2$, $b\eta^2$ litterae α, β scribuntur

$$(4.)\qquad \frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2} = 1,$$

cujus aequationis biquadraticae radices signis u_1, u_2, u_3, u_4 denotentur.

Ut relationes inter normales inveniantur, quae a coordinatis ξ, η non pendeant, quantitates α et β inter aequationes

$$(5.)\quad \begin{cases} \dfrac{\alpha}{(a+u_1)^2}+\dfrac{\beta}{(b+u_1)^2}=1 \\ \dfrac{\alpha}{(a+u_2)^2}+\dfrac{\beta}{(b+u_2)^2}=1 \\ \dfrac{\alpha}{(a+u_3)^2}+\dfrac{\beta}{(b+u_3)^2}=1 \end{cases}$$

eliminandae sunt. Aequatio, quam nota eliminationis regula obtinemus, more auctorum recentiorum, qui theoriam determinantium coluerunt, hoc modo exhiberi potest

$$(6.)\quad \begin{vmatrix} \dfrac{1}{(a+u_1)^2} & \dfrac{1}{(b+u_1)^2} & 1 \\ \dfrac{1}{(a+u_2)^2} & \dfrac{1}{(b+u_2)^2} & 1 \\ \dfrac{1}{(a+u_3)^2} & \dfrac{1}{(b+u_3)^2} & 1 \end{vmatrix} = 0.$$

Quum autem haec aequatio factorem supervacaneum $(u_1-u_2)(u_1-u_3)(u_2-u_3)$ manifesto contineat alia eruenda erit eliminationis methodus cujus ope ad relationem ab hoc incommodo liberam perveniatur.

Designando per $\varphi(a)$ functionem primi vel secundi gradus, atque ponendo

$$(a+u_1)(a+u_2)(a+u_3) = A,$$
$$(b+u_1)(b+u_2)(b+u_3) = B,$$

habemus formulam notam

$$\frac{\varphi(a)}{A} = \frac{\varphi(-u_1)}{u_2-u_1 . u_3-u_1}\,\frac{1}{a+u_1}+\frac{\varphi(-u_2)}{u_1-u_2 . u_3-u_2}\,\frac{1}{a+u_2}+\frac{\varphi(-u_3)}{u_1-u_3 . u_2-u_3}\,\frac{1}{a+u_3}$$

quae differentiatione respectu quantitatis a in hanc

$$(7.)\quad \frac{\varphi(a)A'-\varphi'(a)A}{A^2}$$

$$= \frac{\varphi(-u_1)}{u_2-u_1 . u_3-u_1}\,\frac{1}{(a+u_1)^2}+\frac{\varphi(-u_2)}{u_1-u_2 . u_3-u_2}\,\frac{1}{(a+u_2)^2}+\frac{\varphi(-u_3)}{u_1-u_3 . u_2-u_3}\,\frac{1}{(a+u_3)^2}$$

transit. Jam multiplicando aequationes (5.) per quantitates

$$\frac{\varphi(-u_1)}{u_2-u_1 . u_3-u_1},\qquad \frac{\varphi(-u_2)}{u_1-u_2 . u_3-u_2},\qquad \frac{\varphi(-u_3)}{u_1-u_3 . u_2-u_3}$$

atque addendo, obtinemus ope aequationis (7.)

$$(8.)\quad \frac{\alpha}{A^2}\{\varphi(a)A'-\varphi'(a)A\}+\frac{\beta}{B^2}\{\varphi(b)B'-\varphi'(b)B\}$$
$$=\frac{\varphi(-u_1)}{u_2-u_1\,.\,u_3-u_1}+\frac{\varphi(-u_2)}{u_1-u_2\,.\,u_3-u_2}+\frac{\varphi(-u_3)}{u_1-u_3\,.\,u_2-u_3}.$$

Sit $\varphi(a)$ functio linearis $=a+\lambda$, ubi λ quantitatem constantem denotat. Pars dextra aequationis praecedentis secundum formulam Eulerianam evanescere debet, atque habemus

$$(9.)\quad \frac{\alpha}{A^2}\{\varphi(a)A'-\varphi'(a)A\}+\frac{\beta}{B^2}\{\varphi(b)B'-\varphi'(b)B\}=0.$$

Sed quantitas arbitraria λ ita determinari potest, ut sit

$$(10.)\quad \varphi(a)A'-\varphi'(a)A=0,\quad \text{sive } \frac{1}{a+\lambda}=\frac{A'}{A}$$

qua relatione (9.) in hancce transit

$$(11.)\quad \varphi(b)B'-\varphi'(b)B=0,\quad \text{sive } \frac{1}{b+\lambda}=\frac{B'}{B}.$$

Substitutis iterum valoribus functionum A et B, loco duarum aequationum praecedentium habentur sequentes

$$(12.)\quad \begin{cases} \dfrac{1}{a+\lambda}=\dfrac{1}{a+u_1}+\dfrac{1}{a+u_2}+\dfrac{1}{a+u_3} \\ \dfrac{1}{b+\lambda}=\dfrac{1}{b+u_1}+\dfrac{1}{b+u_2}+\dfrac{1}{b+u_3}. \end{cases}$$

His calculis eliminatio duarum quantitatum α et β e tribus aequationibus (5.) ad eliminationem unius quantitatis λ e duabus aequationibus (12.) reducta est. Jam subtrahendo aequationes (12.) obtinemus, suppresso factore $a-b$,

$$(13.)\quad \frac{1}{(a+\lambda)(b+\lambda)}=\frac{1}{(a+u_1)(b+u_1)}+\frac{1}{(a+u_2)(b+u_2)}+\frac{1}{(a+u_3)(b+u_3)}$$

et e combinatione satis obvia aequationum (12.) et (13.) haec aequatio finalis respectu radicum u_1, u_2, u_3 omnino symmetrica

$$(14.)\quad \frac{1}{(a+u_2)(b+u_3)}+\frac{1}{(a+u_3)(b+u_2)}+\frac{1}{(a+u_3)(b+u_1)}$$
$$+\frac{1}{(a+u_1)(b+u_3)}+\frac{1}{(a+u_1)(b+u_2)}+\frac{1}{(a+u_2)(b+u_1)}=0$$

derivatur. Calculo praecedenti igitur nacti sumus

Theorema I. Designando quantitatem $\dfrac{1}{(a+u_1)(b+u_2)}+\dfrac{1}{(a+u_2)(b+u_1)}$ characteristica $(1,2)$, inter ternas radices u_1, u_2, u_3 aequationis biquadraticae

$$(4.)\qquad \frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2} = 1$$

relatio intercedit

$$(14^*.)\qquad (2,3)+(1,3)+(1,2) = 0.$$

Qua ex aequatione tres aliae, illi (14*.) analogae, permutatione radicum deduci possunt, scilicet

$$\begin{array}{rl} (15.) & (2,4)+(1,4)+(1,2) = 0, \\ (15^*.) & (1,4)+(1,3)+(3,4) = 0, \\ (15^{**}.) & (2,4)+(2,3)+(3,4) = 0. \end{array}$$

Formando combinationem (14*.) + (15.) = (15*.) + (15**.) prodit alia aequatio haud inelegans

$$(16.)\qquad (1,2) = (3,4)$$

unde fluit

Theorema II. Inter quatuor radices aequationis $\frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2}=1$ valet relatio

$$\frac{1}{(a+u_1)(b+u_2)}+\frac{1}{(a+u_2)(b+u_1)} = \frac{1}{(a+u_3)(b+u_4)}+\frac{1}{(a+u_4)(b+u_3)}.$$

Cujus theorematis adjumento sex expressiones

$$(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4),$$

quum sit $(1,2)=(3,4)$, $(1,3)=(2,4)$, $(1,4)=(2,3)$ ad tres diversas $(1,2)$, $(1,3)$, $(1,4)$ reducuntur quae tanquam radices aequationis cubicae, formae

$$U^3+\lambda' U+\lambda'' = 0$$

inveniri possunt, ubi λ', λ'' ipsarum α, β, a, b sunt functiones algebraicae facili negotio eruendae.

Observatio. Aequatio fundamentalis (4.) in formam trigonometricam redigi potest, ponendo

$$\frac{\sqrt{\alpha}}{a+u} = \cos\varphi,\qquad \frac{\sqrt{\beta}}{b+u} = \sin\varphi,$$

unde prodit

$$\frac{\sqrt{\alpha}}{\cos\varphi}-\frac{\sqrt{\beta}}{\sin\varphi} = a-b,$$

sive

$$l\sin\varphi+m\cos\varphi = \sin\varphi\cos\varphi,\quad \text{ubi}\quad l=\frac{\sqrt{\alpha}}{a-b},\ m=-\frac{\sqrt{\beta}}{a-b}.$$

Quae igitur de aequatione biquadratica (4.) supra invenimus theoremata, nunc ita audiunt.

„Inter radices φ_1, φ_2, φ_3, φ_4 aequationis $l\sin\varphi + m\cos\varphi = \sin\varphi\cos\varphi$ „intercedunt relationes

$$\sin(\varphi_1+\varphi_2)+\sin(\varphi_1+\varphi_3)+\sin(\varphi_2+\varphi_3) = 0$$

$$\varphi_1+\varphi_2+\varphi_3+\varphi_4 = (2n+1)\pi; \quad n = \text{num. int.}"$$

Interpretationem geometricam relationis (14.) infra examinabimus.

II.

Perinde problema ducendi lineas normales a puncto (ξ,η,ζ) ad superficiem

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 1$$

a resolutione aequationis sexti gradus

$$(17.) \quad \frac{a\xi^2}{(a+u)^2}+\frac{b\eta^2}{(b+u)^2}+\frac{c\zeta^2}{(c+u)^2} = 1$$

sive

$$(18.) \quad \frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2}+\frac{\gamma}{(c+u)^2} = 1$$

pendet, ubi brevitatis gratia α, β, γ loco ipsarum $a\xi^2$, $b\eta^2$, $c\zeta^2$ scripsimus. Ut proprietates normalium per idem punctum (ξ,η,ζ) transeuntium detegantur, inter quatuor aequationes

$$(19.) \quad \begin{cases} \dfrac{\alpha}{(a+u_1)^2}+\dfrac{\beta}{(b+u_1)^2}+\dfrac{\gamma}{(c+u_1)^2} = 1 \\ \dfrac{\alpha}{(a+u_2)^2}+\dfrac{\beta}{(b+u_2)^2}+\dfrac{\gamma}{(c+u_2)^2} = 1 \\ \dfrac{\alpha}{(a+u_3)^2}+\dfrac{\beta}{(b+u_3)^2}+\dfrac{\gamma}{(c+u_3)^2} = 1 \\ \dfrac{\alpha}{(a+u_4)^2}+\dfrac{\beta}{(b+u_4)^2}+\dfrac{\gamma}{(c+u_4)^2} = 1 \end{cases}$$

quantitates α, β, γ eliminandae sunt, ita quidem ut aequatio finalis a producto differentiarum $u_1-u_2 \,.\, u_1-u_3 \,.\, u_1-u_4 \,.\, u_2-u_3 \,.\, u_2-u_4 \,.\, u_3-u_4$ sit libera. Scribendo

$$(20.) \quad \begin{cases} (a+u_1)(a+u_2)(a+u_3)(a+u_4) = A \\ (b+u_1)(b+u_2)(b+u_3)(b+u_4) = B \\ (c+u_1)(c+u_2)(c+u_3)(c+u_4) = C \end{cases}$$

atque statuendo $\varphi(a) = (a+\lambda)(a+\mu)$, habetur

$$\frac{\varphi(a)}{A} = \frac{\varphi(-u_1)}{u_2-u_1.u_3-u_1.u_4-u_1}\,\frac{1}{a+u_1}+\cdots$$

unde, differentiatione respectu ipsius a instituta, prodit aequatio

$$(21.)\quad \frac{\varphi(a)A'-\varphi'(a)A}{A^2} = \Sigma\frac{\varphi(-u_1)}{u_2-u_1.u_3-u_1.u_4-u_1}\,\frac{1}{(a+u_1)^2}.$$

Cujus formulae ope, si aequationes (19.) ex ordine factoribus

$$\frac{\varphi(-u_1)}{u_2-u_1.u_3-u_1.u_4-u_1},\quad \frac{\varphi(-u_2)}{u_1-u_2.u_3-u_2.u_4-u_2},\quad \frac{\varphi(-u_3)}{u_1-u_3.u_2-u_3.u_4-u_3},$$
$$\frac{\varphi(-u_4)}{u_1-u_4.u_2-u_4.u_3-u_4}$$

multiplicantur, obtinemus addendo

$$\frac{\alpha}{A^2}\{\varphi(a)A'-\varphi'(a)A\}+\frac{\beta}{B^2}\{\varphi(b)B'-\varphi'(b)B\}+\frac{\gamma}{C^2}\{\varphi(c)C'-\varphi'(c)C\}$$
$$= \Sigma\frac{\varphi(-u_1)}{u_2-u_1.u_3-u_1.u_4-u_1}.$$

Quum functio φ secundum gradum non transgrediatur, pars dextra hujus formulae evanescit; habemus itaque

$$(22.)\quad \frac{\alpha}{A^2}\{\varphi(a)A'-\varphi'(a)A\}+\frac{\beta}{B^2}\{\varphi(b)B'-\varphi'(b)B\}+\frac{\gamma}{C^2}\{\varphi(c)C'-\varphi'(c)C\} = 0.$$

Quantitates λ et μ, quae in functione φ inveniuntur, quippe quae omnino arbitrariae sint, ita determinari possunt, ut ipsarum $\frac{\alpha}{A^2}$, $\frac{\beta}{B^2}$ coefficientes evanescant. Quo facto, aequatio (22.) in tres alias discerpitur,

$$\varphi(a)A'-\varphi'(a)A = 0,\quad \varphi(b)B'-\varphi'(b)B = 0,\quad \varphi(c)C'-\varphi'(c)C = 0$$

sive in

$$(23.)\quad \begin{cases} \frac{1}{a+\lambda}+\frac{1}{a+\mu} = \frac{1}{a+u_1}+\frac{1}{a+u_2}+\frac{1}{a+u_3}+\frac{1}{a+u_4} \\ \frac{1}{b+\lambda}+\frac{1}{b+\mu} = \frac{1}{b+u_1}+\frac{1}{b+u_2}+\frac{1}{b+u_3}+\frac{1}{b+u_4} \\ \frac{1}{c+\lambda}+\frac{1}{c+\mu} = \frac{1}{c+u_1}+\frac{1}{c+u_2}+\frac{1}{c+u_3}+\frac{1}{c+u_4}. \end{cases}$$

Reduximus itaque eliminationem ipsarum α, β, γ e quatuor aequationibus (19.) ad eliminationem duarum quantitatum λ et μ e tribus aequationibus (23.). Quae eliminatio, quamvis illa, quam supra (I.) tractavimus, multo complicatior, haud ineleganter hoc modo perficitur.

Subtrahendo binas aequationes (23.) habemus

$$
(24.)\quad \begin{cases}
\frac{1}{(a+\lambda)(b+\lambda)}+\frac{1}{(a+\mu)(b+\mu)} \\
=\frac{1}{(a+u_1)(b+u_1)}+\frac{1}{(a+u_2)(b+u_2)}+\frac{1}{(a+u_3)(b+u_3)}+\frac{1}{(a+u_4)(b+u_4)} \\
\frac{1}{(b+\lambda)(c+\lambda)}+\frac{1}{(b+\mu)(c+\mu)} \\
=\frac{1}{(b+u_1)(c+u_1)}+\frac{1}{(b+u_2)(c+u_2)}+\frac{1}{(b+u_3)(c+u_3)}+\frac{1}{(b+u_4)(c+u_4)} \\
\frac{1}{(a+\lambda)(c+\lambda)}+\frac{1}{(a+\mu)(c+\mu)} \\
=\frac{1}{(a+u_1)(c+u_1)}+\frac{1}{(a+u_2)(c+u_2)}+\frac{1}{(a+u_3)(c+u_3)}+\frac{1}{(a+u_4)(c+u_4)}
\end{cases}
$$

et multiplicando aequationes (23.) ex ordine per

$$\frac{1}{b-a.c-a},\qquad \frac{1}{a-b.c-b},\qquad \frac{1}{a-c.b-c}$$

atque addendo, habetur

$$(25.)\quad \frac{1}{(a+\lambda)(b+\lambda)(c+\lambda)}+\frac{1}{(a+\mu)(b+\mu)(c+\mu)}$$
$$=\frac{1}{(a+u_1)\,(b+u_1)\,(c+u_1)}+\cdots+\frac{1}{(a+u_4)\,(b+u_4)\,(c+u_4)}.$$

Jam procul dubio lectorem accuratius examinantem non fugit, calculos, quorum adjumento ex aeq. (12.) et (13.) quantitatem λ eliminavimus, hac formula concinne exhiberi posse

$$(26.)\quad (\xi_1+\xi_2+\cdots+\xi_n)(\eta_1+\eta_2+\cdots+\eta_n)-(\xi_1\eta_1+\xi_2\eta_2+\cdots+\xi_n\eta_n)$$
$$=\Sigma(\xi_1\eta_2+\xi_2\eta_1).$$

Si $n=1$, pars laeva in nihilum abit; si autem $n \gtreqless 2$, summa ad dextram $\frac{n(n-1)}{2}$ summas binarias ut $\xi_1\eta_2+\xi_2\eta_1$ continet.

Est autem formula (26.) prima in serie quadam infinita formularum, quarum secunda hoc modo scribi potest:

$$(27.)\quad \Sigma\xi_1\Sigma\eta_1\Sigma\zeta_1-\Sigma\xi_1\Sigma\eta_1\zeta_1-\Sigma\eta_1\Sigma\xi_1\zeta_1-\Sigma\zeta_1\Sigma\xi_1\eta_1+2\Sigma\xi_1\eta_1\zeta_1$$
$$=\Sigma(\xi_1\eta_2\zeta_3+\xi_1\eta_3\zeta_2+\xi_2\eta_3\zeta_1+\xi_2\eta_1\zeta_3+\xi_3\eta_1\zeta_2+\xi_3\eta_2\zeta_1).$$

Summae Σ in membro primo aequationis (27.) n terminos, summa autem in membro secundo $\frac{n(n-1)(n-2)}{6}$ summas partiales senarias continet; si autem $n=1$, aut $n=2$, primum membrum evanescit.

Habemus ex. gr. pro $n=3$

$$(28.)\quad \begin{aligned}&(\xi_1+\xi_2+\xi_3)(\eta_1+\eta_2+\eta_3)(\zeta_1+\zeta_2+\zeta_3)-(\xi_1+\xi_2+\xi_3)(\eta_1\zeta_1+\eta_2\zeta_2+\eta_3\zeta_3)\\ &-(\eta_1+\eta_2+\eta_3)(\xi_1\zeta_1+\xi_2\zeta_2+\xi_3\zeta_3)-(\zeta_1+\zeta_2+\zeta_3)(\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3)\\ &+2(\xi_1\eta_1\zeta_1+\xi_2\eta_2\zeta_2+\xi_3\eta_3\zeta_3)\\ &=\xi_1\eta_2\zeta_3+\xi_1\eta_3\zeta_2+\xi_2\eta_1\zeta_3+\xi_2\eta_3\zeta_1+\xi_3\eta_1\zeta_2+\xi_3\eta_2\zeta_1,\end{aligned}$$

unde fluit aequatio identica

$$(29.)\quad \begin{aligned}&(\xi_1+\xi_2)(\eta_1+\eta_2)(\zeta_1+\zeta_2)-(\xi_1+\xi_2)(\eta_1\zeta_1+\eta_2\zeta_2)-(\eta_1+\eta_2)(\xi_1\zeta_1+\xi_2\zeta_2)\\ &-(\zeta_1+\zeta_2)(\xi_1\eta_1+\xi_2\eta_2)+2(\xi_1\eta_1\zeta_1+\xi_2\eta_2\zeta_2)=0.\end{aligned}$$

Quibus expositis ad eliminationem ipsarum λ et μ progrediamur. Formetur hunc ad finem expressio

$$\begin{aligned}&\left(\frac{1}{a+\lambda}+\frac{1}{a+\mu}\right)\left(\frac{1}{b+\lambda}+\frac{1}{b+\mu}\right)\left(\frac{1}{c+\lambda}+\frac{1}{c+\mu}\right)\\ &-\left(\frac{1}{a+\lambda}+\frac{1}{a+\mu}\right)\left(\frac{1}{(b+\lambda)(c+\lambda)}+\frac{1}{(b+\mu)(c+\mu)}\right)\\ &-\left(\frac{1}{b+\lambda}+\frac{1}{b+\mu}\right)\left(\frac{1}{(a+\lambda)(c+\lambda)}+\frac{1}{(a+\mu)(c+\mu)}\right)\\ &-\left(\frac{1}{c+\lambda}+\frac{1}{c+\mu}\right)\left(\frac{1}{(a+\lambda)(b+\lambda)}+\frac{1}{(a+\mu)(b+\mu)}\right)\\ &+2\left(\frac{1}{(a+\lambda)(b+\lambda)(c+\lambda)}+\frac{1}{(a+\mu)(b+\mu)(c+\mu)}\right),\end{aligned}$$

quae secundum formulam (29.) identice evanescit. Substitutis autem valoribus summarum $\frac{1}{a+\lambda}+\frac{1}{a+\mu}$, $\frac{1}{b+\lambda}+\frac{1}{b+\mu}$, etc., supra inventis (cf. formulas (23.), (24), (25.)), obtinemus ope formulae (27.) quaesitam aequationem finalem respectu radicum u_1, u_2, u_3, u_4 omnino symmetricam, quae brevitatis causa

$$(30.)\quad 0=\Sigma\frac{1}{(a+u_1)(b+u_2)(c+u_3)}$$

scribi potest. Signum Σ viginti quatuor terminos continet, e termino primo $\frac{1}{(a+u_1)(b+u_2)(c+u_3)}$ permutatione radicum u_1, u_2, u_3, u_4 deducendos. Quae nunc inventa sunt, hoc modo enuntiari possunt.

Theorema III. Designando quantitatem

$$\begin{aligned}&\frac{1}{(a+u_1)(b+u_2)(c+u_3)}+\frac{1}{(a+u_1)(b+u_3)(c+u_2)}+\frac{1}{(a+u_2)(b+u_1)(c+u_3)}\\ &+\frac{1}{(a+u_2)(b+u_3)(c+u_1)}+\frac{1}{(a+u_3)(b+u_1)(c+u_2)}+\frac{1}{(a+u_3)(b+u_2)(c+u_1)}\end{aligned}$$

characteristica (1, 2, 3), inter quaternas radices u_1, u_2, u_3, u_4 aequationis

$$(18.)\quad \frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2}+\frac{\gamma}{(c+u)^2} = 1$$

intercedit relatio

$$(31.)\quad (2,3,4)+(1,3,4)+(1,2,4)+(1,2,3) = 0.$$

Quae propositio illi analoga est quam supra demonstravimus. Similiter, ut sex quantitates in (I.) signis (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) denotatas ad tres diversas reduci posse docuimus, nunc reductionem viginti qnantitatum (1, 2, 3), (1, 2, 4)... (4, 5, 6) ad decem explicabimus.

Derivantur mutatione radicum e formula (31.) relationes

$$\begin{array}{ll}
(31^a.) & (2,3,5)+(1,3,5)+(1,2,5)+(1,2,3) = 0\\
(31^b.) & (2,4,5)+(1,4,5)+(1,2,5)+(1,2,4) = 0\\
(31^c.) & (3,4,5)+(1,4,5)+(1,3,5)+(1,3,4) = 0\\
(31^d.) & (2,3,4)+(2,3,5)+(2,4,5)+(3,4,5) = 0\\
(31^e.) & (1,2,3)+(1,2,6)+(1,3,6)+(2,3,6) = 0.
\end{array}$$

Formando combinationem $(31.)+(31^a.) = (31^b.)+(31^c.)+(31^d.)$, habemus

$$\begin{array}{lll}
(32.) & (1,2,3) = (1,4,5)+(2,4,5)+(3,4,5), & \text{unde}\\
 & (1,2,3) = (1,4,6)+(2,4,6)+(3,4,6) & \\
 & (1,2,3) = (1,5,6)+(2,5,6)+(3,5,6), & \text{atque similiter}\\
 & (4,5,6) = (1,2,4)+(1,2,5)+(1,2,6) & \\
 & (4,5,6) = (1,3,4)+(1,3,5)+(1,3,6) & \\
 & (4,5,6) = (2,3,4)+(2,3,5)+(2,3,6). &
\end{array}$$

Quarum aequationum ratione habita, summa $(31.)+(31^a.)+(31^e.)$ mutatur in

$$(33.)\quad (1,2,3)+(4,5,6) = 0.$$

Nacti igitur sumus hanc propositionem

Theorema IV. Denotando per u_1, u_2, u_3 tres radices aequationis sexti gradus

$$(10.)\quad \frac{\alpha}{(a+u)^2}+\frac{\beta}{(b+u)^2}+\frac{\gamma}{(c+u)^2} = 1,$$

aequatio vigesimi gradus, cui omnes valores diversi expressionis

$$U = \frac{1}{(a+u_1)(b+u_2)(c+u_3)}+\frac{1}{(a+u_1)(b+u_3)(c+u_2)}+\frac{1}{(a+u_2)(b+u_1)(c+u_3)}$$
$$+\frac{1}{(a+u_2)(b+u_3)(c+u_1)}+\frac{1}{(a+u_3)(b+u_1)(c+u_2)}+\frac{1}{(a+u_3)(b+u_2)(c+u_1)}$$

satisfaciunt, potestatibus imparibus ipsius U caret.

Antequam aequationis generalis $\frac{\alpha_1}{(a_1+u)^2}+\frac{\alpha_2}{(a_2+u)^2}+\cdots+\frac{\alpha_n}{(a_n+u)^2}=1$ proprietates deducemus, nonnulla, ne formularum nexus nimis interrumpatur, de transformatione summae $\Sigma\alpha_1\beta_2\gamma_3\ldots\mu_n$ addantur.

III.

Lemma. „Datis n seriebus quantitatum

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \ldots & \alpha_i \\ \beta_1 & \beta_2 & \ldots & \beta_i \\ \vdots & & & \\ \lambda_1 & \lambda_2 & \ldots & \lambda_i \\ \mu_1 & \mu_2 & \ldots & \mu_i \end{array}$$

summa $\Sigma\alpha_1\beta_2\gamma_3\ldots\mu_n$

„quae $i(i-1)(i-2)\ldots(i-n+1)$ terminos continet, ope summarum

$$(34.)\quad \begin{cases} \Sigma\alpha_1 \quad \Sigma\beta_1 \quad \Sigma\gamma_1 \quad \ldots \quad \Sigma\mu_1 \\ \Sigma\alpha_1\beta_1 \quad \Sigma\alpha_1\gamma_1 \quad \ldots \quad \Sigma\lambda_1\mu_1 \\ \Sigma\alpha_1\beta_1\gamma_1 \quad \Sigma\alpha_1\beta_1\delta_1 \quad \ldots \quad \Sigma\varkappa_1\lambda_1\mu_1 \\ \vdots \\ \Sigma\alpha_1\beta_1\gamma_1\ldots\mu_1 \end{cases}$$

„quarum singulae i terminis compositae sunt, exprimi potest, ita ut sit

$$(35.)\quad \Sigma\alpha_1\beta_2\gamma_3\ldots\mu_n = J,$$

„ubi J summarum (34.) functionem integram denotat. Dicimus insuper, „expressionem J pro $i \gtreqless n-1$ in nihilum abire.

Quod lemma, facile ad inveniendum difficile autem ad explicandum, cum formula de reductione expressionis $\Sigma x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$ ad summas potestatum radicum $x_1, x_2, \ldots x_i$, si ad modum demonstrandi solum spectas, prorsus congruit. Evolutio casuum $i=2, =3, =4$, quorum priores jam supra indicavimus, hic sufficiet (cf. form. 26 et 27). Habetur

$$(26.)\quad \Sigma\alpha_1\beta_2 = \Sigma\alpha_1\Sigma\beta_1 - \Sigma\alpha_1\beta_1$$

unde mutliplicando per $\Sigma\gamma_1$

$$(36.)\quad \Sigma\alpha_1\beta_2\Sigma\gamma_1 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma\alpha_1\beta_1\Sigma\gamma_1.$$

Est autem

$$(36^*.)\quad \Sigma\alpha_1\beta_2\Sigma\gamma_1 = \Sigma\alpha_1\beta_2\gamma_3 + \Sigma\alpha_1\gamma_1\beta_2 + \Sigma\alpha_2\beta_1\gamma_1,$$

atque iisdem, quibus formula (26.) demonstratur, ratiociniis

$$\Sigma\alpha_1\gamma_1\beta_2 = \Sigma\alpha_1\gamma_1\Sigma\beta_1 - \Sigma\alpha_1\beta_1\gamma_1$$
$$\Sigma\alpha_2\beta_1\gamma_1 = \Sigma\beta_1\gamma_1\Sigma\alpha_1 - \Sigma\alpha_1\beta_1\gamma_1.$$

His valoribus in formulis (36.) et (36*.) substitutis, derivatur

$$(37.)\quad \Sigma\alpha_1\beta_2\gamma_3 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma\alpha_1\Sigma\beta_1\gamma_1 - \Sigma\beta_1\Sigma\gamma_1\alpha_1 - \Sigma\gamma_1\Sigma\alpha_1\beta_1 + 2\,\Sigma\alpha_1\beta_1\gamma_1.$$

Perinde reductio ipsius $\Sigma\alpha_1\beta_2\gamma_3\delta_4$ peragitur. Nam multiplicando aequationem praecedentem per $\Sigma\delta_1$, obtinemus

$$(37^*.)\quad \Sigma\delta_1\Sigma\alpha_1\beta_2\gamma_3 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1\Sigma\delta_1 - \Sigma\alpha_1\Sigma\delta_1\Sigma\beta_1\gamma_1 - \Sigma\beta_1\Sigma\delta_1\Sigma\gamma_1\alpha_1$$
$$\Sigma\gamma_1\Sigma\delta_1\Sigma\alpha_1\beta_1 + 2\,\Sigma\delta_1\Sigma\alpha_1\beta_1\gamma_1.$$

Est autem

$$(37^{**}.)\quad \Sigma\delta_1\Sigma\alpha_1\beta_2\gamma_3 = \Sigma\alpha_1\beta_2\gamma_3\delta_4 + \Sigma\alpha_1\delta_1\beta_2\gamma_3 + \Sigma\beta_1\delta_1\alpha_2\gamma_3 + \Sigma\gamma_1\delta_1\alpha_2\beta_3;$$

eademque via, quam ad stabiliendam formulam (37.) secuti sumus, invenitur

$$\Sigma\alpha_1\delta_1\beta_2\gamma_3 = \Sigma\alpha_1\delta_1\Sigma\beta_1\Sigma\gamma_1 - \Sigma\alpha_1\delta_1\Sigma\beta_1\gamma_1 - \Sigma\beta_1\Sigma\alpha_1\gamma_1\delta_1 - \Sigma\gamma_1\Sigma\alpha_1\beta_1\delta_1 + 2\,\Sigma\alpha_1\beta_1\gamma_1\delta_1$$
$$\Sigma\beta_1\delta_1\alpha_2\gamma_3 = \Sigma\beta_1\delta_1\Sigma\alpha_1\Sigma\gamma_1 - \Sigma\beta_1\delta_1\Sigma\alpha_1\gamma_1 - \Sigma\alpha_1\Sigma\beta_1\gamma_1\delta_1 - \Sigma\gamma_1\Sigma\alpha_1\beta_1\delta_1 + 2\,\Sigma\alpha_1\beta_1\gamma_1\delta_1$$
$$\Sigma\gamma_1\delta_1\alpha_2\beta_3 = \Sigma\gamma_1\delta_1\Sigma\alpha_1\Sigma\beta_1 - \Sigma\gamma_1\delta_1\Sigma\alpha_1\beta_1 - \Sigma\alpha_1\Sigma\beta_1\gamma_1\delta_1 - \Sigma\beta_1\Sigma\alpha_1\gamma_1\delta_1 + 2\,\Sigma\alpha_1\beta_1\gamma_1\delta_1.$$

Quibus valoribus in (37*.) atque (37**.) substitutis, derivatur

$$(38.)\quad \Sigma\alpha_1\beta_2\gamma_3\delta_4 = \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1\Sigma\delta_1 - \Sigma\alpha_1\Sigma\beta_1\Sigma\gamma_1\delta_1 - \Sigma\alpha_1\Sigma\gamma_1\Sigma\beta_1\delta_1$$
$$- \Sigma\alpha_1\Sigma\delta_1\Sigma\beta_1\gamma_1 - \Sigma\gamma_1\Sigma\delta_1\Sigma\alpha_1\beta_1 - \Sigma\beta_1\Sigma\delta_1\Sigma\alpha_1\gamma_1 - \Sigma\beta_1\Sigma\gamma_1\Sigma\alpha_1\delta_1$$
$$+ 2\,\Sigma\alpha_1\Sigma_1\beta_1\gamma_1\delta_1 + 2\,\Sigma\beta_1\Sigma\alpha_1\gamma_1\delta_1 + 2\,\Sigma\gamma_1\Sigma\alpha_1\beta_1\delta_1 + 2\,\Sigma\delta_1\Sigma\alpha_1\beta_1\gamma_1$$
$$- 6\,\Sigma\alpha_1\beta_1\gamma_1\delta_1.$$

Si singulae summae $\Sigma\alpha_1$, $\Sigma\beta_1$, $\Sigma\gamma_1$, $\Sigma\delta_1$ quatuor continent terminos, ita ut sit $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ etc., pars laeva aequationis (38.), iisdem viginti quatuor terminis composita atque determinans systematis

$$\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{matrix}$$

in eo tantum a determinante differt, quod omnes ejus termini sunt positivi. Unde fluit, partem dextram formulae (38.) evanescere si singulae summae $\Sigma\alpha_1$, $\Sigma\beta_1$, $\Sigma\gamma_1$, $\Sigma\delta_1$ tres aut duos contineant terminos, aut modo unum.

Eodem modo, ope formulae (38.) ad reductionem summae $\Sigma\alpha_1\beta_2\gamma_3\delta_4\varepsilon_5$ adscenditur, et sic porro.

Formulas speciales (26.), (37.), (38.) ill. **Binet** in diario a schola Polytechnica Parisiensi edito (Cah. XVI. pag. 285) publici juris fecit.

IV.

Inter $(n+1)$ radices diversas $u_1, u_2, \dots u_{n+1}$ aequationis $2n^{ti}$ gradus

$$(39.)\qquad \frac{\alpha_1}{(a_1+u)^2}+\frac{\alpha_2}{(a_2+u)^2}+\dots+\frac{\alpha_n}{(a_n+u)^2}=1$$

manifesto relatio intercedit a constantibus $\alpha_1, \alpha_2, \dots \alpha_n$ libera. Quae ut inveniatur n quantitates α inter $(n+1)$ aequationes eliminandae sunt, quarum prima

$$(40.)\qquad \left|\frac{\alpha_1}{(a_1+u_1)^2}+\frac{\alpha_2}{(a_2+u_1)^2}+\dots+\frac{\alpha_n}{(a_n+u_1)^2}=1\right|$$

brevitatis causa loco totius systematis scribatur. Methodo, quam in art. I. et II. explicavimus, eliminatio ipsarum $\alpha_1, \alpha_2, \dots \alpha_n$ ad eliminationem $(n-1)$ quantitatum $v_1, v_2, \dots v_{n-1}$ e systemate n aequationum

$$(41.)\qquad \begin{cases} \dfrac{1}{a_1+v_1}+\dfrac{1}{a_1+v_2}+\dots+\dfrac{1}{a_1+v_{n-1}} \\ =\dfrac{1}{a_1+u_1}+\dfrac{1}{a_1+u_2}+\dfrac{1}{a_1+u_3}+\dots+\dfrac{1}{a_1+u_{n+1}} \\ \dfrac{1}{a_2+v_1}+\dfrac{1}{a_2+v_2}+\dots+\dfrac{1}{a_2+v_{n-1}} \\ =\dfrac{1}{a_2+u_1}+\dfrac{1}{a_2+u_2}+\dfrac{1}{a_2+u_3}+\dots+\dfrac{1}{a_2+u_{n+1}} \\ \vdots \\ \dfrac{1}{a_n+v_1}+\dfrac{1}{a_n+v_2}+\dots+\dfrac{1}{a_n+v_{n-1}} \\ =\dfrac{1}{a_n+u_1}+\dfrac{1}{a_n+u_2}+\dfrac{1}{a_n+u_3}+\dots+\dfrac{1}{a_n+u_{n+1}} \end{cases}$$

reduci potest. Quo ex systemate derivantur $(n-1)$ alia, quorum primum $\frac{n(n-1)}{2}$ aequationes continens ita repraesentari potest:

$$(42.)\qquad \left|\begin{array}{l} \dfrac{1}{(a_1+v_1)(a_2+v_1)}+\dfrac{1}{(a_1+v_2)(a_2+v_2)}+\dots+\dfrac{1}{(a_1+v_{n-1})(a_2+v_{n-1})} \\ =\dfrac{1}{(a_1+u_1)(a_2+u_1)}+\dfrac{1}{(a_1+u_2)(a_2+u_2)}+\dots+\dfrac{1}{(a_1+u_n)(a_2+u_n)} \\ \qquad+\dfrac{1}{(a_1+u_{n+1})(a_2+u_{n+1})} \end{array}\right|;$$

secundum autem, $\frac{n(n-1)(n-2)}{1.2.3}$ aequationes continens,

$$(43.)\quad \left|\begin{array}{l} \frac{1}{(a_1+v_1)(a_2+v_1)(a_3+v_1)}+\dots+\frac{1}{(a_1+v_{n-1})(a_2+v_{n-1})(a_3+v_{n-1})} \\ = \frac{1}{(a_1+u_1)(a_2+u_1)(a_3+u_1)}+\dots+\frac{1}{(a_1+u_n)(a_2+u_n)(a_3+u_n)} \\ \quad +\frac{1}{(a_1+u_{n+1})(a_2+u_{n+1})(a_3+u_{n+1})} \end{array}\right|$$

etc., usque ad ultimum systema pervenitur quod unam tantummodo aequationem complectitur

$$(44.)\quad \sum_{v_1}^{v_{n-1}}\frac{1}{(a_1+v_1)(a_2+v_1)\dots(a_n+v_1)} = \sum_{u_1}^{u_{n+1}}\frac{1}{(a_1+u_1)(a_2+u_1)\dots(a_n+u_1)},$$

quarum aequationum deductio e systemate primitivo (41.) satis obvia est. Jam consideremus summam $1.2.3\dots(n+1)$ terminorum e termino primo permutatione quantitatum u_1, u_2, u_3, ... u_{n+1} deducendorum

$$\Sigma\frac{1}{(a_1+u_1)(a_2+u_2)\dots(a_n+u_n)}.$$

Quae summa secundum lemma in III. expositum tanquam summarum simplicium

$$(45.)\quad \left\{\begin{array}{l} \sum_{u_1}^{u_{n+1}}\frac{1}{a_1+u_1},\quad \Sigma\frac{1}{a_2+u_1},\quad \dots\quad \Sigma\frac{1}{a_n+u_1} \\ \sum_{u_1}^{u_{n+1}}\frac{1}{(a_1+u_1)(a_2+u_1)},\quad \Sigma\frac{1}{(a_1+u_1)(a_3+u_1)},\quad \dots\quad \Sigma\frac{1}{(a_{n-1}+u_1)(a_n+u_1)} \\ \quad\vdots \\ \sum_{u_1}^{u_{n+1}}\frac{1}{(a_1+u_1)(a_2+u_1)\dots(a_n+u_1)} \end{array}\right.$$

functio rationalis integra J repraesentari potest. Substituendo autem in J summarum (45.) valores supra inventos, scilicet summas analogas

$$(46.)\quad \left\{\begin{array}{l} \sum_{v_1}^{v_{n-1}}\frac{1}{a_1+v_1},\quad \Sigma\frac{1}{a_2+v_1},\quad \dots\quad \Sigma\frac{1}{a_n+v_1} \\ \sum_{v_1}^{v_{n-1}}\frac{1}{(a_1+v_1)(a_2+v_1)},\quad \Sigma\frac{1}{(a_1+v_1)(a_3+v_1)},\quad \dots\quad \Sigma\frac{1}{(a_{n-1}+v_1)(a_n+v_1)} \\ \quad\vdots \\ \sum_{v_1}^{v_{n-1}}\frac{1}{(a_1+v_1)(a_2+v_1)\dots(a_n+v_1)} \end{array}\right.$$

expressio J in nihilum abit, quum expressiones (46.) $(n-1)$ tantummodo terminis constent. Hinc prodit

Theorema V. Designando characteristica $(1, 2, 3, \ldots n)$ summam $1.2.3\ldots n$ terminorum, qui permutatione quantitatum $u_1, u_2, u_3, \ldots u_n$ e termino $\frac{1}{(a_1+u_1)(a_2+u_2)\ldots(a_n+u_n)}$ derivantur, inter $(n+1)$ radices diversas $u_1, u_2, u_3, \ldots u_{n+1}$ aequationis $2n^{\text{ti}}$ gradus

$$(39.)\qquad \frac{\alpha_1}{(a_1+u)^2}+\frac{\alpha_2}{(a_2+u)^2}+\cdots+\frac{\alpha_n}{(a_n+u)^2} = 1$$

intercedit relatio

$$(47.)\ (2,3,4,\ldots n+1)+(1,3,4,\ldots n+1)+(1,2,4,\ldots n+1)+\cdots+(1,2,3,\ldots n)=0$$

quae aequatio una cum reliquis e (47.) permutatione radicum $u_1, u_2, u_3, \ldots u_{2n}$ deducendis systema $\frac{2n.2n-1.2n-2\ldots n+2}{1.\;2.\;3\ldots n-1}$ relationum format.

E magna copia relationum quae a formula (39.) derivantur, nunc unam tantummodo eligemus, quae demonstratione muniatur, scilicet sequentem

$$(48.)\qquad (1,2,3,\ldots n)\mp(n+1, n+2, n+3,\ldots 2n) = 0,$$

valente signo superiori vel inferiori prout n par est vel impar (cf. casus speciales (16.) et (33.)). Designemus signo $[1,2,3,\ldots n, n+1]$, quod $n+1$ indices continet partem laevam formulae (47.), ita ut sit

$$[1,2,3,\ldots n, n+1] = (2,3,4,\ldots n+1)+(1,3,4,\ldots n+1)+\cdots+(1,2,3,\ldots n).$$

Distribuantur indices $1,2,3,\ldots 2n-1$, $2n$ in duo systemata, cujus alterum indices $1, 2, 3, \ldots n$; alterum autem indices $n+1, n+2, n+3, \ldots 2n$ comprehendat et designetur per A_1 summa omnium [ ], quae n indices primi unum autem secundi systematis, per A_2 summa omnium [ ], quae $n-1$ indices primi, duos autem secundi systematis contineat, etc., usque ad A_n pervenietur, qua littera summam omnium [ ] designabimus, quae unum indicem primi, n autem indices secundi systematis comprehendunt. Jam dico expressionem

$$(49.)\quad A_1-\frac{1}{n-1}A_2+\frac{1\;2}{n-1.n-2}A_3-\frac{1.2.3}{n-1.n-2.n-3}A_4+\cdots+(-1)^{n-1}A_n$$

substitutis valoribus ipsarum $A_1, A_2, \ldots A_n$ in

$$(50.)\qquad n\{(1,2,3,\ldots n)+(-1)^{n-1}(n+1, n+2, n+3,\ldots 2n)\}$$

abire. Manifesto enim expressio $(1,2,3,\ldots n)$ in n terminis solis, quibus A_1

composita est, invenitur; scilicet in

$$[1, 2, \ldots n-1, n, n+1], \quad [1, 2, \ldots n-1, n, n+2],$$
$$[1, 2, \ldots n-1, n, n+3], \ \ldots \ [1, 2, \ldots n-1, n, 2n];$$

similiter $(n+1, n+2, n+3, \ldots 2n)$ in n terminis ipsius A_n tantum occurrit. Omnes alios terminos in (49.) se mutuo destruere, hoc modo probari potest. Consideremus simplicitatis causa expressionem

$$(51.) \qquad (1, 2, 3, \ldots n-i, n+1, n+2, n+3, \ldots n+i),$$

quum eadem ratiocinia pro omni expressione ( ) valeant, quae $n-i$ indices primi atque i indices secundi systematis contineat. Quae expressio (51.), ut sine negotio vides, invenitur

$$\begin{array}{ll} \text{in} & [1, 2, 3, \ldots n-i, n-i+1, n+1, n+2, \ldots n+i], \\ & [1, 2, 3, \ldots n-i, n-i+2, n+1, n+2, \ldots n+i], \\ & [1, 2, 3, \ldots n-i, n-i+3, n+1, n+2, \ldots n+i], \\ & \vdots \\ & [1, 2, 3, \ldots n-i, n, n+1, n+2, \ldots n+i], \\ \text{atque in} & [1, 2, 3, \ldots n-i, n+1, n+2, n+3, \ldots n+i, n+i+1], \\ & [1, 2, 3, \ldots n-i, n+1, n+2, n+3, \ldots n+i, n+i+2], \\ & [1, 2, 3, \ldots n-i, n+1, n+2, n+3, \ldots n+i, n+i+3], \\ & \vdots \\ & [1, 2, 3, \ldots n-i, n+1, n+2, n+3, \ldots n+i, n+i, 2n]. \end{array}$$

Jam i expressiones primae seriei ad A_i pertinent, itaque in (49.) factore $\pm\frac{1.2.3\ldots i-1}{n-1.n-2.n-3\ldots(n-i+1)}$ multiplicatae sunt; atque $(n-i)$ expressiones seriei secundae ad A_{i+1} pertinent, itaque in factorem $\mp\frac{1.2.3\ldots i}{n-1.n-2.n-3\ldots n-i}$ ductae sunt. Est igitur in (49.) coefficiens expressionis (51.)

$$= \pm\left\{i\frac{1.2\ldots i-1}{n-1.n-2\ldots n-i+1} - (n-i)\frac{1.2\ldots i}{n-1.n-2\ldots n-i}\right\} = 0, \quad \text{q. e. d.}$$

Quum autem omnes quantitates [ ] secundum theorema V. evanescant, summa (49.) sive summa aequivalens (50.) in nihilum abit. Inde prodit formula (48.) sive

Theorema VI. Designando, ut supra, characteristica $(1, 2, 3, \ldots n)$ summam $1.2.3\ldots n$ terminorum $\Sigma\frac{1}{(a_1+u_1)(a_2+u_2)\ldots(a_n+u_n)}$ qui permutatione quantitatum $u_1, u_2, \ldots u_n$ a termino primo derivantur, inter $2n$

radices $u_1, u_2, \ldots u_{2n}$ aequationis

$$(39.)\quad \frac{\alpha_1}{(a_1+u)^2}+\frac{\alpha_2}{(a_2+u)^2}+\cdots+\frac{\alpha_n}{(a_n+u)^2} = 1$$

relatio intercedit haecce

$$(48^*.)\quad (1, 2, 3, \ldots n) = (-1)^n(n+1, n+2, n+3, \ldots 2n),$$

ita ut resolutione aequationis $\frac{1}{2}\frac{1.2.3\ldots 2n}{(1.2.3\ldots n)^2}$ gradus $\frac{1.2.3\ldots 2n}{(1.2.3\ldots n)^2}$ quantitates $(1, 2, 3, \ldots n)$ ipsae aut earum quadrata inveniri possint.

V.

Jam methodum a nobis adhibitam cum formula e theoria determinantium deducta comparemus. Scribendo

$$\begin{aligned}(a_1+u_1)(a_1+u_2)\ldots(a_1+u_{n+1}) &= A_1\\ (a_2+u_1)(a_2+u_2)\ldots(a_2+u_{n+1}) &= A_2\\ &\vdots\\ (a_n+u_1)(a_n+u_2)\ldots(a_n+u_{n+1}) &= A_n\end{aligned}$$

habemus *)

$$(52.)\quad \text{det.}\left\{\frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \cdots \frac{1}{(a_n+u)^2}, 1\right\}_{u=u_1, =u_2, =u_3 \ldots =u_{n+1}} = \frac{V}{A_1^2 A_2^2 A_3^2 \ldots A_n^2}$$

ubi

$$(53.)\quad V = \text{det.}\{(a_2+u)^2(a_3+u)^2\ldots(a_n+u)^2; \ldots (a_1+u)^2(a_2+u)^2\ldots(a_{n-1}+u)^2; (a_1+u)^2(a_2+u)^2\ldots(a_n+u)^2\}_{u=u_1, =u_2, =u_3 \ldots =u_{n+1}}.$$

Peracta multiplicatione habemus

$$\begin{aligned}(a_2+u)^2(a_3+u)^2\ldots(a_n+u)^2 &= \lambda_1^{(0)}+\lambda_1^{(1)}u+\cdots+\lambda_1^{(2n-3)}u^{2n-3}+u^{2n-2}\\ (a_1+u)^2(a_3+u)^2\ldots(a_n+u)^2 &= \lambda_2^{(0)}+\lambda_2^{(1)}u+\cdots+\lambda_2^{(2n-3)}u^{2n-3}+u^{2n-2}\\ &\vdots\\ (a_1+u)^2(a_2+u)^2\ldots(a_{n-1}+u)^2 &= \lambda_n^{(0)}+\lambda_n^{(1)}u+\cdots+\lambda_n^{(2n-3)}u^{2n-3}+u^{2n-2}\\ (a_1+u)^2(a_2+u)^2\ldots(a_{n-1}+u)^2(a_n+u)^2 &= \mu^0+\mu^{(1)}u+\cdots+\mu^{(2n-1)}u^{2n-1}+u^{2n}\end{aligned}$$

ubi quantitates λ et μ ipsarum $a_1, a_2, \ldots a_n$ sunt functionis integrae. Inde sequitur adjumento notissimi theorematis circa determinantium compositionem; ut V tanquam summam $(2n-1)^n(2n+1)$ determinantium partialium hujusce formae

*) Denotatio determinantium, qua utimur, explicatione vix eget.

$$(54.)\qquad \nu\,\text{det.}\begin{vmatrix} u_1^{\alpha_1} & u_1^{\alpha_2} & \dots & u_1^{\alpha_n} & u_1^{\alpha_{n+1}} \\ u_2^{\alpha_1} & u_2^{\alpha_2} & \dots & u_2^{\alpha_n} & u_2^{\alpha_{n+1}} \\ & & \vdots & & \\ u_{n+1}^{\alpha_1} & u_{n+1}^{\alpha_2} & \dots & u_{n+1}^{\alpha_n} & u_{n+1}^{\alpha_{n+1}} \end{vmatrix}$$

exhiberi possit, ubi ν ipsarum $a_1, a_2, \dots a_n$ functio integra sit. Quum autem determinans (54.) evanescat, quoties duo exponentium $\alpha_1, \alpha_2, \dots \alpha_n, \alpha_{n+1}$ inter se aequales fiunt, determinans V sive (53.) respectu quantitatum $u_1, u_2, \dots u_{n+1}$ altiorem ordinem adipisci nequit, quam determinans partialis

$$\text{det.}\begin{vmatrix} u_1^{n-1} & u_1^n & u_1^{n+1} & \dots & u_1^{2n-3} & u^{2n-2} & u_1^{2n} \\ u_2^{n-1} & u_2^n & u_2^{n+1} & \dots & u_2^{2n-3} & u_2^{2n-2} & u_2^{2n} \\ & & & \vdots & & & \\ u_{n+1}^{n-1} & u_{n+1}^n & u_{n+1}^{n+1} & \dots & u_{n+1}^{2n-3} & u_{n+1}^{2n-2} & u_{n+1}^{2n} \end{vmatrix}$$

functio V igitur $(n-1)+n+(n+1)\dots+(2n-3)+(2n-2)+2n=\frac{n(3n+1)}{2}^{\text{tum}}$ ordinem non transgreditur. Quum autem V productum ex omnibus ipsarum u differentiis conflatum contineat, scilicet productum

$$(55.)\qquad \begin{aligned} \varDelta(u_1, u_2, u_3, \dots u_{n+1}) = (u_1-u_2)(u_1-u_3)\dots(u_1-u_n)\,(u_1-u_{n+1}) \\ (u_2-u_3)\dots(u_2-u_n)\,(u_2-u_{n+1}) \\ \vdots\qquad\qquad \\ (u_{n-1}-u_n)(u_{n-1}-u_{n+1}) \\ (u_n-u_{n+1}) \end{aligned}$$

quod $\frac{n(n+1)}{2}^{\text{ti}}$ ordinis est, manifesto poni potest

$$(56.)\qquad V = \varDelta(u_1, u_2, u_3, \dots u_{n+1})\,V_1,$$

denotando per V_1 functionem, quae $n^{2\,\text{tum}}$ ordinem non transgrediatur.

Consideremus expressionem supra signo $[1, 2, 3, \dots n+1]$ notatam, scilicet summam $1.2.3\dots n+1$ terminorum qui e termino $\frac{1}{(a_1+u_1)(a_2+u_2)\dots(a_n+u_n)}$ permutatione radicum $u_1, u_2, u_3, \dots u_{n+1}$ derivantur. Quae summa manifesto in $\frac{U}{A_1A_2A_3\dots A_n}$ transformari potest, ubi U est functio, respectu ipsarum $u_1, u_2, \dots u_{n+1}$ non minoris quam $n^{2\,\text{ti}}$ ordinis, quae factores u_1-u_2, $u_1-u_3, \dots u_n-u_{n+1}$ non contineat. Quum autem eliminatione quantitatum α e systemate $(n+1)$ aequationum

$$(40.)\qquad \left|\frac{\alpha_1}{(a_1+u)^2}+\frac{\alpha_2}{(a_2+u)^2}+\dots+\frac{\alpha_n}{(a_n+u)^2}-1 = 0\right|_{u=u_1,\ =u_2,\ \dots,\ =u_{n+1}}$$

methodo nota aequatio $V=0$, sive $\Delta(u_1, u_2, \dots u_{n+1}) V_1 = 0$, nostra autem methodo aequatio $U=0$, obtinetur; expressiones V_1 atque U, quum posterior priorem, quae, ut facile probatur, identice non evanescit, metiri debeat, nisi factore a solis quantitatibus $a_1, a_2, \dots a_n$ dependenti differre nequeunt. Habemus igitur formulam

$$(57.)\quad \text{det.}\left\{\frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \dots \frac{1}{(a_n+u)^2}, 1\right\}_{u=u_1, =u_2, \dots, =u_{n+1}}$$
$$= \varrho[1, 2, 3 \dots n+1] \frac{\Delta(u_1, u_2, \dots u_{n+1})}{A_1 . A_2 . A_3 \dots A_n}$$

denotando per ϱ functionem rationalem ipsarum $a_1, a_2, \dots a_n$.

VI.

Quum quinque abhinc annis theoremata eaque demonstrandi methodos, quae in articulis praecedentibus leguntur, invenissem, a determinatione *generali* ipsius ϱ propter calculorum, in quos incideram, prolixitatem abhorrui; pro casibus autem *specialibus* $n=2, =3$, qui soli in illis applicationibus geometricis occurrunt, de quibus postea disseremus, valores ipsius ϱ sine magno labore erui potuerunt. Jam ut de novo quaestionem intacte relictam susciperem, theorema pulcherrimum me commovit, a cl°. ***Borchardt*** inventum, atque, cum luculenta demonstratione, in commentatione de functionibus symmetricis illustri Academiae Berolinensi praeterito anno tradita publicatum *). Propositio, inter elegantissimas, quae de determinantibus traduntur, formulas censenda hoc modo exhiberi potest

$$(58.)\quad \frac{\text{det.}\left\{\frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \dots \frac{1}{(a_n+u)^2}\right\}}{\text{det.}\left\{\frac{1}{a_1+u}, \frac{1}{a_2+u}, \dots \frac{1}{a_n+u}\right\}}_{u=u_1, =u_2, \dots, =u_n}$$
$$= \Sigma \frac{1}{(a_1+u_1)(a_2+u_2)\dots(a_n+u_n)} = (1, 2, 3, \dots n)$$

(De vi characteristicae $(1, 2, 3, \dots n)$ vide theorema V.).

Permagna hujus formulae atque relationis (57.) similitudo ad novas disquisitiones instituendas me induxit, et adjumento nonnullorum artificiorum, quae olim neglexeram, ad formulam illi (58.) omnino analogam perveni. Quas disquisitiones hic addere non inutile fore credo, ne articuli praecedentis evo-

---

*) Cf. Relat. menstr. Acad. Berol. pro anno 1855.

lutiones incompletae maneant; solutionis autem merita, si quae adsint, relationis (58.) sagacissimo auctori sunt attribuenda.

Multiplicando aequationem (57.) per productum $(a_1+u_1)^2(a_2+u_2)^2\dots(a_n+u_n)^2$ atque ponendo deinde $u_1=-a_1$, $u_2=-a_2$, $\dots$ $u_n=-a_n$ valor ipsius ϱ facile eruitur. Quibus calculis enim pars laeva aequationis (57.) sive

$$\text{det.}\begin{vmatrix} \frac{1}{(a_1+u_1)^2} & \frac{1}{(a_2+u_1)^2} & \dots & \frac{1}{(a_n+u_1)^2} & 1 \\ \frac{1}{(a_1+u_2)^2} & \frac{1}{(a_2+u_2)^2} & \dots & \frac{1}{(a_n+u_2)^2} & 1 \\ & & \vdots & & \\ \frac{1}{(a_1+u_n)^2} & \frac{1}{(a_2+u_n)^2} & \dots & \frac{1}{(a_n+u_n)^2} & 1 \\ \frac{1}{(a_1+u_{n+1})^2} & \frac{1}{(a_2+u_{n+1})^2} & \dots & \frac{1}{(a_n+u_{n+1})^2} & 1 \end{vmatrix}$$

mutatur in

$$\text{det.}\begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \dots & 1 & 0 \\ \frac{1}{(a_1+u_{n+1})^2} & \frac{1}{(a_2+u_{n+1})^2} & \dots & \frac{1}{(a_n+u_{n+1})^2} & 1 \end{vmatrix} = 1.$$

In secundo membro aequationis (57.)

$$\varDelta(u_1, u_2, \dots u_{n+1}) = (u_1-u_{n+1})(u_2-u_{n+1})\dots(u_n-u_{n+1})\varDelta(u_1, u_2, u_3, \dots u_n)\text{*)}$$

transit in $(-1)^n.(a_1+u_{n+1})(a_2+u_{n+1})\dots(a_n+u_{n+1})(-1)^{\frac{n(n-1)}{2}}\varDelta(a_1, a_2, \dots a_n)$;

atque $\frac{[1, 2, 3, \dots n+1]}{A_1 A_2 A_3 \dots A_n}$, multiplicatione facta, substitutisque valoribus, quos ipsis u_1, u_2, $\dots$ u_n tribuimus, mutatur in fractionem, cujus numerator unitas, denominator autem

$$\begin{aligned} &= (a_1-a_2)(a_1-a_3)\dots(a_1-a_n)(a_1+u_{n+1}) \\ &\times (a_2-a_1)(a_2-a_3)\dots(a_2-a_n)(a_2+u_{n+1}) \\ &\times (a_3-a_1)(a_3-a_2)\dots(a_3-a_n)(a_3+u_{n+1}) \\ &\qquad\vdots \\ &\times (a_n-a_1)(a_n-a_2)\dots(a_n-a_{n-1})(a_n+u_{n+1}) \end{aligned}$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{1}{\varDelta(a_1, a_2, \dots a_n)^2 (a_1+u_{n+1})(a_2+u_{n+1})\cdots(a_n+u_{n+1})}.$$

*) In producto differentiarum, quod signo $\varDelta$ denotamus, singulae differentiae $u_\alpha - u_\beta$ ita sumendae sunt ut $\alpha < \beta$.

Quibus valoribus substitutis aequatio (57.) transit in

$$1 = \varrho \frac{(-1)^n}{\Delta(a_1, a_2, \dots a_n)}, \quad \text{unde } \varrho = (-1)^n \Delta(a_1, a_2, \dots a_n).$$

Hinc prodit denique e (57.)

$$(59.) \quad \det. \left\{ \frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \dots \frac{1}{(a_n+u)^2}, 1 \right\}_{u=u_1, =u_2, \dots =u_{n+1}}$$
$$= (-1)^n \frac{\Delta(a_1, a_2, \dots a_n)\, \Delta(u_1, u_2, \dots u_{n+1})}{A_1 A_2 A_3 \dots A_n} [1, 2, 3, \dots n+1].$$

Simili modo ut supra, videlicet comparatione dimensionum, invenitur esse

$$(60.) \quad \det. \left\{ \frac{1}{a_1+u}, \frac{1}{a_2+u}, \dots \frac{1}{a_n+u}, 1 \right\}_{u=u_1, =u_2, \dots =u_{n+1}} = \frac{\sigma \Delta(u_1, u_2, \dots u_{n+1})}{A_1 A_2 A_3 \dots A_n}$$

ubi σ ipsarum $a_1, a_2, \dots a_n$ sit functio rationalis. Multiplicando aequationem praecedentem per $(a_1+u_1)(a_2+u_2)\dots(a_n+u_n)$, atque ponendo $u_1 = -a_1$, $u_2 = -a_2, \dots u_n = -a_n$, obtinemus $\sigma = \varrho = (-1)^n \Delta(a_1, a_2, \dots a_n)$, unde

$$(61.) \quad \det. \left\{ \frac{1}{a_1+u}, \frac{1}{a_2+u}, \dots \frac{1}{a_n+u}, 1 \right\}_{u=u_1, =u_2, =u_3, \dots =u_{n+1}}$$
$$= (-1)^n \frac{\Delta(a_1, a_2, a_3, \dots a_n)\, \Delta(u_1, u_2, u_3, \dots u_{n+1})}{A_1 A_2 A_3 \dots A_n}.$$

E combinatione aequationum (59.) et (61.) prodit

$$(62.) \quad \frac{\det. \left\{ \frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \dots \frac{1}{(a_n+u)^2}, 1 \right\}}{\det. \left\{ \frac{1}{a_1+u}, \frac{1}{a_2+u}, \dots \frac{1}{a_n+u}, 1 \right\}}_{u=u_1, =u_2, =u_3, \dots =u_{n+1}} = [1, 2, 3, \dots n+1].$$

Faciendo u_{n+1} = quantitati infinite magnae, aequatio (62.) in relationem a cl. *Borchardt* inventam transit, scilicet in

$$(58.) \quad \frac{\det. \left\{ \frac{1}{(a_1+u)^2}, \frac{1}{(a_2+u)^2}, \dots \frac{1}{(a_n+u)^2} \right\}}{\det. \left\{ \frac{1}{a_1+u}, \frac{1}{a_2+u}, \dots \frac{1}{a_n+u} \right\}}_{u=u_1, =u_2, \dots =u_n}$$
$$= (1, 2, 3, \dots n) = \Sigma \frac{1}{(a_1+u_1)(a_2+u_2)\dots(a_n+u_n)}.$$

Simili modo ut e formula (62.) relationem (58.) deduximus, a posteriori ad priorem calculo persimplici perveniri potest. Insuper relationis (58.) demonstratio exstat directa, quae prorsus similibus, atque supra adhibita sunt, nititur ratiociniis.

VII.

Ad applicationes geometricas transeamus, quae casibus $n=2$ et $n=3$ respondent. Puncta curvae

$$(1.)\qquad \frac{x^2}{a}+\frac{y^2}{b}-1 = 0,$$

quorum normales per datum punctum (ξ,η) transeunt, ut supra vidimus, formulis data sunt

$$(2^*.)\qquad x_1=\frac{a\xi}{a+u_1},\qquad y=\frac{b\eta}{b+u_1},$$

littera u_1 denotante radicem aequationis

$$(3.)\qquad \frac{a\xi^2}{(a+u)^2}+\frac{b\eta^2}{(b+u)^2} = 1.$$

Statuendo porro $x_2=\frac{a\xi}{a+u_2}$, $y_2=\frac{b\eta}{b+u_2}$, etc., aequatio (14.) sive $[1,2,3]=0$, transformari potest in

$$(63.)\qquad x_2y_3+x_3y_2+x_1y_3+x_3y_1+x_1y_2+x_2y_1 = 0.$$

Si igitur puncta (x_1y_1), (x_2y_2), (x_3y_3) conicae (1.) aequationi (63.) satisfaciunt, normales curvae in his tribus punctis erectae per idem punctum transeunt, sive

puncta conicae (1.), quorum normales cum normalibus punctorum conicae (x_1,y_1), (x_2,y_2) in idem punctum concurrunt, in recta

$$(64.)\qquad x(y_1+y_2)+y(x_1+x_2)+x_1y_2+x_2y_1 = 0$$

sita sunt.

Considerationibus geometricis ductus constructionem rectae (64.) hancce dedimus (cf. Comment. de normalibus ellipsis atque ellipsoidae, t. XXVI hujus diarii). Sit (p,q) polus rectae puncta (x_1,y_1), (x_2,y_2) jungentis; habetur

$$\frac{px_1}{a}+\frac{qy_1}{b}=1,\qquad \frac{x_1^2}{a}+\frac{y_1^2}{b}=1,$$

$$\frac{px_2}{a}+\frac{qy_2}{b}=1,\qquad \frac{x_2^2}{a}+\frac{y_2^2}{b}=1,$$

ergo

$$\frac{p}{a}=\frac{y_2-y_1}{x_1y_2-x_2y_1},\qquad \frac{1}{a}=\frac{y_2^2-y_1^2}{x_1^2y_2^2-x_2^2y_1^2}$$

unde (65.)

$$p = \frac{x_1^2 y_2^2 - x_2^2 y_1^2}{x_1 y_2 - x_2 y_1} \frac{y_2 - y_1}{y_2^2 - y_1^2}$$

sive (66.)

$$p = \frac{x_1 y_2 + x_2 y_1}{y_1 + y_2} \quad \text{atque similiter} \quad q = \frac{x_1 y_2 + x_2 y_1}{x_1 + x_2}.$$

Hinc prodit theorema sequens, quod in commentatione supra laudata invenitur:

Designando per p et q coordinatas rectangulares poli lineae rectae, quae puncta h_1, h_2 sectionis conicae $\frac{x^2}{a} + \frac{y^2}{b} - 1 = 0$ jungit, recta $\frac{x}{p} + \frac{y}{q} + 1 = 0$ conicam in duabus aliis punctis h_3, h_4 (sive realibus, sive imaginariis), secabit ita ut normales curvae in punctis h_1, h_2, h_3, h_4 ductae in eodem puncto concurrant.

Jam ad superficiem $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0$ progrediamur.

Supponamus a puncto quodam h ad superficiem sex normales ductas esse hh_1, hh_2, hh_3, hh_4, hh_5, hh_6. Designando per ξ, η, ζ coordinatas ipsius h, per x_1, y_1, z_1 coordinatas puncti h_1, quod in superficie ipsa situm est, etc. habetur

$$x_1 = \frac{a\xi}{a+u_1}, \quad y_1 = \frac{b\eta}{b+u_1}, \quad z_1 = \frac{c\zeta}{c+u_1},$$
$$x_2 = \frac{a\xi}{a+u_2}, \quad y_2 = \frac{b\eta}{b+u_2}, \quad z_2 = \frac{c\zeta}{c+u_2}, \quad \text{etc.}$$

ubi u_1, u_2, ... u_6 sunt radices aequationis

$$(17.) \quad \frac{a\xi^2}{(a+u)^2} + \frac{b\eta^2}{(b+u)^2} + \frac{c\zeta^2}{(c+u)^2} - 1 = 0.$$

Quibus valoribus relatio (31.) transit in

$$\begin{aligned}(67.) \quad & x_2 y_3 z_4 + x_2 y_4 z_3 + x_3 y_4 z_2 + x_3 y_2 z_4 + x_4 y_2 z_3 + x_4 y_3 z_2 \\ & + x_1 y_3 z_4 + x_1 y_4 z_3 + x_3 y_4 z_1 + x_3 y_1 z_4 + x_4 y_1 z_3 + x_4 y_3 z_1 \\ & + x_1 y_2 z_4 + x_1 y_4 z_2 + x_2 y_4 z_1 + x_2 y_1 z_4 + x_4 y_1 z_2 + x_4 y_2 z_1 \\ & + x_1 y_2 z_3 + x_1 y_3 z_2 + x_2 y_3 z_1 + x_2 y_1 z_3 + x_3 y_1 z_2 + x_3 y_2 z_1 = 0,\end{aligned}$$

quae est conditio necesssaria (neutiquam autem sufficiens), si normales in punctis snperficiei (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) ductae per idem punctum h transeunt. Formula (67.) etiam hoc modo enuntiari potest:

Si normales in punctis (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) superficiei $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - 1 = 0$ in eodem puncto h concurrunt, tria alia puncta, quorum normales per h transeunt, in plano

$$(68.)\quad \begin{aligned} & x\{y_2z_3+y_3z_2+y_1z_3+y_3z_1+y_1z_2+y_2z_1\} \\ & +y\{z_2x_3+z_3x_2+z_1x_3+z_3x_1+z_1x_2+z_2x_1\} \\ & +z\{x_2y_3+x_3y_2+x_1y_3+x_3y_1+x_1y_2+x_2y_1\} \\ & +x_1y_2z_3+x_1y_3z_2+x_2y_1z_3+x_2y_3z_1+x_3y_1z_2+x_3y_2z_1 = 0 \end{aligned}$$

sita sunt.

Sit (p, q, r) polus plani, tria puncta (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) jungentis, habentur sex aequationes

$$\frac{px_1}{a}+\frac{qy_1}{b}+\frac{rz_1}{c} = 1; \qquad \frac{x_1^2}{a}+\frac{y_1^2}{b}+\frac{z_1^2}{c} = 1.$$

$$\frac{px_2}{a}+\frac{qy_2}{b}+\frac{rz_2}{c} = 1; \qquad \frac{x_2^2}{a}+\frac{y_2^2}{b}+\frac{z_2^2}{c} = 1.$$

$$\frac{px_3}{a}+\frac{qy_3}{b}+\frac{rz_3}{c} = 1; \qquad \frac{x_3^2}{a}+\frac{y_3^2}{b}+\frac{z_3^2}{c} = 1;$$

unde

$$\frac{p}{a} = \frac{\text{det.}\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}}{\text{det.}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}} \qquad \frac{1}{a} = \frac{\text{det.}\begin{vmatrix} 1 & y_1^2 & z_1^2 \\ 1 & y_2^2 & z_2^2 \\ 1 & y_3^2 & z_3^2 \end{vmatrix}}{\text{det.}\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \end{vmatrix}}; \text{ atque dividendo}$$

$$p = \frac{\text{det.}\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}}{\text{det.}\begin{vmatrix} 1 & y_1^2 & z_1^2 \\ 1 & y_2^2 & z_2^2 \\ 1 & y_3^2 & z_3^2 \end{vmatrix}} \quad \frac{\text{det.}\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \end{vmatrix}}{\text{det.}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}.$$

Generaliter loquendo hic valor ipsius p reductionem non admittit. Substitutis autem valoribus

$$x_1 = \frac{a\xi}{a+u_1}, \qquad y_1 = \frac{b\eta}{b+u_1}, \qquad z_1 = \frac{c\zeta}{c+u_1} \text{ etc.}$$

habetur secundum formulam (62.)

$$\frac{\det.\begin{vmatrix}1 & y_1^2 & z_1^2\\ 1 & y_2^2 & z_2^2\\ 1 & y_3^2 & z_3^2\end{vmatrix}}{\det.\begin{vmatrix}1 & y_1 & z_1\\ 1 & y_2 & z_2\\ 1 & y_3 & z_3\end{vmatrix}} = bc\eta\zeta\frac{\det.\left\{1,\ \frac{1}{(b+u)^2},\ \frac{1}{(c+u)^2}\right\}}{\det.\left\{1,\ \frac{1}{(b+u)},\ \frac{1}{(c+u)}\right\}}_{u=u_1,\ =u_2,\ =u_3}$$

$$= bc\eta\zeta\left\{\frac{1}{(b+u_2)(c+u_3)}+\frac{1}{(b+u_3)(c+u_2)}+\cdots+\frac{1}{(b+u_2)(c+u_1)}\right\}$$

$$= y_2z_3+y_3z_2+y_3z_1+y_1z_3+y_1z_2+y_2z_1;$$

atque secundum (58.),

$$\frac{\det.\begin{vmatrix}x_1^2 & y_1^2 & z_1^2\\ x_2^2 & y_2^2 & z_2^2\\ x_3^2 & y_3^2 & z_3^2\end{vmatrix}}{\det.\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}} = abc\xi\eta\zeta\frac{\det.\left\{\frac{1}{(a+u)^2},\ \frac{1}{(b+u)^2},\ \frac{1}{(c+u)^2}\right\}}{\det.\left\{\frac{1}{a+u},\ \frac{1}{b+u},\ \frac{1}{c+u}\right\}}_{u=u_1,\ =u_2,\ =u_3}$$

$$= abc\xi\eta\zeta\Sigma\frac{1}{(a+u_1)(b+u_2)(c+u_3)}$$

$$= x_1y_2z_3+x_1y_3z_2+x_2y_1z_3+x_2y_3z_1+x_3y_1z_2+x_3y_2z_1,$$

itaque

$$p = \frac{x_1y_2z_3+x_1y_3z_2+x_2y_1z_3+x_2y_3z_1+x_3y_1z_2+x_3y_2z_1}{y_2z_3+y_3z_2+y_1z_3+y_3z_1+y_1z_2+y_2z_1};$$

perinde ipsarum q et r valores eruendi sunt. Ratione igitur habita aequationis (68.) hanc propositionem nacti sumus:

Sint h_1, h_2, h_3 tria puncta superficiei $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$, quorum normales in eodem puncto h concurrunt; designando per p, q et r coordinatas poli plani $h_1h_2h_3$, tria alia puncta h_4, h_5, h_6 superficiei, quorum normales per h transeunt, in plano

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}+1=0$$

sita sunt.

Permultas alias propositiones geometricas e formulis praecedentibus deducendas, quippe quae a nostro fine alienae sunt, nunc praetereamus.

Carlottae-fontibus, m. Sept. 1856.