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Werk

Titel: Journal für die reine und angewandte Mathematik

Verlag: de Gruyter

Jahr: 1966

Kollektion: Mathematica

Werk Id: PPN243919689_0222

PURL: http://resolver.sub.uni-goettingen.de/purl?PID=PPN243919689_0222|LOG_0021

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Germany
Email: gdz@sub.uni-goettingen.de

On the representation of rational numbers as a sum of a fixed number of unit fractions

By *Yehuda Rav* at Hempstead, Long Island, N. Y.

1. Introduction

The representation of rational numbers as sums of distinct unit fractions, i. e., fractions with numerator one, goes back to antiquity ([25], pp. 19—30 and p. 48). Lambert, in 1770 ([18], p. 351), illustrated the expansion of *real* numbers in series of unit fractions, but it was Sylvester [23] who gave a new impetus to the study of unit fraction expansions. While some recent studies [6], [8], [24] are concerned with the representation of rational numbers as sums of reciprocals which belong to certain classes of sequences of positive integers, the number of terms in the sum being immaterial, this paper deals with the solubility in integers of the Diophantine equation

$$(*)\qquad m/n = 1/x_1 + 1/x_2 + \cdots + 1/x_k$$

where k *is fixed*, and m, n are given positive coprime integers. After introducing the concept of decomposing a set of non-zero elements $M_1, \ldots, M_k$ of a unique factorization domain into *relative irreducibles*, necessary and sufficient conditions for the solubility of equation (*) are obtained. Furthermore, an effective algorithm is described to determine whether equation (*) is soluble, and if it is, the algorithm yields all solutions.

2. Decomposition into relative irreducibles in a UFD.

Let U be a UFD (unique factorization domain), $M_1, \ldots, M_k$ a set of non-zero elements in U. We shall define inductively a family of elements $\{X_{(\nu_1,\ldots,\nu_m)}\}$, where $1 \leqq \nu_1 < \cdots < \nu_m \leqq k$ and $(\nu_1, \ldots, \nu_m)$ ranges over all ordered m-tuples from the set $\{1, \ldots, k\}$ for all $1 \leqq m \leqq k$, yielding a total of $\binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} = 2^k - 1$ elements. To simplify notation, set $(\nu_1, \ldots, \nu_m) = \nu(m)$ and abbreviate "$i \in \{\nu_1, \ldots, \nu_m\}$" by "$i \in \nu(m)$". Furthermore, for any $a, b \in U$, the symbol (a, b) will designate the g. c. d. of a and b, *not* the ideal generated by a and b. Similarly, if more than two elements are involved. Let $X_{\nu(k)} = X_{(1,\ldots,k)} = (M_1, \ldots, M_k)$. Setting

$$(2.1)\qquad M_j = X_{\nu(k)} M_j^{(k)}, \qquad (j = 1, \ldots, k)$$

then $(M_1^{(k)}, \ldots, M_k^{(k)}) = 1$. Let $X_{\nu(k-1)} = (M_{\nu_1}^{(k)}, \ldots, M_{\nu_{k-1}}^{(k)})$.

Clearly, any two distinct $X_{\nu(k-1)}$ are relatively prime, since any common divisor of them divides $(M_1^{(k)}, \ldots, M_k^{(k)})$. Thus, one may set

$$M_j^{(k)} = \Big[\prod_{j \in \nu(k-1)} X_{\nu(k-1)}\Big] M_j^{(k-1)}, \qquad (j = 1, \ldots, k)$$

where the product extends over all ordered $(k-1)$-tuples $(\nu_1, \ldots, \nu_{k-1})$ containing j. It follows from the choice of $X_{\nu(k-1)}$ as the g. c. d. of $M_{\nu_1}^{(k)}, \ldots, M_{\nu_{k-1}}^{(k)}$ that

$$(M_{\mu_1}^{(k-1)}, \ldots, M_{\mu_{k-1}}^{(k-1)}) = 1$$

for any ordered $(k-1)$-tuple $(\mu_1, \ldots, \mu_{k-1})$.

Assume that the $X_{\nu(k-s+1)}$ and $M^{(k-s+1)}$ have been defined and that any $k-s+1$ distinct $M^{(k-s+1)}$ are relatively prime. Let $X_{\nu(k-s)} = (M_{\nu_1}^{(k-s+1)}, \ldots, M_{\nu_{k-s}}^{(k-s+1)})$. Then any two distinct $X_{\nu(k-s)}$ are relatively prime and one may set

$$\text{(2. 2)} \qquad M_j^{(k-s+1)} = \Big[\prod_{j \in \nu(k-s)} X_{\nu(k-s)}\Big] M_j^{(k-s)}, \qquad (j = 1, \ldots, k).$$

From the choice of $X_{\nu(k-s)}$ it follows that any $k-s$ distinct $M^{(k-s)}$ are relatively prime, and thus the induction is carried from $s-1$ to s. In particular, if $s = k-2$, equation (2. 2) yields that $M_j^{(3)} = \Big[\prod_{j \in \nu(2)} X_{\nu(2)}\Big] M_j^{(2)}$ and $(M_u^{(2)}, M_v^{(2)}) = 1$ if $u \neq v$. Setting $M_j^{(2)} = X_j$ and substituting successively all expressions for the $M_j^{(i)}$ into (2. 1), we obtain:

$$\text{(2. 3)} \qquad M_j = \prod_{s=0}^{k-1} \prod_{j \in \nu(k-s)} X_{\nu(k-s)}, \qquad (j = 1, \ldots, k).$$

As the g. c. d. of any set of elements in a UFD is unique up to associates, and the preceding construction depends solely on the successive extraction of g. c. d.'s, it follows that the family $\{X_{(\nu_1, \ldots, \nu_m)}\}$ is uniquely determined up to associates by $M_1, \ldots, M_k$. Obviously, some members of the family may be units. The $X_{(\nu_1, \ldots, \nu_m)}$ will be called the *relative irreducibles* of $M_1, \ldots, M_k$ and (2. 3) *the decomposition of* $M_1, \ldots, M_k$ *into relative irreducibles.*

Remark 1. $(X_{(\mu_1, \ldots, \mu_s)}, X_{(\nu_1, \ldots, \nu_t)}) = 1$, *where* $s \leqq t$, *if at least one* $\mu_i \notin \{\nu_1, \ldots, \nu_t\}$.

Proof. Suppose there exists an irreducible element p in U which divides $X_{(\mu_1, \ldots, \mu_s)}$ and $X_{(\nu_1, \ldots, \nu_t)}$. By definition,

$$X_{(\nu_1, \ldots, \nu_t)} = (M_{\nu_1}^{(t+1)}, \ldots, M_{\nu_t}^{(t+1)}) \text{ and } X_{(\mu_1, \ldots, \mu_s)} = (M_{\mu_1}^{(s+1)}, \ldots, M_{\mu_s}^{(s+1)}).$$

Since by construction, $M_j^{(a)}$ divides $M_j^{(b)}$ if $a \leqq b$, it follows that $M_{\mu_i}^{(s+1)}$ divides $M_{\mu_i}^{(t+1)}$ and p divides $(M_{\nu_1}^{(t+1)}, \ldots, M_{\nu_t}^{(t+1)}, M_{\mu_i}^{(t+1)})$. But if $\mu_i \notin \{\nu_1, \ldots, \nu_t\}$, then

$$(M_{\nu_1}^{(t+1)}, \ldots, M_{\nu_t}^{(t+1)}, M_{\mu_i}^{(t+1)}) = 1$$

as any m distinct $M^{(m)}$ are relatively prime. Thus, p cannot be an irreducible in U.

Remark 2. *If* $i \in \{\nu_1, \ldots, \nu_t\}$ *and* n_i *is such that* $(n_i, M_i) = 1$, *then* $(X_{(\nu_1, \ldots, \nu_t)}, n_i) = 1$.

Proof. According to (2. 3), $i \in \{\nu_1, \ldots, \nu_t\}$ implies that $X_{(\nu_1, \ldots, \nu_t)}$ divides M_i, and the conclusion follows.

3. Main results

In the sequel, all symbols will denote rational integers, unless otherwise stipulated.

Theorem 3. 1. *The Diophantine equation*

$$\text{(3. 1)} \qquad m/n = \sum_{i=1}^{k} 1/x_i, \qquad (m, n) = 1, \quad k \text{ fixed},$$

is equivalent to the Diophantine equation

$$(3.\,2) \qquad m \prod_{s=0}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)} = \sum_{i=1}^{k} n_i \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}$$

where the n_i divide n, $(n_1, \ldots, n_k) = 1$.

Specifically, if $\{X_{(\nu_1,\ldots,\nu_t)}\}$ is a family of $2^k - k - 1$ non-zero integers, with $1 \leqq \nu_1 < \cdots < \nu_t \leqq k$ and $(\nu_1, \ldots, \nu_t) = \nu(t)$ ranges over all t-tuples from the set $\{1, \ldots, k\}$, $2 \leqq t \leqq k$, such that (3. 2) is satisfied, where the product $\prod_{\nu(k-s)}$ extends over *all* ordered $(k-s)$-tuples and the product $\prod_{i \notin \nu(k-s)}$ extends over those not containing i, then upon setting

$$(3.\,3) \qquad x_i = n/n_i \prod_{s=0}^{k-2} \prod_{i \in \nu(k-s)} X_{\nu(k-s)} \qquad (i = 1, \ldots, k)$$

a solution of (3. 1) is obtained. Conversely, any solution $x_1, \ldots, x_k$ of (3. 1) determines divisors n_i of n with $(n_1, \ldots, n_k) = 1$ and a family $\{X_{(\nu_1,\ldots,\nu_t)}\}$ of non-zero integers such that (3. 2) and (3. 3) are satisfied. Moreover, $(X_{(\mu_1,\ldots,\mu_s)}, X_{(\nu_1,\ldots,\nu_t)}) = 1$ $(s \leqq t)$ if at least one $\mu_i \notin \{\nu_1, \ldots, \nu_t\}$ and $(X_{(\nu_1,\ldots,\nu_t)}, n_i) = 1$ if $i \in \{\nu_1, \ldots, \nu_t\}$.

Proof. Suppose $2^k - k - 1$ numbers $X_{(\nu_1,\ldots,\nu_t)}$ exist satisfying (3. 2) for some divisors $n_1, \ldots, n_k$ of n. Define x_i as in (3. 3) and set

$$M = \prod_{s=0}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)}, \; N_i = \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}, \; B_i = \prod_{s=0}^{k-2} \prod_{i \in \nu(k-s)} X_{\nu(k-s)}.$$

Then $M = B_i N_i$ $(i = 1, \ldots, k)$ and (3. 2) becomes $mM = \sum_{i=1}^{k} n_i N_i$. Hence

$$m/n = \sum_{i=1}^{k} \frac{n_i N_i}{nM} = \sum_{i=1}^{k} \frac{n_i}{nB_i} = \sum_{i=1}^{k} 1/x_i.$$

Conversely, suppose $x_1, \ldots, x_k$ satisfy (3. 1). We note in passing that (3. 1) is not soluble if $m > kn$. Crossmultiplying and rearranging terms in (3. 1) yields

$$(3.\,4) \qquad x_1 \cdots x_{k-1} (m x_k - n) = n x_k \sum_{i=1}^{k-1} \frac{x_1 \ldots x_{k-1}}{x_i}.$$

Setting

$$(3.\,5) \qquad A_i = (n, x_i), \; x_i = A_i B_i, \; n = A_i n_i; \; (B_i, n_i) = 1 \qquad (i = 1, 2, \ldots, k)$$

and substituting into (3. 4), we obtain after dividing by $A_1 \cdots A_k$ that

$$(3.\,6) \qquad B_1 \cdots B_{k-1} (m B_k - n_k) = B_k \sum_{i=1}^{k-1} \frac{B_1 \ldots B_{k-1}}{B_i} n_i.$$

Let $\{X_{(\nu_1,\ldots,\nu_t)}\}$ be the family of relative irreducibles of $B_1, \ldots, B_k$ (as defined in Sect. 2) and let

$$(3.\,7) \qquad B_j = \prod_{s=0}^{k-1} \prod_{j \in \nu(k-s)} X_{\nu(k-s)} \qquad (j = 1, \ldots, k)$$

be the corresponding decomposition of $B_1, \ldots, B_k$ into relative irreducibles. Then

$$B_1 \cdots B_{k-1} = \prod_{j=1}^{k-1} \prod_{s=0}^{k-1} \prod_{j \in \nu(k-s)} X_{\nu(k-s)}.$$

To simplify the preceding expression, consider first the case when $s = 0$. The factor $X_{\nu(k)} = X_{(1,2,\ldots k)}$ occurs in each B_j and thus appears to the $(k-1)$th power. Consider now any $s \geqq 1$ and a fixed factor $X_{\nu(k-s)} = X_{(\nu_1,\ldots \nu_{k-s})}$. If $\nu_{k-s} \neq k$, then $X_{(\nu_1,\ldots,\nu_{k-s})}$ appears in the product $B_1 \cdots B_{k-1}$ to the $(k-s)$th power, for it is a factor of $B_{\nu_1}, \ldots, B_{\nu_{k-s}}$ and no other B_j. On the other hand, if $\nu_{k-s} = k$, then $X_{(\nu_1,\ldots,\nu_{k-s})}$ appears in precisely $k-s-1$ B_j amongst $B_1, \ldots, B_{k-1}$, and thus $B_1 \cdots B_{k-1}$ contains $X_{(\nu_1,\ldots,\nu_{k-s})}$ to the $(k-s-1)$th power. Hence

$$(3.8) \qquad B_1 \cdots B_{k-1} = X_{(1,\ldots,k)}^{k-1} X_1 X_2 \cdots X_{k-1} \prod_{s=1}^{k-2} \Bigl(\prod_{k \notin \nu(k-s)} X_{\nu(k-s)}^{k-s} \cdot \prod_{k \in \nu(k-s)} X_{\nu(k-s)}^{k-s-1} \Bigr).$$

Consider next $(B_1 \cdots B_k)/B_i$, $1 \leqq i \leqq k-1$. In view of (3.7),

$$(B_1 \cdots B_k)/B_i = \prod_{\substack{j=1 \\ j \neq i}}^{k} \prod_{s=0}^{k=1} \prod_{j \in \nu(k-s)} X_{\nu(k-s)}.$$

By the same argument as before, only i replacing k, we obtain that

$$(3.9) \qquad \frac{B_1 \cdots B_k}{B_i} = X_{(1,\ldots,k)}^{k-1} \frac{X_1 \cdots X_k}{X_i} \prod_{s=1}^{k-2} \Bigl[\prod_{i \notin \nu(k-s)} X_{\nu(k-s)}^{k-s} \cdot \prod_{i \in \nu(k-s)} X_{\nu(k-s)}^{k-s-1} \Bigr].$$

But

$$\begin{aligned} &\prod_{i \notin \nu(k-s)} X_{\nu(k-s)}^{k-s} \cdot \prod_{i \in \nu(k-s)} X_{\nu(k-s)}^{k-s-1} \\ &= \prod_{i \notin \nu(k-s)} X_{\nu(k-s)} \cdot \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}^{k-s-1} \cdot \prod_{i \in \nu(k-s)} X_{\nu(k-s)}^{k-s-1} \\ &= \prod_{i \notin \nu(k-s)} X_{\nu(k-s)} \cdot \prod_{\nu(k-s)} X_{\nu(k-s)}^{k-s-1}. \end{aligned}$$

Thus,

$$\sum_{i=1}^{k-1} n_i \frac{B_1 \cdots B_k}{B_i} = X_{(1,\ldots,k)}^{k-1} X_k \prod_{s=1}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)}^{k-s-1} \cdot \left[\sum_{i=1}^{k-1} n_i \frac{X_1 \cdots X_{k-1}}{X_i} \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)} \right].$$

Substituting (3.8) and the preceding into (3.6) and cancelling common factors yields:

$$\begin{aligned} (3.10) \qquad & X_1 \cdots X_{k-1} \cdot \prod_{s=1}^{k-2} \prod_{k \notin \nu(k-s)} X_{\nu(k-s)} \cdot (m B_k - n_k) \\ &= X_k \cdot \sum_{i=1}^{k-1} \frac{X_1 \cdots X_{k-1}}{X_i} \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}. \end{aligned}$$

Now, X_k, being a factor of B_k, does not divide $m B_k - n_k$, since $(B_k, n_k) = 1$ (3.5). And according to Sect. 2, Remark 1, X_k is relatively prime to each factor preceding $(m B_k - n_k)$ in (3.10). Thus $X_k = 1$. Similarly, $X_i = 1$, for $i = 1, \ldots, k-1$. Finally since $B_k = \prod_{s=0}^{k-2} \prod_{k \in \nu(k-s)} X_{\nu(k-s)}$, (3.10) becomes:

$$(3.11) \qquad m \prod_{s=0}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)} = \sum_{i=1}^{k} n_i \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}.$$

Now (3.11) implies that $(n_1, \ldots, n_k) = 1$, for as $(m, n) = 1$, $n_i \mid n$, any prime divisor of $(n_1, \ldots, n_k)$ divides some $X_{(\nu_1,\ldots,\nu_{k-s})}$, which is impossible according to Sect. 2, Remark 2, since $(n_i, B_i) = 1$ in view of (3.5) for all $1 \leqq i \leqq k$. Combining now (3.5) and (3.7), noting that $X_{\nu(1)} = 1$, we obtain that

$$(3.12) \qquad x_i = (n/n_i)\, B_i = (n/n_i) \prod_{s=0}^{k-2} \prod_{i \in \nu(k-s)} X_{\nu(k-s)}, \qquad (i = 1, \ldots, k)$$

where each B_i is a product of $2^{k-1} - 1$ $X_{(\nu_1,\ldots,\nu_t)}$. Q. E. D.

We shall consider next the determination of all integral solutions of (3. 1) if such exist. One may restrict oneself to *positive* integral solutions, as in the general case the argument varies only slightly. Clearly, if $m/n = 1/x_1 + 1/x_2 + \cdots + 1/x_k$ for positive x_i, then not each of the $1/x_i$ can be less than m/nk, hence we may assume that $1/x_1 \geqq m/nk$, therefore $0 < x_1 \leqq nk/m$ and only finitely many values for x_1 have to be considered. For each choice of x_1 one argues similarly about the equation $(mx_1 - n)/nx_1 = 1/x_2 + \cdots + 1/x_k$, and thus, in a finite number of steps all solutions, if such exist, are obtained. However, in view of Theorem 3. 1, a more efficient algorithm can be applied, as will be described in the proof of the following theorem.

Theorem 3. 2. *Let m, n and k be given positive integers*, $(m, n) = 1$. *Then* (3. 1) *has at most a finite number of positive integral solutions $x_1, \ldots, x_k$. Moreover, there exists an algorithm to determine whether* (3. 1) *is soluble, and if it is, the algorithm yields all solutions.*

Proof. For a given n, there exist finitely many positive integers $n_1, \ldots, n_k$ such that $n_i \mid n$ and $(n_1, \ldots, n_k) = 1$. We consider then all equations (3. 2) for all choices of $n_1, \ldots, n_k$. Clearly,

$$\left[\prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}\right] \Big/ \prod_{s=0}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)} = 1 \Big/ \prod_{s=0}^{k-2} \prod_{i \in \nu(k-s)} X_{\nu(k-s)}.$$

Thus, (3. 2) can be written in the form

$$m = \sum_{i=1}^{k} n_i \Big/ \prod_{s=0}^{k-2} \prod_{i \in \nu(k-s)} X_{\nu(k-s)}. \tag{3. 13}$$

As it is impossible that each of the k summands be less than m/k, there exists j, $1 \leqq j \leqq k$, such that

$$n_j \Big/ \prod_{s=0}^{k-2} \prod_{j \in \nu(k-s)} X_{\nu(k-s)} \geqq m/k,$$

hence

$$0 < \prod_{s=0}^{k-2} \prod_{j \in \nu(k-s)} X_{\nu(k-s)} \leqq kn_j/m \leqq kn^*/m, \tag{3. 14}$$

where $n^* = \max(n_1, \ldots, n_k)$. We may assume, by possibly renumbering the n_i, that $j = 1$. Thus (3. 14) determines the possible range for $2^{k-1} - 1$ of the X's. To determine for each choice of the X's in (3. 14) the possible range of the remaining $2^{k-1} - k$ X's, we proceed as follows: Set

$$\left\{\begin{aligned} m_1 &= \left[m \prod_{s=0}^{k-2} \prod_{1 \in \nu(k-s)} X_{\nu(k-s)}\right] - n_1 \\ n_i^{(1)} &= n_i \prod_{s=1}^{k-2} \prod_{\substack{1 \in \nu(k-s) \\ i \notin \nu(k-s)}} X_{\nu(k-s)} \qquad (i = 2, 3, \ldots, k) \end{aligned}\right\} \tag{3. 15}$$

Then the remaining X's must satisfy the equation

$$m_1 \prod_{s=1}^{k-2} \prod_{1 \notin \nu(k-s)} X_{\nu(k-s)} = \sum_{i=2}^{k} n_i^{(1)} \prod_{s=1}^{k-2} \prod_{\substack{1 \notin \nu(k-s) \\ i \notin \nu(k-s)}} X_{\nu(k-s)},$$

or equivalently,

$$m_1 = \sum_{i=2}^{k} n_i^{(1)} \Big/ \prod_{s=1}^{k-2} \prod_{\substack{1 \notin \nu(k-s) \\ i \in \nu(k-s)}} X_{\nu(k-s)}. \tag{3. 16}$$

Repeating the argument following (3. 13), the algorithm can be continued until all sets of admissible X's within the restricted ranges have been examined for a given choice of $n_1, \ldots, n_k$. Q. E. D.

Theorem 3.1 deals with integral solutions, without restrictions to *positive* solutions. It is clear, however, from (3.2) and (3.3) that we may restrict all the $X_{\nu(k-s)}$ with $s \geqq 1$ to be positive integers, while $n_1, \ldots, n_k$ and $X_{\nu(k)}$ can assume any non-zero integral values, and we shall adhere to this convention henceforth. Several special results are noteworthy. If $k = 2$, (3.2) becomes $m X_{(1,2)} = n_1 + n_2$, hence we obtain the following theorem of Nakayama [12] and Kiss [10]:

Corollary 3.1. *m/n admits a* 2-fold *representation if and only if there exist coprime divisors n_1, n_2 of n such that $n_1 + n_2 \equiv 0 \ (mod\ m)$.*

Next, consider the case when $k = 3$. Then (3.2) becomes

$$m X_{(1,2,3)} X_{(1,2)} X_{(1,3)} X_{(2,3)} = n_1 X_{(2,3)} + n_2 X_{(1,3)} + n_3 X_{(1,2)},$$

which we shall rewrite as

$$mxyzt = n_1 x + n_2 y + n_3 z, \tag{3.17}$$

where $(x, y) = (x, z) = (y, z) = (x, n_2 n_3) = (y, n_1 n_3) = (z, n_1 n_2) = 1$ in view of Remarks 1 and 2 of Sect. 2. Equation (3.17) generalizes equations (A) and (B) of Bernstein ([1], p. 3) for an arbitrary n, not necessarily a prime number. The simplicity of Corollary 3.1 suggests a similar condition if $k > 2$. In view of (3.2), we multiply both m and n by $\prod_{s=1}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)}$. Set

$$(3.18) \quad \left\{ \begin{aligned} M &= m \prod_{s=1}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)}; \quad N = n \prod_{s=1}^{k-2} \prod_{\nu(k-s)} X_{\nu(k-s)} \\ t &= X_{\nu(k)} = X_{(1,2,\ldots,k)} \\ N_i &= n_i \prod_{s=1}^{k-2} \prod_{i \notin \nu(k-s)} X_{\nu(k-s)}, \qquad\qquad (i = 1, \ldots, k). \end{aligned} \right.$$

Clearly, (3.2) then becomes $Mt = \sum_{i=1}^{k} N_i$ and $N_i \mid N$. To show that $(N_1, \ldots, N_k) = 1$, we argue as follows. Let p be a prime divisor of $(N_1, \ldots, N_k)$. Since according to Theorem 3.1 $(n_1, \ldots, n_k) = 1$, we may assume that p divides the terms

$$n_1, n_2, \ldots, n_f, X^{(f+1)}, X^{(f+2)}, \ldots, X^{(k)},$$

where $0 \leqq f < k$, and the i-th term is a factor of N_i, having suitably renumbered the N_i. It follows from Remark 1 that if p divides both $X_{(\mu_1,\ldots,\mu_s)}$ and $X_{(\nu_1,\ldots,\nu_t)}$, where $s \leqq t$, then $\{\mu_1, \ldots, \mu_s\} < \{\nu_1, \ldots, \nu_t\}$. If the subscript of $X^{(f+i)}$ is $(\nu_1^{(f+i)}, \ldots, \nu_{s_i}^{(f+i)})$, $s_1 \leqq \cdots \leqq s_{k-f}$, then $\{\nu_1^{(f+1)}, \ldots, \nu_{s_1}^{(f+1)}\} < \{\nu_1^{(f+2)}, \ldots, \nu_{s_2}^{(f+2)}\} < \cdots < \{\nu_1^{(k)}, \ldots, \nu_{s_{k-f}}^{(k)}\}$. But $f + i \notin \{\nu_1^{(f+i)}, \ldots, \nu_{s_i}^{(f+i)}\}$ in view of the definitions of $X^{(f+i)}$ and N_{f+1}. Hence $\{\nu_1^{(f+1)}, \ldots, \nu_{s_1}^{(f+1)}\}$ does not contain any of the numbers $f+1, f+2, \ldots, k$ and thus $\{\nu_1^{(f+1)}, \ldots, \nu_{s_1}^{(f+1)}\} < \{1, 2, \ldots, f\}$. Therefore $f \geqq 1$. Set $\nu_1^{(f+1)} = \mathrm{r}$. Then $1 \leqq r \leqq f$. Since $(n_r, B_r) = 1$ (equation (3.5)), it follows from Remark 2 and (3.7) that $(X^{(f+1)}, n_r) = 1$ $(X^{(f+1)} = X_{(r,\ldots)})$ while p divides $X^{(f+1)}$ and n_r. Consequently $(N_1, \ldots, N_k) = 1$ and we have proven the following generalization of Nakayama's Theorem:

Main Theorem. *The Diophantine equation*

$$m/n = 1/x_1 + 1/x_2 + \cdots + 1/x_k \qquad (m, n) = 1,$$

is soluble if and only if for some M and N, $m/n = M/N$ and there exist N_i dividing N, $(N_1, \ldots, N_k) = 1$, such that

$$N_1 + N_2 + \cdots + N_k \equiv 0 \ (\mathrm{mod}\ M).$$

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Eingegangen 9. März 1965