Werk

Titel: Commentationes Societatis Regiae Scientiarum Gotti

Verlag: Dieterich

Jahr: 1823

Kollektion: Wissenschaftsgeschichte

Werk Id: PPN35283028X_0005_2NS

PURL: http://resolver.sub.uni-goettingen.de/purl?PID=PPN35283028X_0005_2NS|LOG_0021

THEORIA
COMBINATIONIS OBSERVATIONUM
ERRORIBUS MINIMIS OBNOXIAE.

PARS POSTERIOR.

SOCIETATI REGIAE EXHIBITA FEBR. 2, 1823.

23.

Plures adhuc superſunt disquiſitiones, per quas theoria praecedens tum illuſtrabitur tum ampliabitur.

Ante omnia inueſtigare oportet, num negotium eliminationis, cuius adiumento indeterminatae x, y, z etc. per ξ, η, ζ etc. exprimendae ſunt, ſemper ſit poſſibile. Quum multitudo illarum multitudini harum aequalis ſit, e theoria eliminationis in aequationibus linearibus conſtat, illam eliminationem, ſi ξ, η, ζ etc. ab inuicem independentes ſint, certo poſſibilem fore; ſin minus, impoſſibilem. Supponamus aliquantiſper, ξ, η, ζ etc. non eſſe ab inuicem independentes, ſed exſtare inter ipſas aequationem identicam

$$0 = F\xi + G\eta + H\zeta + \text{etc.} + K$$

Habebimus itaque

$$F\Sigma aa + G\Sigma ab + H\Sigma ac + \text{etc.} = 0$$
$$F\Sigma ab + G\Sigma bb + H\Sigma bc + \text{etc} = 0$$
$$F\Sigma ac + G\Sigma bc + H\Sigma cc + \text{etc.} = 0$$

etc., nec non

$$F\Sigma al + G\Sigma bl + H\Sigma cl + \text{etc.} = -K$$

Statuendo porro

$$\left.\begin{array}{l} aF + bG + cH + \text{etc.} = \theta \\ a'F + b'G + c'H + \text{etc.} = \theta' \\ a''F + b''G + c''H + \text{etc.} = \theta'' \end{array}\right\} \text{(I)}$$

etc., eruitur

$$a\theta + a'\theta' + a''\theta'' + \text{etc.} = 0$$
$$b\theta + b'\theta' + b''\theta'' + \text{etc.} = 0$$
$$c\theta + c'\theta' + c''\theta'' + \text{etc.} = 0$$

etc., nec non

$$l\theta + l'\theta' + l''\theta'' + \text{etc.} = -K$$

Multiplicando itaque aequationes (I) reſp. per θ, θ', θ'' etc. et addendo, obtinemus:

$$0 = \theta\theta + \theta'\theta' + \theta''\theta'' + \text{etc.}$$

quae aequatio manifeſto conſiſtere nequit, niſi ſimul fuerit $\theta = 0$, $\theta' = 0$, $\theta'' = 0$ etc. Hinc primo colligimus, neceſſario eſſe debere $K = 0$. Dein aequationes (I) docent, functiones v, v', v'' etc. ita comparatas eſſe, vt ipſarum valores non mutentur, ſi valores quantitatum x, y, z etc. capiant incrementa vel decrementa ipſis F, G, H etc. reſp. proportionalia, idemque manifeſto de functionibus V, V', V'' etc. valebit. Suppoſitio itaque conſiſtere nequit, niſi in caſu tali, vbi vel e valoribus exactis quantitatum V, V', V'' etc. valores incognitarum x, y, z etc. determinare impoſſibile fuiſſet, i. e. vbi problema natura ſua fuiſſet indeterminatum, quem caſum a diſquiſitione noſtra excluſimus.

24.

Denotemus per β, β', β'' etc. multiplicatores, qui eandem relationem habent ad indeterminatam y, quam habent α, α', α'' etc. ad x, puta ſit

$$a[\beta\alpha] + b[\beta\beta] + c[\beta\gamma] + \text{etc.} = \beta$$
$$a'[\beta\alpha] + b'[\beta\beta] + c'[\beta\gamma] + \text{etc.} = \beta'$$
$$a''[\beta\alpha] + b''[\beta\beta] + c''[\beta\gamma] + \text{etc.} = \beta''$$

etc., ita vt fiat indefinite

$$\beta v + \beta' v' + \beta'' v'' + \text{etc.} = y - B$$

Perinde ſint γ, γ', γ'' etc. multiplicatores ſimiles reſpectu indeterminatae z, puta

$$a[\gamma\alpha] + b[\gamma\beta] + c[\gamma\gamma] + \text{etc.} = \gamma$$
$$a'[\gamma\alpha] + b'[\gamma\beta] + c'[\gamma\gamma] + \text{etc.} = \gamma'$$
$$a''[\gamma\alpha] + b''[\gamma\beta] + c''[\gamma\gamma] + \text{etc.} = \gamma''$$

etc., ita vt fiat indefinite

$$\gamma v + \gamma' v' + \gamma'' v'' + \text{etc.} = z - C$$

et ſic porro. Hoc pacto, perinde vt iam in art. 20. inueneramus

$$\Sigma\alpha a = 1,\ \Sigma\alpha b = 0,\ \Sigma\alpha c = 0,\ \text{etc., nec non } \Sigma\alpha l = -A,$$

etiam habebimus

$$\Sigma\beta a = 0,\ \Sigma\beta b = 1,\ \Sigma\beta c = 0 \text{ etc., atque } \Sigma\beta l = -B$$
$$\Sigma\gamma a = 0,\ \Sigma\gamma b = 0,\ \Sigma\gamma c = 1 \text{ etc., atque } \Sigma\gamma l = -C$$

et ſic porro. Nec minus, quemadmodum in art. 20. prodiit $\Sigma\alpha\alpha = [\alpha\alpha]$, etiam erit

$$\Sigma\beta\beta = [\beta\beta],\ \Sigma\gamma\gamma = [\gamma\gamma] \text{ etc.}$$

Multiplicando porro valores ipſorum α, α', α'' etc. (art. 20. IV) reſp. per β, β', β'' etc. et addendo, obtinemus

$$\alpha\beta + \alpha'\beta' + \alpha''\beta'' \text{ etc.} = [\alpha\beta], \text{ ſiue } \Sigma\alpha\beta = [\alpha\beta]$$

Multiplicando autem valores ipſorum β, β', β'' etc. reſp. per α, α', α'' etc., et addendo, perinde prodit

$$\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \text{etc.} = [\beta\alpha], \text{ adeoque } [\alpha\beta] = [\beta\alpha]$$

Prorſus ſimili modo eruitur

$$[\alpha\gamma] = [\gamma\alpha] = \Sigma\alpha\gamma,\ [\beta\gamma] = [\gamma\beta] = \Sigma\beta\gamma \text{ etc.}$$

25.

Denotemus porro per λ, λ', λ'' etc. valores functionum v, v', v'' etc., qui prodeunt, dum pro x, y, z etc. ipſarum valores maxime plauſibiles A, B, C etc. ſubſtituuntur, puta

$$aA + bB + cC + \text{etc.} + l = \lambda$$
$$a'A + b'B + c'C + \text{etc.} + l' = \lambda'$$
$$a''A + b''B + c''C + \text{etc.} + l'' = \lambda''$$

etc.; ſtatuamus praeterea

$$\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.} = M$$

ita vt ſit M valor functionis Ω valoribus maxime plauſibilibus indeterminatarum reſpondens, adeoque per ea, quae in art. 20. demonſtrauimus, valor minimus huius functionis. Hinc erit $a\lambda + a'\lambda' + a''\lambda'' + \text{etc.}$ valor ipſius ξ, valoribus $x = A$, $y = B$, $z = C$ etc. reſpondens, adeoque $= 0$, i. e. habebimus

$$\Sigma a\lambda = 0$$

et perinde fiet

$$\Sigma b\lambda = 0,\ \Sigma c\lambda = 0 \text{ etc.; nec non } \Sigma\alpha\lambda = 0,\ \Sigma\beta\lambda = 0,\ \Sigma\gamma\lambda = 0 \text{ etc.}$$

Denique multiplicando expreſſiones ipſarum λ, λ', λ'' etc. per λ, λ', λ'' etc. reſp., et addendo, obtinemus $l\lambda + l'\lambda' + l''\lambda'' + \text{etc.} = \lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.}$, ſiue

$$\Sigma l\lambda = M.$$

26.

Substituendo in aequatione $v = ax + by + cz + \text{etc.} + l$, pro x, y, z etc. expreſſiones VII. art. 21, prodibit, adhibitis reductionibus ex praecedentibus obuiis,

$$v = \alpha\xi + \beta\eta + \gamma\zeta + \text{etc.} + \lambda$$

et perinde erit indefinite

$$v' = \alpha'\xi + \beta'\eta + \gamma'\zeta + \text{etc.} + \lambda'$$
$$v'' = \alpha''\xi + \beta''\eta + \gamma''\zeta + \text{etc.} + \lambda''$$

etc. Multiplicando vel has aequationes, vel aequationes I art. 20. reſp. per λ, λ', λ'' etc., et addendo, diſcimus eſſe indefinite

$$\lambda v + \lambda' v' + \lambda'' v'' + \text{etc.} = M.$$

27.

Functio Ω indefinite in pluribus formis exhiberi potest, quas euoluere operae pretium erit. Ac primo quidem quadrando aequationes I art. 20. et addendo, statim fit

$$\Omega = xx\,\Sigma aa + yy\,\Sigma bb + zz\,\Sigma cc + \text{etc.} + 2xy\,\Sigma ab + 2xz\,\Sigma ac + 2yz\,\Sigma bc + \text{etc.} + 2x\,\Sigma al + 2y\,\Sigma bl + 2z\,\Sigma cl + \text{etc.} + \Sigma ll$$

quae est forma *prima*.

Multiplicando easdem aequationes resp. per v, v', v'' etc., et addendo, obtinemus:

$$\Omega = \xi x + \eta y + \zeta z + \text{etc.} + lv + l'v' + l''v'' + \text{etc.}$$

atque hinc, substituendo pro v, v', v'' etc. expressiones in art. praec. traditas,

$$\Omega = \xi x + \eta y + \zeta z + \text{etc.} - A\xi - B\eta - C\zeta - \text{etc.} + M$$

siue

$$\Omega = \xi(x - A) + \eta(y - B) + \zeta(z - C) + \text{etc.} + M$$

quae est forma *secunda*.

Substituendo in forma secunda pro $x - A$, $y - B$, $z - C$ etc. expressiones VII. art. 21, obtinemus formam *tertiam*:

$$\Omega = [\alpha\alpha]\xi\xi + [\beta\beta]\eta\eta + [\gamma\gamma]\zeta\zeta + \text{etc.} + 2[\alpha\beta]\xi\eta + 2[\alpha\gamma]\xi\zeta + 2[\beta\gamma]\eta\zeta + \text{etc.} + M.$$

His adiungi potest forma *quarta*, ex forma tertia, atque formulis art. praec. sponte demanans:

$$\Omega = (v - \lambda)^2 + (v' - \lambda')^2 + (v'' - \lambda'')^2 + \text{etc.} + M, \text{ siue}$$
$$\Omega = M + \Sigma(v - \lambda)^2$$

quae forma conditionem minimi directe ob oculos sistit.

28.

Sint e, e', e'' etc. errores in obseruationibus, quae dederunt $V = L$, $V' = L'$, $V'' = L''$ etc. commissi, i. e. sint valores veri functionum V, V', V'' etc. resp. $L - e$, $L' - e'$, $L'' - e''$ etc. adeoque valores veri ipsarum v, v', v'' etc. resp. $-e\sqrt{p}$, $-e'\sqrt{p'}$,

$- e'' \sqrt{p''}$ etc. Hinc valor verus ipſius x erit $= A - \alpha e \sqrt{p} - \alpha' e' \sqrt{p'} - \alpha'' e'' \sqrt{p''}$ etc., ſiue error valoris ipſius x. in determinatione maxime idonea commiſſus, quem per Ex denotare conuenit,

$$= \alpha e \sqrt{p} + \alpha' e' \sqrt{p'} + \alpha'' e'' \sqrt{p''} + \text{etc.}$$

Perinde error valoris ipſius y in determinatione maxime idonea commiſſus, quem per Ey denotabimus, erit

$$= \beta e \sqrt{p} + \beta' e' \sqrt{p'} + \beta'' e'' \sqrt{p''} + \text{etc.}$$

Valor medius quadrati $(Ex)^2$ inuenitur $= mmp(\alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' + \text{etc.}) = mmp[\alpha\alpha]$; valor medius quadrati $(Ey)^2$ perinde $= mmp[\beta\beta]$ etc., vt iam ſupra docuimus. Iam vero etiam valorem medium *producti* $Ex . Ey$ aſſignare licet, quippe qui invenitur

$$= mmp(\alpha\beta + \alpha'\beta' + \alpha''\beta'' + \text{etc.}) = mmp[\alpha\beta].$$

Concinne haec ita quoque exprimi poſſunt. Valores medii quadratorum $(Ex)^2$, $(Ey)^2$ etc. reſp. aequales ſunt productis ex $\frac{1}{2} mmp$ in quotientes differentialium partialium ſecundi ordinis

$$\frac{dd\Omega}{d\xi^2}, \quad \frac{dd\Omega}{d\eta^2} \quad \text{etc.}$$

valorque medius producti talis, vt $Ex . Ey$, aequalis eſt producto ex $\frac{1}{2} mmp$ in quotientem differentialem $\frac{dd\Omega}{d\xi . d\eta}$, quatenus quidem Ω tamquam functio indeterminatarum ξ, η, ζ etc. conſideratur.

29.

Deſignet t functionem datam linearem quantitatum x, y, z etc. puta ſit

$$t = fx + gy + hz + \text{etc.} + k.$$

Valor ipſius t, e valoribus maxime plauſibilibus ipſarum x, y, z etc. prodiens hinc erit $= fA + gB + hC + \text{etc.} + k$, quem per K denotabimus. Qui ſi tamquam valor verus ipſius t adoptatur, error committitur, qui erit

$$= fEx + gEy + hEz + \text{etc.}$$

atque per Et denotabitur. Manifesto valor medius huius erroris fit $= 0$, siue error a parte constante liber erit. At valor medius quadrati $(Et)^2$, siue valor medius aggregati

$$ff(Ex)^2 + 2fg\,Ex.Ey + 2fh\,Ex.Ez + \text{etc.}$$
$$+ gg\,(Ey)^2 + 2gh\,Ey.Ez + \text{etc.}$$
$$+ hh\,(Ez)^2 + \text{etc. etc.}$$

per ea, quae in art. praec. exposuimus, aequalis fit producto ex mmp in aggregatum

$$ff[\alpha\alpha] + 2fg[\alpha\beta] + 2fh[\alpha\gamma] + \text{etc.}$$
$$+ gg[\beta\beta] + 2gh[\beta\gamma] + \text{etc.}$$
$$+ hh[\gamma\gamma] + \text{etc. etc.}$$

siue producto ex mmp in valorem functionis $\Omega - M$, qui prodit per substitutiones

$$\xi = f,\ \eta = g,\ \zeta = h \text{ etc.}$$

Denotando igitur hunc valorem determinatum functionis $\Omega - M$ per ω, error medius metuendus, dum determinationi $t = K$ adhaeremus, erit $= m\sqrt{p\omega}$, siue pondus huius determinationis $= \frac{1}{\omega}$.

Quum indefinite habeatur $\Omega - M = (x - A)\xi + (y - B)\eta + (z - C)\zeta + \text{etc.}$, patet, ω quoque aequalem esse valori determinato expressionis $(x - A)f + (y - B)g + (z - C)h + \text{etc.}$, siue valori determinato ipsius $t - K$, qui prodit, si indeterminatis x, y, z etc. tribuuntur valores ii, qui respondent valoribus ipsarum ξ, η, ζ etc his f, g, h etc.

Denique obseruamus, si t indefinite in formam functionis ipsarum ξ, η, ζ etc. redigatur, ipsius partem constantem necessario fieri $= K$. Quodsi igitur indefinite fit

$$t = F\xi + G\eta + H\zeta + \text{etc.} + K$$

erit $\omega = fF + gG + hH + \text{etc.}$

30.

Functio Ω valorem suum *absolute minimum* M, vt supra vidimus, nanciscitur, faciendo $x = A$, $y = B$, $z = C$ etc., siue $\xi = 0$,

$\eta = 0$, $\zeta = 0$ etc. Si vero alicui illarum quantitatum valor *alius* iam tributus est, e. g. $x = A + \Delta$, variantibus reliquis Ω assequi potest valorem relatiue minimum, qui manifesto obtinetur adiumento aequationum

$$x = A + \Delta, \ \frac{d\,\Omega}{d\,y} = 0, \ \frac{d\,\Omega}{d\,x} = 0 \text{ etc.}$$

Fieri debet itaque $\eta = 0$, $\zeta = 0$ etc., adeoque, quoniam $x = A + [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta +$ etc., $\xi = \frac{\Delta}{[\alpha\alpha]}$. Simul habebitur

$$y = B + \frac{[\alpha\beta]\,\Delta}{[\alpha\alpha]}, \ z = C + \frac{[\alpha\gamma]\,\Delta}{[\alpha\alpha]} \text{ etc.}$$

Valor relatiue minimus ipsius Ω autem fit $= [\alpha\alpha]\,\xi\xi + M = M + \frac{\Delta\Delta}{[\alpha\alpha]}$. Vice versa hinc colligimus, si valor ipsius Ω limitem praescriptum $M + \mu\mu$ non superare debet, valorem ipsius x necessario inter limites $A - \mu\sqrt{[\alpha\alpha]}$ et $A + \mu\sqrt{[\alpha\alpha]}$ contentum esse debere. Notari meretur, $\mu\sqrt{[\alpha\alpha]}$ aequalem fieri errori medio in valore maxime plausibili ipsius x metuendo, si statuatur $\mu = m\sqrt{p}$, i. e. si μ aequalis sit errori medio obseruationum talium, quibus pondus $= 1$ tribuitur.

Generalius inuestigemus valorem minimum ipsius Ω, qui pro valore dato ipsius t locum habere potest, denotante t vt in art. praec. functionem linearem $fx + gy + hz +$ etc. $+ k$, et cuius valor maxime plausibilis $= K$: valor praescriptus ipsius t denotetur per $K + \varkappa$. E theoria maximorum et minimorum constat, problematis solutionem petendam esse ex aequationibus

$$\frac{d\,\Omega}{d\,x} = \theta\,\frac{d\,t}{d\,x}$$
$$\frac{d\,\Omega}{d\,y} = \theta\,\frac{d\,t}{d\,y}$$
$$\frac{d\,\Omega}{d\,z} = \theta\,\frac{d\,t}{d\,z} \text{ etc.}$$

ſiue $\xi = \theta f$, $\eta = \theta g$, $\zeta = \theta h$ etc., deſignante θ multiplicatorem adhuc indeterminatum. Quare ſi, vt in art. praec., ſtatuimus, eſſe *indefinite*

$$t = F\xi + G\eta + H\zeta + \text{etc.} + K$$

habebimus

$$K + \varkappa = \theta(fF + gG + hH + \text{etc.}) + K, \text{ ſiue}$$

$$\theta = \frac{\varkappa}{\omega}$$

accipiendo ω in eadem ſignificatione vt in art. praec. Et quum $\Omega - M$, indefinite, ſit functio homogenea ſecundi ordinis indeterminatarum ξ, η, ζ etc., ſponte patet, eius valorem pro $\xi = \theta f$, $\eta = \theta g$, $\zeta = \theta h$ etc. fieri $= \theta\theta\omega$, et proin valorem minimum, quem Ω pro $t = K + \varkappa$ obtinere poteſt, fieri $= M + \theta\theta\omega = M + \frac{\varkappa\varkappa}{\omega}$. Vice verſa, ſi Ω debet valorem aliquem praeſcriptum $M + \mu\mu$ non ſuperare, valor ipſius t neceſſario inter limites $K - \mu\sqrt{\omega}$ et $K + \mu\sqrt{\omega}$ contentus eſſe debet, vbi $\mu\sqrt{\omega}$ aequalis ſit errori medio in determinatione maxime plauſibili ipſius t metuendo, ſi pro μ accipitur error medius obſeruationum, quibus pondus $= 1$ tribuitur.

31.

Quoties multitudo quantitatum x, y, z etc. paullo maior eſt, determinatio numerica valorum A, B, C etc. ex aequationibus $\xi = 0$, $\eta = 0$, $\zeta = 0$ etc. per eliminationem vulgarem ſatis moleſta euadit. Propterea in Theoria Motus Corporum Coeleſtium art. 182 algorithmum peculiarem addigitauimus, atque in *Disquiſitione de elementis ellipticis Palladis* (Comm. recent. Soc. Gotting. Vol. I.) copioſe explicauimus, per quem labor ille ad tantam quantam quidem res fert ſimplicitatem euehitur. Reducenda ſcilicet eſt functio Ω ſub formam talem:

$$\frac{u^\circ u^\circ}{\mathfrak{A}^\circ} + \frac{u' u'}{\mathfrak{B}'} + \frac{u'' u''}{\mathfrak{C}''} + \frac{u''' u'''}{\mathfrak{D}'''} + \text{etc.} + M$$

vbi diuifores $\mathfrak{A}^\circ$, $\mathfrak{B}'$, $\mathfrak{C}''$, $\mathfrak{D}'''$ etc. funt quantitates determinatae; u°, u', u'', u''' etc. autem functiones lineares ipfarum x, y, z etc. quarum tamen fecunda u' libera eft ab x, tertia u'' libera ab x et y, quarta libera ab x, y et z, et fic porro, ita vt vltima $u^{(\pi-1)}$ folam vltimam indeterminatarum x, y, z etc. implicet; denique coëfficientes, per quos x, y, z etc. refp. multiplicatae funt in u°, u', u'' etc., refp. aequales funt ipfis $\mathfrak{A}^\circ$, $\mathfrak{B}'$, $\mathfrak{C}''$ etc. Quibus ita factis ftatuendum eft $u^\circ = 0$, $u' = 0$, $u'' = 0$, $u''' = 0$ etc., vnde valores incognitarum x, y, z etc. inuerfo ordine commodiffime elicientur. Haud opus videtur, algorithmum ipfum, per quem haec transformatio functionis Ω abfoluitur, hic denuo repetere.

Sed multo adhuc magis prolixum calculum requirit eliminatio indefinita, cuius adiumento illarum determinationum pondera inuenire oportet. Pondus quidem determinationis incognitae vltimae (quae fola vltimam $u^{(\pi-1)}$ ingreditur) per ea, quae in Theoria Motus Corporum Coeleftium demonftrata funt, facile invenitur aequale termino vltimo in ferie diuiforum $\mathfrak{A}^\circ$, $\mathfrak{B}'$, $\mathfrak{C}''$ etc.; quapropter plures calculatores, vt eliminationem illam moleftam euitarent, deficientibus aliis fubfidiis, ita fibi confuluerunt, vt algorithmum de quo diximus pluries, mutato quantitatum x, y, z etc., ordine, repeterent, fingulis deinceps vltimum locum occupantibus. Gratum itaque geometris fore fperamus, fi modum nouum pondera determinationum calculandi, e penitiori argumenti perfcrutatione hauftum hic exponamus, qui nihil amplius defiderandum relinquere videtur.

32.

32.

Statuamus itaque esse (I)

$$u^\circ = \mathfrak{A}^\circ x + \mathfrak{B}^\circ y + \mathfrak{C}^\circ z + \text{etc.} + \mathfrak{L}^\circ$$
$$u' = \mathfrak{B}' y + \mathfrak{C}' z + \text{etc} + \mathfrak{L}'$$
$$u'' = \mathfrak{C}'' z + \text{etc.} + \mathfrak{L}''$$
etc.

Hinc erit indefinite

$$\tfrac{1}{2}\,\mathrm{d}\,\Omega = \xi\,\mathrm{d}x + \eta\,\mathrm{d}y + \zeta\,\mathrm{d}z + \text{etc.}$$
$$= \frac{u^\circ\,\mathrm{d}u^\circ}{\mathfrak{A}^\circ} + \frac{u'\,\mathrm{d}u'}{\mathfrak{B}'} + \frac{u''\,\mathrm{d}u''}{\mathfrak{C}''} + \text{etc.}$$
$$= u^\circ\left(\mathrm{d}x + \frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ}\,\mathrm{d}y + \frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ}\,\mathrm{d}z + \text{etc.}\right)$$
$$+ u'\left(\mathrm{d}y + \frac{\mathfrak{C}'}{\mathfrak{B}'}\,\mathrm{d}z + \text{etc.}\right) + u''(\mathrm{d}z + \text{etc.}) + \text{etc.}$$

vnde colligimus (II)

$$\xi = u^\circ$$
$$\eta = \frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ}\,u^\circ + u'$$
$$\zeta = \frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ}\,u^\circ + \frac{\mathfrak{C}'}{\mathfrak{B}'}\,u' + u''$$
etc.

Supponamus, hinc deriuari formulas ſequentes (III)

$$u^\circ = \xi$$
$$u' = A'\xi + \eta$$
$$u'' = A''\xi + B''\eta + \zeta$$
etc.

Iam e differentiali completo aequationis

$$\Omega = \xi(x - A) + \eta(y - B) + \zeta(z - C) + \text{etc.} + M$$

ſubtracta aequatione

$$\tfrac{1}{2}\,\mathrm{d}\,\Omega = \xi\,\mathrm{d}x + \eta\,\mathrm{d}y + \zeta\,\mathrm{d}z + \text{etc.}$$

ſequitur

$$\tfrac{1}{2}\,\mathrm{d}\,\Omega = (x - A)\,\mathrm{d}\xi + (y - B)\,\mathrm{d}\eta + (z - C)\,\mathrm{d}\zeta + \text{etc.}$$

quae expreſſio identica eſſe debet cum hac ex III demanante:

$$\frac{u^\circ}{\mathfrak{A}^\circ}\,\mathrm{d}\xi + \frac{u'}{\mathfrak{B}'}(A'\mathrm{d}\xi + \mathrm{d}\eta) + \frac{u''}{\mathfrak{C}''}(A''\mathrm{d}\xi + B''\mathrm{d}\eta + \mathrm{d}\zeta) + \text{etc.}$$

Hinc colligimus (IV)

$$x = \frac{u^\circ}{\mathfrak{A}^\circ} + A'.\frac{u'}{\mathfrak{B}'} + A''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + A$$

$$y = \frac{u'}{\mathfrak{B}'} + B''.\frac{u''}{\mathfrak{C}''} + \text{etc.} + B$$

$$z = \frac{u''}{\mathfrak{C}''} + \text{etc.} + C$$

etc.

Subſtituendo in his expreſſionibus pro u°, u', u'' etc. valores earum ex III depromtos, eliminatio indefinita abſoluta erit. Et quidem ad pondera determinanda habebimus (V)

$$[\alpha\alpha] = \frac{1}{\mathfrak{A}^\circ} + \frac{A'A'}{\mathfrak{B}'} + \frac{A''A''}{\mathfrak{C}''} + \frac{A'''A'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\beta\beta] = \frac{1}{\mathfrak{B}'} + \frac{B''B''}{\mathfrak{C}''} + \frac{B'''B'''}{\mathfrak{D}'''} + \text{etc.}$$

$$[\gamma\gamma] = \frac{1}{\mathfrak{C}''} + \frac{C'''C'''}{\mathfrak{D}'''} + \text{etc.}$$

etc.

quarum formularum ſimplicitas nihil deſiderandum relinquit. Ceterum etiam pro coëfficientibus reliquis $[\alpha\beta]$, $[\alpha\gamma]$, $[\beta\gamma]$ etc. formulae aeque ſimplices prodeunt, quas tamen, quum illorum vſus ſit rarior, hic apponere ſuperſedemus.

33.

Propter rei grauitatem, et vt omnia ad calculum parata ſint, etiam formulas explicitas ad determinationem coëfficientium A', A'', A''' etc. B'', B''' etc. etc. hic adſcribere viſum eſt. Duplici modo hic calculus adornari poteſt, quum aequationes identicae prodire debeant, tum ſi valores ipſarum u°, u', u'' etc. ex III depromti in II ſubſtituuntur, tum ex ſubſtitutione valorum ipſarum ξ, η, ζ etc. ex II in III. Prior modus haec formularum ſyſtemata ſubminiſtrat:

$$\frac{\mathfrak{B}^\circ}{\mathfrak{A}^\circ} + A' = 0$$

$$\frac{\mathfrak{C}^\circ}{\mathfrak{A}^\circ} + \frac{\mathfrak{C}'}{\mathfrak{A}'}.A' + A'' = 0$$

$$\frac{\mathfrak{D}^\circ}{\mathfrak{A}^\circ} + \frac{\mathfrak{D}'}{\mathfrak{B}'}.A' + \frac{\mathfrak{D}''}{\mathfrak{C}''}.A'' + A''' = 0$$

etc. vnde inueniuntur A', A'', A''' etc.

$$\frac{\mathfrak{C}'}{\mathfrak{B}'} + B'' = 0$$

$$\frac{\mathfrak{D}'}{\mathfrak{B}'} + \frac{\mathfrak{D}''}{\mathfrak{C}''} B'' + B''' = 0$$

etc. vnde inueniuntur B'', B''' etc.

$$\frac{\mathfrak{D}''}{\mathfrak{C}''} + C''' = 0$$

etc. vnde inueniuntur C''' etc. Et ſic porro.

Alter modus has formulas ſuggerit:

$$\mathfrak{A}^\circ A' + \mathfrak{B}^\circ = 0$$

vnde habetur A'.

$$\mathfrak{A}^\circ A'' + \mathfrak{B}^\circ B'' + \mathfrak{C}^\circ = 0$$
$$\mathfrak{B}' B'' + \mathfrak{C}' = 0$$

vnde inueniuntur B'' et A''.

$$\mathfrak{A}^\circ A''' + \mathfrak{B}^\circ B''' + \mathfrak{C}^\circ C''' + \mathfrak{D}^\circ = 0$$
$$\mathfrak{B}' B''' + \mathfrak{C}' C''' + \mathfrak{D}' = 0$$
$$\mathfrak{C}'' C''' + \mathfrak{D}'' = 0$$

vnde inueniuntur C''', B''', A'''. Et ſic porro.

Vterque modus aeque fere commodus eſt, ſi pondera determinationum cunctarum x, y, z etc. deſiderantur; quoties vero e quantitatibus $[\alpha\alpha]$, $[\beta\beta]$, $[\gamma\gamma]$ etc. vna tantum vel altera requiritur, manifeſto ſyſtema prius longe praeferendum erit.

Ceterum combinatio aequationum I cum IV ad easdem formulas perducit, insuperque calculum duplicem ad eruendos valores maxime plausibiles A, B, C etc. ipsos suppeditat, puta *primo*

$$A = -\frac{\xi^\circ}{\mathfrak{A}^\circ} - A'\frac{\xi'}{\mathfrak{B}'} - A''\frac{\xi''}{\mathfrak{C}''} - A'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$
$$B = \qquad\qquad -\frac{\xi'}{\mathfrak{B}'} - B''\frac{\xi''}{\mathfrak{C}''} - B'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$
$$C = \qquad\qquad\qquad\qquad -\frac{\xi''}{\mathfrak{C}''} - C'''\frac{\xi'''}{\mathfrak{D}'''} - \text{etc.}$$

etc.

Calculus alter identicus est cum vulgari, vbi statuitur $u^\circ = 0$, $u' = 0$, $u'' = 0$ etc.

34.

Quae in art. 32. exposuimus, sunt tantummodo casus speciales theorematis generalioris, quod ita se habet:

THEOREMA. *Designet t functionem linearem indeterminatarum x, y, z etc. hanc*

$$t = fx + gy + hz + \text{etc.} + k,$$

quae transmutata in functionem indeterminatarum u°, u', u'' etc. fiat

$$t = k^\circ u^\circ + k'u' + k''u'' + \text{etc.} + K$$

Quibus ita se habentibus erit K valor maxime plausibilis ipsius t, atque pondus huius determinationis

$$= \frac{1}{\mathfrak{A}^\circ k^\circ k^\circ + \mathfrak{B}'k'k' + \mathfrak{C}''k''k'' + \text{etc.}}$$

Dem. Pars prior theorematis inde patet, quod valor maxime plausibilis ipsius t valoribus $u^\circ = 0$, $u' = 0$, $u'' = 0$ etc. respondere debet. Ad posteriorem demonstrandam obseruamus, quoniam $\frac{1}{2}d\Omega = \xi dx + \eta dy + \zeta dz +$ etc., atque $dt = f dx + g dy + h dz +$ etc., esse, pro $\xi = f$, $\eta = g$, $\zeta = h$ etc., independenter a valoribus differentialium dx, dy, dz etc.

$$d\Omega = 2\,dt$$

Hinc vero ſequitur, pro iisdem valoribus $\xi = f$, $\eta = g$, $\zeta = h$ etc., fieri

$$\frac{u^\circ}{\mathfrak{A}^\circ}\, \mathrm{d}\, u^\circ + \frac{u'}{\mathfrak{B}'}\, \mathrm{d}\, u' + \frac{u''}{\mathfrak{C}''}\, \mathrm{d}\, u'' + \text{etc.} = k^\circ\, \mathrm{d}\, u^\circ + k'\, \mathrm{d}\, u' \; k''\, \mathrm{d}\, u'' + \text{etc.}$$

Iam facile perſpicitur, ſi $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ etc. ſint ab inuicem independentes, etiam $\mathrm{d}u^\circ$, $\mathrm{d}u'$, $\mathrm{d}u''$ etc., ab inuicem independentes eſſe; vnde colligimus, pro $\xi = f$, $\eta = g$, $\zeta = h$ etc. eſſe

$$u^\circ = \mathfrak{A}^\circ k^\circ,\; u' = \mathfrak{B}' k',\; u'' = \mathfrak{C}'' k'' \text{ etc}$$

Quamobrem valor ipſius Ω, iisdem valoribus reſpondens erit

$$= \mathfrak{A}^\circ k^\circ k^\circ + \mathfrak{B}' k' k' + \mathfrak{C}'' k'' k'' + \text{etc.} + M.$$

vnde per art. 29. theorematis noſtri veritas protinus demanat.

Ceterum ſi transformationem functionis t immediate, i. e. absque cognitione ſubſtitutionum IV. art. 32, perficere cupimus, praeſto ſunt formulae:

$$f = \mathfrak{A}^\circ k^\circ$$
$$g = \mathfrak{B}^\circ k^\circ + \mathfrak{B}' k'$$
$$h = \mathfrak{C}^\circ k^\circ + \mathfrak{C}' k' + \mathfrak{C}'' k''$$

etc., vnde coëfficientes k°, k', k'' etc. deinceps determinabuntur, tandemque habebitur

$$K = -\mathfrak{L}^\circ k^\circ - \mathfrak{L}' k' - \mathfrak{L}'' k'' - \text{etc.}$$

35.

Tractatione peculiari dignum eſt problema ſequens, tum propter vtilitatem practicam, tum propter ſolutionis concinnitatem.

Inuenire mutationes valorum maxime plauſibilium incognitarum ab acceſſione aequationis nouae productas, nec non pondera nouarum determinationum.

Retinebimus deſignationes in praecedentibus adhibitas, ita vt aequationes primitiuae, ad pondus = 1 reductae, ſint hae

$v = 0$, $v' = 0$, $v'' = 0$ etc.; aggregatum indefinitum $vv + v'v' + v''v''$ etc. $= \Omega$; porro vt ξ, η, ζ etc. ſint quotientes differentiales partiales

$$\frac{d\,\Omega}{2\,d\,x},\ \frac{d\,\Omega}{2\,d\,y},\ \frac{d\,\Omega}{2\,d\,z}\ \text{etc.}$$

denique vt ex eliminatione indefinita ſequatur

$$\left.\begin{array}{l} x = A + [\alpha\alpha]\,\xi + [\alpha\beta]\,\eta + [\alpha\gamma]\,\zeta + \text{etc.} \\ y = B + [\alpha\beta]\,\xi + [\beta\beta]\,\eta + [\beta\gamma]\,\zeta + \text{etc.} \\ z = C + [\alpha\gamma]\,\xi + [\beta\gamma]\,\eta + [\gamma\gamma]\,\zeta + \text{etc.} \end{array}\right\}\ (\text{I})$$

Iam ſupponamus, accedere aequationem nouam $v^* = 0$ (proxime veram, et cuius pondus $= 1$), et inquiramus, quantas mutationes hinc nacturi ſint tum valores incognitarum maxime plauſibiles A, B, C etc., tum coëfficientes $[\alpha\alpha]$, $[\alpha\beta]$ etc.

Statuamus $\Omega + v^*v^* = \Omega^*$,

$$\frac{d\,\Omega^*}{2\,d\,x} = \xi^*,\ \frac{d\,\Omega^*}{2\,d\,y} = \eta^*,\ \frac{d\,\Omega^*}{2\,d\,z} = \zeta^*\ \text{etc.}$$

ſupponamusque, hinc per eliminationem ſequi

$$x = A^* + [\alpha\alpha^*]\,\xi^* + [\alpha\beta^*]\,\eta^* + [\alpha\gamma^*]\,\zeta^*\ \text{etc.}$$

Denique ſit

$$v^* = fx + gy + hz + \text{etc.} + k$$

prodeat inde, ſubſtitutis pro x, y, z etc. valoribus ex (I),

$$v^* = F\xi + G\eta + H\zeta + \text{etc.} + K$$

ſtatuaturque $Ff + Gg + Hh + \text{etc.} = \omega$.

Manifeſto K erit valor maxime plauſibilis functionis v^*, quatenus ex aequationibus primitiuis ſequitur, ſine reſpectu valoris 0 quem obſeruatio acceſſoria praebuit, atque $\frac{1}{\omega}$ pondus iſtius determinationis.

Iam habemus

$$\xi^* = \xi + fv^*,\ \eta^* = \eta + gv^*,\ \zeta^* = \zeta + hv^*\ \text{etc.}$$

adeoque

$$F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K = v^*(1 + Ff + Gg + Hh + \text{etc.})$$

fiue $v^* = \dfrac{F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K}{1+\omega}$

Perinde fit

$$x = A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - v^*(f[\alpha\alpha] + g[\alpha\beta] + h[\alpha\gamma] + \text{etc.})$$
$$= A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - Fv^*$$
$$= A + [\alpha\alpha]\xi^* + [\alpha\beta]\eta^* + [\alpha\gamma]\zeta^* + \text{etc.} - \frac{F}{1+\omega}(F\xi^* + G\eta^* + H\zeta^* + \text{etc.} + K)$$

Hinc itaque colligimus

$A^* = A - \dfrac{FK}{1+\omega}$, qui erit valor maxime plaufibilis ipfius x ex *omnibus* obferuationibus;

$$[\alpha\alpha^*] = [\alpha\alpha] - \frac{FF}{1+\omega}$$

adeoque pondus iftius determinationis

$$= \frac{1}{[\alpha\alpha] - \dfrac{FF}{1+\omega}}$$

Prorfus eodem modo inuenitur valor maxime plaufibilis ipfius y, *omnibus* obferuationibus fuperftructus

$$B^* = B - \frac{GK}{1+\omega}$$

atque pondus huius determinationis

$$= \frac{1}{[\beta\beta] - \dfrac{GG}{1+\omega}}$$

et fic porro. *Q. E. I.*

Liceat huic folutioni quasdam annotationes adiicere.

I. Subftitutis his nouis valoribus A^*, B^*, C^* etc., functio v^* obtinet valorem maxime plaufibilem

$$K - \frac{K}{1+\omega}(Ff + Gg + Hh + \text{etc.}) = \frac{K}{1+\omega}.$$ Et quum indefinite ſit

$$v^* = \frac{F}{1+\omega}\cdot\xi^* + \frac{G}{1+\omega}\cdot\eta^* + \frac{H}{1+\omega}\cdot\zeta^* + \text{etc.} + \frac{K}{1+\omega}$$

pondus iſtius determinationis per principia art. 29. eruitur

$$= \frac{1+\omega}{Ff + Gg + Hh + \text{etc.}} = \frac{1}{\omega} + 1.$$

Eadem immediate reſultant ex applicatione regulae in fine art. 21. traditae; ſcilicet complexus aequationum primitiuarum praebuerat determinationem $v^* = K$ cum pondere $= \frac{1}{\omega}$, dein obſeruatio noua dedit determinationem aliam, ab illa independentem, $v^* = 0$, cum pondere $= 1$, quibus combinatis prodit determinatio $v^* = \frac{K}{1+\omega}$ cum pondere $= \frac{1}{\omega} + 1$.

II. Hinc porro ſequitur, quum pro $x = A^*$, $y = B^*$, $z = C^*$ etc. eſſe debeat $\xi^* = 0$, $\eta^* = 0$, $\zeta^* = 0$ etc., pro iisdem valoribus fieri

$$\xi = -\frac{fK}{1+\omega},\ \eta = -\frac{gK}{1+\omega},\ \zeta = -\frac{hK}{1+\omega}\ \text{etc.}$$

nec non, quoniam indefinite $\Omega = \xi(x - A) + \eta(y - B) + \zeta(z - C) + \text{etc.} + M$,

$$\Omega = \frac{KK}{(1+\omega)^2}(Ff + Gg + Hh + \text{etc.}) + M = M + \frac{\omega KK}{(1+\omega)^2};$$

denique, quoniam indefinite $\Omega^* = \Omega + v^* v^*$,

$$\Omega^* = M + \frac{\omega KK}{(1+\omega)^2} + \frac{KK}{(1+\omega)^2} = M + \frac{KK}{1+\omega}$$

III. Comparando haec cum iis quae in art. 30. docuimus, animaduertimus, functionem Ω hic valorem minimum obtinere, quem pro valore determinato functionis $v^* = \frac{K}{1+\omega}$ accipere poteſt.

36.

Problematis alius, praecedenti affinis, puta

Inuestigare mutationes valorum maxime plausibilium incognitarum, a mutato pondere vnius ex obseruationibus primitiuis oriundas, nec non pondera nouarum determinationum

solutionem tantummodo hic adscribemus, demonstrationem, quae ad instar art. praec. facile absoluitur, breuitatis caussa supprimentes.

Supponamus, peracto demum calculo animaduerti, alicui obseruationum pondus seu nimis paruum, seu nimis magnum tributum esse, e g. obseruationi primae, quae dedit $V = L$, loco ponderis p in calculo adhibiti rectius tribui pondus $= p^*$. Tunc haud opus erit calculum integrum repetere, sed commodius correctiones per formulas sequentes computare licebit.

Valores incognitarum maxime plausibiles correcti erunt hi:

$$x = A - \frac{(p^* + p)\alpha\lambda}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc.})}$$

$$y = B - \frac{(p^* - p)\beta\lambda}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc.})}$$

$$z = C - \frac{(p^* - p)\gamma\lambda}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc})}$$

etc. ponderaque harum determinationum inuenientur, diuidendo vnitatem resp. per

$$[\alpha\alpha] - \frac{(p^* - p)\alpha\alpha}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc})}$$

$$[\beta\beta] - \frac{(p^* - p)\beta\beta}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc})}$$

$$[\gamma\gamma] - \frac{(p^* - p)\gamma\gamma}{p + (p^* - p)(a\alpha + b\beta + c\gamma + \text{etc.})} \text{ etc.}$$

Haec solutio simul complectitur casum, vbi peracto calculo percipitur, vnam ex obseruationibus omnino reiici debuisse, quum hoc idem sit ac si facias $p^* = 0$; et perinde valor $p^* = \infty$ refertur ad casum eum, vbi aequatio $V = L$, quae in calculo tamquam approximata tractata erat, reuera praecisione absoluta gaudet.

Ceterum quoties vel aequationibus, quibus calculus superstructus erat, *plures* nouae accedunt, vel *pluribus* ex illis pondera erronea tributa esse percipitur, computus correctionum nimis complicatus euaderet; quocirca in tali casu calculum ab integro reficere praestabit.

37.

In art. 15, 16. methodum explicauimus, obseruationum praecisionem proxime determinandi *). Sed haec methodus supponit, errores, qui reuera occurrerint, satis multos exacte cognitos esse, quae conditio, stricte loquendo, rarissime, ne dicam numquam, locum habebit. Quodsi quidem quantitates, quarum valores approximati per obseruationes innotuerunt, secundum legem cognitam, ab vna pluribusue quantitatibus incognitis pendent, harum valores maxime plausibiles per methodum quadratorum minimorum eruere licebit, ac dein valores quantitatum, quae obseruationum obiecta fuerant, illinc computati perparum a valoribus veris discrepare censebuntur, ita vt ipsorum differentias a valoribus obseruatis eo maiori iure tamquam obseruationum errores veros adoptare liceat, quo maior fuerit harum multitudo. Hanc praxin sequuti sunt omnes calculatores, qui obseruationum praecisionem in casibus concretis a posteriori aestimare susceperunt: sed manifesto illa theoretice erronea est, et quamquam in casibus multis ad vsus practicos sufficere possit, tamen

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*) Disquisitio de eodem argumento, quam in commentatione anteriori (*Zeitschrift für Astronomie und verwandte Wissenschaften* Vol. I, p. 185.) *tradideramus*, eidem hypotheli circa indolem functionis probabilitatem errorum exprimentis innixa erat, cui in Theoria motus corporum coelestium methodum quadratorum minimorum superstruxeramus (vid. art. 9, III.).

in aliis enormiter peccare poteſt. Summopere itaque hoc argumentum dignum eſt, quod accuratius enodetur.

Retinebimus in hac disquiſitione deſignationes inde ab art. 19. adhibitas. Praxis ea de qua diximus, quantitates A, B, C etc. tamquam valores veros ipſarum x, y, z conſiderat, et proin ipſas λ, λ', λ'' etc. tamquam valores veros functionum v, v', v'' etc. Si omnes obſeruationes aequali praeciſione gaudent, ipſarumque pondus $p = p' = p''$ etc. pro vnitate acceptum eſt, eaedem quantitates, ſignis mutatis, in illa ſuppoſitione obſeruationum errores exhibent, vnde praecepta art. 16, praebent obſeruationum errorem medium m

$$= \sqrt{\frac{\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.}}{\pi}} = \sqrt{\frac{M}{\pi}}$$

Si obſeruationum praeciſio non eſt eadem, quantitates $-\lambda$, $-\lambda'$, $-\lambda''$ etc. exhiberent obſeruationum errores per radices quadratas e ponderibus multiplicatos, praeceptaque art. 16. ad eandem formulam $\sqrt{\frac{M}{\pi}}$ perducerent, iam errorem medium talium obſeruationum, quibus pondus $= 1$ tribuitur, denotantem. Sed manifeſto calculus exactus requireret, vt loco quantitatum λ, λ', λ'' etc. valores functionum v, v', v'' etc. e valoribus veris ipſarum x, y, z etc. prodeuntes adhiberentur, i. e. loco ipſius M, valor functionis Ω valoribus veris ipſarum x, y, z etc. reſpondens. Qui quamquam aſſignari nequeat, tamen certi ſumus, eum eſſe maiorem quam M (quippe qui eſt minimus poſſibilis), excipiendo caſum infinite parum probabilem, vbi incognitarum valores maxime plauſibiles exacte cum veris quadrant. In genere itaque affirmare poſſumus, praxin vulgarem errorem medium iuſto minorem producere, ſiue obſeruationibus praeciſionem nimis magnam tribuere. Videamus iam, quid doceat theoria rigoroſa.

38.

Ante omnia inueſtigare oportet, quonam modo M ab obſeruationum erroribus veris pendeat. Denotemus hos, vt in art. 28, per e, e', e'' etc., ſtatuamusque ad maiorem ſimplicitatem

$$e\sqrt{p}=\varepsilon,\ e'\sqrt{p'}=\varepsilon',\ e''\sqrt{p''}=\varepsilon'' \text{ etc., nec non}$$
$$m\sqrt{p}=m'\sqrt{p'}=m''\sqrt{p''} \text{ etc. } =\mu$$

Porro ſint valores veri ipſarum x, y, z etc. reſp. $A-x^\circ$, $B-y^\circ$, $C-z^\circ$ etc., quibus reſpondeant valores ipſarum ξ, η, ζ etc. hi $-\xi^\circ$, $-\eta^\circ$, $-\zeta^\circ$ etc. Manifeſto iisdem reſpondebunt valores ipſarum v, v', v'' etc. hi $-\varepsilon$, $-\varepsilon'$, $-\varepsilon''$ etc. ita vt habeatur

$$\xi^\circ=a\varepsilon+a'\varepsilon'+a''\varepsilon''+\text{ etc.}$$
$$\eta^\circ=b\varepsilon+b'\varepsilon'+b''\varepsilon''+\text{ etc.}$$
$$\zeta^\circ=c\varepsilon+c'\varepsilon'+c''\varepsilon''+\text{ etc.}$$

etc. nec non

$$x^\circ=\alpha\varepsilon+\alpha'\varepsilon'+\alpha''\varepsilon''+\text{ etc.}$$
$$y^\circ=\beta\varepsilon+\beta'\varepsilon'+\beta''\varepsilon''+\text{ etc.}$$
$$z^\circ=\gamma\varepsilon+\gamma'\varepsilon'+\gamma''\varepsilon''+\text{ etc.}$$

Denique ſtatuemus

$$\Omega^\circ=\varepsilon\varepsilon+\varepsilon'\varepsilon'+\varepsilon''\varepsilon''+\text{ etc.}$$

ita vt ſit Ω° aequalis valori functionis Ω valoribus veris ipſarum x, y, z etc. reſpondenti. Hinc quum habeatur indefinite $\Omega=M+(x-A)\xi+(y-B)\eta+(z-C)\zeta+$ etc., erit etiam

$$M=\Omega^\circ-x^\circ\xi^\circ-y^\circ\eta^\circ-z^\circ\zeta^\circ-\text{ etc.}$$

Hinc manifeſtum eſt, M, euolutione facta eſſe functionem homogeneam ſecundi ordinis errorum e, e', e'' etc., quae, pro diuerſis errorum valoribus maior minorue euadere poterit. Sed quatenus errorum magnitudo nobis incognita manet, functionem hanc indefinite conſiderare, imprimisque ſecundum principia calculi probabilitatis eius valorem medium aſſignare conueniet. Quem inueniemus, ſi loco quadratorum ee, $e'e'$, $e''e''$ etc. reſp. ſcribimus mm, $m'm'$, $m''m''$ etc., producta vero ee', ee'', $e'e''$ etc.

omnino omittimus, vel quod idem eſt, ſi loco cuiusuis quadrati $\varepsilon\varepsilon$, $\varepsilon'\varepsilon'$, $\varepsilon''\varepsilon''$ etc. ſcribimus $\mu\mu$, productis $\varepsilon\varepsilon'$, $\varepsilon\varepsilon''$ $\varepsilon'\varepsilon''$ etc. prorſus neglectis. Hoc modo e termino $\Omega°$ manifeſto prouenit $\pi\mu\mu$; terminus $-x°\xi°$ producet

$$-(a\alpha + a'\alpha' + a''\alpha'' + \text{etc.})\,\mu\mu = -\mu\mu$$

et ſimiliter ſingulae partes reliquae praebebunt $-\mu\mu$, ita vt valor medius totalis fiat $= (\pi - \rho)\,\mu\mu$, denotante π multitudinem obſeruationum, ρ multitudinem incognitarum. Valor verus quidem ipſius M, prout fors errores obtulit, maior minorue medio fieri poteſt, ſed diſcrepantia eo minoris momenti erit, quo maior fuerit obſeruationum multitudo, ita vt pro valore approximato ipſius μ accipere liceat

$$\sqrt{\frac{M}{\pi - \rho}}$$

Valor itaque ipſius μ, ex praxi erronea, de qua in art. praec. loquuti ſumus, prodiens, augeri debet in ratione quantitatis $\sqrt{(\pi - \rho)}$ ad $\sqrt{\pi}$.

39.

Quo clarius eluceat, quanto iure valorem fortuitum ipſius M medio aequiparare liceat, adhuc inueſtigare oportet errorem medium metuendum, dum ſtatuimus $\frac{M}{\pi - \rho} = mm$. Iſte error medius aequalis eſt radici quadratae e valore medio quantitatis

$$\left(\frac{\Omega° - x°\xi° - y°\eta° - z°\zeta° - \text{etc.} - (\pi - \rho)\,mm}{\pi - \rho}\right)^2$$

quam ita exhibebimus

$$\left(\frac{\Omega° - x°\xi° - y°\eta° - z°\zeta° - \text{etc.}}{\pi - \rho}\right)^2$$

$$-\frac{2\mu\mu}{\pi - \rho}\left(\Omega° - x°\xi° - y°\eta° - z°\zeta° - \text{etc.} - (\pi - \rho)\,\mu\mu\right) - \mu^4$$

et quum manifeſto valor medius termini ſecundi fiat $= 0$, res in eo vertitur, vt indagemus valorem medium functionis

$$\Psi' = (\Omega^\circ - x^\circ \xi^\circ - y^\circ \eta^\circ - z^\circ \zeta^\circ - \text{etc.})^2$$

quo inuento et per N defignato, error medius quaefitus erit

$$= \sqrt{\left(\frac{N}{(\pi - \varrho)^2} - \mu^4\right)}$$

Expreffio Ψ' euoluta manifefto eft functio homogenea fiue errorum e, e', e'' etc., fiue quantitatum $\varepsilon, \varepsilon', \varepsilon''$ etc., eiusque valor medius inuenietur, fi

1° pro biquadratis e^4, e'^4, e''^4 etc. fubftituuntur eorum valores medii

2° pro fingulis productis e binis quadratis vt $e e e' e'$, $e e e'' e''$, $e' e' e'' e''$ etc. producta ex ipforum valoribus mediis, puta $m m m' m'$, $m m m'' m''$, $m' m' m'' m''$ etc.

3° partes vero reliquae, quae implicabunt vel factorem talem $e^3 e'$, vel talem $e e e' e''$, omnino omittuntur. Valores medios biquadratorum e^4, e'^4, e''^4 etc. ipfis biquadratis m^4, m'^4, m''^4 etc. proportionales fupponemus (vid. art. 16), ita vt illi fint ad haec vt ν^4 ad μ^4, adeoque ν^4 denotet valorem medium biquadratorum obferuationum talium quarum pondus $= 1$. Hinc praecepta praecedentia ita quoque exprimi poterunt: Loco fingulorum biquadratorum $\varepsilon^4, \varepsilon'^4, \varepsilon''^4$ etc. fcribendum erit ν^4, loco fingulorum productorum e binis quadratis vt $\varepsilon\varepsilon\varepsilon'\varepsilon'$, $\varepsilon\varepsilon\varepsilon''\varepsilon''$, $\varepsilon'\varepsilon'\varepsilon''\varepsilon''$ etc., fcribendum erit μ^4, omnesque reliqui termini, qui implicabunt factores tales vt $\varepsilon^3 \varepsilon'$, vel $\varepsilon\varepsilon\varepsilon'\varepsilon''$, vel $\varepsilon\varepsilon'\varepsilon''\varepsilon'''$ erunt fupprimendi.

His probe intellectis facile patebit

I. Valorem medium quadrati $\Omega^\circ \Omega^\circ$ effe $\pi\nu^4 + (\pi\pi - \pi)\mu^4$

II. Valor medius producti $\varepsilon\varepsilon x^\circ \xi^\circ$ fit $= a\alpha\nu^4 + (a'\alpha' + a''\alpha'' + \text{etc.})\mu^4$, fiue quoniam $a\alpha + a'\alpha' + a''\alpha'' + \text{etc.} = 1$,

$$= a\alpha(\nu^4 - \mu^4) + \mu^4$$

Et quum perinde valor medius producti $\varepsilon'\varepsilon' x^\circ \xi^\circ$ fiat $=$

$a'\alpha'(\nu^4-\mu^4)+\mu^4$, valor medius producti $\varepsilon''\varepsilon''x^\circ\xi^\circ$ autem $=a''\alpha''(\nu^4-\mu^4)+\mu^4$ et ſic porro, patet, valorem medium producti $(\varepsilon\varepsilon+\varepsilon'\varepsilon'+\varepsilon''\varepsilon''+\text{etc.})\,x^\circ\xi^\circ$ ſiue $\Omega^\circ x^\circ\xi^\circ$ eſſe

$$=\nu^4-\mu^4+\pi\mu^4$$

Eundem valorem medium habebunt producta $\Omega^\circ y^\circ\eta^\circ$, $\Omega^\circ z^\circ\zeta^\circ$ etc. Quapropter valor medius producti $\Omega^\circ(x^\circ\xi^\circ+y^\circ\eta^\circ+z^\circ\zeta^\circ+\text{etc.})$ fit

$$=\varrho\nu^4+\varrho(\pi-1)\mu^4$$

III. Ne euolutiones reliquae nimis prolixae euadant, idonea denotatio introducenda erit. Vtemur itaque characteriſtica Σ ſenſu aliquantum latiori quam ſupra paſſim factum eſt, ita vt denotet aggregatum termini, cui praefixa eſt, cum omnibus ſimilibus ſed non identicis inde per omnes obſeruationum permutationes oriundis Hoc pacto e. g. habemus $x^\circ=\Sigma\alpha\varepsilon$, $x^\circ x^\circ=\Sigma\alpha\alpha\varepsilon\varepsilon+2\Sigma\alpha\alpha'\varepsilon\varepsilon'$. Colligendo itaque valorem medium producti $x^\circ x^\circ\xi^\circ\xi^\circ$ per partes, habemus primo valorem medium producti $\alpha\alpha\varepsilon\varepsilon\xi^\circ\xi^\circ$

$$=aa\alpha\alpha\nu^4+\alpha\alpha(a'a'+a''a''+\text{etc.})\,\mu^4$$
$$=aa\alpha\alpha(\nu^4-\mu^4)+\alpha\alpha\mu^4\Sigma aa$$

Perinde valor medius producti $\alpha'\alpha'\varepsilon'\varepsilon'\xi^\circ\xi^\circ$ fit $=a'a'\alpha'\alpha'(\nu^4-\mu^4)+\alpha'\alpha'\mu^4\Sigma aa$ et ſic porro, adeoque valor medius producti $\xi^\circ\xi^\circ\Sigma\alpha\alpha\varepsilon\varepsilon$

$$=(\nu^4-\mu^4)\Sigma aa\alpha\alpha+\mu^4\Sigma aa\,.\,\Sigma\alpha\alpha$$

Porro valor medius producti $\alpha\alpha'\varepsilon\varepsilon'\xi^\circ\xi^\circ$ fit $=2\alpha\alpha'aa'\mu^4$, valor medius producti $\alpha\alpha''\varepsilon\varepsilon''\xi^\circ\xi^\circ$ perinde $=2\alpha\alpha''aa''\mu^4$ etc., vnde facile concluditur, valorem medium producti $\xi^\circ\xi^\circ\Sigma\alpha\alpha'\varepsilon\varepsilon'$ fieri

$$=2\mu^4\Sigma\alpha a\alpha'a'=\mu^4((\Sigma a\alpha)^2-\Sigma aa\alpha\alpha)=\mu^4(1-\Sigma aa\alpha\alpha)$$

His collectis habemus valorem medium producti $x^\circ x^\circ\xi^\circ\xi^\circ$

$$=(\nu^4-3\mu^4)\Sigma aa\alpha\alpha+2\mu^4+\mu^4\Sigma aa\;\Sigma\alpha\alpha.$$

IV Haud absimili modo inuenitur valor medius producti $x^\circ y^\circ \xi^\circ \eta^\circ$

$$= \nu^4 \Sigma a b \alpha \beta + \mu^4 \Sigma a \alpha b' \beta' + \mu^4 \Sigma a b \alpha' \beta' + \mu^4 \Sigma a \beta b' \alpha'$$

Sed habetur

$$\Sigma a \alpha b' \beta' = \Sigma a \alpha . \Sigma b \beta - \Sigma a \alpha b \beta$$
$$\Sigma a b \alpha' \beta' = \Sigma a b . \Sigma \alpha \beta - \Sigma a b \alpha \beta$$
$$\Sigma a \beta b' \alpha' = \Sigma a \beta . \Sigma b \alpha - \Sigma a \beta b \alpha$$

vnde valor ille medius fit, propter $\Sigma a \alpha = 1$, $\Sigma b \beta = 1$, $\Sigma a \beta = 0$, $\Sigma b \alpha = 0$,

$$= (\nu^4 - 3\mu^4) \Sigma a b \alpha \beta + \mu^4 (1 + \Sigma a b . \Sigma \alpha \beta)$$

V. Quum prorsus eodem modo valor medius producti $x^\circ z^\circ \xi^\circ \zeta^\circ$ fiat

$$= (\nu^4 - 3\mu^4) \Sigma a c \alpha \gamma + \mu^4 (1 + \Sigma a c . \Sigma \alpha \gamma)$$

et sic porro, additio valorem medium producti $x^\circ \xi^\circ (x^\circ \xi^\circ + y^\circ \eta^\circ + z^\circ \zeta^\circ + \text{etc.})$ suppeditat

$$= (\nu^4 - 3\mu^4) \Sigma (a \alpha (a \alpha + b \beta + c \gamma + \text{etc})) + (\varrho + 1) \mu^4$$
$$+ \mu^4 (\Sigma a a . \Sigma \alpha \alpha + \Sigma a b . \Sigma \alpha \beta + \Sigma a c . \Sigma \alpha \gamma + \text{etc.})$$
$$= (\nu^4 - 3\mu^4) \Sigma (a \alpha (a \alpha + b \beta + c \gamma + \text{etc.})) + (\varrho + 2) \mu^4$$

VI. Prorsus eodem modo valor medius producti $y^\circ \eta^\circ (x^\circ \xi^\circ + z^\circ \zeta^\circ + \text{etc.})$ eruitur

$$= (\nu^4 - 3\mu^4) \Sigma (b \beta (a \alpha + b \beta + c \gamma + \text{etc.})) + (\varrho + 2) \mu^4$$

dein valor medius producti $z^\circ \zeta^\circ (x^\circ \xi^\circ + y^\circ \eta^\circ + z^\circ \zeta^\circ + \text{etc.})$

$$= (\nu^4 - 3\mu^4) \Sigma (c \gamma (a \alpha + b \beta + c \gamma + \text{etc.})) + (\varrho + 2) \mu^4$$

et sic porro. Hinc per additionem prodit valor medius quadrati $(x^\circ \xi^\circ + y^\circ \eta^\circ + z^\circ \zeta^\circ + \text{etc.})^2$

$$= (\nu^4 - 3\mu^4) \Sigma ((a \alpha + b \beta + c \gamma + \text{etc})^2) + (\varrho \varrho + 2 \varrho) \mu^4$$

VII. Omnibus tandem rite collectis eruitur

$$N = (\pi - 2\varrho) \nu^4 + (\pi \pi - \pi - 2 \pi \varrho + 4 \varrho + \varrho \varrho) \mu^4 +$$
$$(\nu^4 - 3\mu^4) \Sigma ((a \alpha + b \beta + c \gamma + \text{etc.})^2)$$

$$= (\pi - \varrho)$$

$$= (\pi - \varrho)(\nu^4 - \mu^4) + (\pi - \varrho)^2 \mu^4 - (\nu^4 - 3\mu^4)(\varrho - \Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2))$$

Error itaque medius in determinatione ipſius $\mu\mu$ per formulam

$$\mu\mu = \frac{M}{\pi - \varrho}$$

metuendus erit

$$= \sqrt{\left\{\frac{\nu^4 - \mu^4}{\pi - \varrho} - \frac{\nu^4 - 3\mu^4}{(\pi - \varrho)^2} \cdot (\varrho - \Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2))\right\}}$$

40.

Quantitas $\Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2)$, quae in expreſſionem modo inuentam ingreditur, generaliter quidem ad formam ſimpliciorem reduci nequit; nihilominus duo limites aſſignari poſſunt, inter quos ipſius valor neceſſario iacere debet. *Primo* ſcilicet e relationibus ſupra euolutis facile demonſtratur eſſe

$$(a\alpha + b\beta + c\gamma + \text{etc.})^2 + (a\alpha' + b\beta' + c\gamma' + \text{etc.})^2 + (a\alpha'' + b\beta'' + c\gamma'' + \text{etc.})^2 + \text{etc.} = a\alpha + b\beta + c\gamma + \text{etc.}$$

vnde concludimus, $a\alpha + b\beta + c\gamma + \text{etc.}$ eſſe quantitatem poſitiuam vnitate minorem (ſaltem non maiorem). Idem valet de quantitate $a'\alpha' + b'\beta' + c'\gamma' + \text{etc.}$, quippe cui aggregatum

$$(a'\alpha + b'\beta + c'\gamma + \text{etc.})^2 + (a'\alpha' + b'\beta' + c'\gamma' + \text{etc.})^2 + (a'\alpha'' + b'\beta'' + c'\gamma'' \text{ etc.})^2 + \text{etc.}$$

aequale inuenitur; ac perinde $a''\alpha'' + b''\beta'' + c''\gamma'' + \text{etc.}$ vnitate minor erit, et ſic porro. Hinc $\Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2)$ neceſſario eſt minor quam π. *Secundo* habetur $\Sigma(a\alpha + b\beta + c\gamma + \text{etc.}) = \varrho$, quoniam ſit $\Sigma a\alpha = 1$, $\Sigma b\beta = 1$, $\Sigma c\gamma = 1$ etc.; vnde facile deducitur, ſummam quadratorum $\Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2)$ eſſe maiorem quam $\frac{\varrho\varrho}{\pi}$, vel ſaltem non minorem. Hinc terminus

$$\frac{\nu^4 - 3\mu^4}{(\pi-\varrho)^2}\cdot(\varrho - \Sigma((a\alpha + b\beta + c\gamma + \text{etc.})^2))$$

neceſſario iacet inter limites $-\frac{\nu^4 - 3\mu^4}{\pi-\varrho}$ et $\frac{\nu^4 - 3\mu^4}{\pi-\varrho}\cdot\frac{\varrho}{\pi}$, vel, ſi latiores praeferimus, inter hos $-\frac{\nu^4 - 3\mu^4}{\pi-\varrho}$ et $+\frac{\nu^4 - 3\mu^4}{\pi-\varrho}$, et proin erroris medii in valore ipſius $\mu\mu = \frac{M}{\pi-\varrho}$ metuendi quadratum inter limites $\frac{2\nu^4 - 4\mu^4}{\pi-\varrho}$ et $\frac{2\mu^4}{\pi-\varrho}$, ita vt praeciſionem quantamuis aſſequi liceat, ſi modo obſeruationum multitudo fuerit ſatis magna.

Valde memorabile eſt, in hypotheſi ea (art 9, III.), cui theoria quadratorum minimorum olim ſuperſtructa fuerat, illum terminum omnino excidere, et ſicuti, ad eruendum valorem approximatum erroris medii obſeruationum μ, in omnibus caſibus aggregatum $\lambda\lambda + \lambda'\lambda' + \lambda''\lambda'' + \text{etc.} = M$ ita tractare oportet, ac ſi eſſet aggregatum $\pi - \varrho$ errorum fortuitorum, ita in illa hypotheſi etiam praeciſionem ipſam huius determinationis aequalem fieri ei, quam determinationi ex $\pi - \varrho$ erroribus veris tribuendam eſſe in art. 15. inuenimus.