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On Certain Definite Integrals.

(By *John C. Malet*, Queens College Cork.)

It may be proved by well-known transformations of elliptic functions or by direct substitution, that the differential equation

$$\frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}} + \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = 0$$

is satisfied by the relation

$$x - \alpha = (\gamma - \alpha) \frac{y - \beta}{y - \gamma}.$$

Let us now make this substitution in the definite integral

$$\int_{\beta}^{\alpha} \frac{\log(x-\alpha) dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}},$$

and remarking that when $x = \alpha$, $y = \beta$ and vice versa; writing also for shortness

$$(x-\alpha)(x-\beta)(x-\gamma) \equiv X,$$

we get the identity

$$(1.) \quad \left\{ \begin{array}{l} \int_{\beta}^{\alpha} \frac{\log(x-\alpha)}{\sqrt{X}} dx - \int_{\beta}^{\alpha} \frac{\log(x-\beta)}{\sqrt{X}} dx \\ + \int_{\beta}^{\alpha} \frac{\log(x-\gamma)}{\sqrt{X}} dx - \log(\gamma-\alpha) \int_{\beta}^{\alpha} \frac{dx}{\sqrt{X}} \end{array} \right. \equiv 0.$$

In this equation interchanging α and β we get

$$(2.) \quad \left\{ \begin{array}{l} \int_{\alpha}^{\beta} \frac{\log(x-\beta)}{\sqrt{X}} dx - \int_{\alpha}^{\beta} \frac{\log(x-\alpha)}{\sqrt{X}} dx \\ + \int_{\alpha}^{\beta} \frac{\log(x-\gamma)}{\sqrt{X}} dx - \log(\gamma-\beta) \int_{\alpha}^{\beta} \frac{dx}{\sqrt{X}} \end{array} \right. \equiv 0;$$

hence subtracting (2.) from (1.) we get the formula

$$(3.) \int_{\beta}^{\alpha} \frac{\log(x-\gamma)}{\sqrt{X}} dx = \frac{1}{2} \{ \log(\gamma-\alpha) + \log(\gamma-\beta) \} \int_{\beta}^{\alpha} \frac{dx}{\sqrt{X}}.$$

As each side of this equation may contain both real and imaginary parts, it is necessary to examine the resulting formulae in two cases, viz. when γ is the greatest or least of the quantities α, β, γ which I suppose to be all real. The case of γ being between α and β is easily seen to be included in these cases.

Ist. Let $\alpha > \beta > \gamma$, then multiplying each side of (3.) by $\sqrt{-1}$ and taking the real part of each side, we deduce the formula

$$(4.) \int_{\beta}^{\alpha} \frac{\log(x-\gamma)}{\sqrt{-X}} dx = \log \sqrt{(\alpha-\gamma)(\beta-\gamma)} \int_{\beta}^{\alpha} \frac{dx}{\sqrt{-X}}.$$

II^d. Let $\gamma > \beta > \alpha$, we get in a similar manner

$$\int_{\alpha}^{\beta} \frac{\log(\gamma-x)}{\sqrt{-X}} dx = \log \sqrt{(\gamma-\beta)(\gamma-\alpha)} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{-X}}.$$

Hence we see that the definite integrals

$$\int_{\alpha}^{\beta} \frac{\log(x-\gamma)}{\sqrt{-X}} dx \quad \text{or} \quad \int_{\alpha}^{\beta} \frac{\log(\gamma-x)}{\sqrt{-X}} dx$$

where

$$X = (x-\alpha)(x-\beta)(x-\gamma)$$

and γ is the least or the greatest of the three real quantities α, β, γ , can be expressed each as an elliptic function of the first kind.

Let us now consider the case when X has four factors. Let

$$X = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

We find as before that the differential equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0$$

(where $Y = (y-\alpha)(y-\beta)(y-\gamma)(y-\delta)$)

is satisfied by the relation

$$(\alpha-\gamma) \frac{x-\beta}{x-\alpha} = (\beta-\delta) \frac{y-\gamma}{y-\delta}.$$

Hence making this substitution in

$$\int_{\gamma}^{\beta} \frac{\log(x-\beta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\alpha)}{\sqrt{X}} dx,$$

we get the identity

$$(5.) \quad \left\{ \begin{array}{l} \int_{\gamma}^{\beta} \frac{\log(x-\beta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\alpha)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\gamma)}{\sqrt{X}} dx \\ + \int_{\gamma}^{\beta} \frac{\log(x-\delta)}{\sqrt{X}} dx - \log \frac{\beta-\delta}{\alpha-\gamma} \int_{\gamma}^{\beta} \frac{dx}{\sqrt{X}} \equiv 0. \end{array} \right.$$

In this identity interchange β and γ , and we have

$$(6.) \quad \left\{ \begin{array}{l} \int_{\beta}^{\gamma} \frac{\log(x-\gamma)}{\sqrt{X}} dx - \int_{\beta}^{\gamma} \frac{\log(x-\alpha)}{\sqrt{X}} dx - \int_{\beta}^{\gamma} \frac{\log(x-\beta)}{\sqrt{X}} dx \\ + \int_{\beta}^{\gamma} \frac{\log(x-\delta)}{\sqrt{X}} dx - \log \frac{\gamma-\delta}{\alpha-\beta} \int_{\beta}^{\gamma} \frac{dx}{\sqrt{X}} \equiv 0. \end{array} \right.$$

Subtracting (6.) from (5.) there results

$$(7.) \quad \int_{\gamma}^{\beta} \frac{\log(x-\delta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\alpha)}{\sqrt{X}} dx = \log \sqrt{\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\beta)}} \int_{\gamma}^{\beta} \frac{dx}{\sqrt{X}} = 0;$$

we have again by adding (5.) and (6.)

$$(8.) \quad \int_{\gamma}^{\beta} \frac{\log(x-\beta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\gamma)}{\sqrt{X}} dx = \log \sqrt{\frac{(\beta-\delta)(\alpha-\beta)}{(\alpha-\gamma)(\gamma-\delta)}} \int_{\gamma}^{\beta} \frac{dx}{\sqrt{X}}.$$

From (7.) and (8.) we get the formula

$$(9.) \quad \left\{ \begin{array}{l} \log \left(\frac{(\beta-\delta)(\alpha-\beta)}{(\alpha-\gamma)(\gamma-\delta)} \right) \left\{ \int_{\gamma}^{\beta} \frac{\log(x-\delta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\alpha)}{\sqrt{X}} dx \right\} \\ = \log \left(\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)} \right) \left\{ \int_{\gamma}^{\beta} \frac{\log(x-\beta)}{\sqrt{X}} dx - \int_{\gamma}^{\beta} \frac{\log(x-\gamma)}{\sqrt{X}} dx \right\} \end{array} \right.$$

where

$$X = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

In formulae (7.), (8.) and (9.) certain modifications are necessary according to the values of β and γ with respect to α and δ , in each case however we may suppose β and γ to have consecutive values and also β to be greater than γ .

Ist. Suppose $\alpha > \beta > \gamma > \delta$. In this case we merely change $\log(x-\alpha)$ to $\log(\alpha-x)$ and $\log(x-\beta)$ to $\log(\beta-x)$ leaving the rest of the formulae unchanged.

II^d. Suppose $\beta > \gamma > \alpha > \delta$.

Change $\log(x-\beta)$ to $\log(\beta-x)$, $\alpha-\beta$ to $\beta-\alpha$,
 $\alpha-\gamma$ to $\gamma-\alpha$ and X to $-X$.

III^d. Let $\alpha > \delta > \beta > \gamma$.

Change $\log(x-\beta)$ to $\log(\beta-x)$,
 $\log(x-\alpha)$ to $\log(\alpha-x)$,
 $\log(x-\delta)$ to $\log(\delta-x)$,
 $\beta-\delta$ to $\delta-\beta$, $\gamma-\delta$ to $\delta-\gamma$ and X to $-X$.

It may be interesting to examine the formulae already arrived at when the integrals are reduced to the standard form of elliptic functions. Consider the integral

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\Delta(\lambda, \theta)} d\theta$$

where, as in the usual notation

$$\Delta(\lambda, \theta) = \sqrt{1-\lambda^2 \sin^2 \theta}.$$

If we make

$$\sin \theta = \frac{\cos \varphi}{\Delta(\lambda, \varphi)},$$

we get the identity

$$(10.) \quad \lambda \int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\Delta(\lambda, \theta)} d\theta - \lambda \int_0^{\frac{\pi}{2}} \frac{\log \cos \theta}{\Delta(\lambda, \theta)} d\theta + \lambda \int_0^{\frac{\pi}{2}} \frac{\log \Delta(\lambda, \theta)}{\Delta(\lambda, \theta)} d\theta \equiv 0.$$

In this identity suppose $\lambda > 1 = \frac{1}{k}$, where $k < 1$; then transform each integral by the substitution

$$\sin \theta = k \sin \varphi$$

from which

$$\frac{\lambda d\theta}{\Delta(\lambda, \theta)} = \frac{d\varphi}{\Delta(k, \varphi)}.$$

We find

$$\begin{aligned} & \lambda \int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\Delta(\lambda, \theta)} d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\log \sin \varphi}{\Delta(k, \varphi)} d\varphi + \int_{\frac{\pi}{2}}^{\sin^{-1} \frac{1}{k}} \frac{\log \sin \varphi}{\Delta(k, \varphi)} d\varphi + \log k \int_0^{\sin^{-1} \frac{1}{k}} \frac{d\varphi}{\Delta(k, \varphi)} \end{aligned}$$

and

$$\lambda \int_0^{\frac{\pi}{2}} \frac{\log \frac{\cos \theta}{\Delta(\lambda, \theta)}}{\Delta(\lambda, \theta)} d\theta = \int_0^{\frac{\pi}{2}} \frac{\log \frac{\Delta(k, \varphi)}{\cos \varphi}}{\Delta(k, \varphi)} d\varphi + \int_{\frac{\pi}{2}}^{\sin^{-1} \frac{1}{k}} \frac{\log \frac{\Delta(k, \varphi)}{\cos \varphi}}{\Delta(k, \varphi)} d\varphi.$$

Now for convenience let us call the integrals

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\Delta(k, \theta)} d\theta, \quad \int_0^{\frac{\pi}{2}} \frac{\log \cos \theta}{\Delta(k, \theta)} d\theta, \quad \int_0^{\frac{\pi}{2}} \frac{\log \Delta(k, \theta)}{\Delta(k, \theta)} d\theta$$

L , M and N , and let L' , M' , N' denote the values of L , M and N when k is changed to k' where $k^2 + k'^2 = 1$; also, after *Jacobi*, let K and K' be the complete elliptic functions of the first kind with k and k' for moduli.

Now by the transformation

$$\sin \varphi = \frac{1}{\Delta(k', \psi)}$$

we get

$$\int_{\frac{\pi}{2}}^{\sin^{-1} \frac{1}{k}} \frac{\log \sin \varphi}{\Delta(k, \varphi)} d\varphi = \sqrt{-1} N',$$

$$\int_0^{\sin^{-1} \frac{1}{k}} \frac{d\varphi}{\Delta(k, \varphi)} = K - \sqrt{-1} K',$$

$$\int_{\frac{\pi}{2}}^{\sin^{-1} \frac{1}{k}} \frac{\log \frac{\Delta(k, \varphi)}{\cos \varphi}}{\Delta(k, \varphi)} d\varphi = \sqrt{-1} \{L' - M' + \log(\sqrt{-1} K')\}.$$

Hence we have

$$\lambda \int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\Delta(\lambda, \theta)} d\theta = L + \sqrt{-1} N' + \log k (K - \sqrt{-1} K'),$$

$$\lambda \int_0^{\frac{\pi}{2}} \frac{\log \frac{\cos \theta}{\Delta(\lambda, \theta)}}{\Delta(\lambda, \theta)} d\theta = N - M + \sqrt{-1} (L' - M' + \log(\sqrt{-1} K')).$$

Hence substituting in identity (10.) we have the identity

$$L + M - N + K \log k - \sqrt{-1} \log(\sqrt{-1} K') + \sqrt{-1} (N' + M' - L' - K' \log k) \equiv 0.$$

Hence equating to 0 separately the real and imaginary parts of this expression we get

$$L + M - N + K \log k - \sqrt{-1} \log(\sqrt{-1} K') = 0,$$

$$L' - M' - N' + K' \log k = 0,$$

or in this latter equation changing k' to k , and remembering identity (10.) we have for the determination of L , M and N the three equations

$$\begin{aligned} L + M - N + K \log k - K' \sqrt{-1} \log \sqrt{-1} &= 0, \\ L - M - N + K \log k' &= 0, \\ L - M + N &= 0. \end{aligned}$$

Hence we have

$$\begin{aligned} N &= \frac{1}{2} K \log k', \\ M &= \frac{1}{2} K \log \frac{k'}{k} + \frac{\sqrt{-1} \log(-1)}{4} K', \\ L &= -\frac{1}{2} K \log k + \frac{\sqrt{-1} \log(-1)}{4} K'. \end{aligned}$$

To determine what value of $\log(-1)$ must be taken, let $k = 1$ in the last equation, and we find

$$\frac{\sqrt{-1} \log(-1)}{4} \cdot \frac{\pi}{2} = \int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\cos \theta} d\theta = \int_0^1 \frac{\log x}{1-x^2} dx = -\frac{\pi^2}{8}.$$

Hence we have the formulae

$$\begin{aligned} (a.) \quad \int_0^{\frac{\pi}{2}} \frac{\log \sin \theta}{\sqrt{1-k^2 \sin^2 \theta}} d\theta &= -\frac{1}{2} K \log k - \frac{\pi}{4} K', \\ (b.) \quad \int_0^{\frac{\pi}{2}} \frac{\log \cos \theta}{\sqrt{1-k^2 \sin^2 \theta}} d\theta &= \frac{1}{2} K \log \frac{k'}{k} - \frac{\pi}{4} K', \\ (c.) \quad \int_0^{\frac{\pi}{2}} \frac{\log \sqrt{1-k^2 \sin^2 \theta}}{\sqrt{1-k^2 \sin^2 \theta}} d\theta &= \frac{1}{2} K \log k'. \end{aligned}$$

Equation (a.) is a known formula of use in the theory of elliptic functions (see *Schlömlich's* Elliptic Functions).

I may add that there are other limits of integration besides $\frac{\pi}{2}$ and 0 for which the integrals L , M and N can be expressed in terms of elliptic functions. For example the integral

$$\int_{\beta}^{\alpha} \frac{\log \sqrt{1-k^2 \sin^2 \theta}}{\sqrt{1-k^2 \sin^2 \theta}} d\theta$$

can be expressed in terms of an elliptic function of the first kind if

$$\sin \beta = \frac{\cos \alpha}{\sqrt{1-k^2 \sin^2 \alpha}}.$$

This follows at once by transforming the integral

$$\int_0^{\alpha} \frac{\log \mathcal{A}(k, \theta)}{\mathcal{A}(k, \theta)} d\theta$$

by the transformation

$$\sin \theta = \frac{\cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

from which we have

$$\int_{\beta}^{\alpha} \frac{\log \mathcal{A}(k, \theta)}{\mathcal{A}(k, \theta)} d\theta = \frac{1}{2} \log k' \int_{\beta}^{\alpha} \frac{d\theta}{\mathcal{A}(k, \theta)}$$

when β has the value given above.

January 3^d, 1882.
