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On subseries

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This paper studies several questions concerning the notion of subseries and their asymptotic density in the original series and also some properties (topological and metrical) of the sets

$$C\left(\sum_1^\infty d_n\right), \quad D\left(\sum_1^\infty d_n\right)$$

which are defined in the theory of subseries.

1. Asymptotic densities of subseries

Let

$$\sum_{n=1}^\infty d_n = d_1 + d_2 + \cdots + d_n + \cdots \tag{1}$$

be an infinite series, let

$$k_1 < k_2 < \cdots < k_n < \cdots$$

be (some) increasing sequence of natural numbers. The series

$$\sum_{n=1}^\infty d_{k_n} = d_{k_1} + d_{k_2} + \cdots + d_{k_n} + \cdots \tag{2}$$

is called the subseries of the series (1).

If we express the numbers of the interval $(0, 1\rangle$ in their dyadic expansions with infinitly many digits equal to 1, then to each $x \in (0, 1\rangle$,

$$x = \sum_{k=1}^\infty \varepsilon_k(x)\, 2^{-k} \tag{3}$$

[$\varepsilon_k(x) = 0$ or 1, for an infinite number of k $\varepsilon_k(x) = 1$] we can associate an infinite series

$$(x) = \sum_{k=1}^\infty \varepsilon_k(x)\, d_k \tag{4}$$

[this is a subseries of the series (1)] and also conversely, every subseries (2) of the series (1) may be expressed in the form (4) if we put $\varepsilon_{k_n}(x) = 1$ $(n = 1, 2, 3, \ldots)$ and $\varepsilon_k(x) = 0$ $(k \neq k_n,\ n = 1, 2, 3, \ldots)$.

In such a way we get a transformation of the set of all subseries of a given series into $(0, 1\rangle$. Briefly we shall say, that a number x expressed by means of the expression (3) corresponds to the subseries (x) [see (4)] and conversely.

This correspondence makes possible "to measure" certain subsets of the set of all subseries of a given series. Thus we say that some property V is valid for almost all subseries of a given series if the set of all $x \in (0, 1\rangle$, corresponding to the subseries having the property V, has the Lebesgue measure 1. Let us note the well-known classical result (see e.g. [2], p. 404, or [1]), according to which almost all subseries of a divergent series are divergent. An analogous result is true also for the subsequences of a given divergent sequence.

In the following we introduce this notation. Let (4) be a subseries of the series (1), let

$$p(n, x) = \sum_{k=1}^{n} \varepsilon_k(x).$$

Then the numbers

$$p_1(x) = \liminf_{n\to\infty} \frac{p(n, x)}{n}, \qquad p_2(x) = \limsup_{n\to\infty} \frac{p(n, x)}{n}$$

will be called lower and upper asymptotic density of the series (4) in the series (1). If there exists

$$p(x) = \lim_{n\to\infty} \frac{p(n, x)}{n},$$

then the last number will be called asymptotic density of the series (4) in the series (1). Obviously $p(x), p_1(x), p_2(x) \in \langle 0, 1\rangle$.

In this part of the paper we shall prove two theorems on subseries of divergent series with non-negative terms.

In paper [5] and [6] it is proved that if $d_n = 1/n$ $(n = 1, 2, 3, \ldots)$ and the series (4) is convergent, then $p(x) = 0$. This result is generalized in the following theorem.

Theorem 1. *Let* $d_n \downarrow 0$ *and* $\liminf_{n\to\infty} n d_n > 0$. *Let the subseries*

$$(x) = \sum_{k=1}^{\infty} \varepsilon_k(x)\, d_k$$

of the series $\sum_{n=1}^{\infty} d_n$ *be convergent. Then*

$$p(x) = \lim_{n\to\infty} \frac{p(n, x)}{n} = 0.$$

Proof. Obviously it suffices to prove that

$$p_2(x) = \limsup_{n\to\infty} \frac{p(n, x)}{n} > 0 \tag{5}$$

implies: (x) is divergent.

Let (5) be valid. Then there exists $\delta' > 0$ such that for an infinite number of n $p(n, x) > \delta' n$. Further, according to the proposition of the theorem there exists $\delta > 0$ such that $n d_n \geqq \delta$ for all $n = 1, 2, 3, \ldots$.

Let now N be an arbitrary natural number. Let us put $\varepsilon=\frac{1}{2}\delta\delta'>0$ and choose the natural number n_0 such that

1. $n_0>N$,
2. $p(n_0, x)>\delta' n_0$,
3. $N d_{n_0}<\frac{1}{2}\delta\delta'=\varepsilon$.

This is evidently possible. Then for such n_0 we get by means of simple estimation

$$\sum_{r=N+1}^{n_0}\varepsilon_r(x)\, d_r \geqq (\delta' n_0-N)\, d_{n_0} \geqq \delta\delta' - N d_{n_0} > \varepsilon .$$

Consequently for the series (x) the Cauchy-Bolzano condition is not fulfilled. This completes the proof.

Note 1. The condition

$$d_1\geqq d_2\geqq\cdots\geqq d_n\geqq\cdots$$

is in the foregoing theorem substantial as it is shown in the following example.

$$\sum_1^\infty d_n=\underbrace{\frac{1}{1}+\frac{1}{2}+\frac{1}{2^2}}+\underbrace{\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^4}}+\cdots+\underbrace{\frac{1}{n}+\frac{1}{2^{2n-1}}+\frac{1}{2^{2n}}}+\cdots$$

The subseries $\sum_{n=1}^{\infty} 1/2^n$ of this series is convergent and its asymptotic density in the original series is obviously $\frac{2}{3}$.

Note 2. We shall show by an example that the condition

$$\liminf_{n\to\infty} n\, d_n>0$$

in the foregoing theorem cannot be substituted by the weaker condition

$$\sum_{n=1}^{\infty} d_n=+\infty .$$

Let us put for fixed $n (n=1, 2, \ldots)$

$$d_{n^n+k(n)}=\frac{1}{n^{n+2}},$$

where $k(n)$ is an integer, $0\leqq k(n)<(n+1)^{n+1}-n^n$. Evidently $d_1\geqq d_2\geqq\cdots\geqq d_n\geqq\cdots$, and when n is fixed, then for each $k(n)$, $0\leqq k(n)<(n+1)^{n+1}-n^n$

$$(n^n+k(n))\, d_{n^n+k(n)}\leqq (n+1)^{n+1}\frac{1}{n^{n+2}}=\frac{1}{n}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n.$$

From this it is seen that $\lim_{l\to\infty} l\, d_l=0$. According to ($n$ is fixed)

$$\sum_{0\leqq k(n)<(n+1)^{n+1}-n^n} d_{n^n+k(n)}=\frac{(n+1)^{n+1}-n^n}{n^{n+2}},$$

$$(n+1)^{n+1}-n^n=(n+1)\, n^n\left\{\left(1+\frac{1}{n}\right)^n-1\right\},$$

$(n+1)^{n+1}-n^n>cn^{n+1}$, $c>0$ independent of n, we get $\sum_{l=1}^{\infty} d_l=+\infty$.

Let us define now

$$x\in(0,1\rangle, x=\sum_{k=1}^{\infty}\varepsilon_k(x)\,2^{-k}$$

(a dyadic expansion) in this way: for fixed n we put

$$\varepsilon_{n^n+l\,(n)}(x)=1\,, \quad \text{if} \quad 0\leqq l(n)\leqq n^n\,,$$
$$\varepsilon_{n^n+l\,(n)}(x)=0\,, \quad \text{if} \quad n^n<l(n)<(n+1)^{n+1}-n^n:$$

Let us construct the subseries

$$(x)=\sum_{k=1}^{\infty}\varepsilon_k(x)\,d_k\,.$$

For fixed n it is

$$\sum_{s\leqq 2n^n}\varepsilon_s(x)\,d_s=\sum_{l\leqq n}\frac{l^l+1}{l^{l+2}}\leqq 2\sum_{l\leqq n}\frac{1}{l^2}\,.$$

From this it is seen that (x) is convergent. Further, according to the notation of Theorem 1, we have

$$p(2n^n,x)\geqq n^n(n=1,2,3,\dots)\,,$$

consequently $p_2(x)\geqq\frac{1}{2}$ and the conclusion of Theorem 1 is false. Let us note that if we take $p((n+1)^{n+1}-1,\,x)$ $(n=1,\,2,\,\dots)$, then it is not difficult to see that $p_1(x)=0$.

Theorem 2. *Let* $\sum_{n=1}^{\infty} d_n=+\infty$, *let there exist* s *such that*

$$d_s\geqq d_{s+1}\geqq\cdots\geqq d_{s+n}\geqq\cdots$$

Let

$$(x)=\sum_{k=1}^{\infty}\varepsilon_k(x)\,d_k<+\infty\,.$$

Then

$$p_1(x)=\liminf_{n\to\infty}\frac{p(n,x)}{n}=0\,.$$

Proof. Evidently it suffices to prove, that if $p_1(x)>0$, then (x) is divergent. So let $p_1(x)>0$. There exists $\delta>0$ such that for all

$$n\geqq r\geqq s\,, \qquad r\geqq 2\,, \qquad p(n,x)\geqq n\,\delta>0\,.$$

By means of ABEL's partial summation we get

$$\begin{aligned}\sum_{k=r}^{r+t}\varepsilon_k(x)\,d_k=&-p(r-1,x)\,d_r+p(r,x)\,(d_r-d_{r+1})+\\&+p(r+1,x)\,(d_{r+1}-d_{r+2})+\cdots+p(r+t-1,x)\,(d_{r+t-1}-d_{r+t})+\\&+p(r+t,x)\,d_{r+t}\geqq-p(r-1,x)\,d_r+\delta(d_r+d_{r+1}+\cdots+d_{r+t})\end{aligned}$$

and the right side of the last inequality has for $t\to\infty$ the limit $+\infty$. This ends the proof.

2. The sets $C\left(\sum_1^\infty d_n\right)$, $D\left(\sum_1^\infty d_n\right)$

Let $\sum_1^\infty d_n$ in this part of the paper denote a series with real terms. The set of all those

$$x\in(0,1\rangle,\qquad x=\sum_{k=1}^\infty \varepsilon_k(x)\,2^{-k}$$

(a dyadic expansion of x), for which the subseries

$$(x)=\sum_{k=1}^\infty \varepsilon_k(x)\,d_k$$

is convergent (divergent) will be denoted

$$C\left(\sum_1^\infty d_n\right)\qquad \left(D\left(\sum_1^\infty d_n\right)\right).$$

Already in the preface of this paper we have mentioned the classical result according to which, if $\sum_1^\infty d_n$ is a divergent series,

$$\left|C\left(\sum_1^\infty d_n\right)\right|=0\quad\text{and consequently}\quad\left|D\left(\sum_1^\infty d_n\right)\right|=1$$

($|H|$ denotes the Lebesgue measure of the set H).

In this part of the paper the properties of the sets

$$C\left(\sum_1^\infty d_n\right),\qquad D\left(\sum_1^\infty d_n\right)$$

for various classes of the series $\sum_1^\infty d_n$ will be studied more precisely.

For the following purposes let us introduce this notation: for fixed n the interval $(0,1\rangle$ is a union of 2^n pairly disjoint intervals of the order n:

$$\left(\frac{s}{2^n},\frac{s+1}{2^n}\right\rangle, \tag{6}$$

where $s=0,1,\dots 2^n-1$. All the numbers of an interval (6) have on the first n places in their dyadic expansions the same numbers $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$. Briefly we say that (6) belongs to the finite sequence $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$.

In what follows M^0 denotes the set of all condensation points of the set M.

Theorem 3. *Let $\sum_1^\infty |d_n|=+\infty$, let 0 be a limit point of the sequence $\{d_n\}_1^\infty$. Then*

$$C\left(\sum_1^\infty d_n\right)^0=D\left(\sum_1^\infty d_n\right)^0=\langle 0,1\rangle.$$

Proof. Let

$$x=\sum_{k=1}^{\infty}\varepsilon_k(x)\,2^{-k}\in(0,1\rangle .$$

We shall show, that x is the condensation point of the set

$$C\left(\sum_1^{\infty} d_n\right).$$

Let

$$S(x,\delta)=(x-\delta,x+\delta)\cap(0,1\rangle,\qquad \delta>0 .$$

According to the assumption of the theorem there exists a sequence

$$k_1<k_2<\cdots<k_r<\cdots$$

so that $d_{k_r}\to 0$, which implies that there exists a sequence of natural numbers

$$l_1<l_2<\cdots<l_n<\cdots$$

such that

$$\sum_{n=1}^{\infty}|d_{l_n}|<+\infty .$$

Let us choose a natural number r such that $(0\leqq s\leqq 2^{l_r}-1)$

$$\left(\frac{s}{2^{l_r}},\frac{s+1}{2^{l_r}}\right\rangle\subset S(x,\delta).$$

Evidently this is possible. Let M denote the set of all those

$$y\in\left(\frac{s}{2^{l_r}},\frac{s+1}{2^{l_r}}\right\rangle$$

whose dyadic expansions are of the form

$$y=\sum_{k=1}^{\infty}\varepsilon_k(y)\,2^{-k};\qquad \varepsilon_{l_i}(y)=0\qquad\text{or } 1 \text{ for } i=r+1,r+2,\dots,$$

for an infinite number of $i>r$ is $\varepsilon_{l_i}(y)=1$ and $\varepsilon_k(y)=0$ for each $k\neq l_i$, $i=r+1,r+2,\dots$. Obviously M is an uncountable set of the power of the continuum,

$$M\subset S(x,\delta),\qquad M\subset C\left(\sum_1^{\infty}d_n\right),\quad\text{consequently}\quad x\in C\left(\sum_1^{\infty}d_n\right)^0.$$

We shall show that x is also an element of

$$D\left(\sum_1^{\infty}d_n\right)^0.$$

From the condition

$$\sum_1^{\infty}|d_n|=+\infty$$

it follows, that the series constructed from the non-negative terms of the sequence $\{d_n\}_1^\infty$ or the series constructed from the negative terms of the mentioned sequence is divergent. Let e.g.

$$\sum_1^\infty \max(d_n, 0) = +\infty$$

(in the other case the proof is analogous). There exists a sequence of natural numbers

$$r_1 < r_2 < \cdots < r_k < \cdots$$

such that

$$d_{r_i} \geqq 0 \qquad (i = 1, 2, \ldots), \qquad \sum_{n=1}^\infty d_{r_n} = +\infty .$$

According to the foregoing part of the proof there exists a sequence of natural numbers

$$l_1 < l_2 < \cdots < l_n < \cdots$$

such that

$$\sum_{n=1}^\infty |d_{l_n}| < +\infty .$$

Let R denote the set of all those terms of the sequence $\{r_n\}_{n=1}^\infty$ which do not appear in $\{l_n\}_{n=1}^\infty$. Evidently R is an infinite set. Let us form an ascending sequence from its elements

$$s_1 < s_2 < \cdots < s_n < \cdots .$$

Evidently

$$\sum_{n=1}^\infty d_{s_n} = +\infty .$$

Let $S(x, \delta)$ have the previous meaning. Let us choose t such that

$$\left(\frac{s}{2^t}, \frac{s+1}{2^t} \right) \subset S(x, \delta), \qquad 0 \leqq s \leqq 2^t - 1.$$

Let M' denote the set of all those

$$y = \sum_{k=1}^\infty \varepsilon_k(y)\, 2^{-k} \in \left(\frac{s}{2^t}, \frac{s+1}{2^t} \right)$$

for which the following is valid: $\varepsilon_{l_i}(y) = 0$ or 1 for $l_i > t$, $\varepsilon_{s_i}(y) = 1$ for $s_i > t$ and $\varepsilon_l(y) = 0$ for $l > t$, $l \neq s_i (i = 1, 2, \ldots)$, $l \neq l_i (i = 1, 2, \ldots)$. Obviously M' is uncountable of the power of the continuum,

$$M' \subset S(x, \delta), \qquad M' \subset D\left(\sum_1^\infty d_n\right), \quad \text{consequently} \quad x \in D\left(\sum_1^\infty d_n\right)^0 .$$

In an analogical way we can see that also 0 is an element of both sets

$$C\left(\sum_1^\infty d_n\right)^0, \quad D\left(\sum_1^\infty d_n\right)^0$$

and
$$C\left(\sum_1^\infty d_n\right)^0 \subset \langle 0, 1\rangle, \quad D\left(\sum_1^\infty d_n\right)^0 \subset \langle 0, 1\rangle.$$

This completes the proof.

The character of the sets
$$C\left(\sum_1^\infty d_n\right)$$
depends on the series $\sum_1^\infty d_n$. If 0 is not a limit point of the sequence $\{d_n\}_1^\infty$, then
$$C\left(\sum_1^\infty d_n\right)=\emptyset \quad \text{and if} \quad \sum_1^\infty |d_n|<+\infty, \quad \text{then} \quad C\left(\sum_1^\infty d_n\right)=(0, 1\rangle.$$
Those are extrem and from our point of view uninteresting cases.

We have seen, that if $\sum_1^\infty d_n$ fulfills the properties

(a) $\sum_1^\infty |d_n|=+\infty$,

(b) 0 is the limit point of the sequence $\{d_n\}_1^\infty$,

then
$$C\left(\sum_1^\infty d_n\right)$$
has the following properties

(a′) $\left|C\left(\sum_1^\infty d_n\right)\right|=0$,

(b′) $C\left(\sum_1^\infty d_n\right)^0=\langle 0, 1\rangle$.

The following question arises: Does there exist to every set $P\subset(0, 1\rangle$ with the properties

(a′) $|P|=0$,

(b′) $P^0=\langle 0, 1\rangle$

a series $\sum_1^\infty d_n$ with the properties (a), (b) such that
$$P=C\left(\sum_1^\infty d_n\right)?$$
It is not difficult to show that the answer is negative.

Theorem 4. *There exists a system $\mathfrak{A}$ of the power 2^c (c is the power of the continuum) of sets $P\subset(0, 1\rangle$ with properties*

(a′) $|P|=0$,

(b′) $P^0=\langle 0, 1\rangle$

such that for each series $\sum_1^\infty d_n$ *with real terms and for each* $P \in \mathfrak{A}$ *is*

$$C\left(\sum_1^\infty d_n\right) \neq P.$$

Proof. Let A' denote the Cantor discontinuum constructed on $\langle 0, 1\rangle$, let $A = A' - \{0\}$. If r is rational, $r \in \langle 0, 1)$, let us put $A(r) = [x \in (0, 1\rangle : x = y + r$ where $y \in A]$. Let us put $E = \bigcup_{0 \leq r < 1} A(r)$. Evidently $|E| = 0$.

Now, let X run through the system of all subset of the set A. Let us associate to each X the set $P(X) = E - X$. If $X' \neq X''$, then evidently $P(X') \neq P(X'')$ and each of the sets $P(X)$ has the properties (a′), (b′). The power of all the sets $P(X)$ is 2^c and the power of all

$$C\left(\sum_1^\infty d_n\right)$$

is not greater than the power of all the series with real terms, consequently not greater than c. Let $\mathfrak{A}'(\mathfrak{B}')$ denote the system of all $P(X)$, for which there does not exists (there exists)

$$\sum_1^\infty d_n \quad \text{such that} \quad C\left(\sum_1^\infty d_n\right) = P(X).$$

Then

$$\overline{\overline{\mathfrak{A}'}} + \overline{\overline{\mathfrak{B}'}} = 2^c$$

($\overline{\overline{M}}$ is the cardinal number of the set M) and as

$$a = \overline{\overline{\mathfrak{B}'}} \leq c, \qquad z = \overline{\overline{\mathfrak{A}'}} \leq 2^c,$$

we get

$$2^c = z + a \leq z + c \leq 2^c + c = 2^c$$

(see [8], p. 168–170). From the last we get $z + c = 2^c$, $z = 2^c$.

The proof is complete.

Let $\sum_1^\infty d_n$ be a convergent series, let $\sum_1^\infty |d_n| = +\infty$, then (see [3]) for all $x \in (0, 1\rangle$ with the exception of a set of the first category

$$\liminf_{n\to\infty} \sum_{k=1}^n \varepsilon_k(x)\, d_k = -\infty, \qquad \limsup_{n\to\infty} \sum_{k=1}^n \varepsilon_k(x)\, d_k = +\infty.$$

It is seen from the proof of this theorem that in the case of the unboundedness of the partial sums of the series

$$\sum_1^\infty d_n, \quad \text{the set} \quad C\left(\sum_1^\infty d_n\right)$$

is a set of the first category in $(0, 1\rangle$.

In view of

(7) $$\left|C\left(\sum_1^\infty d_n\right)\right|=0 \quad \text{if} \quad \sum_1^\infty d_n$$

is divergent, therefore it is a natural problem to examine the Hausdorff dimensions of these sets. As to it, the following theorem is an improvement of the classical result (7) in the case of the series with non-negative terms which form an ascending sequence.

Let us note that in what follows the Hausdorff dimension will be taken with respect to the system of measure functions

$$\mu^{(\alpha)}(t)=t^\alpha, \qquad t\in\langle 0, +\infty), \qquad \alpha\in(0, 1)$$

(see [*7*], [*9*]).

Theorem 5. *Let* $\sum_1^\infty d_n=+\infty$, *let there exist* s *such that*

(8) $$d_s\geqq d_{s+1}\geqq\cdots\geqq d_{s+n}\geqq\cdots$$

Then

$$\dim C\left(\sum_1^\infty d_n\right)=0.$$

Proof. Let us denote by the symbol M (0) the set of all those

$$x\in(0, 1\rangle, \qquad x=\sum_{k=1}^\infty \varepsilon_k(x)\,2^{-k}$$

(a dyadic expansion) for which

$$p_1(x)=\liminf_{n\to\infty}\frac{p(n, x)}{n}=0.$$

As VL. KNICHAL has proved (see [*4*] or [*9*]) $\dim M(0)=0$ holds. According to Theorem 2 of the present paper

$$C\left(\sum_1^\infty d_n\right)\subset M(0)$$

and from this fact

$$\dim C\left(\sum_1^\infty d_n\right)=0$$

follows.

Note 3. If we take in account that the series $\sum_1^\infty d_n=+\infty$ fulfilling the condition (8) are of extremly various kind and some of them (see note 2) fulfill so much as the necessary condition for the convergence

(9) $$\lim_{n\to\infty} n\, d_n=0,$$

one could expect that among a great number of these series [especially among those, where the condition (9) is valid] the sets

$$C\left(\sum_1^\infty d_n\right)$$

will be sufficiently rich and that their Hausdorff dimensions will be positive. As to the above mentioned, Theorem 5 is relatively surprising.

Theorem 5 leads us to the question, whether it is possible, that the set

$$C\left(\sum_1^\infty d_n\right), \quad \text{where} \quad \sum_1^\infty d_n$$

is divergent, has a positive Hausdorff dimension. This question will be answered in what will follow (see the corollary of Theorem 6 and the note 4), where we show (under certain restricting hypothesis on $\sum_1^\infty d_n$) the upper and lower estimation of

$$\dim C\left(\sum_1^\infty d_n\right).$$

Theorem 6. *Let $\sum_1^\infty d_n$ be a series with positive terms, let*

$$(x_0)=\sum_{k=1}^\infty \varepsilon_k(x_0)\,d_k<+\infty.$$

Then

$$\dim C\left(\sum_1^\infty d_n\right)\geqq p_1(x_0).$$

Corollary. *If $p_1(x_0)=1$, then*

$$\dim C\left(\sum_1^\infty d_n\right)=1.$$

Example. Let us put $\varepsilon_k(x_0)=0$ if k is a prime number, $\varepsilon_k(x_0)=1$ in other cases. Further let $d_n=1/n$ if n is a prime number and $d_n=1/2^n$ in other cases. Then

$$p_1(x_0)=p(x_0)=1 \quad \text{and} \quad \sum_{n=1}^\infty d_n=+\infty$$

as it is known from the number theory. Consequently

$$\left|C\left(\sum_1^\infty d_n\right)\right|=0, \qquad \dim C\left(\sum_1^\infty d_n\right)=1.$$

Proof of Theorem 6. If $p_1(x_0)=0$, the theorem is trivial. Let us suppose that $p_1(x_0)>0$. It will be proved by means of the Theorem 1 of the paper [7].

Let us denote for n natural by the symbol I_n' (I_n) the system of all the intervals of the order n

$$\left(\frac{s}{2^n}, \frac{s+1}{2^n}\right)\left(\left\langle\frac{s}{2^n}, \frac{s+1}{2^n}\right\rangle\right)$$

belonging to such sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, for which $\varepsilon_i=0$ if $\varepsilon_i(x_0)=0$ and $\varepsilon_i=0$ or 1 if $\varepsilon_i(x_0)=1$ $(i=1, 2, \ldots n)$. By means of the same symbol I_n' (I_n) we shall denote also the union of such intervals.

The length of each interval of the system I_n is $\lambda_n = 1/2^n$, consequently

$$\lambda_n \to 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = \frac{1}{2} > 0.$$

The intervals $i \neq i'$, $i, i' \in I_n$ have no common inner points and each of the intervals $i \in I_n$ contains the same number of the intervals $i \in I_{n+1}$ [this number is 1, if $\varepsilon_{n+1}(x_0) = 0$, and 2, if $\varepsilon_{n+1}(x_0) = 1$].

Now if we show that for $0 < \alpha < p_1(x_0)$

$$\lim_{n\to\infty} g_n \mu^{(\alpha)}(\lambda_n) = +\infty \tag{10}$$

[g_n denotes the number of the intervals of the system I_n and $\mu^{(\alpha)}(t) = t^\alpha$), from the Theorem 1 of the paper [7] we get

$$\dim M = \dim \bigcap_{n=1}^{\infty} I_n \geqq p_1(x_0) \qquad \left(M = \bigcap_{n=1}^{\infty} I_n\right). \tag{11}$$

We shall prove (10). Evidently $g_n = 2^{p(n, x_0)}$. Let $0 < \alpha < p_1(x_0)$. Then there exists $\varepsilon > 0$ and a natural number n_0 such that for

$$n > n_0 \quad \text{is} \quad \frac{p(n, x_0)}{n} > \alpha + \varepsilon$$

and so (for $n > n_0$)

$$g_n \mu^{(\alpha)}(\lambda_n) = 2^{p(n, x_0)} \frac{1}{2^{n\alpha}} > 2^{n\varepsilon}.$$

From this it is obvious that (10) is valid and so is (11).

Now let us put

$$M' = \bigcap_{n=1}^{\infty} I'_n.$$

Evidently $M' \subset M$ and $M - M'$ is a countable set, consequently (see [7] Lemma 3) is $\dim M' = \dim M \geqq p_1(x_0)$. Obviously from the definition of the set M' we have

$$M' \subset C\left(\sum_1^{\infty} d_n\right)$$

and therefore also

$$\dim C\left(\sum_1^{\infty} d_n\right) \geqq p_1(x_0).$$

This ends the proof.

For the following purposes let us define the function $d(\zeta)$ in this way: $d(0) = d(1) = 0$ and for $\zeta \in (0, 1)$ we put

$$d(\zeta) = \frac{\zeta \log \zeta + (1-\zeta)\log(1-\zeta)}{\log \frac{1}{2}}.$$

It is not difficult to see, that for $\zeta\in(0, \frac{1}{2}\rangle$ we have $d(\zeta)>\zeta$. It follows from the fact, that for $\zeta\in(0, \frac{1}{3})$ we have (as we can easily see) $d'(\zeta)>1$, further $d(\frac{1}{3})>\frac{1}{2}$ and $d'(\zeta)>0$ on the interval $\langle\frac{1}{3}, \frac{1}{2})$. Further we can see that $d(\frac{1}{2})=1$.

Theorem 7. *Let*

$$\sum_{1}^{\infty} d_n=+\infty, \qquad d_n>0,$$

$$(x_0)=\sum_{k=1}^{\infty}\varepsilon_k(x_0)\,d_k<+\infty .$$

Let $p_2(x_0)\in\langle 0, \frac{1}{2}\rangle$. *Let us put*

$$B=[n:\varepsilon_n(x_0)=0]=\{l_1<l_2<\dots<l_n<\dots\}.$$

Let

$$d_{l_n}\downarrow 0 \quad \textit{and} \quad \liminf_{n\to\infty} n\,d_{l_n}>0 .$$

Then

$$\dim C\left(\sum_{1}^{\infty} d_n\right)\leqq d(p_2(x_0)).$$

Proof. Let

$$x\in C\left(\sum_{1}^{\infty} d_n\right), \qquad x=\sum_{k=1}^{\infty}\varepsilon_k(x)\,2^{-k},$$

let $A=N-B$, N is the set of all natural numbers. Take in account that

$$(12)\qquad \begin{cases} p(n,x)=\sum\limits_{k=1}^{n}\varepsilon_k(x)=\sum\limits_{k\leqq n,\,k\in A}\varepsilon_k(x)+\sum\limits_{k\leqq n,\,k\in B}\varepsilon_k(x)\leqq\sum\limits_{k\leqq n,\,k\in A}1+\sum\limits_{k\leqq n,\,k\in B}\varepsilon_k(x)\\ \quad =p(n,x_0)+\sum\limits_{k\leqq n,\,k\in B}\varepsilon_k(x). \end{cases}$$

All natural numbers $l_i\in B$ appearing in the sequence 1, 2, ... n are $l_1, l_2, \dots, l_{B(n)}$ ($B(n)$ denotes the number of those $l\in B$, for which $l\leqq n$), therefore

$$\sum_{k\leqq n,\,k\in B}\varepsilon_k(x)=\sum_{i=1}^{B(n)}\varepsilon_{l_i}(x).$$

From the convergence of the series

$$\sum_{k=1}^{\infty}\varepsilon_k(x)\,d_k$$

taking into account that $d_k>0 (k=1, 2, \dots)$ the convergence of the series

$$\sum_{i=1}^{\infty}\varepsilon_{l_i}(x)\,d_{l_i}$$

follows and as the series $\sum\limits_{i=1}^{\infty} d_{l_i}$ fulfills the assumptions of the Theorem 2

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{B(n)}\varepsilon_{l_i}(x)}{B(n)}=0$$

and obviously

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{B(n)} \varepsilon_{l_i}(x)}{n} = 0.$$

From the last and from (12) we get

$$\limsup_{n\to\infty} \frac{p(n,x)}{n} \leqq \limsup_{n\to\infty} \frac{p(n,x_0)}{n} + \lim_{n\to\infty} \frac{\sum_{i=1}^{B(n)} \varepsilon_{l_i}(x)}{n} = p_2(x_0).$$

Consequently it was show, that

$$C\left(\sum_1^\infty d_n\right) \subset R(p_2(x_0)),$$

where $R(\zeta)$, $\zeta \in (0, \frac{1}{2}\rangle$ denotes the set of all these $x \in (0, 1\rangle$ for which the sequence

$$\left\{\frac{p(n,x)}{n}\right\}_{n=1}^\infty$$

has some limit point η, $\eta \leqq \zeta$.

On the basis of a certain result of B. VOLKMANN (see [9]) we get

$$\dim C\left(\sum_1^\infty d_n\right) \leqq \dim R(p_2(x_0)) = d(p_2(x_0)).$$

The proof is complete.

Sumarizing Theorem 6 and 7 we get the following theorem, which gives for a certain class of series with positive terms $\sum_1^\infty d_n$ a lower and an upper estimation of

$$\dim C\left(\sum_1^\infty d_n\right).$$

Theorem 8. *Let*

$$\sum_1^\infty d_n = +\infty, \qquad d_n > 0,$$

let

$$(x_0) = \sum_{k=1}^\infty \varepsilon_k(x_0)\, d_k < +\infty.$$

Let us assume that there exists $p(x_0) \in (0, \frac{1}{2})$. *Let us put*

$$B = [n : \varepsilon_n(x_0) = 0] = \{l_1 < l_2 < \cdots < l_n < \cdots\},$$

let

$$d_{l_n} \downarrow 0 \quad \textit{and} \quad \liminf_{n\to\infty} n\, d_{l_n} > 0.$$

Then

$$0 < p(x_0) \leqq \dim C\left(\sum_1^\infty d_n\right) \leqq d(p(x_0)) < 1.$$

Note 4. From Theorem 8 it follows that there exists a certain class of divergent series with positive terms $\sum_1^\infty d_n$, for which

$$0<\dim C\left(\sum_1^\infty d_n\right)<1.$$

Example. Let us put

$$d_{3k}=\frac{1}{(3k)^2}\,(k=1,2,3,\ldots)$$

and $d_n=1/n$ for other $n \neq 3k$. Let us define

$$x_0=\sum_{l=1}^\infty \varepsilon_l(x_0)\,2^{-l}$$

such that $\varepsilon_l(x_0)=1$, if $l=3k(k=1,2,3,\ldots)$ and $\varepsilon_l(x_0)=0$, if $l=3k+1$ or $3k+2(k=0,1,2,\ldots)$. Then evidently

$$(x_0)=\sum_{k-1}^\infty \varepsilon_k(x_0)\,d_k<+\infty, \qquad p(x_0)=\tfrac{1}{3}$$

and using the notation of Theorem 8 we have $l_{2k+1}=3k+1$, $l_{2k+2}=3k+2$. It is not difficult to see that

$$d_{l_n}\downarrow 0 \quad \text{and} \quad \liminf_{n\to\infty} n\,d_{l_n}=\tfrac{2}{3}>0.$$

So on basis of Theorem 8

$$\frac{1}{3}\leqq \dim C\left(\sum_1^\infty d_n\right)\leqq d\left(\frac{1}{3}\right)=\frac{\log(3/\sqrt[8]{4})}{\log 2}<1.$$

Now we show, that we can get for a class of divergent series with non-negative terms an upper estimation of

$$\dim C\left(\sum_1^\infty d_n\right)$$

with the help of the same idea, on which the proof of Theorem 5 is based.

In what follows N denotes the set of all natural numbers and if $T\subset N$, then $T(n)$ denotes the number of all those $t\in T$, for which $t\leqq n$ is true.

Theorem 9. *Let*

$$\sum_{n=1}^\infty d_n=+\infty, \qquad d_n\geqq 0,$$

let there exist

$$K=\{k_1<k_2<\cdots<k_n<\cdots\}$$

such that

(α) $$\liminf_{n\to\infty}\frac{K(n)}{n}=\delta, \qquad \frac{1}{2}\leqq\delta\leqq 1,$$

(β) $$\sum_{n=1}^\infty d_{k_n}=+\infty; \qquad d_{k_1}\geqq d_{k_2}\geqq\cdots\geqq d_{k_n}\geqq\cdots$$

Then

$$\dim C\left(\sum_1^\infty d_n\right) \leqq d(\delta)$$

$$\left(d(\zeta) = \frac{\zeta \log \zeta + (1-\zeta)\log(1-\zeta)}{\log \frac{1}{2}} \text{ for } \zeta \in (0,1),\, d(0) = d(1) = 0\right).$$

Note 5. If we put in the foregoing theorem $\delta = 1$, then we get an improvement of Theorem 5.

Proof of Theorem 9. If

$$x = \sum_{k=1}^\infty \varepsilon_k(x) 2^{-k} \in (0,1\rangle,$$

then we put

$$p'(l,x) = \sum_{i=1}^{l} \varepsilon_{k_i}(x).$$

We put $K^* = N - K$, let

$$x \in C\left(\sum_{n=1}^\infty d_n\right), \qquad x = \sum_{k=1}^\infty \varepsilon_k(x) 2^{-k},$$

then from

$$\sum_{k=1}^\infty \varepsilon_k(x) d_k < +\infty$$

it follows that

$$\sum_{i=1}^\infty \varepsilon_{k_i}(x) d_{k_i} < +\infty$$

and from this and from the Theorem 2 (with respect to (β)) we have

$$\liminf_{l\to\infty} \frac{p'(l,x)}{l} = 0. \tag{13}$$

Let s be a natural number, let us denote by $k_1, k_2, \dots k_l$ all the numbers $k_i \in K$ for which $k_i \leqq s$. Then (the notation is the same as in Theorem 5) we have

$$p(s,x) = \sum_{k=1}^{s} \varepsilon_k(x) = \sum_{i=1}^{l} \varepsilon_{k_i}(x) + \sum_{r \leqq s,\, r \in K^*} \varepsilon_r(x) \leqq p'(l,x) + K^*(s).$$

From this we get

$$\frac{p(s,x)}{s} \leqq \frac{p'(l,x)}{l}\frac{l}{s} + \frac{K^*(s)}{s}. \tag{14}$$

From (13) it follows, that there exists a sequence of natural numbers

$$l_1 < l_2 < \dots < l_n < \cdots$$

such that

$$\frac{p'(l_n,x)}{l_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Let us put $s_n = k_{l_n}$ $(n=1, 2, 3, \ldots)$, then from (14) we get

$$\frac{p(s_n, x)}{s_n} \leqq \frac{p'(l_n, x)}{l_n} \frac{l_n}{s_n} + \frac{K^*(s_n)}{s_n}. \tag{15}$$

If we consider that

$$\limsup_{n\to\infty} \frac{K^*(s_n)}{s_n} \leqq \limsup_{s\to\infty} \frac{K^*(s)}{s} = 1 - \liminf_{s\to\infty} \frac{K(s)}{s} = 1 - \delta$$

and $l_n \leqq s_n$ $(n=1, 2, \ldots)$, we get from (15)

$$\limsup_{n\to\infty} \frac{p(s_n, x)}{s_n} \leqq 1 - \delta$$

so the sequence

$$\left\{\frac{p(s, x)}{s}\right\}_{s=1}^{\infty}$$

has a limit point ζ, for which $\zeta \leqq 1 - \delta \leqq \frac{1}{2}$ is valid and so $x \in R(1-\delta)$ $\big(R(\zeta)$ has the same meaning as in the proof of the Theorem 7$\big)$.

So we get

$$C\left(\sum_1^{\infty} d_n\right) \subset R(1-\delta)$$

and with the help of the mentioned result of B. VOLKMANN (see [9]) we get

$$\dim C\left(\sum_1^{\infty} d_n\right) \leqq \dim R(1-\delta) = d(1-\delta) = d(\delta).$$

The proof is complete.

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