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On the Compactification of Strongly Pseudoconvex Surfaces III.

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To the memory of my father.

Throughout, we shall freely use the definitions and notations introduced in [9a, b]. In particular, all $\mathbb{C}$-analytic surfaces (compact or non compact) are assumed to be *minimal*.

1. Situation of the Problem

Definition 1 [2, 9a, b]. A non compact $\mathbb{C}$-analytic surface X is said to be *strongly pseudoconvex* if

i) X is holomorphically convex;

ii) there exists a compact analytic curve $E \subset X$ such that $T \subseteq E$ for any irreducible compact analytic curve $T \subset X$.

E is called the *exceptional* curve of X.

In the special case where $E = \emptyset$, X is called a *Stein* surface.

Definition 2. A non compact $\mathbb{C}$-analytic surface X is called *compactifiable* if there exist

i) a compact $\mathbb{C}$-analytic surface M;

ii) a compact analytic curve $\Gamma \subset M$ such that X is biholomorphically equivalent to $M \backslash \Gamma$.

M is called an *algebraic* (resp. a *non algebraic*) compactification if M is an algebraic (resp. a non algebraic) surface.

In this paper we continue to investigate the global structure of

i) compactifiable Stein surfaces;

ii) compactifiable strongly pseudoconvex surfaces which are not Stein (i.e. $E \neq \emptyset$).

Remark 1. By definition, Stein surfaces are just special cases of strongly pseudoconvex surfaces; so one might ask why the treatment of those two surfaces

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has to be dealt with separately. In fact, one of the main purposes of this paper is to point out the sharp contrast between those two surfaces from the view-point of compactification. So from now on, as we did in [9a, b] strongly pseudoconvex surfaces are meant to be Non Stein!

Our present investigation is motivated by the following results:

Theorem 1 [4, 1c]. *Let M be a compactification of some Stein surface X. Then M is either*

i) *an algebraic surface;*

ii) *a non elliptic Hopf surface containing exactly one compact analytic curve (see Definition 3) or*

iii) *a parabolic Inoue surface containing exactly one compact analytic curve (see Definition 5).*

Theorem 2 [9a, b]. *Let M be a compactification of some strongly pseudoconvex surface X. Then M is either*

i) *an algebraic surface or*

ii) *a parabolic Inoue surface containing exactly two connected compact analytic curves (see Definition 5).*

Remark 2. Notice that all the alternatives in Theorems 1 and 2 do indeed occur [1b, 4, 9a].

Theorem 3 [9b, c]. (*Existence of compactifications.*)

Let X be a strongly pseudoconvex surface (resp. a Stein surface). Then X is compactifiable iff X is algebraic.

Naturally, Theorem 3 leads us to the question of uniqueness of compactification for strongly pseudoconvex surfaces (resp. Stein surfaces); however, Theorems 1 and 2 guide us to the following precise formulations:

Problem 1. Let M_1 and M_2 be two algebraic (resp. non algebraic) compactifications of some Stein surface X. Do M_1 and M_2 birationally (resp. bimeromorphically) equivalent?

Problem 2. Let M_1 and M_2 be two algebraic (resp. non algebraic) compactifications of some strongly pseudoconvex surface X. Do M_1 and M_2 birationally (resp. bimeromorphically) equivalent?

As explained in Remark 1, the outcomes for these two problems turn out to be at the two extreme ends of the logic scale. In fact, the answer for Problem 1 is no which is the main topic of Sect. 2. Meanwhile in Sect. 3, an affirmative answer for Problem 2 will be given. Finally, the affine structure (resp. strongly pseudo affine structure) of compactifiable Stein surfaces (resp. compactifiable strongly pseudoconvex surfaces) will be taken up in Sect. 4.

2. Compactifiable Stein Surfaces

We are now in a position to exhibit counterexamples for Problem 1.

A) Let us consider the following construction which is due to Serre (see [3a]). Let Δ be a non singular elliptic curve and let $\mathbb{E} \to \Delta$ be a holomorphic vector bundle of rank 2 which is not a trivial extension of O_Δ by itself, namely one has the following exact sequence

$$0 \to O_\Delta \to \mathbb{E} \to O_\Delta \to 0.$$

Let $Z := P(\mathbb{E})$ be the associated elliptic ruled surface with its natural projection $\pi: Z \to \Delta$. Now one can check that the line bundle $O_X(1)$ defines an effective divisor Y on Z which is the unique section of π with $Y^2 = 0$; furthermore, $Z \setminus Y \simeq \mathbb{C}^* \times \mathbb{C}^*$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Now let $M_1 := Z$ and $\Gamma_1 := Y$ and let $M_2 := \mathbb{P}_2$ and $\Gamma_2 :=$ the union of three lines in general position. Then

$$M_1 \setminus \Gamma_1 \simeq \mathbb{C}^* \times \mathbb{C}^* \simeq M_2 \setminus \Gamma_2.$$

Hence one obtains a Stein surface $X := \mathbb{C}^* \times \mathbb{C}^*$ which admits two algebraic compactifications which are not birationally equivalent: M_1 an irrational surface and M_2 a rational surface.

B) Let $U := \mathbb{C}^2 \setminus \{0, 0\}$ and let $g: U \to U$ be an automorphism of U defined by

$$g(z_1, z_2) := (\alpha^m z_1 + \lambda z_2^m, \alpha z_2)$$

where m is a fixed positive integer, $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$.

Now one can check that [6] the cyclic group $\langle g \rangle$ is properly discontinuous and the quotient space $\mathbb{H}_\alpha := U/\langle g \rangle$ is a compact $\mathbb{C}$-analytic surface with

$$b_1(\mathbb{H}_\alpha) = 1 \quad \text{and} \quad b_2(\mathbb{H}_\alpha) = a(\mathbb{H}_\alpha) = 0.$$

Furthermore, the punctured line $U \cap \{z_2 = 0\}$ is invariant under g, so it is mapped by the projection $\pi: U \to \mathbb{H}_\alpha$ onto a non singular elliptic curve $\Gamma_\alpha := \mathbb{C}^*/\langle \alpha \rangle$ which is the only compact $\mathbb{C}$-analytic curve in $\mathbb{H}_\alpha$.

Definition 3. $\mathbb{H}_\alpha$ is called a *non elliptic Hopf* surface containing exactly one compact analytic curve Γ_α.

Now one can check easily the following

Lemma 1. *$\mathbb{H}_\alpha$ is biholomorphically equivalent to $\mathbb{H}_\beta$ iff $\alpha = \beta$.*

Remark 3 [4]. Let us consider the following functions

$$f_1(z_1, z_2) := \exp\left(\frac{2\pi i \alpha^m z_1}{\lambda z_2^m}\right)$$

and

$$f_2(z_1, z_2) := (1/z_2) \exp\left(\frac{\alpha^m z_1}{\lambda z_2^m} \log \alpha\right)$$

which are defined and holomorphic on $U \setminus \{z_2 = 0\}$.

One can check that $f := (f_1, f_2)$ maps $U \setminus \{z_2 = 0\}$ onto $\{\mathbb{C}^2 \mid z_1 z_2 \neq 0\}$ and $f(z) = f(z')$ iff $z' = g^k(z)$ for some k. Consequently $\mathbb{H}_\alpha \setminus \Gamma_\alpha \simeq \{\mathbb{C}^2 \mid z_1 z_2 \neq 0\} \simeq \mathbb{C}^* \times \mathbb{C}^*$.

Now let $M_1 := \mathbb{H}_\alpha$, $\Gamma_1 := \Gamma_\alpha$ and let $M_2 := \mathbb{H}_\beta$, $\Gamma_2 := \Gamma_\beta$ with $\alpha \neq \beta$. Then Remark 3 tells us that

$$M_1 \backslash \Gamma_1 \simeq M_2 \backslash \Gamma_2 \simeq \mathbb{C}^* \times \mathbb{C}^*.$$

Hence one obtains a Stein surface $X := \mathbb{C}^* \times \mathbb{C}^*$ which admits two non algebraic compactifications M_1 and M_2 which, in view of Lemma 1, are not bimeromorphically equivalent.

3. Compactifiable Strongly Pseudoconvex Surfaces

In this section, we shall provide positive answers for Problem 2.

A) **Definition 4** [5, 7, 8]. Let M be a compact $\mathbb{C}$-algebraic surface, let Γ be a compact analytic curve in M and let $X = M \backslash \Gamma$. Now let L be a holomorphic line bundle on M. For a positive integer m, let

$$\phi_{|mL|}: M \to \mathbb{P}_N$$

be a meromorphic map defined by

$$w \mapsto [\phi_0(w): \ldots : \phi_N(w)]$$

where $\{\phi_i\}$ is a basis for $H^0(M, O(mL))$.

Let $N(L, M) := \{m > 0 \mid \dim H^0(M, O(mL)) > 0\}$. Then one defines

$$\kappa(L, M) := \begin{cases} -\infty & \text{if } N(L, M) = \emptyset \\ \max_m \{\dim \phi_{|mL|}(M)\} & \text{if } N(L, M) \neq \emptyset. \end{cases}$$

Now the number $\kappa(X) := \kappa(\mathbb{K} + \Gamma, M)$ is called the *logarithmic Kodaira dimension* of X, where $\mathbb{K}$ is the canonical bundle of M.

Remark 4. i) By definition, $\kappa(X)$ can assume only the following values $-\infty$, 0, 1 or 2.

ii) If $X = M$ and $\Gamma = \emptyset$, the logarithmic Kodaira dimension coincides with the notion of Kodaira dimension for compact surfaces; in fact $\kappa(M) = \kappa(\mathbb{K}, M)$.

iii) One always has $\kappa(X) \geqq \kappa(M)$.

Now some basic ingredients are in order.

Lemma 2. *Let M be a complete algebraic surface with a complete curve Γ in M and let $X := M \backslash \Gamma$ with $\kappa(X) = 0$ or 1. If $\kappa(M) \geqq 0$ then X is neither Stein nor strongly pseudoconvex.*

Proof. i) Cases where $\kappa(X) = 0$.

By hypothesis, one must have $\kappa(M) = 0$; therefore, it follows from [6] that M is either an abelian surface, a hyperelliptic surface, a K_3 surface or an Enriques surface. Now if M is either an abelian surface or a hyperelliptic surface, it follows from [5] (Example 4) that $\kappa(X) \geqq 1$ which is excluded in our current situation.

On the other hand, if M is either a K_3 surface or an Enriques surface, again it follows from [5] (Example 4) that $\Gamma^2<0$ which in turn implies that X is not holomorphically convex (see e.g. Theorem B of [9a]).

ii) Cases where $\kappa(X)=1$.

By hypothesis $\kappa(M)\geqq 0$, hence a result in [7] (Theorem 2.3.1) tells us that M has a structure of an elliptic surface $\pi: M\to \Delta$ over a non singular complete curve Δ and Γ consists of finitely many fibres of π. Consequently X contains infinitely many compact analytic curves; hence X can be neither strongly pseudoconvex nor Stein. Q.E.D.

Corollary 3. *Let M be an algebraic compactification of some Stein surface (resp. some strongly pseudoconvex surface) X. Then $\kappa(M)=-\infty$ if $\kappa(X)<2$.*

We are now in a position to provide a positive answer for the first half of Problem 2.

Theorem 4. *Let M_1 and M_2 be two arbitrary algebraic compactifications of some strongly pseudoconvex surfaces X. Then M_1 and M_2 are birationally equivalent.*

Proof [1]. Let E be the exceptional curve in X and let M be an algebraic compactification of X.

i) If $\kappa(X)=-\infty$, 0 or 1, in view of Corollary 3, $\kappa(M)=-\infty$, i.e. following [6] M is either a $\mathbb{P}_2$ or a ruled surface. Since X is strongly pseudoconvex, M cannot be a $\mathbb{P}_2$. Hence M is a ruled surface. Let $\pi: M\to B$ be a surjective morphism onto some compact $\mathbb{C}$-analytic curve B. It is known that $M\simeq \mathbb{P}(\mathbb{E})$ where $\mathbb{E}$ is a locally free sheaf of rank 2 on B. Let ε be the divisor on B corresponding to $\Lambda^2\mathbb{E}$, and let $e:=-\deg\varepsilon\cdot e$ is an invariant for M.

Now let us fix a section Ξ of M with $O_M(\Xi)\simeq O_{\mathbb{P}(\mathbb{E})}(1)$ and let us denote the fibres of M by F. It is shown [3b] that

$$\Xi^2=-e. \tag{*}$$

Claim. $E\simeq\Xi$.

In fact, since $H^2(M,\mathbb{Z})$ is generated by Ξ and F, one can write

$$\theta=a\Xi+bF \tag{**}$$

where $a, b\in\mathbb{Z}$ and θ is some irreducible component of E. Since E is exceptional, hence

$$\theta^2<0. \tag{†}$$

Now let us assume that $\theta\neq\Xi, F$; in view of (*) and (**) $\theta^2=2ab-a^2e$.

Case 1. For $e\geqq 0$, one has (see [3b] Prop. V.2.20) $a>0$, $b\geqq ae$. Therefore $\theta^2\geqq 2a^2e-a^2e=a^2e\geqq 0$, contradicting (†).

Case 2. For $e<0$, one has (see [3b] Prop. V.2.21) either

i) $a=1, b\geqq 0$ or

ii) $a\geqq 2$, $b\geqq 1/2\,ae$ which in either cases, imply that $\theta^2\geqq 0$, contradicting (†).

[1] We would like to thank the referee who pointed out a fatal error in our previous proof

Since θ is exceptional so $\theta \neq F$. Therefore E is irreducible and our claim is proved..

In particular M is birationally equivalent to $E \times \mathbb{P}_1$. Hence it follows readily that M_1 and M_2, as ruled surfaces, are birationally equivalent.

ii) If $\kappa(X)=2$, then a result in [8] (Sect. 4) tells us that M_1 and M_2 are birationally equivalent. Q.E.D.

Remarks. i) Although E is a section for the ruled surface M, $\Gamma := M \setminus X$ in general, is not, as shown by the following example which was communicated to us by the referee:

Let M be a Hirzebruch surface with a section E such that $E^2 = -2$. Let Γ be an irreducible compact $\mathbb{C}$-analytic curve in M such that $\Gamma \equiv 2E + 4F$. Since $\Gamma^2 > 0$ and since $\Gamma \cdot E = 0$, $X := M \setminus \Gamma$ is a strongly pseudoconvex surface with exceptional curve E, but Γ is not a section of M.

ii) In parallel with Theorem 4, let us consider the following class of non compact $\mathbb{C}$-analytic surfaces:

Let X be a non compact $\mathbb{C}$-analytic surface satisfying the following condition:

(†) there exists an irreducible compact $\mathbb{C}$-analytic curve E on X such that the normal bundle $N_{E/X}$ is ample.

Notice that such surfaces exist; in fact let E be a hyperplane section in $\mathbb{P}_2$ and let $x \in \mathbb{P}_2 \setminus E$. Then $X := \mathbb{P}_2 \setminus \{x\}$ is a non compact $\mathbb{C}$-analytic surface satisfying (†). In fact one can show that X is *strongly pseudoconcave* in the sense of Andreotti-Grauert (see e.g. [3a]). Notice also that strongly pseudoconcave surfaces are in duality with strongly pseudoconvex surfaces.

Now in view of important results by Hironaka and Matsumura on formal meromorphic functions along E (see [3a] for precise references) an analogue for Theorem 4, for non compact $\mathbb{C}$-analytic surfaces satisfying (†) can be stated as follows:

Theorem 4′. *Let X be a non compact $\mathbb{C}$-analytic surfaces satisfying* (†) *and let M_1 and M_2 be 2 algebraic compactifications of X.*

Then M_1 and M_2 are birationally equivalent.

B) **Definition 5.** In [1a] Enoki constructed compact $\mathbb{C}$-analytic surfaces, denoted by $S_{n,\alpha,t}$ where $n>0$, $0<|\alpha|<1$ and $t := (t_0, \ldots, t_{n-1}) \in \mathbb{C}^n$. Those surfaces are completely characterised by the following intrinsic properties: (see [1a] for complete details)

i) $S_{n,\alpha,t}$ is of class VII_0

ii) $b_2(S_{n,\alpha,t}) = n$

iii) $S_{n,\alpha,t}$ contains a connected compact analytic curve $D_{n,\alpha,t}$ with $(D_{n,\alpha,t})^2 = 0$.

The surfaces $S_{n,\alpha,t}$ are called *parabolic Inoue surfaces.*

Following closely an idea in [1a, b] an affirmative answer for the second half of Problem 2 can be stated as follows:

Theorem 5. *Let M_1 and M_2 be two arbitrary non algebraic compactifications of some strongly pseudoconvex surface X.*

Then M_1 and M_2 are biholomorphically equivalent.

Proof. Let E be the exceptional curve in X and let M_i with $i=1$ or 2, be some non algebraic compactifications for X.

In view of Theorem 2, $M_i \simeq S_{n,\alpha,t}$ for some fixed n, α and t. Let Γ_i be a compact analytic curve in M_i such that X is biholomorphic to $M_i \setminus \Gamma_i$. Since X is strongly pseudoconvex and $a(M_i)=0$, Γ_i is necessarily connected and $\Gamma_i^2=0$; hence in view of the construction of the $S_{n,\alpha,t}$, one has $\Gamma_i \simeq D_{n,\alpha,t}$.

Claim:

i) $t=0$

ii) $n=-E^2$ and

iii) $E=\mathbb{C}^*/\langle\alpha\rangle$.

In fact, by construction [1a] one has a biholomorphic map

$$\tau: \mathbb{C}\times\mathbb{C}^*/\langle g_{n,\alpha,t}\rangle \xrightarrow{\sim} S_{n,\alpha,t}\setminus D_{n,\alpha,t} \simeq X$$

where

$$g_{n,\alpha,t}: \mathbb{C}\times\mathbb{C}^* \to \mathbb{C}\times\mathbb{C}^*$$

$$(z,w)\mapsto\left(w^n z+\sum_{i=0}^{n-1} t_i w^i, \alpha w\right) \tag{†}$$

is an automorphism of $\mathbb{C}\times\mathbb{C}^*$.

The natural projection $\mathbb{C}\times\mathbb{C}^*\to\mathbb{C}^*$ induced a map

$$\pi: \mathbb{C}\times\mathbb{C}^*/\langle g_{n,\alpha,t}\rangle =: \mathbb{A} \to \Delta := \mathbb{C}^*/\langle\alpha\rangle$$

which, in turn, provides $\mathbb{A}$ a structure of an affine $\mathbb{C}$-bundle over an elliptic curve Δ.

By hypothesis, the compact analytic curve $\tilde{E} := \tilde{\tau}^1(E)$ sits in $\mathbb{A}$. So let $\{a_1, \ldots, a_r\} := \tilde{\pi}^1(x)\cap\tilde{E}$ for some general point $x\in\Delta$ and let $\sigma(x) := \sum_{i=1}^{r} a_i$. Then σ defines a section for π.

Consequently (†) implies that $\sum_{i=0}^{n-1} t_i w^i = 0$ i.e. $t=0$.

Therefore $\mathbb{A}$ is actually a holomorphic line bundle of degree $=-n$ over Δ and E is a section for π; in particular, $E\simeq\Delta\simeq\mathbb{C}^*/\langle\alpha\rangle$ and $E^2=-n$. Hence our claim is proved.

Now it follows readily from this claim that M_1 and M_2 are biholomorphically equivalent. Q.E.D.

Remark 5. In retrospect the counterexamples exhibited in Sect. 2 are in some extent unique. In fact, with arguments similar to the ones in Theorems 4 and 5 one can easily establish the following:

Theorem 6. *Let M_i with $i=1$ or 2 be two algebraic (resp. non algebraic) compactifications of some Stein surface X.*

Let us assume that M_i are not ruled surfaces (resp. not non elliptic Hopf surfaces).

Then M_i are birationally equivalent (resp. biholomorphically equivalent).

We refer to [9c] for complete proof of this result and further detailed accounts on compactifiable Stein surfaces.

4. The Affine Structure

Recently Enoki raised the following

Problem 3. Let X be a compactifiable Stein surface. Does X always admit some affine structure?

Analogously, we would like to consider the following

Problem 4. Let X be a compactifiable strongly pseudoconvex surface. Does X always admit some strongly pseudo affine structure?

Definition 6 [3a]. A non complete $\mathbb{C}$-algebraic variety Z is called a "modification" (of some affine variety) if

i) there exists an affine variety Y and a proper morphism $\pi: Z \to Y$ inducing an isomorphism $\pi_*(O_Z) \simeq O_Y$;

ii) the set $B := \{y \in Y \mid \dim_{\mathbb{C}} \pi^1(y) > 0\}$ is finite.

Remark 6. Notice that for a given $\mathbb{C}$-algebraic variety Z which is a modification, Z is affine iff Z does not contain complete algebraic varieties of positive dimension.

Definition 7. Let X be a $\mathbb{C}$-analytic space. We say that X admits a *strongly pseudo affine structure* if there exists a $\mathbb{C}$-algebraic variety Z which is a modification such that X is biholomorph to Z_{an}, the analytic space associated to Z.

In order to establish the connection between Problems 3 and 4, we would like to mention the following results:

Theorem A [4]. *Let M be a complete $\mathbb{C}$-algebraic variety and let $Y \subset M$ be a closed subvariety.*

If there exists an effective ample divisor D on M such that $\operatorname{supp}(D) = Y$, *then $X := M \setminus Y$ is affine.*

Theorem B [2]. *Let D be an effective divisor on a compact $\mathbb{C}$-analytic surface M. Let us assume that $D^2 > 0$. Then D is ample iff $M \setminus D$ contains no compact analytic curves.*

In view of Theorems A and B, Problems 3 and 4 are equivalent to the following

Problem 5. Let X be a compactifiable Stein (resp. strongly pseudoconvex) surface. Does X always admit some algebraic compactification M such that $D^2 > 0$ where D is an effective divisor in M with $\operatorname{supp}(D) = M \setminus X$?

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Note added in proof.

Let us use the same notations as in Definition 4 for logarithmic Kodaira dimension $\kappa(X)$; there, if one replaces the vector space $H^0(M, O(mL))$ by $H^0(M, O(m\mathbb{K}+(m-1)\Gamma))$, then the number $\bar{\kappa}(X) := \kappa(L, M)$ is called the analytic Kodaira dimension for X (see [8] and [9c] for more details).

Now, by using similar arguments as in the first part of the proof of Theorem 4, our Corollary 3 can be strengthened as follows:

Theorem 3′. *Let X be a compactifiable Stein surface. Then $\bar{\kappa}(X) = -\infty$ or 2.*

Certainly this result is false, if one replaces the analytic Kodaira dimension $\bar{\kappa}(X)$ by the logarithmic Kodaira dimension $\kappa(X)$. We refer to [7] section 5 for counterexamples.

Furthermore, it would be interesting to find out whether Theorem 3′ still holds for compactifiable strongly pseudoconvex surfaces.