DISQVISITIONES GENERALES

CIRCA SERIEM INFINITAM

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}xx+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$

AVCTORE

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PARS I.

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INTRODVCTIO.

1.

Series, quam in hac commentatione perſcrutari ſuſcipimus, tamquam functio quatuor quantitatum α, β, γ, x ſpectari poteſt, quas ipſius *elementa* vocabimus, ordine ſuo elementum primum α, ſecundum β, tertium γ, quartum x diſtinguentes. Manifeſto elementum primum cum ſecundo permutare licet: quodſi itaque breuitatis cauſſa ſeriem noſtram hoc ſigno $F(\alpha, \beta, \gamma, x)$ denotamus, habebimus $F(\beta, \alpha, \gamma, x) = F(\alpha, \beta, \gamma, x)$.

2.

Tribuendo elementis α, β, γ valores determinatos, ſeries noſtra in functionem vnicae variabilis x tranſit, quae manifeſto poſt terminum $1-\alpha^{\text{tum}}$ vel $1-\beta^{\text{tum}}$ abrumpitur, ſi $\alpha-1$ vel $\beta-1$ eſt numerus integer negatiuus, in caſibus reliquis vero in infinitum excur-

rit.

rit. In caſu priori ſeries exhibet functionem algebraicam rationalem, in poſteriori autem plerumque functionem transſcendentem. Elementum tertium γ debet eſſe neque numerus negatiuus integer neque $= 0$, ne ad terminos infinite magnos delabamur.

3.

Coëfficientes poteſtatum x^m, x^{m+1} in ſerie noſtra ſunt vt $1 + \frac{\gamma+1}{m} + \frac{\gamma}{mm} : 1 + \frac{\alpha+\beta}{m} + \frac{\alpha\beta}{mm}$, adeoque ad rationem aequalitatis eo magis accedunt, quo maior aſſumitur m. Si itaque etiam elemento quarto x valor determinatus tribuitur, ab huius indole conuergentia ſeu diuergentia pendebit. Quoties ſcilicet ipſi x tribuitur valor realis, poſitiuus ſeu negatiuus, vnitate minor, ſeries certo, ſi non ſtatim ab initio, tamen poſt certum interuallum, conuergens erit, atque ad ſummam finitam ex aſſe determinatam perducet. Idem eueniet per valorem imaginarium ipſius x formae $a + b\sqrt{-1}$, quoties $aa + bb < 1$. Contra pro valore ipſius x reali vnitateque maiori, vel pro imaginario formae $a + b\sqrt{-1}$, quoties $aa + bb > 1$, ſeries ſi non ſtatim tamen poſt certum interuallum neceſſario diuergens erit, ita vt de ipſius *ſumma* ſermo eſſe nequeat. Denique pro valore $x = 1$ (ſeu generalius pro valore formae $a + b\sqrt{-1}$, quoties $aa + bb = 1$) ſeriei conuergentia ſeu diuergentia ab ipſarum α, β, γ indole pendebit, de qua, atque in ſpecie de ſumma ſeriei pro $x = 1$, in Sect. tertia loquemur.

Patet itaque, quatenus functio noſtra tamquam ſumma ſeriei definita ſit, disquiſitionem natura ſua reſtrictam eſſe ad caſus eos, vbi ſeries reuera conuergat, adeoque quaeſtionem ineptam eſſe, quisnam ſit valor ſeriei pro valore ipſius x vnitate maiori. Infra autem, inde a Sectione quarta, functionem noſtram altiori principio ſuperſtruemus, quod applicationem generaliſſimam patiatur.

4.

Differentiatio ſeriei noſtrae, conſiderando ſolum elementum quartum x tamquam variabile, ad functionem ſimilem perducit, quum manifeſto habeatur

$$\frac{dF(\alpha,\beta,\gamma,x)}{dx} = \frac{\alpha\beta}{\gamma} F(\alpha+1,\beta+1,\gamma+1,x).$$

Idem valet de differentiationibus repetitis.

5.

5.

Operae pretium erit, quasdam functiones, quas ad seriem nostram reducere licet, quarumque vsus in tota analysi est frequentissimus, hic apponere.

I. $(t+u)^n = t^n F(-n, \beta, \beta, -\frac{u}{t})$

vbi elementum β est arbitrarium.

II. $(t+u)^n + (t-u)^n = 2t^n F(-\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}, \frac{uu}{tt})$

III. $(t+u)^n + t^n = 2t^n F(-n, \omega, 2\omega, -\frac{u}{t})$

denotante ω quantitatem infinite paruam.

IV. $(t+u)^n - (t-u)^n = 2nt^{n-1}u\, F(-\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+1, \frac{3}{2}, \frac{uu}{tt})$

V. $(t+u)^n - t^n = nt^{n-1}u\, F(1-n, 1, 2, -\frac{u}{t})$

VI. $\log(1+t) = t F(1, 1, 2, -t)$

VII. $\log \frac{1+t}{1-t} = 2t\, F(\frac{1}{2}, 1, \frac{3}{2}, tt)$

VIII. $e^t = F(1, k, 1, \frac{t}{k}) = 1 + t\, F(1, k, 2, \frac{t}{k}) = 1 + t + \frac{1}{2}tt F(1, k, 3, \frac{t}{k})$ etc.

denotante e basin logarithmorum hyperbolicorum, k numerum infinite magnum.

IX. $e^t + e^{-t} = 2F(k, k', \frac{1}{2}, \frac{tt}{4kk'})$.

denotantibus k, k' numeros infinite magnos.

X. $e^t - e^{-t} = 2t\, F(k, k', \frac{3}{2}, \frac{tt}{4kk'})$

XI. $\sin t = t\, F(k, k', \frac{3}{2}, -\frac{tt}{4kk'})$

XII. $\cos t = F(k, k', \frac{1}{2}, -\frac{tt}{4kk'})$

XIII. $t = \sin t.\, F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \sin t^2)$

XIV. $t = \sin t.\, \cos t.\, F(1, 1, \frac{3}{2}, \sin t^2)$

XV. $t = \operatorname{tang} t.\, F(\frac{1}{2}, 1, \frac{3}{2}, -\operatorname{tang} t^2)$

XVI. $\sin nt = n \sin t.\, F(\frac{1}{2}n+\frac{1}{2}, -\frac{1}{2}n+\frac{1}{2}, \frac{3}{2}, \sin t^2)$

XVII. $\sin nt = n \sin t.\, \cos t.\, F(\frac{1}{2}n+1, -\frac{1}{2}n+1, \frac{3}{2}, \sin t^2)$

XVIII.

XVIII. $\sin nt = n \sin t . \cos t^{n-1} F(-\frac{1}{2}n + 1, -\frac{1}{2}n + \frac{1}{2}, \frac{3}{2}, -\text{tang}\, t^2)$
XIX. $\sin nt = n \sin t . \cos t^{-n-1} F(\frac{1}{2}n + 1, \frac{1}{2}n + \frac{1}{2}, \frac{3}{2}, -\text{tang}\, t^2)$
XX. $\cos nt = F(\frac{1}{2}n, -\frac{1}{2}n, \frac{1}{2}, \sin t^2)$
XXI. $\cos nt = \cos t . F(\frac{1}{2}n + \frac{1}{2}, -\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, \sin t^2)$
XXII. $\cos nt = \cos t^n F(-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}, -\text{tang}\, t^2)$
XXIII. $\cos nt = \cos t^{-n} F(\frac{1}{2}n + \frac{1}{2}, \frac{1}{2}n, \frac{1}{2}, -\text{tang}\, t^2)$

6.

Functiones praecedentes sunt algebraicae atque transscendentes a logarithmis circuloque pendentes. Neutiquam vero *harum* caussa disquisitionem nostram *generalem* suscipimus, sed potius in gratiam theoriae functionum transscendentium altiorum promouendae, quarum genus amplissimum series nostra complectitur. Huc, inter infinita alia, pertinent coëfficientes ex euolutione functionis $(aa + bb - 2ab \cos \varphi)^{-n}$ in seriem secundum cosinus angulorum φ, 2φ, 3φ etc. progredientem orti, de quibus *in specie* alia occasione fusius agemus. Ad formam seriei nostrae autem illi coëfficientes pluribus modis reduci possunt. Scilicet statuendo

$$(aa + bb - 2ab \cos \varphi)^{-n} = \Omega = A + 2A' \cos \varphi + 2A'' \cos 2\varphi + 2A''' \cos 3\varphi + \text{etc.},$$

habemus *primo*

$$A = a^{-2n} F(n, n, 1, \frac{bb}{aa})$$

$$A' = n a^{-2n-1} b F(n, n + 1, 2, \frac{bb}{aa})$$

$$A'' = \frac{n(n+1)}{1 . 2} a^{-2n-2} bb F(n, n + 2, 3, \frac{bb}{aa})$$

$$A''' = \frac{n(n+1)(n+2)}{1 . 2 . 3} a^{-2n-3} b^3 F(n, n + 3, 4, \frac{bb}{aa})$$

etc.

Si enim $aa + bb - 2ab \cos \varphi$ consideratur tamquam productum ex $a - br$ in $a - br^{-1}$ (designante r quantitatem $\cos \varphi + \sin \varphi . \sqrt{-1}$), fit Ω aequalis producto

ex a^{-2n}

in $$1 + n \frac{br}{a} + \frac{n(n+1)}{1 . 2} . \frac{bbrr}{aa} + \frac{n(n+1)(n+2)}{1 . 2 . 3} . \frac{b^3 r^3}{a^3} + \text{etc.}$$

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in $1 + n\frac{br^{-1}}{a} + \frac{n(n+1)}{1\,.\,2}\cdot\frac{bbr^{-2}}{aa} + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\cdot\frac{b^3r^{-3}}{a^3} + \text{etc.}$
Quod productum quum identicum esse debeat cum $A + A'(r+r^{-1}) + A''(rr+r^{-2}) + A'''(r^3+r^{-3})$, valores supra dati sponte prodeunt.

Porro habemus *secundo*

$$A = (aa+bb)^{-n}F(\tfrac{1}{2}n, \tfrac{1}{2}n+\tfrac{1}{2}, 1, \frac{4aabb}{(aa+bb)^2})$$

$$A' = n(aa+bb)^{-n-1}\,ab\,F(\tfrac{1}{2}n+\tfrac{1}{2}, \tfrac{1}{2}n+1, 2, \frac{4aabb}{(aa+bb)^2})$$

$$A'' = \frac{n(n+1)}{1\,.\,2}(aa+bb)^{-n-2}\,aabb\,F(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{3}{2}, 3, \frac{4aabb}{(aa+bb)^2})$$

$$A''' = \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}(aa+bb)^{-n-3}\,a^3b^3\,F(\tfrac{1}{2}n+\tfrac{3}{2}, \tfrac{1}{2}n+2, 4, \frac{4aabb}{(aa+bb)^2})$$

etc.
qui valores facile deducuntur ex

$$\Omega(aa+bb)^n = 1 + n(r+r^{-1})\frac{ab}{aa+bb} + \frac{n(n+1)}{1\,.\,2}(r+r^{-1})^2\frac{aabb}{(aa+bb)^2} + \text{etc.}$$

Tertio fit

$$A = (a+b)^{-2n}F(n, \tfrac{1}{2}, 1, \frac{4ab}{(a+b)^2})$$

$$A' = n(a+b)^{-2n-2}\,ab\,F(n+1, \tfrac{3}{2}, 3, \frac{4ab}{(a+b)^2})$$

$$A'' = \frac{n(n+1)}{1\,.\,2}(a+b)^{-2n-4}\,aabb\,F(n+2, \tfrac{5}{2}, 5, \frac{4ab}{(a+b)^2})$$

$$A''' = \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}(a+b)^{-2n-6}\,a^3b^3\,F(n+3, \tfrac{7}{2}, 7, \frac{4ab}{(a+b)^2})$$

etc.

Denique fit *quarto*

$$A = (a-b)^{-2n}F(n, \tfrac{1}{2}, 1, -\frac{4ab}{(a-b)^2})$$

$$A' = n(a-b)^{-2n-2}\,ab\,F(n+1, \tfrac{3}{2}, 3, -\frac{4ab}{(a-b)^2})$$

$A'' =$

$$A'' = \frac{n(n+1)}{1.2}(a-b)^{-2n-4}\, aabb\, F(n+2, \tfrac{5}{2}, 5, -\frac{4ab}{(a-b)^2})$$

$$A''' = \frac{n(n+1)(n+2)}{1.2.3}(a-b)^{-2n-6}\, a^3 b^3\, F(n+3, \tfrac{7}{2}, 7, -\frac{4ab}{(a-b)^2})$$

etc.

Valores illi atque hi facile eruuntur ex

$$\Omega\,(a+b)^{2n} = (1 - \frac{4ab\cos\frac{1}{2}\phi^2}{(a+b)^2})^{-n} = 1 + n\frac{ab}{(a+b)^2}(r^{\frac{1}{2}} + r^{-\frac{1}{2}})^2 + \frac{n(n+1)}{1.2}\cdot\frac{aabb}{(a+b)^4}(r^{\frac{1}{2}} + r^{-\frac{1}{2}})^4 + \text{etc.}$$

$$\Omega\,(a-b)^{2n} = (1 + \frac{4ab\sin\frac{1}{2}\phi^2}{(a-b)^2})^{-n} = 1 + n\frac{ab}{(a-b)^2}(r^{\frac{1}{2}} - r^{-\frac{1}{2}})^2 + \frac{n(n+1)}{1.2}\cdot\frac{aabb}{(a-b)^4}(r^{\frac{1}{2}} - r^{-\frac{1}{2}})^4 + \text{etc.}$$

SECTIO

SECTIO PRIMA.

Relationes inter functiones contiguas.

7.

Functionem ipsi $F(\alpha, \beta, \gamma, x)$ *contiguam* vocamus, quae ex illa oritur dum elementum primum, secundum, vel tertium vnitate vel augetur vel diminuitur, manentibus tribus reliquis elementis. Functio itaque primaria $F(\alpha, \beta, \gamma, x)$ sex contiguas suppeditat, inter quarum binas ipsamque primariam aequatio persimplex linearis datur. Has aequationes, numero quindecim, hic in conspectum producimus, brevitatis gratia elementum quartum quod semper subintelligitur $= x$ omittentes, functionemque primariam simpliciter per F denotantes.

$$[1]\quad 0 = (\gamma - 2\alpha - (\beta - \alpha)x)\, F + \alpha(1 - x)\, F(\alpha + 1, \beta, \gamma) - (\gamma - \alpha)\, F(\alpha - 1, \beta, \gamma)$$

$$[2]\quad 0 = (\beta - \alpha)\, F + \alpha\, F(\alpha + 1, \beta, \gamma) - \beta F(\alpha, \beta + 1, \gamma)$$

$$[3]\quad 0 = (\gamma - \alpha - \beta)\, F + \alpha(1 - x)\, F(\alpha + 1, \beta, \gamma) - (\gamma - \beta)\, F(\alpha, \beta - 1, \gamma)$$

$$[4]\quad 0 = \gamma(\alpha - (\gamma - \beta)x)\, F - \alpha\gamma(1 - x)\, F(\alpha + 1, \beta, \gamma) + (\gamma - \alpha)(\gamma - \beta)\, F(\alpha, \beta, \gamma + 1)$$

$$[5]\quad 0 = (\gamma - \alpha - 1)\, F + \alpha F(\alpha + 1, \beta, \gamma) - (\gamma - 1)\, F(\alpha, \beta, \gamma - 1)$$

$$[6]\quad 0 = (\gamma - \alpha - \beta)\, F - (\gamma - \alpha)\, F(\alpha - 1, \beta, \gamma) + \beta(1 - x)\, F(\alpha, \beta + 1, \gamma)$$

$$[7]\quad 0 = (\beta - \alpha)(1 - x)\, F - (\gamma - \alpha)\, F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)\, F(\alpha, \beta - 1, \gamma)$$

$$[8]\quad 0 = \gamma(1 - x)\, F - \gamma F(\alpha - 1, \beta, \gamma) + (\gamma - \beta)x\, F(\alpha, \beta, \gamma + 1)$$

$$[9]\quad 0 = (\alpha - 1 - (\gamma - \beta - 1)x)\, F + (\gamma - \alpha)\, F(\alpha - 1, \beta, \gamma) - (\gamma - 1)(1 - x)\, F(\alpha, \beta, \gamma - 1)$$

$$[10]\quad 0 = (\gamma - 2\beta + (\beta - \alpha)x)\, F + \beta(1 - x)\, F(\alpha, \beta + 1, \gamma) - (\gamma - \beta)\, F(\alpha, \beta - 1, \gamma)$$

$$[11]\quad 0 = \gamma(\beta - (\gamma - \alpha)x)\, F - \beta\gamma(1 - x)\, F(\alpha, \beta + 1, \gamma) + (\gamma - \alpha)(\gamma - \beta)\, F(\alpha, \beta, \gamma + 1)$$

$$[12]\quad 0 = (\gamma - \beta - 1)\, F + \beta F(\alpha, \beta + 1, \gamma) - (\gamma - 1)\, F(\alpha, \beta, \gamma - 1)$$

$$[13]\quad 0 = \gamma(1 - x)\, F - \gamma F(\alpha, \beta - 1, \gamma) + (\gamma - \alpha)x\, F(\alpha, \beta, \gamma + 1)$$

$$[14]\quad 0 = (\beta - 1 - (\gamma - \alpha - 1)x)\, F + (\gamma - \beta)\, F(\alpha, \beta - 1, \gamma) - (\gamma - 1)(1 - x)\, F(\alpha, \beta, \gamma - 1)$$

$$[15]\quad 0 = \gamma(\gamma - 1 - (2\gamma - \alpha - \beta - 1)x)\, F + (\gamma - \alpha)(\gamma - \beta)x\, F(\alpha, \beta, \gamma + 1) - \gamma(\gamma - 1)(1 - x)\, F(\alpha, \beta, \gamma - 1)$$

8.

Ecce iam demonstrationem harum formularum. Statuendo

$$\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+m-1)\,\beta(\beta+1)\ldots\ldots(\beta+m-2)}{1\,.\,2\,.\,3\ldots\ldots m\,.\,\gamma(\gamma+1)\ldots\ldots(\gamma+m-1)}=M$$

erit coëfficiens potestatis x^m

in F . . . . $\alpha(\beta+m-1)M$

in $F(\alpha,\beta-1,\gamma)$ . . $\alpha(\beta-1)M$

in $F(\alpha+1,\beta,\gamma)$ . . $(\alpha+m)(\beta+m-1)M$

in $F(\alpha,\beta,\gamma-1)$ . . $\dfrac{\alpha(\beta+m-1)(\gamma+m-1)M}{\gamma-1}$

coëfficiens autem potestatis x^{m-1} in $F(\alpha+1,\beta,\gamma)$, seu coëfficiens potestatis x^m in $xF(\alpha+1,\beta,\gamma)$

$$=m(\gamma+m-1)M$$

Hinc statim demanat veritas formularum 5 et 3; permutando α cum β, oritur ex 5 formula 12, atque ex his duabus per eliminationem 2. Perinde per eandem permutationem ex 3 oritur 6; ex 6 et 12 combinatis oritur 9, hinc per permutationem 14, quibus combinatis habetur 7; denique ex 2 et 6 eruitur 1, atque hinc permutando 10. Formula 8 simili modo vt supra formulae 5 et 3, e consideratione coëfficientium deriuari potest (eodemque modo si placeret *omnes* 15 formulae erui possent), vel elegantius ex iam notis sequenti modo. Mutando in formula 5 elementum α in $\alpha-1$, atque γ in $\gamma+1$, prodit

$$0=(\gamma-\alpha+1)\;F(\alpha-1,\beta,\gamma+1)+(\alpha-1)\;F(\alpha,\beta,\gamma+1)-\gamma F(\alpha-1,\beta,\gamma)$$

Mutando vero in formula 9 tantummodo γ in $\gamma+1$, fit

$$0=(\alpha-1-(\gamma-\beta)x)\;F(\alpha,\beta,\gamma+1)+(\gamma-\alpha+1)\;F(\alpha-1,\beta,\gamma+1)-\gamma(1-x)\;F(\alpha,\beta,\gamma)$$

E subtractione harum formularum statim oritur 8, atque hinc per permutationem 13. Ex 1 et 8 prodit 4, hincque permutando 11. Denique ex 8 et 9 deducitur 15.

9.

Si $\alpha'-\alpha$, $\beta'-\beta$, $\gamma'-\gamma$, nec non $\alpha''-\alpha$, $\beta''-\beta$, $\gamma''-\gamma$ sunt numeri integri (positiui seu negatiui), a functione $F(\alpha,\beta,\gamma)$ ad functionem $F(\alpha',\beta',\gamma')$, et perinde ab hac vsque ad functionem $F(\alpha'',\beta'',\gamma'')$ transire licet per seriem similium functionum, ita vt quaelibet conti-

gua

qua ſit antecedenti et conſequenti, mutando ſcilicet primo elementum vnum e. g. α continuo vnitate, donec a $F(\alpha, \beta, \gamma)$ peruentum ſit ad $F(\alpha', \beta, \gamma)$, dein mutando elementum ſecundum donec peruentum ſit ad $F(\alpha', \beta', \gamma)$, denique mutando elementum tertium donec peruentum ſit ad $F(\alpha', \beta', \gamma')$, et perinde ab hac vsque ad $F(\alpha'', \beta'', \gamma'')$. Quum itaque per art. 7 habeantur aequationes lineares inter functionem primam, ſecundam atque tertiam, et generaliter inter ternas quascunque conſequentes huius ſeriei, facile perſpicitur, hinc per eliminationem deduci poſſe aequationem linearem inter functiones $F(\alpha, \beta, \gamma)$, $F(\alpha', \beta', \gamma')$, $F(\alpha'', \beta'', \gamma'')$, ita vt generaliter loquendo e duabus functionibus, quarum tria elementa prima numeris integris differunt, quamlibet aliam functionem eadem proprietate gaudentem deriuare liceat, ſiquidem elementum quartum idem maneat. Ceterum hic nobis ſufficit, hanc veritatem inſignem generaliter ſtabiliuiſſe, neque hic compendiis immoramur, per quae operationes ad hunc finem neceſſariae quam breuiſſimae reddantur.

10.

Propoſitae ſint e. g. functiones $F(\alpha, \beta, \gamma)$, $F(\alpha+1, \beta+1, \gamma+1)$, $F(\alpha+2, \beta+2, \gamma+2)$, inter quas aequationem linearem inuenire oporteat. Iungamus ipſas per functiones contiguas ſequenti modo:

$F(\alpha, \beta, \gamma) = F$

$F(\alpha+1, \beta, \gamma) = F'$

$F(\alpha+1, \beta+1, \gamma) = F''$

$F(\alpha+1, \beta+1, \gamma+1) = F'''$

$F(\alpha+2, \beta+1, \gamma+1) = F^{\mathrm{IV}}$

$F(\alpha+2, \beta+2, \gamma+1) = F^{\mathrm{V}}$.

$F(\alpha+2, \beta+2, \gamma+2) = F^{\mathrm{VI}}$

Habemus itaque quinque aequationes lineares (e formulis 6, 13, 5 art. 7):

I. $0 = (\gamma-\alpha-1)F - (\gamma-\alpha-1-\beta)F' - \beta(1-x)F''$

II. $0 = \gamma F' - \gamma(1-x)F'' - (\gamma-\alpha-1)xF'''$

III. $0 = \gamma F'' - (\gamma-\alpha-1)F''' - (\alpha+1)F^{\mathrm{IV}}$

IV. $0 = (\gamma-\alpha-1)F''' - (\gamma-\alpha-2-\beta)F^{\mathrm{IV}} - (\beta+1)(1-x)F^{\mathrm{V}}$

V. $0 = (\gamma+1)F^{\mathrm{IV}} - (\gamma+1)(1-x)F^{\mathrm{V}} - (\gamma-\alpha-1)xF^{\mathrm{VI}}$

Ex I et II prodit, eliminando F',

VI. $0 = \gamma F - \gamma(1-x)F'' - (\gamma-\alpha-\beta-1)xF'''$

Hinc atque ex III, eliminando F''

VII. $0 = \gamma F - (\gamma-\alpha-1-\beta x)F''' - (\alpha+1)(1-x)F^{\mathrm{IV}}$

Porro ex IV atque V, eliminando F^{V}

VIII. $0 = (\gamma + 1) F''' - (\gamma + 1) F^{IV} + (\beta + 1) x F^{VI}$

Hinc atque ex VII, eliminando F^{IV},

IX. $0 = \gamma(\gamma + 1) F - (\gamma + 1)(\gamma - (\alpha + \beta + 1) x) F''' - (\alpha + 1)(\beta + 1) x (1 - x) F^{VI}$

11.

Si omnes relationes inter ternas functiones $F(\alpha, \beta, \gamma)$, $F(\alpha + \lambda, \beta + \mu, \gamma + \nu)$, $F(\alpha + \lambda', \beta + \mu', \gamma + \nu')$, in quibus $\lambda, \mu, \nu, \lambda', \mu', \nu'$ vel $= 0$ vel $= +1$ vel $= -1$, exhaurire vellemus, formularum multitudo vsque ad 325 aſcenderet. Haud inutilis foret talis collectio, ſaltem ſimpliciorum ex his formulis: hoc vero loco ſufficiat, paucas tantummodo appoſuiſſe, quas vel ex formulis art. 7, vel ſi magis placet, ſimili modo vt duae priores ex illis in art. 8 erutae ſunt, quiuis nullo negotio ſibi demonſtrare poterit.

$$[16]\ F(\alpha, \beta, \gamma) - F(\alpha, \beta, \gamma - 1) = -\frac{\alpha\beta x}{\gamma(\gamma - 1)} F(\alpha + 1, \beta + 1, \gamma + 1)$$

$$[17]\ F(\alpha, \beta + 1, \gamma) - F(\alpha, \beta, \gamma) = \frac{\alpha x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1)$$

$$[18]\ F(\alpha + 1, \beta, \gamma) - F(\alpha, \beta, \gamma) = \frac{\beta x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1)$$

$$[19]\ F(\alpha, \beta + 1, \gamma + 1) - F(\alpha, \beta, \gamma) = \frac{\alpha(\gamma - \beta) x}{\gamma(\gamma + 1)} F(\alpha + 1, \beta + 1, \gamma + 2)$$

$$[20]\ F(\alpha + 1, \beta, \gamma + 1) - F(\alpha, \beta, \gamma) = \frac{\beta(\gamma - \alpha) x}{\gamma(\gamma + 1)} F(\alpha + 1, \beta + 1, \gamma + 2)$$

$$[21]\ F(\alpha - 1, \beta + 1, \gamma) - F(\alpha, \beta, \gamma) = \frac{(\alpha - \beta - 1) x}{\gamma} F(\alpha, \beta + 1, \gamma + 1)$$

$$[22]\ F(\alpha + 1, \beta - 1, \gamma) - F(\alpha, \beta, \gamma) = \frac{(\beta - \alpha - 1) x}{\gamma} F(\alpha + 1, \beta, \gamma + 1)$$

$$[23]\ F(\alpha - 1, \beta + 1, \gamma) - F(\alpha + 1, \beta - 1, \gamma) = \frac{(\alpha - \beta) x}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1)$$

SECTIO

SECTIO SECVNDA.

Fractiones continuae.

12.

Designando

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} \text{ per } G(\alpha, \beta, \gamma, x)$$

fit

$$\frac{F(\alpha+1, \beta, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = \frac{F(\beta, \alpha+1, \gamma+1, x)}{F(\beta, \alpha, \gamma, x)} = G(\beta, \alpha, \gamma, x)$$

et proin, diuidendo aequationem 19 per $F(\alpha, \beta+1, \gamma+1, x)$,

$$1 - \frac{1}{G(\alpha, \beta, \gamma, x)} = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x \; G(\beta+1, \alpha, \gamma+1, x)$$

siue

$$[24] \quad G(\alpha, \beta, \gamma, x) = \cfrac{1}{1 - \cfrac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x \; G(\beta+1, \alpha, \gamma+1, x)}$$

et quum perinde fiat

$$G(\beta+1, \alpha, \gamma+1, x) = \cfrac{1}{1 - \cfrac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)} x \; G(\alpha+1, \beta+1, \gamma+2, x)}$$

etc., resultabit pro $G(\alpha, \beta, \gamma, x)$ fractio continua

$$[25] \quad \frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{ax}{1 - \cfrac{bx}{1 - \cfrac{cx}{1 - \cfrac{dx}{1 - \text{etc.}}}}}}$$

vbi

$$a = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}, \quad b = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)}$$

$$c = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)}, \quad d = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)}$$

$e =$

$$e = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)}, \quad f = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)}$$

etc., cuius lex progressionis obuia est.

Porro ex aequationibus 17, 18, 21, 22 sequitur

[26] $$\frac{F(\alpha, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{\alpha x}{\gamma} G(\beta+1, \alpha, \gamma, x)}$$

[27] $$\frac{F(\alpha+1, \beta, \gamma, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{\beta x}{\gamma} G(\alpha+1, \beta, \gamma, x)}$$

[28] $$\frac{F(\alpha-1, \beta+1, \gamma, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{(\alpha-\beta-1)x}{\gamma} G(\beta+1, \alpha-1, \gamma, x)}$$

[29] $$\frac{F(\alpha+1, \beta-1, \gamma, x)}{F(\alpha, \beta, \gamma, x)} = \cfrac{1}{1 - \cfrac{(\beta-\alpha-1)x}{\gamma} G(\alpha+1, \beta-1, \gamma, x)}$$

vnde substitutis pro functione G eius valoribus in fractionibus continuis, totidem fractiones continuae nouae prodeunt.

Ceterum sponte patet, fractionem continuam in formula 25 abrumpi, si e numeris $\alpha, \beta, \gamma-\alpha, \gamma-\beta$ aliquis fuerit integer negatiuus, alioquin vero in infinitum excurrere.

13.

Fractiones continuae in art. praec. erutae maximi sunt momenti, asserique potest, vix vllas fractiones continuas secundum legem obviam progredientes ab analystis hactenus erutas esse, quae sub nostris tamquam casus speciales non sint contentae. Imprimis memorabilis est casus is, vbi in formula 25 statuitur $\beta = 0$, vnde $F(\alpha, \beta, \gamma, x) = 1$, adeoque, scribendo $\gamma - 1$ pro γ

[30] $$F(\alpha, 1, \gamma) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}xx + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

$$= \frac{1}{1-\alpha x}$$

$$= \cfrac{1}{1 - \cfrac{ax}{1 - \cfrac{bx}{1 - \cfrac{cx}{1 - \cfrac{dx}{1 - \text{etc.}}}}}}$$

vbi

$$a = \frac{\alpha}{\gamma}, \qquad b = \frac{\gamma - \alpha}{\gamma(\gamma + 1)}$$

$$c = \frac{(\alpha + 1)\gamma}{(\gamma + 1)(\gamma + 2)}, \qquad d = \frac{2(\gamma + 1 - \alpha)}{(\gamma + 2)(\gamma + 3)}$$

$$e = \frac{(\alpha + 2)(\gamma + 1)}{(\gamma + 3)(\gamma + 4)}, \qquad f = \frac{3(\gamma + 2 - \alpha)}{(\gamma + 4)(\gamma + 5)}$$

etc.

14.

Operae pretium erit, quosdam casus speciales huc adscripsisse. Ex formula I art. 5 sequitur, statuendo $t = 1$, $\beta = 1$

$$[31]\quad (1 + u)^n = \cfrac{1}{1 - \cfrac{nu}{1 + \cfrac{\frac{n+1}{2}u}{1 - \cfrac{\frac{n-1}{2.3}u}{1 + \cfrac{\frac{2(n+2)}{3.4}u}{1 - \frac{2(n-2)}{4.5}u \text{ etc.}}}}}}$$

E formulis VI, VII art. 5 sequitur

$$[32]\quad \log(1 + t) = \cfrac{t}{1 + \cfrac{\frac{1}{2}t}{1 + \cfrac{\frac{1}{6}t}{1 + \cfrac{\frac{2}{6}t}{1 + \cfrac{\frac{2}{10}t}{1 + \cfrac{\frac{3}{10}t}{1 + \frac{3}{14}t \text{ etc.}}}}}}}$$

[33]

[33] $$\log\frac{1+t}{1-t}=\cfrac{2t}{1-\cfrac{\frac{1}{3}tt}{1-\cfrac{\frac{2.2}{3.5}tt}{1-\cfrac{\frac{3.3}{5.7}tt}{1-\frac{4.4}{7.9}tt\quad\text{etc.}}}}}$$

Mutando hic ſigna $-$ in $+$ prodit fractio continua pro arc. tang t

Porro habemus

[34] $$e^t=\cfrac{1}{1-\cfrac{t}{1+\cfrac{\frac{1}{2}t}{1-\cfrac{\frac{1}{6}t}{1+\cfrac{\frac{1}{6}t}{1-\cfrac{\frac{1}{10}t}{1+\frac{1}{10}t\quad\text{etc.}}}}}}}$$

[35] $$t=\cfrac{\sin t\cos t}{1-\cfrac{\frac{1.2}{1.3}\sin t^2}{1-\cfrac{\frac{1.2}{3.5}\sin t^2}{1-\cfrac{\frac{3.4}{5.7}\sin t^2}{1-\cfrac{\frac{3.4}{7.9}\sin t^2}{1-\frac{5.6}{9.11}\sin t^2\quad\text{etc.}}}}}}$$

Statuendo $\alpha=3$, $\gamma=\frac{5}{2}$, e formula 30 ſponte ſequitur fractio continua in *Theoria motus corporum coeleſtium* p. 97 propoſita. Ibidem duae aliae fractiones continuae prolatae ſunt, quarum euolutionem hacce occaſione ſupplere viſum eſt. Statuendo

$Q=1-$

$$Q = 1 - \cfrac{\frac{5.8}{7.9}x}{1 - \cfrac{\frac{1.4}{9.11}x}{1 - \frac{7.10}{11.13}x \text{ etc.}}}$$

fit l. c. $x - \xi = \cfrac{x}{1 + \frac{2x}{35Q}} = \cfrac{xQ}{Q + \frac{2}{35}x}$, adeoque

$$\xi = \cfrac{\frac{2}{35}xx}{Q + \frac{2}{35}x}$$

quae est formula prior: posterior sequenti modo eruitur. Statuendo

$$R = 1 - \cfrac{\frac{1.4}{7.9}x}{1 - \cfrac{\frac{5.8}{9.11}x}{1 - \cfrac{\frac{3.6}{11.13}x}{1 - \frac{7.10}{13.15}x \text{ etc.}}}}$$

erit per formulam 25

$\frac{1}{R} = G(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x)$, atque $\frac{1}{Q} = G(\tfrac{5}{2}, -\tfrac{1}{2}, \tfrac{7}{2}, x)$

Hinc

$RF(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x)$

$QF(\tfrac{5}{2}, \tfrac{1}{2}, \tfrac{9}{2}, x) = F(\tfrac{5}{2}, -\tfrac{1}{2}, \tfrac{7}{2}, x)$

siue permutando elementum primum cum secundo

$QF(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x)$

Sed per aequationem 21 habemus

$F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x) - F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x) = -\tfrac{4}{7}x\, F(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x)$

vnde fit $Q = R - \frac{4}{7}x$, quo valore in formula supra data substituto prodit

$$\xi = \frac{\frac{2}{35}xx}{R - \frac{18}{35}x}$$

quae est formula posterior.

Statuendo in formula 30, $\alpha = \frac{m}{n}$, $x = -\gamma nt$, fit pro valore infinite magno ipsius γ

[36] $F(\frac{m}{n}, 1, \gamma, -\gamma nt) = 1 - mt + m(m+n)tt - m(m+n)(m+2n)t^3 +$ etc.

$$= \cfrac{1}{1 + \cfrac{mt}{1 + \cfrac{nt}{1 + \cfrac{(m+n)t}{1 + \cfrac{2nt}{1 + \cfrac{(m+2n)t}{1 + 3nt \text{ etc.}}}}}}}$$

SECTIO TERTIA.

De summa seriei nostrae statuendo elementum quartum = 1, vbi simul quaedam aliae functiones transscendentes discutiuntur.

15.

Quoties elementa α, β, γ omnia sunt quantitates positiuae, omnes coëfficientes potestatum elementi quarti x positiui euadunt: quoties vero ex illis elementis vnum alterumue negatiuum est, saltem inde ab aliqua potestate x^m omnes coëfficientes eodem signo affecti erunt, si modo m accipitur maior quam valor absolutus elementi negatiui maximi. Porro hinc sponte patet, seriei summam pro $x = 1$ finitam esse non posse, nisi coëfficientes saltem post certum terminum in infinitum decrescant, vel, vt secundum morem analystarum loquamur, nisi coëfficiens termini x^∞ sit $= 0$. Ostendemus autem, et quidem, in gratiam eorum qui methodis rigorosis antiquorum geometrarum fauent, omni rigore,

primo

primo, coëfficientes (siquidem series non abrumpatur), in infinitum crescere, quoties fuerit $\alpha+\beta-\gamma-1$ quantitas positiua.

secundo, coëfficientes versus limitem finitum continuo conuergere, quoties fuerit $\alpha+\beta-\gamma-1=0$.

tertio, coëfficientes in infinitum decrescere, quoties fuerit $\alpha+\beta-\gamma-1$ quantitas negatiua.

quarto, summam seriei nostrae pro $x=1$, non obstante conuergentia in casu tertio, infinitam esse, quoties fuerit $\alpha+\beta-\gamma$ quantitas positiua vel $=0$.

quinto, summam vero *finitam* esse, quoties $\alpha+\beta-\gamma$ fuerit quantitas negatiua.

16.

Hanc disquisitionem generalius adaptabimus seriei infinitae M, M', M'', M''' etc ita formatae, vt quotientes $\frac{M'}{M}$, $\frac{M''}{M'}$, $\frac{M'''}{M''}$ etc. resp. sint valores fractionis

$$\frac{t^\lambda+At^{\lambda-1}+Bt^{\lambda-2}+Ct^{\lambda-3}+\text{ etc.}}{t^\lambda+at^{\lambda-1}+bt^{\lambda-2}+ct^{\lambda-3}+\text{ etc.}}$$

pro $t=m$, $t=m+1$, $t=m+2$ etc. Breuitatis caussa huius fractionis numeratorem per P, denominatorem per p denotabimus: praeterea supponemus, P, p non esse identicas, siue differentias $A-a$, $B-b$, $C-c$ etc. non omnes simul euanescere.

I. Quoties e differentiis $A-a$, $B-b$, $C-c$ etc. prima quae non euanescit est positiua, assignari poterit limes aliquis l, quem simulac egressus est valor ipsius t, valores functionum P et p certo semper euadent positiui, atque $P>p$. Manifestum est, hoc euenire, si pro l accipiatur radix maxima realis aequationis $p(P-p)=0$; si vero haec aequatio nullas omnino radices reales habeat, proprietatem illam pro omnibus valoribus ipsius t locum habere. Quapropter in serie $\frac{M'}{M}$, $\frac{M''}{M'}$, $\frac{M'''}{M''}$ etc. saltem post certum interuallum (si non ab initio) omnes termini erunt positiui atque maiores vnitate; quodsi itaque nullus neque $=0$ neque infinite magnus euadit, perspicuum est,

seriem M, M', M'', M''' etc si non ab initio tamen post certum interuallum omnes suos terminos eodem signo affectos continuoque crescentes habituram esse.

Eadem ratione, si e differentiis $A-a$, $B-b$, $C-c$ etc. prima quae non euanescit est negatiua, series M, M', M'', M''' etc. si non ab initio tamen post certum interuallum omnes suos terminos eodem signo affectos continuoque decrescentes habebit.

II. Si iam coëfficientes A, a sunt inaequales, termini seriei M, M', M'', M''' etc. vltra omnes limites siue in infinitum vel crescent vel decrescent, prout differentia $A-a$ est positiua vel negatiua: hoc ita demonstramus. Si $A-a$ est quantitas positiua, accipiatur numerus integer h ita, vt fiat $h(A-a) > 1$, statuaturque $\frac{M^h}{m} = N$, $\frac{M'^h}{m+1} = N'$, $\frac{M''^h}{m+2} = N''$, $\frac{M'''^h}{m+3} = N'''$ etc., nec non $tP^h = Q$, $(t+1)p^h = q$. Tunc patet, $\frac{N'}{N}$, $\frac{N''}{N'}$, $\frac{N'''}{N''}$ etc. esse valores fractionis $\frac{Q}{q}$ ponendo $t=m$, $t=m+1$, $t=m+2$ etc., ipsas Q, q vero esse functiones algebraicas formae huius

$$Q = t^{\lambda h} + hAt^{\lambda h-1} + \text{etc.}$$
$$q = t^{\lambda h} + (ha+1)t^{\lambda h-1} + \text{etc.}$$

Quare quum per hyp. differentia $hA-(ha+1)$ sit quantitas positiua, termini seriei N, N', N'', N''' etc. si non ab initio tamen post certum interuallum continuo crescent (per I); hinc termini seriei $mN, (m+1)N'$, $(m+2)N''$, $(m+3)N'''$ etc. necessario vltra omnes limites crescent, et proin etiam termini seriei M, M', M'', M''' etc., quippe quorum potestates exponente h illis sunt aequales. *Q. E. P.*

Si $A-a$ est quantitas negatiua, accipere oportet integrum h ita, vt $h(a-A)$ fiat maior quam 1, vnde per ratiocinia similia termini seriei

$$mM^h,\ (m+1)M'^h,\ (m+2)M''^h,\ (m+3)M'''^h \text{ etc.}$$

post certum interuallum continuo decrescent. Quamobrem termini seriei M^h, M'^h, M''^h etc. adeoque etiam termini huius M, M', M'', M''' etc. necessario in infinitum decrescent. *Q. E. S.*

III. Si vero coëfficientes primi A, a sunt aequales, termini seriei M, M', M'', M''' etc. versus limitem finitum continuo conuergent, quod ita demonstramus. Supponamus primo, terminos seriei post certum interuallum continuo crescere, siue e differentiis $B-b$, $C-c$ etc. primam quae non euanescat esse positiuam. Sit h integer talis vt $h+b-B$ fiat quantitas positiua, statuamusque

M

$$M\left(\frac{m}{m-1}\right)^h = N,\ M'\left(\frac{m+1}{m}\right)^h = N',\ M''\left(\frac{m+2}{m+1}\right)^h = N'' \text{ etc.}$$

atque $(tt-1)^h P = Q$, $t^{2h} p = q$, ita vt $\frac{N'}{N}$, $\frac{N''}{N'}$ etc. sint valores fractionis $\frac{Q}{q}$ ponendo $t = m$, $t = m+1$ etc. Quum itaque habeatur

$$Q = t^{\lambda+2h} + At^{\lambda+2h-1} + (B-h)\,t^{\lambda+2h-2} \text{ etc.}$$
$$q = t^{\lambda+2h} + At^{\lambda+2h-1} + bt^{\lambda+2h-2} \text{ etc.}$$

atque per hyp. $B-h-b$ sit quantitas negatiua, termini seriei N, N', N'', N''' etc. post certum saltem interuallum continuo decrescent, adeoque termini seriei M, M', M'', M''' etc., qui illis resp. semper sunt minores, dum continuo crescunt, tantummodo versus limitem finitum conuergere possunt. *Q. E. P.*

Si termini seriei M, M', M'', M''' etc. post certum interuallum continuo decrescunt, accipere oportet pro h integrum talem, vt $h+B-b$ sit quantitas positiua, euinceturque per ratiocinia prorsus similia, terminos seriei

$$M\left(\frac{m-1}{m}\right)^h,\quad M'\left(\frac{m}{m+1}\right)^h,\quad M''\left(\frac{m+1}{m+2}\right)^h \quad \text{etc.}$$

post certum interuallum continuo crescere, vnde termini seriei M, M', M'' etc., qui illis resp. semper sunt maiores, dum continuo decrescunt, necessario tantummodo versus limitem finitum decrescere possunt. *Q. E. S.*

IV. Denique quod attinet ad *summam* seriei, cuius termini sunt M, M', M'', M''' etc., in casu eo vbi hi in infinitum decrescunt, supponamus primo $A-a$ cadere inter o et -1, siue $A+1-a$ esse vel quantitatem positiuam vel $=0$. Sit h integer positiuus, arbitrarius in casu eo vbi $A+1-a$ est quantitas positiua, vel talis qui reddat quantitatem $h+m+A+B-b$ positiuam in casu eo vbi $A+1-a=0$. Tunc erit

$$P(t-(m+h-1)) = t^{\lambda+1} + (A+1-m-h)t^{\lambda} + (B-A(m+h-1))\,t^{\lambda-1} \text{ etc.}$$
$$p(t-(m+h)) = t^{\lambda+1} + (a-m-h)t^{\lambda} + (b-a(m+h))\,t^{\lambda-1} \text{ etc.}$$

vbi vel $A+1-m-h-(a-m-h)$ erit quantitas positiua, vel, si haec sit $=0$, saltem $B-A(m+h-1)-(b-a(m+h))$ positiua erit. Hinc (per I) pro quantitate t assignari poterit valor aliquis l, quem simulac transgressa est, valores fractionis $\frac{P(t-(m+h-1))}{p(t-(m+h))}$

semper

ſemper fient poſitiui atque vnitate maiores. Sit n integer maior quam l ſimulque maior quam h, ſintque termini ſeriei M, M', M'', M''' etc., qui reſpondent valoribus $t=m+n$, $t=m+n+1$, $t=m+n+2$ etc. hi N, N', N'', N''' etc. Erunt itaque $\frac{(n+1-h)N'}{(n-h)N}$, $\frac{(n+2-h)N''}{(n+1-h)N'}$, $\frac{(n+3-h)N'''}{(n+2-h)N''}$ etc. quantitates poſitiuae vnitate maiores, vnde $N' > \frac{(n-h)N}{n+1-h}$, $N'' > \frac{(n-h)N}{n+2-h}$, $N''' > \frac{(n-h)N}{n+3-h}$ etc., adeoque ſumma ſeriei $N+N'+N''+N'''+$ etc. maior ſumma ſeriei

$$(n-h)N\left(\frac{1}{n-h}+\frac{1}{n+1-h}+\frac{1}{n+2-h}+\frac{1}{n+3-h}+\text{etc.}\right)$$

quotcunque termini colligantur. Sed poſterior ſeries, creſcente terminorum numero in infinitum, omnes limites egreditur, quum ſumma ſeriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ etc. quam infinitam eſſe conſtat etiam infinita maneat, ſi ab initio termini $1+\frac{1}{2}+\frac{1}{3}+$ etc. $+\frac{1}{n-1-h}$ reſcindantur. Quare ſumma ſeriei $N+N'+N''+N'''+$ etc., adeoque etiam ſumma huius $M+M'+M''+M'''+$ etc., cuius pars eſt illa, vltra omnes limites creſcit.

V. Quoties autem $A-a$ eſt quantitas negatiua abſolute maior quam vnitas, ſumma ſeriei $M+M'+M''+M'''+$ etc. in infinitum continuatae certo erit finita. Sit enim h quantitas poſitiua minor quam $a-A-1$, demonſtrabiturque per ratiocinia ſimilia, aſſignari poſſe valorem aliquem l quantitatis t, vltra quem fractio $\frac{Pt}{p(t-h-1)}$ ſemper adipiſcatur valores poſitiuos vnitate minores. Quodſi iam pro n accipitur numerus integer ipſis l, m, $h+1$ maior, terminique ſeriei M, M', M'', M''' etc., valoribus $t=n$, $t=n+1$, $t=n+2$ etc. reſpondentes, deſignantur per N, N', N'' etc., erit $N' < \frac{n-h-1}{n}.N$, $N'' < \frac{(n-h-1)(n-h)}{n(n+1)}.N$ etc., adeoque ſumma ſeriei $N+N'+N''+$ etc. quotcunque termini colligantur, minor producto ex N in ſummam totidem terminorum ſeriei

$$1+\frac{n-h-1}{n}+\frac{(n-h-1)(n-h)}{n(n+1)}+\frac{(n-h-1)(n-h)(n-h+1)}{n(n+1)(n+2)}$$

etc.

Huius

Huius vero ſumma pro quolibet terminorum numero facile aſſignari poteſt; eſt ſcilicet

terminus primus $= \frac{n-1}{h} - \frac{n-h-1}{h}$

ſumma duorum terminorum $= \frac{n-1}{h} - \frac{(n-h-1)(n-h)}{hn}$

ſumma trium terminorum $= \frac{n-1}{h} - \frac{(n-h-1)(n-h)(n-h+1)}{hn(n+1)}$

etc. et quum pars altera (per II) formet ſeriem vltra omnes limites decreſcentem, ſumma illa in infinitum continuata ſtatui debet $= \frac{n-1}{h}$. Hinc $N + N' + N''$ etc. in infinitum continuata ſemper manebit minor quam $\frac{N(n-1)}{h}$, et proin $M + M' + M''$ etc. certo ad ſummam finitam conuerget. Q. E. D.

VI. Vt ea, quae generaliter de ſerie M, M', M'' etc. demonſtravimus, ad coëfficientes poteſtatum x^m, x^{m+1}, x^{m+2} etc. in ſerie $F(\alpha, \beta, \gamma, x)$, applicentur, ſtatuere oportebit $\lambda = 2$, $A = \alpha + \beta$, $B = \alpha\beta$, $a = \gamma + 1$, $b = \gamma$, vnde quinque aſſertiones in art. praec. propoſitae ſponte demanant.

17.

Disquiſitio itaque de indole ſummae ſeriei $F(\alpha, \beta, \gamma, 1)$ natura ſua reſtringitur ad caſum, quo $\gamma - \alpha - \beta$ eſt quantitas poſitiua, vbi illa ſumma ſemper exhibet quantitatem finitam. Praemittimus autem obſeruationem ſequentem. Si coëfficientes ſeriei $1 + ax + bxx + cx^3 +$ etc. $= S$ inde a certo termino vltra omnes limites decreſcunt, productum

$$(1-x)S = 1 + (a-1)x + (b-a)xx + (c-b)x^3 + \text{etc.}$$

pro $x = 1$ ſtatuere oportet $= 0$, etiamſi ſumma ipſius ſeriei S infinite magna euadat. Quoniam enim collectis duobus terminis ſumma fit $= a$, collectis tribus $= b$, collectis quatuor $= c$ etc., limes ſummae in infinitum continuatae eſt $= 0$. Quoties itaque $\gamma - \alpha - \beta$ eſt quantitas poſitiua, ſtatuere oportet $(1-x) F(\alpha, \beta, \gamma - 1, x) = 0$ pro $x = 1$, vnde per aequationem 15 art. 7 habebimus

$$0 = \gamma(\alpha + \beta - \gamma) F(\alpha, \beta, \gamma, 1) + (\gamma - \alpha)(\gamma - \beta) F(\alpha, \beta, \gamma + 1, 1), \text{ ſiue}$$

$$[37] \quad F(\alpha,\beta,\gamma,1) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)} F(\alpha,\beta,\gamma+1,1)$$

Quare quum perinde fiat

$$F(\alpha,\beta,\gamma+1,1) = \frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)} F(\alpha,\beta,\gamma+2,1)$$

$$F(\alpha,\beta,\gamma+2,1) = \frac{(\gamma+2-\alpha)(\gamma+2-\beta)}{(\gamma+2)(\gamma+2-\alpha-\beta)} F(\alpha,\beta,\gamma+3,1)$$

et sic porro, erit generaliter, k denotante integrum positiuum quemcunque

$F(\alpha,\beta,\gamma,1)$ aequalis producto ex $F(\alpha,\beta,\gamma+k,1)$
in $(\gamma-\alpha)(\gamma+1-\alpha)(\gamma+2-\alpha)\ldots(\gamma+k-1-\alpha)$
in $(\gamma-\beta)(\gamma+1-\beta)(\gamma+2-\beta)\ldots(\gamma+k-1-\beta)$
diuiso per productum
ex $\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+k-1)$
in $(\gamma-\alpha-\beta)(\gamma+1-\alpha-\beta)(\gamma+2-\alpha-\beta)\ldots(\gamma+k-1-\alpha-\beta)$

18.

Introducamus abhinc sequentem notationem:

$$[38] \quad \Pi(k,z) = \frac{1 \cdot 2 \cdot 3 \ldots k}{(z+1)(z+2)(z+3)\ldots(z+k)} k^z$$

vbi k natura sua subintelligitur designare integrum positiuum, qua restrictione $\Pi(k,z)$ exhibet functionem duarum quantitatum k, z prorsus determinatam. Hoc modo facile intelligetur, theorema in fine art. praec. propositum ita exhiberi posse

$$[39] \quad F(\alpha,\beta,\gamma,1) = \frac{\Pi(k,\gamma-1).\,\Pi(k,\gamma-\alpha-\beta-1)}{\Pi(k,\gamma-\alpha-1).\,\Pi(k,\gamma-\beta-1)} \cdot F(\alpha,\beta,\gamma+k,1)$$

19.

Operae pretium erit, indolem functionis $\Pi(k,z)$ accuratius perpendere. Quoties z est integer negatiuus, functio manifesto valorem infinite magnum obtinet, simulac ipsi k tribuitur valor satis magnus. Pro valoribus integris ipsius z non negatiuis autem habemus

$$\Pi(k,0) = 1$$

$$\Pi(k,1) = \frac{1}{1+\frac{1}{k}}$$

$\Pi(k,$

$$\Pi(k,2) = \frac{1 \cdot 2}{(1+\frac{1}{k})(1+\frac{2}{k})}$$

$$\Pi(k,3) = \frac{1 \cdot 2 \cdot 3}{(1+\frac{1}{k})(1+\frac{2}{k})(1+\frac{3}{k})}$$

etc. siue generaliter

[40] $$\Pi(k,z) = \frac{1 \cdot 2 \cdot 3 \ldots\ldots z}{(1+\frac{1}{k})(1+\frac{2}{k})(1+\frac{3}{k})\ldots(1+\frac{z}{k})}$$

Generaliter autem pro *quouis* valore ipsius z habemus

[41] $$\Pi(k,z+1) = \Pi(k,z) \cdot \frac{1+z}{1+\frac{1+z}{k}}$$

[42] $$\Pi(k+1,z) = \Pi(k,z) \cdot \left\{\frac{(1+\frac{1}{k})^{z+1}}{1+\frac{1+z}{k}}\right\}$$

adeoque, quum $\Pi(1,z) = \frac{1}{z+1}$,

[43] $$\Pi(k,z) = \frac{1}{z+1} \cdot \frac{2^{z+1}}{1^z \cdot (2+z)} \cdot \frac{3^{z+1}}{2^z(3+z)} \cdot \frac{4^{z+1}}{3^z(4+z)} \cdots \frac{k^{z+1}}{(k-1)^z(k+z)}$$

20.

Imprimis vero attentione dignus est *limes*, ad quem pro valore dato ipsius z functio $\Pi(k,z)$ continuo conuerget, dum k in infinitum crescit. Sit primo h valor finitus ipsius k maior quam z, patetque, si k transire supponatur ex h in $h+1$, logarithmum ipsius $\Pi(k,z)$ accipere incrementum, quod per seriem conuergentem sequentem exprimatur

$$\frac{z(1+z)}{2(h+1)^2} + \frac{z(1-zz)}{3(h+1)^3} + \frac{z(1+z^3)}{4(h+1)^4} + \frac{z(1-z^4)}{5(h+1)^5} + \text{etc.}$$

Si itaque k e valore h transit in $h+n$, logarithmus ipsius $\Pi(k,z)$ accipiet incrementum

 $\frac{1}{2}z(1+z)$

$$\tfrac{1}{2}z(1+z)\left(\frac{1}{(h+1)^2}+\frac{1}{(h+2)^2}+\frac{1}{(h+3)^2}+\text{etc.}+\frac{1}{(h+n)^2}\right)$$
$$+\tfrac{1}{3}z(1-zz)\left(\frac{1}{(h+1)^3}+\frac{1}{(h+2)^3}+\frac{1}{(h+3)^3}+\text{etc.}+\frac{1}{(h+n)^3}\right)$$
$$+\tfrac{1}{4}z(1+z^3)\left(\frac{1}{(h+1)^4}+\frac{1}{(h+2)^4}+\frac{1}{(h+3)^4}+\text{etc.}+\frac{1}{(h+n)^4}\right)$$
$$+\text{ etc.}$$

quod ſemper finitum manere, etiamſi n in infinitum creſcat, facile demonſtrari poteſt. Quare niſi iam in $\Pi(h, z)$ factor infinitus affuerit, i. e. niſi z ſit numerus integer negatiuus, limes ipſius $\Pi(k, z)$ pro $k=\infty$ certo erit quantitas finita. Manifeſto itaque $\Pi(\infty, z)$ tantummodo a z pendet, ſiue functionem ipſius z ex aſſe determinatam exhibet, quae abhinc ſimpliciter per Πz denotabitur. Definimus itaque functionem Πz per valorem producti

$$\frac{1\,.\,2\,.\,3\,\ldots\ldots k\,.\,k^z}{(z+1)(z+2)(z+3)\ldots\ldots(z+k)}$$

pro $k=\infty$. aut ſi mauis per limitem producti infiniti

$$\frac{1}{z+1}\cdot\frac{2^{z+1}}{1^z(2+z)}\cdot\frac{3^{z+1}}{2^z(3+z)}\cdot\frac{4^{z+1}}{3^z(4+z)}\text{ etc.}$$

21.

Ex aequatione 41 ſtatim ſequitur aequatio fundamentalis

[44] $\Pi(z+1)=(z+1)\Pi z$

vnde generaliter, deſignante n integrum poſitiuum quemcunque

[45] $\Pi(z+n)=(z+1)(z+2)(z+3)\ldots\ldots(z+n)\Pi z$

Pro valore integro negatiuo ipſius z erit valor functionis Πz infinite magnus; pro valoribus integris non negatiuis habemus

$\Pi 0=1$
$\Pi 1=1$
$\Pi 2=2$
$\Pi 3=6$
$\Pi 4=24$ etc.

atque generaliter

[46] $\Pi z=1.2.3\ldots\ldots z$

Sed male haec proprietas functionis noſtrae tamquam ipſius definitio venditaretur, quippe quae natura ſua ad valores integros reſtringitur, et praeter functionem noſtram infinitis aliis (e. g. cos $2\pi z$. Πz, cos

cos πz^{2n} Πz etc., denotante π semiperipheriam circuli cuius radius $= 1$) communis est.

22.

Functio $\Pi(k, z)$, etiamsi generalior videatur quam Πz, tamen abhinc nobis superflua erit, quum facile ad posteriorem reducatur. Colligitur enim e combinatione aequationum 38, 45, 46

$$[47] \quad \Pi(k, z) = \frac{k^z \, \Pi k . \Pi z}{\Pi(k + z)}$$

Ceterum nexus harum functionum cum iis quas clar. Kramp *facultates numericas* nominauit per se obuius est. Scilicet facultas numerica quam hic auctor per $a^{b|c}$ designat, in signis nostris est

$$= \frac{c^b \, b^{\frac{a}{c} - 1} \, \Pi b}{\Pi(b, \frac{a}{c} - 1)} = \frac{c^b \, \Pi(\frac{a}{c} + b - 1)}{\Pi(\frac{a}{c} - 1)},$$

Sed consultius videtur, functionem *unius* variabilis in analysin introducere, quam functionem trium variabilium, praesertim quum hanc ad illam reducere liceat.

23.

Continuitas functionis Πz interrumpitur, quoties ipsius valor fit infinite magnus, i. e. pro valoribus integris negatiuis ipsius z. Erit itaque illa positiua a $z = -1$ vsque ad $z = \infty$, et quum pro vtroque limite Πz obtineat valorem infinite magnum, inter ipsos dabitur valor minimus, quem esse $= 0{,}8856024$ atque respondere valori $z = 0{,}4616321$ inuenimus. Inter limites $z = -1$ et $z = -2$, valor functionis Πz fit negatiuus, inter $z = -2$ atque $z = -3$ iterum positiuus et sic porro, vti ex aequ. 44 sponte sequitur. Porro patet, si omnes valores functionis Πz inter limites arbitrarios vnitate differentes e. g. a $z = 0$ vsque ad $z = 1$ pro notis habere liceat, valorem functionis pro quouis alio valore reali ipsius z adiumento aequationis 45 facile inde deduci posse. Ad hunc finem construximus *tabulam*, ad calcem huius sectionis annexam, quae ad figuras viginti exhibet logarithmos briggicos functionis Πz, pro $z = 0$ vsque ad $z = 1$ per singulas partes centesimas summa cura computatos, ybi tamen monendum, figuram vltimam vigesimam interdum vna duabusue vnitatibus erroneam esse posse.

24.

Quum limes functionis $F(\alpha, \beta, \gamma + k, 1)$, crescente k in infinitum, manifesto sit vnitas, aequatio 39 transit in hanc

[48] $F(\alpha, \beta, \gamma, 1) = \frac{\Pi(\gamma - 1) \,.\, \Pi(\gamma - \alpha - \beta - 1)}{\Pi(\gamma - \alpha - 1) \,.\, \Pi(\gamma - \beta - 1)}$

quae formula exhibet solutionem completam quaestionis, quae obiectum huius sectionis constituit. Sponte hinc sequuntur aequationes elegantes:

[49] $F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma - \alpha - \beta, 1)$

[50] $F(\alpha, \beta, \gamma, 1) \,.\, F(-\alpha, \beta, \gamma - \alpha, 1) = 1$

[51] $F(\alpha, \beta, \gamma, 1) \,.\, F(\alpha, -\beta, \gamma - \beta, 1) = 1$

in quarum prima γ, in secunda $\gamma - \beta$, in tertia $\gamma - \alpha$ debet esse quantitas positiua.

25.

Applicemus formulam 48 ad quasdam ex aequationibus art. 5. Formula XIII, statuendo $t = 90° = \frac{1}{2}\pi$, fit $\frac{1}{2}\pi = F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1)$, siue aequiualet aequationi notae

$$\frac{1}{2}\pi = 1 + \frac{1 \,.\, 1}{2 \,.\, 3} + \frac{1 \,.\, 1 \,.\, 3}{2 \,.\, 4 \,.\, 5} + \frac{1 \,.\, 1 \,.\, 3 \,.\, 5}{2 \,.\, 4 \,.\, 6 \,.\, 7} + \text{etc.}$$

Quare quum per formulam 48 habeatur $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1) = \frac{\Pi\frac{1}{2} \,.\, \Pi(-\frac{1}{2})}{\Pi 0 \,.\, \Pi 0}$, atque sit $\Pi 0 = 1$, $\Pi\frac{1}{2} = \frac{1}{2}\Pi(-\frac{1}{2})$, fit $\pi = (\Pi(-\frac{1}{2}))^2$ siue

[52] $\Pi(-\frac{1}{2}) = \sqrt{\pi}$

[53] $\Pi\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

Formula XVI art. 5, quae aequiualet aequationi notae

$$\sin nt = n \sin t - \frac{n(nn-1)}{2 \,.\, 3}\sin t^3 + \frac{n(nn-1)(nn-9)}{2 \,.\, 3 \,.\, 4 \,.\, 5}\sin t^5 - \text{etc.}$$

atque generaliter pro quouis valore ipsius n locum habet, si modo t limites $-90°$ et $+90°$ non transgrediatur, dat pro $t = \frac{1}{2}\pi$,

$$\sin\frac{n\pi}{2} = \frac{n\Pi\frac{1}{2} \,.\, \Pi(-\frac{1}{2})}{\Pi(-\frac{1}{2}n) \,.\, \Pi\frac{1}{2}n}$$

vnde deducitur formula elegans

$$\Pi\tfrac{1}{2}n \,.\, \Pi(-\tfrac{1}{2}n) = \frac{\frac{1}{2}n\pi}{\sin\frac{1}{2}n\pi},$$ siue statuendo $n = 2z$

[54

[54] $\Pi(-z) \,.\, \Pi(+z) = \dfrac{z\pi}{\sin z\pi}$

[55] $\Pi(-z) \,.\, \Pi(z-1) = \dfrac{\pi}{\sin z\pi}$

nec non scribendo $z + \frac{1}{2}$ pro z

[56] $\Pi(-\frac{1}{2}+z) \,.\, \Pi(-\frac{1}{2}-z) = \dfrac{\pi}{\cos z\pi}$

E combinatione formulae 54 cum definitione functionis Π sequitur, $\dfrac{z\pi}{\sin z\pi}$ esse limitem producti infiniti

$$\frac{(1\,.\,2\,.\,3\,.\,4\,\ldots\,k)^2}{(1-zz)\,(4-zz)\,(9-zz)\,\ldots\,(kk-zz)}$$

crescente k in infinitum, adeoque

$$\sin z\pi = z\pi\,(1-zz)\,(1-\frac{zz}{4})\,(1-\frac{zz}{9}) \text{ etc. in inf.}$$

similique modo ex 56 deducitur

$$\cos z\pi = (1-4zz)\,(1-\frac{4zz}{9})\,(1-\frac{4zz}{25}) \text{ etc. in inf.}$$

formulae notissimae, quae ab analystis per methodos prorsus diuersas erui solent.

26.

Designante n numerum integrum, valor expressionis

$$\frac{n^{nz}\,\Pi(k,z)\,.\,\Pi(k,z-\frac{1}{n})\,.\,\Pi(k,z-\frac{2}{n})\,\ldots\,\Pi(k,z-\frac{n-1}{n})}{\Pi(nk,nz)}$$

rite collectus inuenitur

$$= \frac{(1\,.\,2\,.\,3\,\ldots\,k)^n\, n^{nk}}{1\,.\,2\,.\,3\,\ldots\,nk\,.\,k^{\frac{1}{2}(n-1)}}$$

adeoque a z est independens, siue idem manebit, quicunque valor ipsi z tribuatur. Exhiberi poterit itaque, quoniam $\Pi(k,0) = \Pi(nk,0) = 1$, per productum

$$\Pi(k,-\frac{1}{n})\,.\,\Pi(k,-\frac{2}{n})\,.\,\Pi(k,-\frac{3}{n})\,\ldots\,\Pi(k,-\frac{n-1}{n})$$

Crescente igitur k in infinitum, nanciscimur

$$\frac{n^{nz}\,\Pi z.\,\Pi(z-\frac{1}{n})\,\Pi(z-\frac{2}{n})\ldots\,\Pi(z-\frac{n-1}{n})}{\Pi nz} =$$

$$\Pi(-\frac{1}{n}).\Pi(-\frac{2}{n}).\Pi(-\frac{3}{n})\ldots\Pi(-\frac{n-1}{n})$$

Productum ad dextram, in se ipsum ordine factorum inuerso multiplicatum, producit, per form. 55,

$$\frac{\pi}{\sin\frac{1}{n}\pi}\cdot\frac{\pi}{\sin\frac{2}{n}\pi}\cdot\frac{\pi}{\sin\frac{3}{n}\pi}\ldots\frac{\pi}{\sin\frac{n-1}{n}\pi} = \frac{(2\pi)^{n-1}}{n})$$

Vnde habemus theorema elegans

$$[57]\quad \frac{n^{nz}\,\Pi z.\,\Pi(z-\frac{1}{n}).\,\Pi(z-\frac{2}{n})\ldots\Pi(z-\frac{n-1}{n})}{\Pi nz} = \frac{(2\pi)^{\frac{1}{2}(n-1)}}{\sqrt{n}}$$

27.

Integrale $\int x^{\lambda-1}(1-x^{\mu})^{\nu}\,dx$, ita acceptum vt euanescat pro $x=0$, exprimitur per seriem sequentem, siquidem λ, μ sunt quantitates positiuae:

$$\frac{x^{\lambda}}{\lambda}-\frac{\nu x^{\mu+\lambda}}{\mu+\lambda}+\frac{\nu(\nu-1)x^{2\mu+\lambda}}{1.2.(2\mu+\lambda)}-\text{etc.}=\frac{x^{\lambda}}{\lambda}F(-\nu,\frac{\lambda}{\mu},\frac{\lambda}{\mu}+1,x^{\mu})$$

Hinc ipsius valor pro $x=1$ erit

$$=\frac{\Pi\frac{\lambda}{\mu}.\,\Pi\nu}{\lambda\Pi(\frac{\lambda}{\mu}+\nu)}.$$

Ex hoc theoremate omnes relationes, quas ill. Euler olim multo labore euoluit, sponte demanant. Ita e g. statuendo

$$\int\frac{dx}{\sqrt{(1-x^4)}}=A,\quad \int\frac{xx\,dx}{\sqrt{(1-x^4)}}=B$$

erit $A=\dfrac{\Pi\frac{1}{4}.\Pi(-\frac{1}{2})}{\Pi(-\frac{1}{4})}$, $B=\dfrac{\Pi\frac{3}{4}\,\Pi(-\frac{1}{2})}{3\Pi\frac{1}{4}}=\dfrac{\Pi(-\frac{1}{4}).\,\Pi(-\frac{1}{2})}{4\Pi\frac{1}{4}}$, adeoque $AB=\frac{1}{4}\pi$. Simul hinc sequitur, quoniam $\Pi\frac{1}{4}.\,\Pi(-\frac{1}{4})=\dfrac{\frac{1}{4}\pi}{\sin\frac{1}{4}\pi}=\dfrac{\pi}{\sqrt{8}}$, $\Pi\frac{1}{4}=$

$$\Pi\tfrac{1}{4} = \sqrt[4]{(\tfrac{1}{8}\pi AA)} = \sqrt[4]{\frac{\pi^3}{128\,BB}},\ \Pi(-\tfrac{1}{4}) = \sqrt[4]{\frac{\pi^3}{8AA}} = \sqrt[4]{(2\pi BB)}$$

Valor numericus ipsius A, computante Stirling, habetur $=$ 1,3110287771 4605987, valor ipsius B, secundum eundem auctorem, $=$ 0,5990701173 6779611; ex nostro calculo, artificio peculiari innixo, $=$ 0,5990701173 6779610372

Generaliter facile ostendi potest, valorem functionis Πz, si z sit quantitas rationalis $= \frac{m}{\mu}$, denotantibus m, μ integros, ex $\mu - 1$ valoribus determinatis talium integralium pro $x = 1$ deduci posse, et quidem permultis modis diuersis. Accipiendo enim pro λ numerum integrum atque pro ν fractionem cuius denominator $= \mu$, valor illius integralis semper reducitur ad tres Πz, vbi z est fractio cum denominatore $= \mu$; quoduis vero huiusmodi Πz vel ad $\Pi(-\frac{1}{\mu})$, vel ad $\Pi(-\frac{2}{\mu})$, vel ad $\Pi(-\frac{3}{\mu})$ etc. vel ad $\Pi(-\frac{\mu-1}{\mu})$ reduci potest per formulam 45, siquidem z reuera est fractio; si enim z est integer, Πz per se constat. Ex illis vero integralium valoribus, generaliter loquendo, quoduis $\Pi(-\frac{m}{\mu})$, si $m < \mu$, per eliminationem erui potest *). Quin adeo semissis talium integralium sufficiet, si formulam 54 simul in auxilium vocamus. Ita e. g. statuendo $\int\frac{dx}{\sqrt[5]{(1-x^5)}} = C$, $\int\frac{dx}{\sqrt[5]{(1-x^5)^2}} = D$, $\int\frac{dx}{\sqrt[5]{(1-x^5)^3}} = E$, $\int\frac{dx}{\sqrt[5]{(1-x^5)^4}} = F$, erit

$$C = \Pi\tfrac{1}{5}\cdot\Pi(-\tfrac{1}{5}),\quad D = \frac{\Pi\frac{1}{5}\cdot\Pi(-\frac{2}{5})}{\Pi(-\frac{1}{5})},\quad E = \frac{\Pi\frac{1}{5}\cdot\Pi(-\frac{3}{5})}{\Pi(-\frac{2}{5})},$$

$$F = \frac{\Pi\frac{1}{5}\cdot\Pi(-\frac{4}{5})}{\Pi(-\frac{3}{5})}$$

Hinc propter $\Pi\frac{1}{5} = \frac{1}{5}\ \Pi(-\frac{4}{5})$, habemus

$$\Pi(-\tfrac{1}{5}) = \sqrt[5]{\frac{5C^4}{DEF}},\ \Pi(-\tfrac{2}{5}) = \sqrt[5]{\frac{25\,C^3D^3}{EEFF}},$$

$$\Pi(-\tfrac{3}{5}) = \sqrt[5]{\frac{125\,CCDDEE}{F^3}},\ \Pi(-\tfrac{4}{5}) = \sqrt[5]{(625\,CDEF)}.$$

Formula

*) Haec eliminatio, si pro quantitatibus ipsis logarithmos introducimus, aequationibus tantummodo linearibus applicanda erit.

Formulae 54, 55 adhuc suppeditant

$$C = \frac{\pi}{5 \sin \frac{1}{5}\pi}, \quad \frac{D}{F} = \frac{\sin \frac{1}{5}\pi}{\sin \frac{2}{5}\pi}$$

ita vt duo integralia D, E, vel E et F sufficiant, ad omnes valores $\Pi(-\frac{1}{5})$, $\Pi(-\frac{2}{5})$ etc. computandos.

28.

Statuendo $y = \nu x$, atque $\mu = 1$, $\frac{\Pi\lambda \,.\, \Pi\nu}{\lambda \Pi(\lambda+\nu)}$ erit valor integralis $\int \frac{y^{\lambda-1}(1-\frac{y}{\nu})^{\nu} dy}{\nu^{\lambda}}$ ab $y = 0$ vsque ad $y = \nu$, siue valor integralis $\int y^{\lambda-1}(1-\frac{y}{\nu})^{\nu} dy$ inter eosdem limites $= \frac{\nu^{\lambda}\Pi\lambda \,.\, \Pi\nu}{\lambda\Pi(\lambda+\nu)} = \frac{\Pi(\nu,\lambda)}{\lambda}$ (form. 47), siquidem ν denotet integrum. Iam crescente ν in infinitum, limes ipsius $\Pi(\nu,\lambda)$ erit $= \Pi\lambda$, limes ipsius $(1-\frac{y}{\nu})^{\nu}$ autem e^{-y}, denotante e basin logarithmorum hyperbolicorum. Quamobrem si λ est positiua, $\frac{\Pi\lambda}{\lambda}$ siue $\Pi(\lambda-1)$ exprimet integrale $\int y^{\lambda-1} e^{-y} dy$ ab $y = 0$ vsque ad $y = \infty$, siue scribendo λ pro $\lambda - 1$, $\Pi\lambda$ est valor integralis $\int y^{\lambda} e^{-y} dy$ ab $y = 0$ vsque ad $y = \infty$, si $\lambda + 1$ est quantitas positiua.

Generalius statuendo $y = z^{\alpha}$, $\alpha\lambda + \alpha - 1 = \beta$, transit $\int y^{\lambda} e^{-y} dy$ in $\int \alpha z^{\beta} \cdot e^{-z^{\alpha}} dz$, quod itaque inter limites $z = 0$ atque $z = \infty$ sumtum, exprimetur per $\Pi(\frac{\beta+1}{\alpha} - 1)$ siue

Valor integralis $\int z^{\beta} e^{-z^{\alpha}} dz$, a $z = 0$ vsque ad $z = \infty$ fit $= \frac{\Pi(\frac{\beta+1}{\alpha} - 1)}{\alpha} = \frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$ si modo α atque $\beta + 1$ sunt quantitates positiuae (si vtraque est negatiua, integrale per $-\frac{\Pi\frac{\beta+1}{\alpha}}{\beta+1}$ exprimetur.

tur). Ita e. g. pro $\beta = 0$, $\alpha = 2$, valor integralis $\int e^{-zz} dz$ inuenitur $= \Pi \frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

29.

III. Euler pro summa logarithmorum $\log 1 + \log 2 + \log 3 +$ etc. $+ \log z$ eruit seriem $(z + \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} -$ etc.
vbi $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$ etc. sunt numeri Bernoulliani. Per hanc itaque seriem exprimitur $\log \Pi z$; etiamsi enim primo aspectu haec conclusio ad valores integros restricta videatur, tamen rem propius contemplando inuenietur, euolutionem ab Eulero adhibitam (Instit Calc. Diff pag. 466) saltem ad valores positiuos fractos eodem iure applicari posse, quo ad integros: supponit enim tantummodo, functionem ipsius z, in seriem euoluendam, esse talem, vt ipsius diminutio, si z transeat in $z - 1$, exhiberi possit per theorema Taylori, simulque vt eadem diminutio sit $= \log z$. Conditio prior innititur *continuitati* functionis, adeoque locum non habet pro valoribus negatiuis ipsius z, ad quos proin seriem illam extendere non licet: conditio posterior autem functioni $\log \Pi z$ generaliter competit sine restrictione ad valores integros ipsius z. Statuemus itaque

[58] $$\log \Pi z = (z + \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2z} - \frac{\mathfrak{B}}{3.4z^3} + \frac{\mathfrak{C}}{5.6z^5} - \frac{\mathfrak{D}}{7.8z^7} + \text{etc.}$$

Quum hinc quoque habeatur

$$\log \Pi 2z = (2z + \tfrac{1}{2}) \log 2z - 2z + \tfrac{1}{2} \log 2\pi + \frac{\mathfrak{A}}{1.2.2z} - \frac{\mathfrak{B}}{3.4.8z^3} + \frac{\mathfrak{C}}{5.6.32z^5} - \frac{\mathfrak{D}}{7.8.128z^7} + \text{etc.}$$

atque per formulam 57, statuendo $n = 2$, $\log \Pi (z - \frac{1}{2}) = \log \Pi 2z - \log \Pi z - (2z + \frac{1}{2}) \log 2 + \frac{1}{2} \log 2\pi$, fit

[59] $$\log \Pi (z - \tfrac{1}{2}) = z \log z - z + \tfrac{1}{2} \log 2\pi - \frac{\mathfrak{A}}{1.2.2z} + \frac{7\mathfrak{B}}{3.4.8z^3} - \frac{31\mathfrak{C}}{5.6.32z^5} + \frac{127\mathfrak{D}}{7.8.128z^7} - \text{etc.}$$

Hae duae series pro valoribus magnis ipsius z ab initio satis promte conuergunt, ita vt summam approximatam commode satisque

 exacte

exacte colligere liceat: attamen probe notandum eſt, pro quouis valore dato ipſius z, quantumuis magno, praecifionem limitatam tantummodo obtineri poſſe, quum numeri Bernoulliani ſeriem hypergeometricam conſtituant, adeoque ſeries illae, ſi modo ſatis longe extendantur, certo e conuergentibus diuergentes euadant. Ceterum negari nequit, theoriam talium ſerierum diuergentium adhuc quibusdam difficultatibus premi, de quibus forſan alia occaſione pluribus commentabimur.

30.

E formula 38 ſequitur

$$\frac{\Pi(k, z+\omega)}{\Pi(k, z)} = \frac{z+1}{z+1+\omega}\cdot\frac{z+2}{z+2+\omega}\cdot\frac{z+3}{z+3+\omega}\cdots\cdots\frac{z+k}{z+k+\omega}\cdot k^{\omega}$$

vnde ſumtis logarithmis, in ſeries infinitas euolutis, prodit

[60] $$\log\Pi(k, z+\omega) = \log\Pi(k, z)$$
$$+\omega\left(\log k - \frac{1}{z+1} - \frac{1}{z+2} - \frac{1}{z+3} - \text{etc} - \frac{1}{z+k}\right)$$
$$+\tfrac{1}{2}\omega\omega\left(\frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \text{etc.} + \frac{1}{(z+k)^2}\right)$$
$$-\tfrac{1}{3}\omega^3\left(\frac{1}{(z+1)^3} + \frac{1}{(z+2)^3} + \frac{1}{(z+3)^3} + \text{etc.} + \frac{1}{(z+k)^3}\right)$$
$$+ \text{etc. in inf.}$$

Series, hic in ω multiplicata, quae, ſi magis placet, ita etiam exhiberi poteſt,

$$-\frac{1}{z+1} + \log 2 - \frac{1}{z+2} + \log\tfrac{3}{2} - \frac{1}{z+3} + \log\tfrac{4}{3} - \frac{1}{z+4} + \log\tfrac{5}{4} - \text{etc.} - \frac{1}{z+k} + \log\frac{k}{k-1}$$

e terminorum multitudine finita conſtat, creſcente autem k in infinitum, ad limitem certum conuerget, qui nouam functionum transſcendentium ſpeciem nobis ſiſtit, in poſterum per Ψz denotandam.

Deſignando porro ſummas ſerierum ſequentium, in *infinitum* extenſarum,

$$\frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \text{etc.}$$
$$\frac{1}{(z+1)^3} + \frac{1}{(z+2)^3} + \frac{1}{(z+3)^3} + \text{etc.}$$

$\frac{1}{(z+1)^4}+\frac{1}{(z+2)^4}+\frac{1}{(z+3)^4}+$ etc.

etc.

resp. per P, Q, R etc. (pro quibus signa functionalia introducere minus necessarium videtur), habebimus

[61] $\log \Pi(z+\omega)=\log \Pi z+\omega \Psi z+\frac{1}{2}\omega\omega P-\frac{1}{3}\omega^3 Q+\frac{1}{4}\omega^4 R-$ etc.

Manifesto functio Ψz erit functio deriuata prima functionis $\log \Pi z$, adeoque

[62] $\frac{d\Pi z}{dz}=\Pi z.\Psi z$

Perinde erit $P=\frac{d\Psi z}{dz}$, $Q=-\frac{dd\Psi z}{2dz^2}$, $R=+\frac{d^3\Psi z}{2.3dz^3}$ etc.

31.

Functio Ψz aeque fere memorabilis est atque functio Πz, quapropter insigniores relationes ad illam spectantes hic colligemus. E differentiatione aequationis 44 fit

[63] $\Psi(z+1)=\Psi z+\frac{1}{z+1}$

vnde

[64] $\Psi(z+n)=\Psi z+\frac{1}{z+1}+\frac{1}{z+2}+\frac{1}{z+3}+$ etc. $+\frac{1}{z+n}$

Huius adiumento a valoribus minoribus ipsius z ad maiores progredi, vel a maioribus ad minores regredi licet: pro valoribus maioribus positiuis ipsius z functionis valores numerici satis commode per formulas sequentes e differentiatione aequationum 58, 59 oriundas computantur, de quibus tamen eadem sunt tenenda, quae in art. 29 circa formulas 58 et 59 monuimus:

[65] $\Psi z=\log z+\frac{1}{2z}-\frac{\mathfrak{A}}{2zz}+\frac{\mathfrak{B}}{4z^4}-\frac{\mathfrak{C}}{6z^6}+$ etc.

[66] $\Psi(z-\frac{1}{2})=\log z+\frac{\mathfrak{A}}{2.2zz}-\frac{7\mathfrak{B}}{4.8z^4}+\frac{31\mathfrak{C}}{6.32z^6}-$ etc.

Ita pro $z=10$ computauimus

$\Psi z=2.35175\,25890\,66721\,10764\,743$,

vnde regredimur ad

$\Psi 0 = -0{,}5772156649\ 0153286060\ 653$ *)

Pro valore integro positiuo ipsius z fit generaliter

[67] $\Psi z = \Psi 0 + 1 + \frac{1}{2} + \frac{1}{3} + \text{etc.} + \frac{1}{z}$

Pro valore integro negatiuo autem manifesto Ψz fit quantitas infinite magna.

32.

Formula 55 nobis suppeditat $\log \Pi(-z) + \log \Pi(z-1) = \log \pi - \log \sin z\pi$, vnde fit per differentiationem

[68] $\Psi(-z) - \Psi(z-1) = \pi \operatorname{cotang} z\pi$

Et quum e definitione functionis Ψ generaliter habeatur

[69] $\Psi x - \Psi y = -\frac{1}{x+1} + \frac{1}{y+1} - \frac{1}{x+2} + \frac{1}{y+2} - \frac{1}{x+3} + \text{etc.}$

oritur series nota

$$\pi \operatorname{cotang} z\pi = \frac{1}{z} - \frac{1}{1-z} + \frac{1}{1+z} - \frac{1}{2-z} + \frac{1}{2+z} - \frac{1}{3-z} + \text{etc.}$$

Simili modo e differentiatione formulae 57 prodit

[70] $\Psi z + \Psi(z - \frac{1}{n}) + \Psi(z - \frac{2}{n}) + \text{etc.} + \Psi(z - \frac{n-1}{n}) = n\Psi nz - n\log n$

adeoque statuendo $z = 0$

[71] $\Psi(-\frac{1}{n}) + \Psi(-\frac{2}{n}) + \Psi(-\frac{3}{n}) + \text{etc.} + \Psi(-\frac{n-1}{n}) = (n-1)\Psi 0 - n\log n$

Ita e. g. habetur

$\Psi(-\frac{1}{2}) = \Psi 0 - 2\log 2 = -1{,}9635100260\ 2142347944\ 099$, vnde porro $\Psi\frac{1}{2} = +0{,}0364899739\ 7857652055\ 901.$

33.

*) Quum hic valor inde a figura vigesima discrepet ab eo quem computauit clar. Mascheroni in Adnotat. ad Euleri Calculum Integr., adhortatus sum Fridericum Bernhardum Gothofredum *Nicolai*, iuuenem in calculo indefessum, vt computum illum repeteret vlteriusque extenderet. Inuenit itaque per calculum duplicem, scilicet descendens tum a $z = 50$ tum a $z = 100$, $\Psi 0 = -0{,}5772156649\ 0153286060\ 6512090082\ 4024310421$

Eidem calculatori exercitatissimo etiam debetur tabulae ad finem huius Sectionis annexae pars altera, exhibens valores functionis Ψz ad 18 figuras (quarum vltima haud certa), pro omnibus valoribus ipsius z a 0 vsque ad 1 per singulas partes centesimas. Ceterum methodi, per quas vtraque tabula constructa est, innituntur partim theorematibus quae hic traduntur, partim calculi artificiis singularibus, quae alia occasione proferemus.

33.

Sicuti in art. praec. $\Psi(-\frac{1}{2})$ ad $\Psi 0$ et logarithmum reduximus, ita generaliter $\Psi(-\frac{m}{n})$, designantibus m, n integros, quorum minor m, ad $\Psi 0$ et logarithmos reducemus. Statuamus $\frac{2\pi}{n} = \omega$, sitque φ alicui angulorum $\omega, 2\omega, 3\omega \ldots (n-1)\omega$ aequalis; vnde $1 = \cos n\varphi = \cos 2n\varphi = \cos 3n\varphi$ etc., $\cos\varphi = \cos(n+1)\varphi =$ ~~$\cos(n+2)\varphi$~~ etc., $\cos 2\varphi = \cos(n+2)\varphi$ etc., nec non $\cos\varphi + \cos 2\varphi + \cos 3\varphi +$ etc. $+ \cos(n-1)\varphi + 1 = 0$. Habemus itaque

$$\cos\varphi.\Psi\frac{1-n}{n} = -n\cos\varphi + \cos\varphi.\log 2 - \frac{n}{n+1}\cos(n+1)\varphi + \cos\varphi.\log\tfrac{3}{2} - \text{etc.}$$

$$\cos 2\varphi.\Psi\frac{2-n}{n} = -\frac{n}{2}\cos 2\varphi + \cos 2\varphi.\log 2 - \frac{n}{n+2}\cos(n+2)\varphi + \cos 2\varphi.\log\tfrac{3}{2} - \text{etc.}$$

$$\cos 3\varphi.\Psi\frac{3-n}{n} = -\frac{n}{3}\cos 3\varphi + \cos 3\varphi.\log 2 - \frac{n}{n+3}\cos(n+3)\varphi + \cos 3\varphi.\log\tfrac{3}{2} - \text{etc.}$$

etc. vsque ad

$$\cos(n-1)\varphi.\Psi(-\frac{1}{n}) = -\frac{n}{n-1}\cos(n-1)\varphi + \cos(n-1)\varphi.\log 2 - \frac{n}{2n-1}\cos(2n-1)\varphi + \cos(n-1)\varphi.\log\tfrac{3}{2} - \text{etc.}$$

$$\Psi 0 = -\frac{n}{n}\cos n\varphi + \log 2 - \frac{n}{2n}\cos.\, 2n\varphi + \log\tfrac{3}{2} - \text{etc.}$$

atque per *summationem*

$$\cos\varphi.\Psi\frac{1-n}{n} + \cos 2\varphi.\Psi\frac{2-n}{n} + \cos 3\varphi.\Psi\frac{3-n}{n} + \text{etc.} + \cos(n-1)\varphi.\Psi(-\frac{1}{n}) + \Psi 0 = -n(\cos\varphi + \tfrac{1}{2}\cos 2\varphi + \tfrac{1}{3}\cos 3\varphi + \tfrac{1}{4}\cos 4\varphi + \text{etc. in infin.})$$

Sed habetur generaliter, pro valore ipsius x vnitate non maiori, $\log(1 - 2x\cos\varphi + xx) = -2(x\cos\varphi + \frac{1}{2}xx\cos 2\varphi + \frac{1}{3}x^3\cos 3\varphi + \text{etc})$ quae quidem series facile sequitur ex euolutione $\log(1-rx) + \log(1-\frac{x}{r})$, denotante r quantitatem $\cos\varphi + \sqrt{-1}.\sin\varphi$. Hinc fit

aequatio

aequatio praecedens

[72] $\cos\varphi.\Psi\frac{1-n}{n}+\cos 2\varphi.\Psi\frac{2-n}{n}+\cos 3\varphi.\Psi\frac{3-n}{n}+\text{etc.}+$
$\cos(n-1)\varphi.\Psi(-\frac{1}{n})$
$=-\Psi 0+\frac{1}{2}n\log(2-2\cos\varphi)$

Statuatur in hac aequatione deinceps $\varphi=\omega$, $\varphi=2\omega$, $\varphi=3\omega$ etc. vsque ad $\varphi=(n-1)\omega$, multiplicentur singulae hae aequationes ordine suo per $\cos m\omega$, $\cos 2m\omega$, $\cos 3m\omega$ etc. vsque ad $\cos(n-1)m\omega$, productorumque aggregato adiiciatur adhuc aequatio 71

$$\Psi\frac{1-n}{n}+\Psi\frac{2-n}{n}+\Psi\frac{3-n}{n}+\text{etc.}+\Psi(-\frac{1}{n})=(n-1)\Psi 0-n\log n$$

Quodsi iam perpenditur, esse

$$1+\cos m\omega.\cos k\omega+\cos 2m\omega.\cos 2k\omega+\cos 3m\omega.\cos 3k\omega+\text{etc.}+\cos(n-1)m\omega.\cos(n-1)k\omega=0$$

denotante k aliquem numerorum 1, 2, 3 .... $(n-1)$ exceptis his duobus m atque $n-m$, pro quibus summa illa fit $=\frac{1}{2}n$, patebit, ex summatione illarum aequationum prodire, post diuisionem per $\frac{n}{2}$,

[73] $\Psi(-\frac{m}{n})+\Psi(-\frac{n-m}{m})=2\Psi 0-2\log n+\cos m\omega.\log(2-2\cos\omega)+$
$\cos 2m\omega.\log(2-2\cos 2\omega)+\cos 3m\omega.\log(2-2\cos 3\omega)+$
$\text{etc.}+\cos(n-1)m\omega.\log(2-2\cos.(n-1)\omega)$

Manifesto terminus vltimus huius aequationis fit $=\cos m\omega.\log(2-2\cos\omega)$, penultimus $=\cos 2m\omega.\log(2-2\cos 2\omega)$ etc., ita vt bini termini semper sint aequales, excepto, si n est par, termino singulari $\cos\frac{n}{2}.m\omega\log(2-2\cos\frac{n}{2}\omega)$, qui fit $=+2\log 2$ pro m pari, vel $=-2\log 2$ pro m impari. Combinando iam cum aequatione 73 hanc

$$\Psi(-\frac{m}{n})-\Psi(-\frac{n-m}{n})=\pi\cot\text{ang}\frac{m}{n}\pi$$

habemus, pro valore impari ipsius n, siquidem m est integer positiuus minor quam n,

$$[74]\ \Psi\left(-\frac{m}{n}\right)=\Psi 0+\tfrac{1}{2}\pi\,\mathrm{cotang}\,\frac{m\pi}{n}-\log n+\cos\frac{2m\pi}{n}.\log\left(2-2\cos\frac{2\pi}{n}\right)$$
$$+\cos\frac{4m\pi}{n}.\log\left(2-2\cos\frac{4\pi}{n}\right)+\cos\frac{6m\pi}{n}.\log\left(2-2\cos\frac{6\pi}{n}\right)$$
$$+\text{etc.}+\cos\frac{(n-1)m\pi}{n}.\log\left(2-2\cos\frac{(n-1)\pi}{n}\right)$$

Pro valore pari ipsius n autem

$$[75]\ \Psi\left(-\frac{m}{n}\right)=\Psi 0+\tfrac{1}{2}\pi\,\mathrm{cotang}\,\frac{m\pi}{n}-\log n+\cos\frac{2m\pi}{n}\log\left(2-2\cos\frac{2\pi}{n}\right)$$
$$+\cos\frac{4m\pi}{n}\log\left(2-2\cos\frac{4\pi}{n}\right)+\text{etc.}+$$
$$\cos\frac{(n-2)m\pi}{n}\log\left(2-2\cos\frac{(n-2)\pi}{n}\right)$$
$$\pm\log 2.$$

vbi signum superius valet pro m pari, inferius pro impari. Ita e. g. inuenitur $\Psi(-\tfrac{1}{4})=\Psi 0+\tfrac{1}{2}\pi-3\log 2$, $\Psi(-\tfrac{3}{4})=\Psi 0-\tfrac{1}{2}\pi-3\log 2$, $\Psi(-\tfrac{1}{3})=\Psi 0+\tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}}-\tfrac{3}{2}\log 3$, $\Psi(-\tfrac{2}{3})=\Psi 0-\tfrac{1}{2}\pi\sqrt{\tfrac{1}{3}}-\tfrac{3}{2}\log 3$.

Ceterum combinatis his aequationibus cum aequatione 64 sponte patet, Ψz generaliter pro *quouis valore rationali* ipsius z, positiuo seu negatiuo per $\Pi 0$ atque logarithmos determinari posse, quod theorema sane maxime est memorabile.

34.

Quum, per art. 28, $\Pi\lambda$ sit valor integralis $\int y^\lambda e^{-y}\,dy$, ab $y=0$ vsque ad $y=\infty$, siquidem $\lambda+1$ est quantitas positiua, fit differentiando secundum λ

$$\frac{d\Pi\lambda}{d\lambda}=\frac{d\int y^\lambda e^{-y}\,dy}{d\lambda}=\int y^\lambda e^{-y}\log y\,dy$$

siue

$$[76)\ \Pi\lambda.\Psi\lambda=\int y^\lambda e^{-y}\log y.dy,\ \text{ab } y=0 \text{ vsque ad } y=\infty$$

Generalius statuendo $y=z^\alpha$, $\alpha\lambda+\alpha-1=\beta$, valor integralis $\int z^\beta e^{-z^\alpha}\log z.dz$, a $z=0$ vsque ad $z=\infty$, fit $=\frac{1}{\alpha\alpha}\Pi\left(\frac{\beta+1}{\alpha}-1\right).$
$\Psi\left(\frac{\beta+1}{\alpha}-1\right)=\frac{1}{\alpha(\beta+1)}\Pi\frac{\beta+1}{\alpha}.\Psi\frac{\beta+1}{\alpha}-\frac{1}{(\beta+1)^2}\Pi\frac{\beta+1}{\alpha}$, siqui-

dem

dem simul $\beta+1$ atque α sunt quantitates positiuae, vel aequalis eidem quantitati cum signo opposito, si vtraque $\beta+1$, α est negatiua.

35.

At non solum productum $\Pi\lambda.\Psi\lambda$, verum etiam ipsa functio $\Psi\lambda$ per integrale determinatum exhiberi potest. Designante k integrum positiuum, patet valorem integralis $\int\frac{x^\lambda - x^{\lambda+k}}{1-x}.dx$, ab $x=0$ vsque ad $x=1$ esse

$$=\frac{1}{\lambda+1}+\frac{1}{\lambda+2}+\frac{1}{\lambda+3}+\text{etc.}+\frac{1}{\lambda+k}$$

Porro quum valor integralis $\int(\frac{1}{1-x}-\frac{kx^{k-1}}{1-x^k})dx$ generaliter sit $=$ Const. $+\log\frac{1-x^k}{1-x}$, idem inter limites $x=0$ atque $x=1$ erit $=\log k$, vnde patet, valorem integralis $S=\int(\frac{1-x^\lambda+x^{\lambda+k}}{1-x}-\frac{kx^{k-1}}{1-x^k})dx$ inter eosdem limites esse

$$=\log k-\frac{1}{\lambda+1}-\frac{1}{\lambda+2}-\frac{1}{\lambda+3}-\text{etc.}-\frac{1}{\lambda+k}$$

quam expressionem denotabimus per Ω. Discerpamus integrale S in duas partes

$$\int(\frac{1-x^\lambda}{1-x})dx+\int(\frac{x^{\lambda+k}}{1-x}-\frac{kx^{k-1}}{1-x^k})dx$$

Pars prima $\int\frac{1-x^\lambda}{1-x}.dx$, statuendo $x=y^k$ mutatur in

$$\int\frac{ky^{k-1}-ky^{\lambda k+k-1}}{1-y^k}dy$$

vnde sponte patet, illius valorem ab $x=0$ vsque ad $x=1$, aequalem esse valori integralis

$$\int\frac{kx^{k-1}-kx^{\lambda k+k-1}}{1-x^k}dx$$

inter eosdem limites, quum manifesto literam y sub hac restrictione in x mutare liceat. Hinc fit integrale S, inter eosdem limites

$$=\int(\frac{x^{\lambda+k}}{1-x}-\frac{kx^{\lambda k+k-1}}{1-x^k})dx$$

Ho

Hoc vero integrale, ſtatuendo $x^k = z$, tranſit in

$$\int \left(\frac{z^{\frac{\lambda+1}{k}}}{k(1-z^{\frac{1}{k}})} - \frac{z^\lambda}{1-z} \right) dz$$

quod itaque inter limites $z = 0$ atque $z = 1$ ſumtum aequale eſt ipſi Ω. Sed creſcente k in infinitum, limes ipſius Ω eſt $\Psi\lambda$, limes ipſius $\frac{\lambda+1}{k}$ eſt 0, limes ipſius $k(1-z^{\frac{1}{k}})$ vero eſt $\log \frac{1}{z}$ ſiue $-\log z$. Quare habemus

$$[77] \quad \Psi\lambda = \int \left(\frac{1}{\log \frac{1}{z}} - \frac{z^\lambda}{1-z} \right) dz = \int \left(-\frac{1}{\log z} - \frac{z^\lambda}{1-z} \right) dz$$

a $z = 0$ vsque ad $z = 1$.

36.

Integralia determinata, per quae ſupra expreſſae ſunt functiones $\Pi\lambda$, $\Pi\lambda . \Psi\lambda$, reſtringere oportuit ad valores ipſius λ tales, vt $\lambda + 1$ euadat quantitas poſitiua: haec reſtrictio ex ipſa deductione demanauit, reueraque facile perſpicitur, pro aliis valoribus ipſius λ illa integralia ſemper fieri infinita, etiamſi functiones $\Pi\lambda$, $\Pi\lambda . \Psi\lambda$ finitae manere poſſint. Veritati formula 77 certo eadem conditio ſubeſſe debet, vt $\lambda + 1$ ſit quantitas poſitiua (alioquin enim integrale certo infinitum euadit, etiamſi functio $\Psi\lambda$ maneat finita): ſed deductio formulae primo aſpectu generalis nullique reſtrictioni obnoxia eſſe videtur. Sed propius attendenti facile patebit, ipſi analyſi per quam formula eruta eſt hanc reſtrictionem iam ineſſe. Scilicet tacite ſuppoſuimus, integrale $\int \frac{1-x^\lambda}{1-x} dx$, cui aequale $\int \frac{kx^{k-1} - kx^{\lambda k+k-1}}{1-x^k} dx$ ſubſtituimus, habere valorem *finitum*, quae conditio requirit, vt $\lambda + 1$ ſit quantitas poſitiua. Ex analyſi noſtra quidem ſequitur, haec duo integralia ſemper eſſe aequalia, ſi hoc extendatur ab $x = 0$ vsque ad $x = 1 - \omega$, illud ab $x = 0$ vsque ad $x = (1-\omega)^k$, quantumuis parua ſit quantitas ω modo non ſit $= 0$: ſed hoc non obſtante in caſu eo vbi $\lambda + 1$ non eſt quantitas poſitiua, duo integralia ab $x = 0$ vsque ad *eundem* terminum $x = 1 - \omega$ extenſa neutiquam ad aequalitatem conuergunt, ſed potius tunc ipſorum differentia, decreſcente ω in infini-

 tum

tum, in infinitum crefcet. Hocce exemplum monftrat, quanta circumfpicientia opus fit in tractandis quantitatibus infinitis, quae in ratiociniis analyticis noftro iudicio eatenus tantum funt admittendae, quatenus ad theoriam limitum reduci poffunt.

37.

Statuendo in formula 77, $z = e^{-u}$, patet illam etiam ita exhiberi poffe

$$\Psi\lambda = -\int\left(\frac{e^{-u}}{u} - \frac{e^{-u\lambda-u}}{1-e^{-u}}\right) du, \text{ ab } u = \infty \text{ vsque ad } u = 0, \text{ i. e.}$$

$$[78]\quad \Psi\lambda = \int\left(\frac{e^{-u}}{u} - \frac{e^{-\lambda u}}{e^{u}-1}\right) du, \text{ ab } u = 0 \text{ vsque ad } u = \infty.$$

(Perinde valor ipfius $\Pi\lambda$ in art. 28 allatus, mutatur ftatuendo $e^{-y} = v$, in fequentem

$$\Pi\lambda = \int\left(\log\frac{1}{v}\right)^{\lambda} dv, \text{ a } v = 0 \text{ vsque ad } v = 1),$$

Porro patet e formula 77, effe

$$[79]\quad \Psi\lambda - \Psi\mu = \int\frac{z^{\mu} - z^{\lambda}}{1-z} dz, \text{ a } z = 0 \text{ vsque ad } z = 1$$

vbi praeter $\lambda + 1$ etiam $\mu + 1$ debet effe quantitas pofitiua.

Statuendo in eadem formula 77, $z = u^{\alpha}$, defignante α quantitatem pofitiuam, fit

$$\Psi\lambda = \int\left(-\frac{u^{\alpha-1}}{\log u} - \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}}\right) du, \text{ ab } u = 0 \text{ vsque ad } u = 1.$$

et quum perinde ftatui poffit, pro valore pofitiuo ipfius β,

$$\Psi\lambda = \int\left(-\frac{u^{\beta-1}}{\log u} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}}\right) du$$

patet fieri

$$0 = \int\left(\frac{u^{\alpha-1} - u^{\beta-1}}{\log u} + \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}} - \frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}}\right) du$$

fiue

$$\int\frac{u^{\alpha-1} - u^{\beta-1}}{\log u} du = \int\left(\frac{\beta u^{\beta\lambda+\beta-1}}{1-u^{\beta}} - \frac{\alpha u^{\alpha\lambda+\alpha-1}}{1-u^{\alpha}}\right) du.$$

integralibus ſemper ab $u = 0$ vsque ad $u = 1$ extenſis. Sed ponendo $\lambda = 0$, integrale poſterius *indefinite* aſſignari poteſt; eſt ſcilicet $= \log \frac{1 - u^\alpha}{1 - u^\beta}$, ſi euaneſcere debet pro $u = 0$; quare quum pro $u = 1$ ſtatuere oporteat $\frac{1 - u^\alpha}{1 - u^\beta} = \frac{\alpha}{\beta}$, erit integrale $\log \frac{\alpha}{\beta} = \int \frac{u^{\alpha - 1} - u^{\beta - 1}}{\log u} du$, ab $u = 0$ vsque ad $u = 1$, quod theorema olim ab ill. Euler per alias methodos erutum eſt.

z	log Πz	Ψz
0.00	0.0000000000 0000000000	— 0.5772156649 01532861
0.01	9.9975287306 5869172624	0.5608854578 68674498
0.02	9.9951278719 8879034144	0.5447893104 56179789
0.03	9.9927964208 8883589748	0.5289210872 85430502
0.04	9.9905334004 0842900595	0.5132748789 16830312
0.05	9.9883378587 9012046216	0.4978449912 99870371
0.06	9.9862088685 5581945437	0.4826259358 14825705
0.07	9.9841455256 3523567773	0.4676124198 67553632
0.08	9.9821469485 3403172902	0.4527993380 01712885
0.09	9.9802122775 3951136603	0.4381817634 95334764
0.10	9.9783406739 6180754713	0.4237549404 11076796
0.11	9.9765313194 0866250820	0.4095142760 71694248
0.12	9.9747834150 9201128963	0.3954553339 34292807
0.13	9.9730961811 6469083029	0.3815738268 38792064
0.14	9.9714688560 8569966779	0.3678656106 07749546
0.15	9.9699006960 1252903489	0.3543266779 76279272
0.16	9.9683909742 1917527943	0.3409531528 32261794
0.17	9.9669389805 3852656982	0.3277412847 48392299
0.18	9.9655440208 2789424567	0.3146874437 88860621
0.19	9.9642054164 5653136262	0.3017881155 74610030
0.20	9.9629225038 1404835193	0.2890398965 92188296
0.21	9.9616946338 3869862929	0.2764394897 32192051
0.22	9.9605211715 6456577252	0.2639837000 44220200
0.23	9.9594014956 8673884734	0.2516694306 96100107
0.24	9.9583349981 4361387302	0.2394936791 25936794
0.25	9.9573210837 1550754011	0.2274535333 76265408
0.26	9.9563591696 3881435774	0.2155461686 00265182
0.27	9.9554486852 3498063412	0.2037688437 30623157
0.28	9.9545890715 5360828076	0.1921188983 02221732
0.29	9.9537797810 2903856417	0.1805937494 20369178
0.30	9.9530202771 4980077695	0.1691908888 66799656
0.31	9.9523100341 4034352140	0.1579078803 36141874
0.32	9.9516485366 5449703876	0.1467423567 95996017
0.33	9.9510352794 8014390879	0.1356920179 64169332
0.34	9.9504697672 5460261315	0.1247546278 97003946
0.35	9.9499515141 9025401627	0.1139280126 83088296

z	log Πz	Ψz
0.35	9.9499515141 9025401627	− 0.1139280126 83088296
0.36	9.9494800438 0996487612	0.1032100582 36977615
0.37	9.9490548886 9188515282	0.0925987081 87861259
0.38	9.9486765902 2321722697	0.0820919618 58406487
0.39	9.9483416983 6257525751	0.0716878728 29281510
0.40	9.9480527714 1057187897	0.0613845445 85116146
0.41	9.9478083757 8828733374	0.0511801337 37897756
0.42	9.9476080858 2329302469	0.0410728433 24024375
0.43	9.9474514835 4291742066	0.0310609236 71447052
0.44	9.9473381584 7445730981	0.0211426703 33530475
0.45	9.9472677074 6205163055	0.0113164225 86445845
0.46	9.9472397344 2994856529	0.0015805619 87083418
0.47	9.9472538503 0190930853	+ 0.0080664890 11364893
0.48	9.9473096727 2650396072	0.0176262683 88849468
0.49	9.9474068259 5806639475	0.0271002758 35486201
0.50	9.9475419406 8308573196	0.0364899739 78576520
0.51	9.9477236538 6182228429	0.0457967895 61914496
0.52	9.9479426085 7494550351	0.0550221145 79551622
0.53	9.9482014538 7500065798	0.0641673073 66077154
0.54	9.9484998446 4251966174	0.0732336936 45365776
0.55	9.9488374414 4659973817	0.0822225675 39644344
0.56	9.9492139104 0978143536	0.0911351925 40635189
0.57	9.9496289230 7706494873	0.0999728024 44444623
0.58	9.9500821562 8891076887	0.1087366022 51781439
0.59	9.9505732920 5807738191	0.1174277690 35011042
0.60	9.9511020174 5015512544	0.1260474527 73476253
0.61	9.9516680244 6766136244	0.1345967771 58445210
0.62	9.9522710099 3756789859	0.1430768403 68980212
0.63	9.9529106754 0213704917	0.1514887158 19958383
0.64	9.9535867270 1294797674	0.1598334528 83415463
0.65	9.9542988754 2799988466	0.1681120775 84327804
0.66	9.9550468357 1178337730	0.1763255932 71894293
0.67	9.9558303272 3821579829	0.1844749812 67329607
0.68	9.9566490735 9634064632	0.1925612014 89132418
0.69	9.9575028024 9869525351	0.2005851930 56747012
0.70	9.9583912456 9225480685	0.2085478748 73493948

z	$\log \Pi z$	Ψz
0.70	9.9583912456 9225480685	+ 0.2085478748 75493948
0.71	9.9593141388 7186450668	0.2164501461 89604789
0.72	9.9602712215 9607519880	0.2242948871 46167521
0.73	9.9612622372 0530119641	0.2320769595 00672792
0.74	9.9622869327 4222225320	0.2398032061 35096466
0.75	9.9633450588 7435456849	0.2474724535 46861164
0.76	9.9644365698 1871920539	0.2550855103 25688536
0.77	9.9655606232 6853798084	0.2626431686 02762795
0.78	9.9667175803 2189101417	0.2701462043 14883540
0.79	9.9679070054 1227146665	0.2775953776 14168016
0.80	9.9691286662 4097614416	0.2849914332 93861642
0.81	9.9703823337 1127271250	0.2923351011 88779580
0.82	9.9716677818 6428658993	0.2996270965 64887544
0.83	9.9729847878 1655271065	0.3068681204 96501033
0.84	9.9743331316 9917940601	0.3140588602 51568639
0.85	9.9757125965 9857361442	0.3211999895 45479708
0.86	9.9771229684 9867851092	0.3282921690 83820641
0.87	9.9785640362 2467644771	0.3353360466 94485409
0.88	9.9800355913 8811182162	0.3423322577 49528903
0.89	9.9815374283 3339013630	0.3492814254 57135499
0.90	9.9830693440 8561111078	0.3561841611 64059720
0.91	9.9846311382 9969520321	0.3630410646 48881125
0.92	9.9862226132 1070437381	0.3698527244 06401469
0.93	9.9878435735 8573930651	0.3766197179 23498793
0.94	9.9894938266 7611664682	0.3833426119 46740214
0.95	9.9911731821 7489109803	0.3900219627 42043086
0.96	9.9928814521 5658844947	0.3966583163 46662402
0.97	9.9946184510 6337679375	0.4032522088 13771306
0.98	9.9963839956 3222432515	0.4098041664 49890838
0.99	9.9981779048 6807320161	0.4163147060 45414956
1.00	0.0000000000 0000000000	0.4227843350 98467139