On Certain Definite Integrals.

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It may be proved by well-known transformations of elliptic functions or by direct substitution, that the differential equation

$$\frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}}+\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = 0$$

is satisfied by the relation

$$x-\alpha = (\gamma-\alpha)\frac{y-\beta}{y-\gamma}.$$

Let us now make this substitution in the definite integral

$$\int_\beta^\alpha \frac{\log(x-\alpha)\,dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)}},$$

and remarking that when $x=\alpha$, $y=\beta$ and vice versa; writing also for shortness

$$(x-\alpha)(x-\beta)(x-\gamma) \equiv X,$$

we get the identity

$$(1.)\quad \left\{\begin{array}{l} \displaystyle\int_\beta^\alpha \frac{\log(x-\alpha)}{\sqrt{X}}\,dx-\int_\beta^\alpha \frac{\log(x-\beta)}{\sqrt{X}}\,dx \\ \displaystyle+\int_\beta^\alpha \frac{\log(x-\gamma)}{\sqrt{X}}\,dx-\log(\gamma-\alpha)\int_\beta^\alpha \frac{dx}{\sqrt{X}} \equiv 0. \end{array}\right.$$

In this equation interchanging α and β we get

$$(2.)\quad \left\{\begin{array}{l} \displaystyle\int_\alpha^\beta \frac{\log(x-\beta)}{\sqrt{X}}\,dx-\int_\alpha^\beta \frac{\log(x-\alpha)}{\sqrt{X}}\,dx \\ \displaystyle+\int_\alpha^\beta \frac{\log(x-\gamma)}{\sqrt{X}}\,dx-\log(\gamma-\beta)\int_\alpha^\beta \frac{dx}{\sqrt{X}} \equiv 0; \end{array}\right.$$

hence subtracting (2.) from (1.) we get the formula

$$(3.)\quad \int_\beta^\alpha \frac{\log(x-\gamma)}{\sqrt{X}}dx = \tfrac{1}{2}\{\log(\gamma-\alpha)+\log(\gamma-\beta)\}\int_\beta^\alpha \frac{dx}{\sqrt{X}}.$$

As each side of this equation may contain both real and imaginary parts, it is necessary to examine the resulting formulae in two cases, viz. when γ is the greatest or least of the quantities α, β, γ which I suppose to be all real. The case of γ being between α and β is easily seen to be included in these cases.

I[st]. Let $\alpha > \beta > \gamma$, then multiplying each side of (3.) by $\sqrt{-1}$ and taking the real part of each side, we deduce the formula

$$(4.)\quad \int_\beta^\alpha \frac{\log(x-\gamma)}{\sqrt{-X}}dx = \log\sqrt{(\alpha-\gamma)(\beta-\gamma)}\int_\beta^\alpha \frac{dx}{\sqrt{-X}}.$$

II[d]. Let $\gamma > \beta > \alpha$, we get in a similar manner

$$\int_\alpha^\beta \frac{\log(\gamma-x)}{\sqrt{-X}}dx = \log\sqrt{(\gamma-\beta)(\gamma-\alpha)}\int_\alpha^\beta \frac{dx}{\sqrt{-X}}.$$

Hence we see that the definite integrals

$$\int_\alpha^\beta \frac{\log(x-\gamma)}{\sqrt{-X}}dx \quad\text{or}\quad \int_\alpha^\beta \frac{\log(\gamma-x)}{\sqrt{-X}}dx$$

where

$$X = (x-\alpha)(x-\beta)(x-\gamma)$$

and γ is the least or the greatest of the three real quantities α, β, γ, can be expressed each as an elliptic function of the first kind.

Let us now consider the case when X has four factors. Let

$$X = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

We find as before that the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}} = 0$$

(where $Y = (y-\alpha)(y-\beta)(y-\gamma)(y-\delta)$)
is satisfied by the relation

$$(\alpha-\gamma)\frac{x-\beta}{x-\alpha} = (\beta-\delta)\frac{y-\gamma}{y-\delta}.$$

Hence making this substitution in

$$\int_\gamma^\beta \frac{\log(x-\beta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\alpha)}{\sqrt{X}}dx,$$

we get the identity

$$(5.)\quad \begin{cases} \int_\gamma^\beta \frac{\log(x-\beta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\alpha)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\gamma)}{\sqrt{X}}dx \\ + \int_\gamma^\beta \frac{\log(x-\delta)}{\sqrt{X}}dx - \log\frac{\beta-\delta}{\alpha-\gamma}\int_\gamma^\beta \frac{dx}{\sqrt{X}} \equiv 0. \end{cases}$$

In this identity interchange β and γ, and we have

$$(6.)\quad \begin{cases} \int_\beta^\gamma \frac{\log(x-\gamma)}{\sqrt{X}}dx - \int_\beta^\gamma \frac{\log(x-\alpha)}{\sqrt{X}}dx - \int_\beta^\gamma \frac{\log(x-\beta)}{\sqrt{X}}dx \\ + \int_\beta^\gamma \frac{\log(x-\delta)}{\sqrt{X}}dx - \log\frac{\gamma-\delta}{\alpha-\beta}\int_\beta^\gamma \frac{dx}{\sqrt{X}} \equiv 0. \end{cases}$$

Subtracting (6.) from (5.) there results

$$(7.)\quad \int_\gamma^\beta \frac{\log(x-\delta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\alpha)}{\sqrt{X}}dx = \log\sqrt{\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\gamma)(\alpha-\beta)}}\int_\gamma^\beta \frac{dx}{\sqrt{X}} = 0;$$

we have again by adding (5.) and (6.)

$$(8.)\quad \int_\gamma^\beta \frac{\log(x-\beta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\gamma)}{\sqrt{X}}dx = \log\sqrt{\frac{(\beta-\delta)(\alpha-\beta)}{(\alpha-\gamma)(\gamma-\delta)}}\int_\gamma^\beta \frac{dx}{\sqrt{X}}.$$

From (7.) and (8.) we get the formula

$$(9.)\quad \begin{cases} \log\left(\frac{(\beta-\delta)(\alpha-\beta)}{(\alpha-\gamma)(\gamma-\delta)}\right)\left\{\int_\gamma^\beta \frac{\log(x-\delta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\alpha)}{\sqrt{X}}dx\right\} \\ = \log\left(\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)}\right)\left\{\int_\gamma^\beta \frac{\log(x-\beta)}{\sqrt{X}}dx - \int_\gamma^\beta \frac{\log(x-\gamma)}{\sqrt{X}}dx\right\} \end{cases}$$

where

$$X = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

In formulae (7.), (8.) and (9.) certain modifications are necessary according to the values of β and γ with respect to α and δ, in each case however we may suppose β and γ to have consecutive values and also β to be greater than γ.

I[st]. Suppose $\alpha > \beta > \gamma > \delta$. In this case we merely change $\log(x-\alpha)$ to $\log(\alpha-x)$ and $\log(x-\beta)$ to $\log(\beta-x)$ leaving the rest of the formulae unchanged.

II[d]. Suppose $\beta > \gamma > \alpha > \delta$.

Change $\log(x-\beta)$ to $\log(\beta-x)$, $\alpha-\beta$ to $\beta-\alpha$,
$\alpha-\gamma$ to $\gamma-\alpha$ and X to $-X$.

III[d]. Let $\alpha > \delta > \beta > \gamma$.

Change $\log(x-\beta)$ to $\log(\beta-x)$,
$\log(x-\alpha)$ to $\log(\alpha-x)$,
$\log(x-\delta)$ to $\log(\delta-x)$,
$\beta-\delta$ to $\delta-\beta$, $\gamma-\delta$ to $\delta-\gamma$ and X to $-X$.

It may be interesting to examine the formulae already arrived at when the integrals are reduced to the standard form of elliptic functions.

Consider the integral

$$\int_0^{\frac{\pi}{2}} \frac{\log\sin\theta}{\Delta(\lambda,\theta)}\,d\theta$$

where, as in the usual notation

$$\Delta(\lambda,\theta) = \sqrt{1-\lambda^2\sin^2\theta}.$$

If we make

$$\sin\theta = \frac{\cos\varphi}{\Delta(\lambda,\varphi)},$$

we get the identity

$$(10.)\quad \lambda\int_0^{\frac{\pi}{2}} \frac{\log\sin\theta}{\Delta(\lambda,\theta)}\,d\theta - \lambda\int_0^{\frac{\pi}{2}} \frac{\log\cos\theta}{\Delta(\lambda,\theta)}\,d\theta + \lambda\int_0^{\frac{\pi}{2}} \frac{\log\Delta(\lambda,\theta)}{\Delta(\lambda,\theta)}\,d\theta \equiv 0.$$

In this identity suppose $\lambda > 1 = \frac{1}{k}$, where $k < 1$; then transform each integral by the substitution

$$\sin\theta = k\sin\varphi$$

from which

$$\frac{\lambda\,d\theta}{\Delta(\lambda,\theta)} = \frac{d\varphi}{\Delta(k,\varphi)}.$$

We find

$$\lambda\int_0^{\frac{\pi}{2}} \frac{\log\sin\theta}{\Delta(\lambda,\theta)}\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\log\sin\varphi}{\Delta(k,\varphi)}\,d\varphi + \int_{\frac{\pi}{2}}^{\sin^{-1}\frac{1}{k}} \frac{\log\sin\varphi}{\Delta(k,\varphi)}\,d\varphi + \log k\int_0^{\sin^{-1}\frac{1}{k}} \frac{d\varphi}{\Delta(k,\varphi)}$$

and

$$\lambda\int_0^{\frac{\pi}{2}} \frac{\log\dfrac{\cos\theta}{\Delta(\lambda,\theta)}}{\Delta(\lambda,\theta)}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{\log\dfrac{\Delta(k,\varphi)}{\cos\varphi}}{\Delta(k,\varphi)}\,d\varphi + \int_{\frac{\pi}{2}}^{\sin^{-1}\frac{1}{k}} \frac{\log\dfrac{\Delta(k,\varphi)}{\cos\varphi}}{\Delta(k,\varphi)}\,d\varphi.$$

Now for convenience let us call the integrals

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin\theta}{\Delta(k,\theta)} d\theta, \quad \int_0^{\frac{\pi}{2}} \frac{\log \cos\theta}{\Delta(k,\theta)} d\theta, \quad \int_0^{\frac{\pi}{2}} \frac{\log \Delta(k,\theta)}{\Delta(k,\theta)} d\theta$$

L, M and N, and let L', M', N' denote the values of L, M and N when k is changed to k' where $k^2+k'^2=1$; also, after *Jacobi*, let K and K' be the complete elliptic functions of the first kind with k and k' for moduli.

Now by the transformation

$$\sin\varphi = \frac{1}{\Delta(k',\psi)}$$

we get

$$\int_{\frac{\pi}{2}}^{\sin^{-1}\frac{1}{k}} \frac{\log\sin\varphi}{\Delta(k,\varphi)} d\varphi = \sqrt{-1}N',$$

$$\int_0^{\sin^{-1}\frac{1}{k}} \frac{d\varphi}{\Delta(k,\varphi)} = K-\sqrt{-1}K',$$

$$\int_{\frac{\pi}{2}}^{\sin^{-1}\frac{1}{k}} \frac{\log\frac{\Delta(k,\varphi)}{\cos\varphi}}{\Delta(k,\varphi)} d\varphi = \sqrt{-1}\{L'-M'+\log(\sqrt{-1})K'\}.$$

Hence we have

$$\lambda\int_0^{\frac{\pi}{2}} \frac{\log\sin\theta}{\Delta(\lambda,\theta)} d\theta = L+\sqrt{-1}N'+\log k(K-\sqrt{-1}K'),$$

$$\lambda\int_0^{\frac{\pi}{2}} \frac{\log\frac{\cos\theta}{\Delta(\lambda,\theta)}}{\Delta(\lambda,\theta)} d\theta = N-M+\sqrt{-1}(L'-M'+\log(\sqrt{-1})K').$$

Hence substituting in identity (10.) we have the identity

$$L+M-N+K\log k-\sqrt{-1}\log(\sqrt{-1})K'+\sqrt{-1}(N'+M'-L'-K'\log k) \equiv 0.$$

Hence equating to 0 separately the real and imaginary parts of this expression we get

$$L+M-N+K\log k-\sqrt{-1}\log(\sqrt{-1})K' = 0,$$
$$L'-M'-N'+K'\log k = 0,$$

or in this latter equation changing k' to k, and remembering identity (10.) we have for the determination of L, M and N the three equations

$$L+M-N+K\log k-K'\sqrt{-1}\log\sqrt{-1} = 0,$$
$$L-M-N+K\log k' = 0,$$
$$L-M+N = 0.$$

Hence we have

$$N = \tfrac{1}{2}K\log k',$$
$$M = \tfrac{1}{2}K\log\frac{k'}{k}+\frac{\sqrt{-1}\log(-1)}{4}K',$$
$$L = -\tfrac{1}{2}K\log k+\frac{\sqrt{-1}\log(-1)}{4}K'.$$

To determine what value of $\log(-1)$ must be taken, let $k=1$ in the last equation, and we find

$$\frac{\sqrt{-1}\log(-1)}{4}\cdot\frac{\pi}{2}=\int_0^{\frac{\pi}{2}}\frac{\log\sin\theta}{\cos\theta}d\theta=\int_0^1\frac{\log x}{1-x^2}dx=-\frac{\pi^2}{8}.$$

Hence we have the formulae

$$(a.)\quad \int_0^{\frac{\pi}{2}}\frac{\log\sin\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta = -\tfrac{1}{2}K\log k-\frac{\pi}{4}K',$$

$$(b.)\quad \int_0^{\frac{\pi}{2}}\frac{\log\cos\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta = \tfrac{1}{2}K\log\frac{k'}{k}-\frac{\pi}{4}K',$$

$$(c.)\quad \int_0^{\frac{\pi}{2}}\frac{\log\sqrt{1-k^2\sin^2\theta}}{\sqrt{1-k^2\sin^2\theta}}d\theta = \tfrac{1}{2}K\log k'.$$

Equation (*a.*) is a known formula of use in the theory of elliptic functions (see *Schlömilch*'s Elliptic Functions).

I may add that there are other limits of integration besides $\frac{\pi}{2}$ and 0 for which the integrals L, M and N can be expressed in terms of elliptic functions. For example the integral

$$\int_\beta^\alpha\frac{\log\sqrt{1-k^2\sin^2\theta}}{\sqrt{1-k^2\sin^2\theta}}d\theta$$

can be expressed in terms of an elliptic function of the first kind if

$$\sin\beta = \frac{\cos\alpha}{\sqrt{1-k^2\sin^2\alpha}}.$$

This follows at once by transforming the integral

$$\int_0^\alpha \frac{\log \Delta(k,\theta)}{\Delta(k,\theta)}\,d\theta$$

by the transformation

$$\sin\theta = \frac{\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

from which we have

$$\int_\beta^\alpha \frac{\log \Delta(k,\theta)}{\Delta(k,\theta)}\,d\theta = \tfrac{1}{2}\log k' \int_\beta^\alpha \frac{d\theta}{\Delta(k,\theta)}$$

when β has the value given above.

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