SUPPLEMENTUM THEORIAE COMBINATIONIS OBSERVATIONUM

ERRORIBUS MINIMIS OBNOXIAE,

AUCTORE

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1.

In tractatione theoriae combinationis obseruationum Voluminis V Commentationum Recentiorum inserta supposuimus, quantitates eas, quarum valores per obseruationes praecisione absoluta non gaudentes propositi sunt, a certis elementis incognitis ita pendere, vt in forma functionum datarum horum elementorum exhibitae sint, reique cardinem in eo verti, vt haec elementa quam exactissime ex obseruationibus deriuentur.

In plerisque quidem casibus suppositio ista immediate locum habet. In aliis vero casibus problematis conditio paullo aliter se offert, ita vt primo aspectu dubium videatur, quonam pacto ad formam requisitam reduci possit. Haud raro scilicet accidit, vt quantitates eae, ad quas referuntur obseruationes, nondum exhibitae sint in forma functionum certorum elementorum, neque etiam ad talem formam reducibiles videantur, saltem non commode vel sine ambagibus: dum, ex altera parte, rei indoles quasdam conditiones suppeditat, quibus valores veri quantitatum obseruatarum exacte et necessario satisfacere debent.

Attamen, re propius considerata, facile perspicitur, hunc casum ab altero reuera essentialiter haud differre, sed ad eundem reduci posse. Designando scilicet multitudinem quantitatum obseruatarum per π, multitudinem aequationum conditionalium autem per σ, eligendoque e prioribus $\pi - \sigma$ ad lubitum, nihil impedit, quominus has ipsas pro elementis accipiamus, reliquasque, quarum multitudo erit σ, adiumento aequationum conditionalium tamquam functiones illarum consideremus, quo pacto res ad suppositionem nostram reducta erit.

Verum enim vero etiamsi haec via in permultis casibus satis commode ad finem propositum perducat, tamen negari non potest, eam minus genuinam, operaeque adeo pretium esse, problema in ista altera forma seorsim tractare, tantoque magis, quod solutionem perelegantem admittit. Quin adeo, quum haec solutio noua ad calculos expeditiores perducat, quam solutio problematis in statu priori, quoties σ est minor quam $\frac{1}{2}\pi$, siue quod idem est, quoties multitudo elementorum in commentatione priori per ρ denotata maior est quam $\frac{1}{2}\pi$, solutionem nouam, quam in commentatione praesente explicabimus, in tali casu praeferre conueniet priori, siquidem aequationes conditionales e problematis indole absque ambagibus depromere licet.

2.

Designemus per v, v', v'' etc. quantitates, multitudine π, quarum valores per obseruationem innotescunt, pendeatque quantitas incognita ab illis tali modo, vt per functionem datam illarum, puta

u, exhibeatur: sint porro l, l', l'' etc. valores quotientium differentialium

$$\frac{\mathrm{d}u}{\mathrm{d}v},\ \frac{\mathrm{d}u}{\mathrm{d}v'},\ \frac{\mathrm{d}u}{\mathrm{d}v''}\ \text{etc.}$$

valoribus veris quantitatum v, v', v'' etc. respondentes. Quemadmodum igitur per substitutionem horum valorum verorum in functione u huius valor verus prodit, ita, si pro v, v', v'' etc. valores erroribus e, e', e'' etc. resp. a veris discrepantes substituuntur, obtinebitur valor erroneus incognitae, cuius error statui potest

$$= le + l'e' + l''e'' + etc.$$

siquidem, quod semper supponemus, errores e, e', e'' etc. tam exigui sunt, vt (pro functione u non lineari) quadrata et producta negligere liceat. Et quamquam magnitudo errorum e, e', e'' etc. incerta maneat, tamen incertitudinem tali incognitae determinationi inhaerentem generaliter aestimare licet, et quidem per errorem medium in tali determinatione metuendum, qui per principia commentationis prioris fit

$$= \sqrt{(llmm + l'l'm'm' + l''l''m''m'' + \text{etc.})}$$

denotantibus m, m', m'' etc. errores medios obseruationum, aut si singulae obseruationes aequali incertitudini obnoxiae sunt,

$$= m\sqrt{(ll + l'l' + l''l'' + \text{etc.})}$$

Manifesto in hoc calculo pro l, l', l'' etc. aequali iure etiam eos valores quotientium differentialium adoptare licebit, qui valoribus obseruatis quantitatum v, v', v'' etc. respondent.

3.

Quoties quantitates v, v', v'' etc. penitus inter se sunt independentes, incognita vnico tantum modo per illas determinari poterit: quamobrem tunc illam incertitudinem nullo modo nec euitare neque diminuere licet, et circa valorem incognitae ex obseruationibus deducendum nihil arbitrio relinquitur.

At longe secus se habet res, quoties inter quantitates v, v', v'' etc. mutua dependentia intercedit, quam per σ aequationes conditionales

$$X = 0,\ Y = 0,\ Z = 0 \text{ etc.}$$

exprimi supponemus, denotantibus X, Y, Z etc. functiones datas indeterminatarum v, v', v'' etc. In hoc casu incognitam nostram infinitis modis diuersis per combinationes quantitatum v, v', v'' etc. determinare licet, quum manifesto loco functionis u adoptari possit quaecunque alia U ita comparata, vt $U - u$ indefinite euanescat, statuendo $X = 0$, $Y = 0$, $Z = 0$ etc.

In applicatione ad casum determinatum nulla quidem hinc prodiret differentia respectu valoris incognitae, si obseruationes absoluta praecisione gauderent: sed quatenus hae erroribus obnoxiae manent, manifesto in genere alia combinatio alium valorem incognitae afferet. Puta, loco erroris

$$le + l'e' + l''e'' + \text{etc.}$$

quem functio u commiserat, iam habebimus

$$Le + L'e' + L''e'' + \text{etc.}$$

si functionem U adoptamus, atque valores quotientium differentialium $\frac{\mathrm{d}U}{\mathrm{d}v}$, $\frac{\mathrm{d}U}{\mathrm{d}v'}$, $\frac{\mathrm{d}U}{\mathrm{d}v''}$ etc. resp. per L, L', L'' etc. denotamus. Et quamquam errores ipsos assignare nequeamus, tamen errores medios in diuersis obseruationum combinationibus metuendos inter se comparare licebit: optimaque combinatio ea erit, in qua hic error medius quam minimus euadit. Qui quum fiat

$$= \sqrt{(LLmm + L'L'm'm' + L''L''m''m'' + \text{etc.})}$$

in id erit incumbendum, vt aggregatum $LLmm + L'L'm'm' + L''L''m''m'' + \text{etc.}$ nanciscatur valorem minimum.

4.

Quum varietas infinita functionum U, quae secundum conditionem in art. praec. enunciatam ipsius u vice fungi possunt, eatenus tantum hic consideranda veniat, quatenus diuersa systemata valorum coëfficientium L, L', L'' etc. inde sequuntur, indagare oportebit ante omnia nexum, qui inter cuncta systemata admissibilia locum habere debet. Designemus valores determinatos quotientium differentialium partialium

$$\frac{\mathrm{d}X}{\mathrm{d}v}, \frac{\mathrm{d}X}{\mathrm{d}v'}, \frac{\mathrm{d}X}{\mathrm{d}v''} \text{ etc.}$$
$$\frac{\mathrm{d}Y}{\mathrm{d}v}, \frac{\mathrm{d}Y}{\mathrm{d}v'}, \frac{\mathrm{d}Y}{\mathrm{d}v''} \text{ etc.}$$
$$\frac{\mathrm{d}Z}{\mathrm{d}v}, \frac{\mathrm{d}Z}{\mathrm{d}v'}, \frac{\mathrm{d}Z}{\mathrm{d}v''} \text{ etc. etc.}$$

quos obtinent, si ipsis v, v', v'' etc. valores veri tribuuntur, resp. per

$$a, \; a', \; a'' \text{ etc.}$$
$$b, \; b', \; b'' \text{ etc.}$$
$$c, \; c', \; c'' \text{ etc. etc.}$$

patetque, si ipsis v, v', v'' etc. accedere concipiantur talia incrementa $\mathrm{d}v$, $\mathrm{d}v'$, $\mathrm{d}v''$ etc. per quae X, Y, Z etc. non mutentur, adeoque singulae maneant $= 0$, i. e. satisfacientia aequationibus

$$0 = a\,\mathrm{d}v + a'\mathrm{d}v' + a''\mathrm{d}v'' + \text{etc.}$$
$$0 = b\,\mathrm{d}v + b'\mathrm{d}v' + b''\mathrm{d}v'' + \text{etc.}$$
$$0 = c\,\mathrm{d}v + c'\mathrm{d}v' + c''\mathrm{d}v'' + \text{etc.}$$
etc.

etiam $u - U$ non mutari debere, adeoque fieri

$$0 = (l-L)\mathrm{d}v + (l'-L')\mathrm{d}v' + (l''-L'')\mathrm{d}v'' + \text{etc.}$$

Hinc facile concluditur, coëfficientes L, L', L'' etc. contentos esse debere sub formulis talibus

$$L = l + ax + by + cz + \text{etc.}$$
$$L' = l' + a'x + b'y + c'z + \text{etc.}$$
$$L'' = l'' + a''x + b''y + c''z + \text{etc.}$$

etc., denotantibus x, y, z etc. multiplicatores determinatos. Vice versa patet, si systema multiplicatorum determinatorum x, y, z etc.

ad lubitum assumatur, semper assignari posse functionem U talem, cui valores ipsorum L, L', L'' etc. his aequationibus conformes respondeant, et quae pro conditione in art. praec. enunciata ipsius u vice fungi possit: quin adeo hoc infinitis modis diuersis effici posse. Modus simplicissimus erit statuere $U = u + xX + yY + zZ +$ etc.; generalius statuere licet $U = u + xX + yY + zZ +$ etc. $+ u'$, denotante u' talem functionem indeterminatarum v, v', v'' etc., quae semper euanescit pro $X = 0$, $Y = 0$, $Z = 0$ etc., et cuius valor in casu determinato de quo agitur fit maximus vel minimus. Sed ad institutum nostrum nulla hinc oritur differentia.

5.

Facile iam erit, multiplicatoribus x, y, z etc. valores tales tribuere, vt aggregatum

$$LLmm + L'L'm'm' + L''L''m''m'' + \text{etc.}$$

assequatur valorem minimum. Manifesto ad hunc finem haud opus est cognitione errorum mediorum m, m', m'' etc. absoluta, sed sufficit ratio, quam inter se tenent. Introducemus itaque ipsorum loco pondera obseruationum p, p', p'' etc., i. e. numeros quadratis mm, $m'm'$, $m''m''$ etc. reciproce proportionales, pondere alicuius obseruationis ad lubitum pro vnitate accepto. Quantitates x, y, z etc. itaque sic determinari debebunt, vt polynomium indefinitum

$$\frac{(ax + by + cz + \text{etc.} + l)^2}{p} + \frac{(a'x + b'y + c'z + \text{etc.} + l')^2}{p'} + \frac{(a''x + b''y + c''z + \text{etc.} + l'')^2}{p''} + \text{etc.}$$

nanciscatur valorem minimum, quod fieri supponemus per valores *determinatos* x°, y°, z° etc.

Introducendo denotationes sequentes

$$\frac{aa}{p} + \frac{a'a'}{p'} + \frac{a''a''}{p''} + \text{etc.} = [aa]$$

$$\frac{ab}{p} + \frac{a'b'}{p'} + \frac{a''b''}{p''} + \text{etc.} = [ab]$$
$$\frac{ac}{p} + \frac{a'c'}{p'} + \frac{a''c''}{p''} + \text{etc.} = [ac]$$
$$\frac{bb}{p} + \frac{b'b'}{p'} + \frac{b''b''}{p''} + \text{etc.} = [bb]$$
$$\frac{bc}{p} + \frac{b'c'}{p'} + \frac{b''c''}{p''} + \text{etc.} = [bc]$$
$$\frac{cc}{p} + \frac{c'c'}{p'} + \frac{c''c''}{p''} + \text{etc.} = [cc]$$

etc. nec non

$$\frac{al}{p} + \frac{a'l'}{p'} + \frac{a''l''}{p''} + \text{etc.} = [al]$$
$$\frac{bl}{p} + \frac{b'l'}{p'} + \frac{b''l''}{p''} + \text{etc.} = [bl]$$
$$\frac{cl}{p} + \frac{c'l'}{p'} + \frac{c''l''}{p''} + \text{etc.} = [cl]$$

etc.

manifesto conditio minimi requirit vt fiat

$$\left.\begin{array}{l} 0 = [aa]x^\circ + [ab]y^\circ + [ac]z^\circ + \text{etc.} + [al] \\ 0 = [ab]x^\circ + [bb]y^\circ + [bc]z^\circ + \text{etc.} + [bl] \\ 0 = [ac]x^\circ + [bc]y^\circ + [cc]z^\circ + \text{etc.} + [cl] \\ \text{etc.} \end{array}\right\} \quad (1)$$

Postquam quantitates x°, y°, z° etc. per eliminationem hinc deriuatae sunt, statuetur

$$\left.\begin{array}{l} ax^\circ + by^\circ + cz^\circ + \text{etc.} + l = L \\ a'x^\circ + b'y^\circ + c'z^\circ + \text{etc.} + l' = L' \\ a''x^\circ + b''y + c''z^\circ + \text{etc.} + l'' = L'' \\ \text{etc.} \end{array}\right\} \quad (2)$$

His ita factis, functio quantitatum v, v' v'' etc. ea ad determinationem incognitae nostrae maxime idonea minimaeque incertitudini obnoxia erit, cuius quotientes differentiales partiales in casu determinato de quo agitur habent valores L, L', L'' etc. resp., pondusque huius determinationis, quod per P denotabimus, erit

$$= \frac{1}{\frac{LL}{p} + \frac{L'L'}{p'} + \frac{L''L''}{p''} + \text{etc.}} \quad (3)$$

siue $\frac{1}{P}$ erit valor polynomii supra allati pro eo systemate valorum quantitatum x, y, z etc., per quod aequationibus (1) satisfit.

6.

In art. praec. eam functionem U dignoscere docuimus, quae determinationi maxime idoneae incognitae nostrae inseruit: videamus iam, quemnam *valorem* incognita hoc modo assequatur. Designetur hic valor per K, qui itaque oritur, si in U valores obseruati quantitatum v, v', v'' etc. substituuntur; per eandem substitutionem obtineat functio u valorem k; denique sit $\varkappa$ valor verus incognitae, qui proin e valoribus veris quantitatum v, v', v'' etc. proditurus esset, si hos vel in U vel in u substituere possemus. Hinc itaque erit

$$k = \varkappa + le + l'e' + l''e'' + \text{etc.}$$
$$K = \varkappa + Le + L'e' + L''l'' + \text{etc.}$$

adeoque

$$K = k + (L - l)e + (L' - l')e' + (L'' - l'')e'' + \text{etc.}$$

Substituendo in hac aequatione pro $L - l$, $L' - l'$, $L'' - l''$ etc. valores ex (2), statuendoque

$$\left.\begin{aligned} ae + a'e' + a''e'' + \text{etc.} &= \mathfrak{A} \\ be + b'e' + b''e'' + \text{etc.} &= \mathfrak{B} \\ ce + c'e' + c''e'' + \text{etc.} &= \mathfrak{C} \end{aligned}\right\} (4)$$

etc., habebimus

$$K = k + \mathfrak{A}x^\circ + \mathfrak{B}y^\circ + \mathfrak{C}z^\circ \text{ etc.} \quad (5)$$

Valores quantitatum 𝔄, 𝔅, ℭ etc. per formulas (4) quidem calculare non possumus, quum errores e, e', e'' etc. maneant incogniti; at sponte manifestum est, illos nihil aliud esse, nisi valores functionum X, Y, Z etc., qui prodeunt, si pro v, v', v'' etc. valores obseruati substituuntur. Hoc modo systema aequationum (1), (3), (5) completam problematis nostri solutionem exhibet, quum ea, quae in fine art. 2. de computo quantitatum l, l', l'' etc., valoribus obseruatis quantitatum v, v', v'' etc. superstruendo monuimus, manifesto aequali iure ad computum quantitatum a, a', a'' etc. b, b', b'' etc. etc. extendere liceat.

7.

Loco formulae (3), pondus determinationis maxime plausibilis exprimentis, plures aliae exhiberi possunt, quas euoluere operae pretium erit.

Primo obseruamus, si aequationes (2) resp. per $\frac{a}{p}$, $\frac{a'}{p'}$, $\frac{a''}{p''}$ etc. multiplicentur et addantur, prodire

$$[aa]x^\circ + [ab]y^\circ + [ac]z^\circ + \text{etc.} = \frac{aL}{p} + \frac{a'L'}{p'} + \frac{a''L''}{p''} + \text{etc.}$$

Pars ad laeuam fit $= 0$, partem ad dextram iuxta analogiam per $[aL]$ denotamus: habemus itaque

$$[aL] = 0, \text{ et prorsus simili modo } [bL] = 0, [cL] = 0 \text{ etc.}$$

Multiplicando porro aequationes (2) deinceps per $\frac{L}{p}$, $\frac{L'}{p'}$, $\frac{L''}{p''}$ etc., et addendo, inuenimus

$$\frac{lL}{p} + \frac{l'L'}{p'} + \frac{l''L''}{p''} + \text{etc.} = \frac{LL}{p} + \frac{L'L'}{p'} + \frac{L''L''}{p''} + \text{etc.}$$

vnde obtinemus expressionem *secundam* pro pondere,

$$P = \frac{1}{\frac{lL}{p} + \frac{l'L'}{p'} + \frac{l''L''}{p''} + \text{etc.}}$$

Denique multiplicando aequationes (2) deinceps per $\frac{l}{p}$, $\frac{l'}{p'}$, $\frac{l''}{p''}$, etc. et addendo, peruenimus ad expressionem *tertiam* ponderis

$$P = \frac{1}{[al]x^\circ + [bl]y^\circ + [cl]z + \text{etc.} + [ll]}$$

si ad instar reliquarum denotationum statuimus

$$\frac{ll}{p} + \frac{l'l'}{p'} + \frac{l''l''}{p''} + \text{etc.} = [ll]$$

Hinc adiumento aequationum (1) facile fit transitus ad *expressionem quartam*, quam ita exhibemus:

$$\frac{1}{P} = [ll] - [aa]x^\circ x^\circ - [bb]y^\circ y^\circ - [cc]z^\circ z^\circ - \text{etc.}$$
$$- 2[ab]x^\circ y^\circ - 2[ac]x^\circ z^\circ - 2[bc]y^\circ z^\circ - \text{etc.}$$

8.

Solutio generalis, quam hactenus explicauimus, ei potissimum casui adaptata est, vbi *vna* incognita a quantitatibus obseruatis pendens determinanda est. Quoties vero plures incognitae ab iisdem obseruationibus pendentes valores maxime plausibiles exspectant, vel quoties adhuc incertum est, quasnam potissimum incognitas ex obseruationibus deriuare oporteat, has alia ratione praeparare conueniet, cuius euolutionem iam aggredimur.

Considerabimus quantitates x, y, z etc. tamquam indeterminatas, statuemus

$$\left.\begin{array}{l} [aa]x + [ab]y + [ac]z + \text{etc.} = \xi \\ [ab]x + [bb]y + [bc]z + \text{etc.} = \eta \\ [ac]x + [bc]y + [cc]z + \text{etc.} = \zeta \end{array}\right\} \quad (6)$$

etc., supponemusque, per eliminationem hinc sequi

$$\left.\begin{array}{l} [\alpha\alpha]\xi + [\alpha\beta]\eta + [\alpha\gamma]\zeta + \text{etc.} = x \\ [\beta\alpha]\xi + [\beta\beta]\eta + [\beta\gamma]\zeta + \text{etc.} = y \\ [\gamma\alpha]\xi + [\beta\gamma]\eta + [\gamma\gamma]\zeta + \text{etc.} = z \end{array}\right\} \quad (7)$$

etc.

Ante omnia hic obseruare oportet, coëfficientes symmetrice positos necessario aequales fieri, puta

$$[\beta\alpha] = [\alpha\beta]$$
$$[\gamma\alpha] = [\alpha\gamma]$$
$$[\gamma\beta] = [\beta\gamma] \text{ etc.}$$

quod quidem e theoria generali eliminationis in aequationibus linearibus sponte sequitur, sed etiam infra, absque illa, directe demonstrabitur.

Habebimus itaque

$$\left.\begin{aligned} x^\circ &= -[\alpha\alpha].[al] - [\alpha\beta].[bl] - [\alpha\gamma].[cl] - \text{etc.} \\ y^\circ &= -[\alpha\beta].[al] - [\beta\beta].[bl] - [\beta\gamma].[cl] - \text{etc.} \\ z^\circ &= -[\alpha\gamma].[al] - [\beta\gamma].[bl] - [\gamma\gamma].[cl] - \text{etc.} \end{aligned}\right\} (8)$$

etc.

vnde, si statuimus

$$\left.\begin{aligned} [\alpha\alpha]\mathfrak{A} + [\alpha\beta]\mathfrak{B} + [\alpha\gamma]\mathfrak{C} + \text{etc.} &= A \\ [\alpha\beta]\mathfrak{A} + [\beta\beta]\mathfrak{B} + [\beta\gamma]\mathfrak{C} + \text{etc.} &= B \\ [\alpha\gamma]\mathfrak{A} + [\beta\gamma]\mathfrak{B} + [\gamma\gamma]\mathfrak{C} + \text{etc.} &= C \end{aligned}\right\} (9)$$

etc., obtinemus

$$K = k - A[al] - B[bl] - C[cl] - \text{etc.}$$

vel si insuper statuimus

$$\left.\begin{aligned} aA + bB + cC + \text{etc.} &= p\varepsilon \\ a'A + b'B + c'C + \text{etc.} &= p'\varepsilon' \\ a''A + b''B + c''C + \text{etc.} &= p''\varepsilon'' \end{aligned}\right\} (10)$$

etc., erit

$$K = k - l\varepsilon - l'\varepsilon' - l''\varepsilon'' - \text{etc.} \quad (11)$$

9.

Comparatio aequationum (7), (9) docet, quantitates auxiliares A, B, C etc. esse valores indeterminatarum x, y, z etc. respondentes valoribus indeterminatarum ξ, η, ζ etc. his $\xi = \mathfrak{A}$, $\eta = \mathfrak{B}$, $\zeta = \mathfrak{C}$ etc., vnde patet haberi

$$\left.\begin{aligned}
[aa]A + [ab]B + [ac]C + \text{etc.} &= \mathfrak{A}\\
[ab]A + [bb]B + [bc]C + \text{etc.} &= \mathfrak{B}\\
[ac]A + [bc]B + [cc]C + \text{etc.} &= \mathfrak{C}
\end{aligned}\right\} \quad (12)$$

etc. Multiplicando itaque aequationes (10) resp. per $\frac{a}{p}$, $\frac{a'}{p'}$, $\frac{a''}{p''}$ etc. et addendo, obtinemus

$$\left.\begin{aligned}
\mathfrak{A} &= a\varepsilon + a'\varepsilon' + a''\varepsilon'' + \text{etc.}\\
&\text{et prorsus simili modo}\\
\mathfrak{B} &= b\varepsilon + b'\varepsilon' + b''\varepsilon'' + \text{etc.}\\
\mathfrak{C} &= c\varepsilon + c'\varepsilon' + c''\varepsilon'' + \text{etc.}
\end{aligned}\right\} \quad (13)$$

etc. Iam quum $\mathfrak{A}$ sit valor functionis X, si pro v, v', v'' etc. valores obseruati substituuntur, facile perspicietur, si his applicentur correctiones $-\varepsilon$, $-\varepsilon'$, $-\varepsilon''$ etc. resp., functionem X hinc adepturam esse valorem 0, et perinde functiones Y, Z etc. hinc ad valorem euanescentem reductum iri. Simili ratione ex aequatione (11) colligitur, K esse valorem functionis u ex eadem substitutione emergentem.

Applicationem correctionum $-\varepsilon$, $-\varepsilon'$, $-\varepsilon''$ etc. ad obseruationes, vocabimus *obseruationum compensationem*, manifestoque deducti sumus ad conclusionem grauissimam, puta, obseruationes eo quem docuimus modo compensatas omnibus aequationibus conditionalibus exacte satisfacere, atque cuilibet quantitati ab obseruationibus quomodocunque pendenti eum ipsum valorem conciliare, qui ex obseruationum non mutatarum combinatione maxime idonea emergeret. Quum itaque impossibile sit, errores ipsos e, e', e'' etc. ex aequationibus conditionalibus eruere, quippe quarum multitudo haud sufficit, saltem *errores maxime plausibiles* nacti sumus, qua denominatione quantitates ε, ε', ε'' etc. designare licebit.

10.

Quum multitudo obseruationum maior esse supponatur multitudine aequationum conditionalium, praeter systema correctionum maxime plausibilium $-\varepsilon$, $-\varepsilon'$, $-\varepsilon''$ etc. infinite multa alia inueniri possunt, quae aequationibus conditionalibus satisfaciant, operaeque pretium est indagare, quomodo haec ad illud se habeant. Constituant itaque $-E$, $-E'$, $-E''$ etc. tale systema a maxime plausibili diuersum, habebimusque

$$aE + a'E' + a''E'' + \text{etc.} = \mathfrak{A}$$
$$bE + b'E' + b''E'' + \text{etc.} = \mathfrak{B}$$
$$cE + c'E' + c''E'' + \text{etc.} = \mathfrak{C}$$

etc. Multiplicando has aequationes resp. per A, B, C etc. et addendo, obtinemus adiumento aequationum (10)

$$p\varepsilon E + p'\varepsilon' E' + p''\varepsilon'' E'' + \text{etc.} = A\mathfrak{A} + B\mathfrak{B} + C\mathfrak{C} + \text{etc.}$$

Prorsus vero simili modo aequationes (13) suppeditant

$$p\varepsilon\varepsilon + p'\varepsilon'\varepsilon' + p''\varepsilon''\varepsilon'' + \text{etc.} = A\mathfrak{A} + B\mathfrak{B} + C\mathfrak{C} + \text{etc.} \quad (14)$$

E combinatione harum duarum aequationum facile deducitur

$$pEE + p'E'E' + p''E''E'' + \text{etc.} = p\varepsilon\varepsilon + p'\varepsilon'\varepsilon' + p''\varepsilon''\varepsilon'' + \text{etc.}$$
$$+ p(E-\varepsilon)^2 + p'(E'-\varepsilon')^2 + p''(E''-\varepsilon'')^2 + \text{etc.}$$

Aggregatum $pEE + p'E'E' + p''E''E'' +$ etc. itaque necessario *maius* erit aggregato $p\varepsilon\varepsilon + p'\varepsilon'\varepsilon' + p''\varepsilon''\varepsilon'' +$ etc., quod enunciari potest tamquam

THEOREMA. Aggregatum quadratorum correctionum, per quas obseruationes cum aequationibus conditionalibus conciliare licet, per pondera obseruationum resp. multiplicatorum, fit minimum, si correctiones maxime plausibiles adoptantur.

Hoc est ipsum principium quadratorum minimorum, ex quo etiam aequationes (12), (10) facile immediate deriuari possunt. Ceterum pro hoc aggregato minimo, quod in sequentibus per S denotabimus, aequatio (14) nobis suppeditat expressionem $\mathfrak{A}A + B\mathfrak{B} + C\mathfrak{C} +$ etc.

11.

Determinatio errorum maxime plausibilium, quum a coëfficientibus l, l', l'' etc. independens sit, manifesto praeparationem commodissimam sistit, ad quemuis vsum, in quem obseruationes vertere placuerit. Praeterea perspicuum est, ad illud negotium haud opus esse eliminatione *indefinita* seu cognitione coëfficientium $[\alpha\alpha]$, $[\alpha\beta]$ etc., nihilque aliud requiri, nisi vt quantitates auxiliares A, B, C etc., quas in sequentibus *correlata* aequationum conditionalium $X = 0$, $Y = 0$, $Z = 0$ etc. vocabimus, ex aequationibus (12) per eliminationem definitam eliciantur atque in formulis (10) substituantur.

Quamquam vero haec methodus nihil desiderandum linquat, quoties quantitatum ab obseruationibus pendentium valores maxime plausibiles tantummodo requiruntur, tamen res secus se habere videtur, quoties insuper pondus alicuius determinationis in votis est, quum ad hunc finem, prout hoc vel illa quatuor expressionum supra traditarum vti placuerit, cognitio quantitatum L, L', L'' etc., vel saltem cognitio harum x°, y°, z° etc. necessaria videatur. Hac ratione vtile erit, negotium eliminationis accuratius perscrutari, vnde via facilior ad pondera quoque inuenienda se nobis aperiet.

12.

Nexus quantitatum in hac disquisitione occurrentium haud parum illustratur per introductionem functionis indefinitae secundi ordinis

$$[aa]xx + 2[ab]xy + 2[ac]xz + \text{etc.} + [bb]yy + 2[bc]yz + \text{etc.} + [cc]zz + \text{etc.}$$

quam per T denotabimus. Primo statim obuium est, hanc functionem fieri

$$\frac{(ax + by + cz + \text{etc.})^2}{p} + \frac{(a'x + b'y + c'z + \text{etc.})^2}{p'} + \frac{(a''x + b''y + c''z + \text{etc.})^2}{p''} + \text{etc.} \qquad (15)$$

Porro patet esse

$$T = x\xi + y\eta + z\zeta + \text{etc.} \quad (16)$$

et si hic denuo x, y, z etc. adiumento aequationum (7) per ξ, η, ζ etc. exprimuntur,

$$T = [\alpha\alpha]\xi\xi + 2[\alpha\beta]\xi\eta + 2[\alpha\gamma]\xi\zeta + \text{etc.} + [\beta\beta]\eta\eta + 2[\beta\gamma]\eta\zeta + \text{etc.} + [\gamma\gamma]\zeta\zeta + \text{etc.}$$

Theoria supra euoluta bina systemata valorum determinatorum quantitatum x, y, z etc., atque ξ, η, ζ etc. continet; priori, in quo $x = x^0$, $y = y^0$, $z = z^0$ etc. $\xi = -[al]$, $\eta = -[bl]$, $\zeta = -[cl]$ etc., respondebit valor ipsius T hic

$$T = [ll] - \frac{1}{P}$$

quod vel per expressionem tertiam ponderis P cum aequatione (16) comparatam, vel per quartam sponte elucet; posteriori, in quo $x = A$, $y = B$, $z = C$ etc., atque $\xi = \mathfrak{A}$, $\eta = \mathfrak{B}$, $\zeta = \mathfrak{C}$ etc., respondet valor $T = S$, vti vel e formulis (10) et (15), vel ex his (14) et (16) manifestum est.

13.

Iam negotium principale consistit in transformatione functionis T ei simili, quam in Theoria Motus Corporum Coelestium art. 182 atque fusius in Disquisitione de elementis ellipticis Palladis exposuimus. Scilicet statuemus (17)

$$[bb, 1] = [bb] - \frac{[ab]^2}{[aa]}$$

$$[bc, 1] = [bc] - \frac{[ab]\,[ac]}{[aa]}$$

$$[bd, 1] = [bd] - \frac{[ab]\,[ad]}{[aa]}$$

etc.

$$[cc, 2] = [cc] - \frac{[ac]^2}{[aa]} - \frac{[bc, 1]^2}{[bb, 1]}$$

$$[cd, 2] = [cd] - \frac{[ac]\,[ad]}{[aa]} - \frac{[bc, 1]\,[bd, 1]}{[bb, 1]}$$

etc.

$$[dd, 3] = [dd] - \frac{[ad]^2}{[aa]} - \frac{[bd, 1]^2}{[bb, 1]} - \frac{[cd, 2]^2}{[cc, 2]}$$

etc. etc. Dein statuendo *)

$$[bb,1]y + [bc, 1]z + [bd, 1]w + \text{etc.} = \eta'$$
$$[cc, 2]z + [cd, 2]w + \text{etc.} = \zeta''$$
$$[dd, 3]w + \text{etc.} = \varphi'''$$

etc., erit

$$T = \frac{\xi\xi}{[aa]} + \frac{\eta'\eta'}{[bb, 1]} + \frac{\zeta''\zeta''}{[cc, 2]} + \frac{\varphi'''\varphi'''}{[dd, 3]} + \text{etc.}$$

quantitatesque η', ζ'', φ''' etc. a ξ, η, ζ, φ etc. pendebunt per aequationes sequentes:

$$\eta' = \eta - \frac{[ab]}{[aa]}\xi$$

$$\zeta'' = \zeta - \frac{[ac]}{[aa]}\xi - \frac{[bc, 1]}{[bb, 1]}\eta'$$

$$\varphi''' = \varphi - \frac{[ad]}{[aa]}\xi - \frac{[bd, 1]}{[bb, 1]}\eta' - \frac{[cd, 2]}{[cc, 2]}\zeta''$$

etc.

Facile iam omnes formulae ad propositum nostrum necessariae hinc desumuntur. Scilicet ad determinationem correlatorum A, B, C etc. statuemus (18)

𝔅′

*) In praecedentibus sufficere poterant ternae literae pro variis systematibus quantitatum ad tres primas aequationes conditionales referendae: hoc vero loco, vt algorithmi lex clarius eluceat, quartam adiungere visum est; et quum in serie naturali literas a, b, c; A, B, C; 𝔄, 𝔅, ℭ sponte sequantur d, D, 𝔇, in serie x, y, z, deficiente alphabeto, apposuimus w, nec non in hac ξ, η, ζ hanc φ.

$$\mathfrak{B}' = \mathfrak{B} - \frac{[ab]}{[aa]}\mathfrak{A}$$

$$\mathfrak{C}'' = \mathfrak{C} - \frac{[ac]}{[aa]}\mathfrak{A} - \frac{[bc,1]}{[bb,1]}\mathfrak{B}'$$

$$\mathfrak{D}''' = \mathfrak{D} - \frac{[ad]}{[aa]}\mathfrak{A} - \frac{[bd,1]}{[bb,1]}\mathfrak{B}' - \frac{[cd,2]}{[cc,2]}\mathfrak{C}''$$

etc., ac dein A, B, C, D etc. eruentur per formulas sequentes, et quidem ordine inuerso, incipiendo ab vltima,

$$\left.\begin{aligned}
[aa]A + [ab]B + [ac]C + [ad]D + \text{etc.} &= \mathfrak{A}\\
[bb,1]B + [bc,1]C + [bd,1]D + \text{etc.} &= \mathfrak{B}'\\
[cc,2]C + [cd,2]D + \text{etc.} &= \mathfrak{C}''\\
[dd,3]D + \text{etc.} &= \mathfrak{D}'''\\
\text{etc.} &
\end{aligned}\right\}(19)$$

Pro aggregato S autem habemus formulam nouam (20)

$$S = \frac{\mathfrak{A}\mathfrak{A}}{[aa]} + \frac{\mathfrak{B}'\mathfrak{B}'}{[bb,1]} + \frac{\mathfrak{C}''\mathfrak{C}''}{[cc,2]} + \frac{\mathfrak{D}'''\mathfrak{D}'''}{[dd,3]} \text{ etc.}$$

Denique si pondus P, quod determinationi maxime plausibili quantitatis per functionem u expressae tribuendum est, desideratur, faciemus (21)

$$[bl,1] = [bl] - \frac{[ab]\,[al]}{[aa]}$$

$$[cl,2] = [cl] - \frac{[ac]\,[al]}{[aa]} - \frac{[bc,1]\,[bl,1]}{[bb,1]}$$

$$[dl,3] = [dl] - \frac{[ad]\,[al]}{[aa]} - \frac{[bd,1]\,[bl,1]}{[bb,1]} - \frac{[cd,2]\,[cl,2]}{[cc,2]}$$

etc., quo facto erit (22)

$$\frac{1}{P} = [ll] - \frac{[al]^2}{[aa]} - \frac{[bl,1]^2}{[bb,1]} - \frac{[cl,2]^2}{[cc,2]} - \frac{[dl,3]^2}{[dd,3]} - \text{etc.}$$

Formulae (17) .... (22), quarum simplicitas nihil desiderandum relinquere videtur, solutionem problematis nostri ab omni parte completam exhibent.

14.

Postquam problemata primaria absoluimus, adhuc quasdam quaestiones secundarias attingemus, quae huic argumento maiorem lucem affundent.

Primo inquirendum est, num eliminatio, per quam x, y, z etc. ex ξ, η, ζ etc. deriuare oportet, vmquam impossibilis fieri possit. Manifesto hoc eueniret, si functiones ξ, η, ζ etc. inter se haud independentes essent. Supponamus itaque aliquantisper, vnam earum per reliquas iam determinari, ita vt habeatur aequatio identica

$$\alpha\xi + \beta\eta + \gamma\zeta + \text{etc.} = 0$$

denotantibus α, β, γ etc. numeros determinatos. Erit itaque

$$\alpha[aa] + \beta[ab] + \gamma[ac] + \text{etc.} = 0$$
$$\alpha[ab] + \beta[bb] + \gamma[bc] + \text{etc.} = 0$$
$$\alpha[ac] + \beta[bc] + \gamma[cc] + \text{etc.} = 0$$

etc., vnde, si statuimus

$$\alpha a + \beta b + \gamma c + \text{etc.} = p\,\Theta$$
$$\alpha a' + \beta b' + \gamma c' + \text{etc.} = p'\,\Theta'$$
$$\alpha a'' + \beta b'' + \gamma c'' + \text{etc.} = p''\,\Theta''$$

etc., sponte sequitur

$$a\Theta + a'\Theta' + a''\Theta'' + \text{etc.} = 0$$
$$b\Theta + b'\Theta' + b''\Theta'' + \text{etc.} = 0$$
$$c\Theta + c'\Theta' + c''\Theta'' + \text{etc.} = 0$$

etc., nec non

$$p\Theta\Theta + p'\Theta'\Theta' + p''\Theta''\Theta'' + \text{etc.} = 0$$

quae aequatio, quum omnes p, p', p'' etc. natura sua sint quantitates positiuae, manifesto consistere nequit, nisi fuerit $\Theta = 0$, $\Theta' = 0$, $\Theta'' = 0$ etc.

Iam consideremus valores differentialium completorum dX, dY, dZ etc., respondentes valoribus iis quantitatum v, v', v'' etc, ad quos referuntur obseruationes. Haec differentialia, puta

$$a\,\mathrm{d}v + a'\,\mathrm{d}v' + a''\,\mathrm{d}v'' + \text{etc.}$$
$$b\,\mathrm{d}v + b'\,\mathrm{d}v' + b''\,\mathrm{d}v'' + \text{etc.}$$
$$c\,\mathrm{d}v + c'\,\mathrm{d}v' + c''\,\mathrm{d}v'' + \text{etc.}$$

etc., per conclusionem, ad quam modo delati sumus, inter se ita dependentia erunt, vt per α, β, γ etc. resp. multiplicata aggregatum identice euanescens producant, siue quod idem est, quoduis ex ipsis (cui quidem respondet multiplicator α, β, γ etc. non euanescens) sponte euanescet, simulac omnia reliqua euanescere supponuntur. Quamobrem ex aequationibus conditionalibus $X = 0$, $Y = 0$, $Z = 0$ etc., vna (ad minimum) pro *superflua* habenda est, quippe cui sponte satisfit, simulac reliquis satisfactum est.

Ceterum si res profundius inspicitur, apparet, hanc conclusionem per se tantum pro ambitu infinite paruo variabilitatis indeterminatarum valere. Scilicet proprie duo casus distinguendi erunt, alter, vbi vna aequationum conditionalium $X = 0$, $Y = 0$, $Z = 0$ etc. absolute et generaliter iamiam in reliquis contenta est, quod facile in quouis casu auerti poterit; alter, vbi, quasi fortuito, pro iis valoribus concretis quantitatum v, v', v'' etc., ad quos obseruationes referuntur, vna functionum X, Y, Z etc. e. g. prima X, valorem maximum vel minimum (vel generalius, stationarium) nanciscitur respectu mutationum omnium, quas quantitatibus v, v', v'' etc., saluis aequationibus $Y = 0$, $Z = 0$ etc., applicare possemus. Attamen quum in disquisitione nostra variabilitas quantitatum tantummodo intra limites tam arctos consideretur, vt ad instar infinite paruae tractari possit, hic casus secundus (qui in praxi vix vmquam occurret) eundem effectum habebit, quem primus, puta vna aequationum conditionalium tamquam superflua reiicienda erit, certique esse possumus, si omnes aequationes conditionales retentae eo sensu quem hic intelligimus ab inuicem independentes sint, eliminationem necessario fore possibilem. Ceterum disquisitionem vberiorem, qua hoc argumentum, propter theoreticam subtilitatem potius quam practicam vtilitatem haud indignum est, ad aliam occasionem nobis reseruare debemus.

15.

In commentatione priori art. 37 sqq. methodum docuimus, obseruationum praecisionem a posteriori quam proxime eruendi. Scilicet si valores approximati π quantitatum per obseruationes aequali praecisione gaudentes innotuerunt, et cum valoribus iis comparantur, qui e valoribus maxime plausibilibus ϱ elementorum, a quibus illae pendent, per calculum prodeunt: differentiarum quadrata addere, aggregatumque per $\pi - \varrho$ diuidere oportet, quo facto quotiens considerari poterit tamquam valor approximatus quadrati erroris medii tali obseruationum generi inhaerentis. Quoties obseruationes inaequali praecisione gaudent, haec praecepta eatenus tantum mutanda sunt, vt quadrata ante additionem per obseruationum pondera multiplicari debeant, errorque medius hoc modo prodiens ad obseruationes referatur, quarum pondus pro vnitate acceptum est.

Iam in tractatione praesente illud aggregatum manifesto quadrat cum aggregato S, differentiaque $\pi - \varrho$ cum multitudine aequationum conditionalium σ, quamobrem pro errore medio obseruationum, quarum pondus $= 1$, habebimus expressionem $\sqrt{\frac{S}{\sigma}}$, quae determinatio eo maiori fide digna erit, quo maior fuerit numerus σ.

Sed operae pretium erit, hoc etiam independenter a disquisitione priori stabilire. Ad hunc finem quasdam nouas denotationes introducere conueniet. Scilicet respondeant valoribus indeterminatarum ξ, η, ζ etc. his

$$\xi = a, \ \eta = b, \ \zeta = c \text{ etc.}$$

valores ipsarum x, y, z etc. hi

$$x = \alpha,\ y = \beta,\ z = \gamma \text{ etc.}$$

ita vt habeatur

$$\alpha = a[\alpha\alpha] + b[\alpha\beta] + c[\alpha\gamma] + \text{etc.}$$
$$\beta = a[\alpha\beta] + b[\beta\beta] + c[\beta\gamma] + \text{etc.}$$
$$\gamma = a[\alpha\gamma] + b[\beta\gamma] + c[\gamma\gamma] + \text{etc.}$$

etc. Perinde valoribus

$$\xi = a',\ \eta = b',\ \zeta = c' \text{ etc.}$$

respondere supponemus hos

$$x = \alpha',\ y = \beta',\ z = \gamma' \text{ etc.}$$

nec non his

$$\xi = a'',\ \eta = b'',\ \zeta = c'' \text{ etc.}$$

sequentes

$$x = \alpha'',\ y = \beta'',\ z = \gamma'' \text{ etc.}$$

et sic porro.

His positis combinatio aequationum (4), (9) suppeditat

$$A = \alpha e + \alpha' e' + \alpha'' e'' + \text{etc.}$$
$$B = \beta e + \beta' e' + \beta'' e'' + \text{etc.}$$
$$C = \gamma e + \gamma' e' + \gamma'' e'' + \text{etc.}$$

etc. Quare quum habeatur $S = \mathfrak{A}A + \mathfrak{B}B + \mathfrak{C}C + \text{etc.}$, patet fieri

$$S = (ae + a'e' + a''e'' + \text{etc.})(\alpha e + \alpha' e' + \alpha'' e'' + \text{etc.})$$
$$+ (be + b'e' + b''e'' + \text{etc.})(\beta e + \beta' e' + \beta'' e'' + \text{etc.})$$
$$+ (ce + c'e' + c''e'' + \text{etc.})(\gamma e + \gamma' e' + \gamma'' e'' + \text{etc.}) + \text{etc.}$$

16.

Institutionem obseruationum, per quas valores quantitatum v, v', v'' etc. erroribus fortuitis e, e', e'' etc. affectos obtinemus, considerare possumus tamquam experimentum, quod quidem singulorum errorum commissorum magnitudinem docere non valet, attamen, praeceptis quae supra explicauimus adhibitis, valorem quantitatis S subministrat, qui per formulam modo inuentam est functio data illorum errorum. In tali experimento errores fortuiti vtique alii maiores alii minores prodire possunt; sed quo plures errores concurrunt, eo maior spes aderit, valorem quantitatis S in experimento singulari a valore suo medio parum deuiaturum esse. Rei cardo itaque in eo vertitur, vt valorem medium quantitatis S stabiliamus. Per principia in commentatione priori exposita, quae hic repetere superfluum esset, inuenimus hunc valorem medium

$$= (a\alpha + b\beta + c\gamma + \text{etc.})\, mm + (a'\alpha' + b'\beta' + c'\gamma' + \text{etc.})\, m'm' + (a''\alpha'' + b''\beta'' + c''\gamma'' + \text{etc.})\, m''m'' + \text{etc.}$$

Denotando errorem medium obseruationum talium, quarum pondus $= 1$, per μ, ita vt sit $\mu\mu = pmm = p'm'm' = p''m''m''$ etc., expressio modo inuenta ita exhiberi potest:

$$\left(\frac{a\alpha}{p} + \frac{a'\alpha'}{p'} + \frac{a''\alpha''}{p''}\ \text{etc.}\right)\mu\mu + \left(\frac{b\beta}{p} + \frac{b'\beta'}{p'} + \frac{b''\beta''}{p''} + \text{etc.}\right)\mu\mu + \left(\frac{c\gamma}{p} + \frac{c'\gamma'}{p'} + \frac{c''\gamma''}{p''} + \text{etc.}\right)\mu\mu + \text{etc.}$$

Sed aggregatum $\frac{a\alpha}{p} + \frac{a'\alpha'}{p'} + \frac{a''\alpha''}{p''} + \text{etc.}$ inuenitur

$$= [aa]\cdot[\alpha\alpha] + [ab]\cdot[\alpha\beta] + [ac]\cdot[\alpha\gamma] + \text{etc.}$$

adeoque $= 1$, vti e nexu aequationum (6), (7) facile intelligitur. Perinde fit

$$\frac{b\beta}{p} + \frac{b'\beta'}{p'} + \frac{b''\beta''}{p''} + \text{etc.} = 1$$

$$\frac{c\gamma}{p} + \frac{c'\gamma'}{p'} + \frac{c''\gamma''}{p''} + \text{etc.} = 1$$

et sic porro.

Hinc tandem valor medius ipsius S fit $= \sigma\mu\mu$, quatenusque igitur valorem fortuitum ipsius S pro medio adoptare licet, erit $\mu = \sqrt{\frac{S}{\sigma}}$.

17.

Quanta fides huic determinationi habenda sit, diiudicare oportet per errorem medium vel in ipsa vel in ipsius quadrato metuendum: posterior erit radix quadrata valoris medii expressionis

$$\left(\frac{S}{\sigma} - \mu\mu\right)^2$$

cuius euolutio absoluetur per ratiocinia similia iis, quae in commentatione priori artt. 39 sqq. exposita sunt. Quibus breuitatis caussa hic suppressis, formulam ipsam tantum hic apponimus. Scilicet error medius in determinatione quadrati $\mu\mu$ metuendus exprimitur per

$$\sqrt{\left(\frac{2\mu^4}{\sigma} + \frac{\nu^4 - 3\mu^4}{\sigma\sigma} \cdot N\right)}$$

denotante ν^4 valorem medium biquadratorum errorum, quorum pondus $= 1$, atque N aggregatum

$$(a\alpha + b\beta + c\gamma + \text{etc.})^2 + (a'\alpha' + b'\beta' + c'\gamma' + \text{etc.})^2 + (a''\alpha'' + b''\beta'' + c''\gamma'' + \text{etc.})^2 \text{ etc.}$$

Hoc aggregatum in genere ad formam simpliciorem reduci nequit, sed simili modo vt in art. 40. prioris commentationis ostendi potest, eius valorem semper contineri intra limites π et $\frac{\sigma\sigma}{\pi}$. In hypothesi ea, cui theoria quadratorum minimorum ab initio superstructa erat, terminus hoc aggregatum continens, propter $\nu^4 = 3\mu^4$, omnino excidit, praecisioque, quae errori medio, per formulam $\sqrt{\frac{S}{\sigma}}$ determinato, tribuenda est, eadem erit, ac si ex σ erroribus exacte cognitis secundum artt. 15, 16 prioris commentationis erutus fuisset.

18.

Ad compensationem obseruationum duo, vt supra vidimus, requiruntur: primum, vt aequationum conditionalium correlata, i. e. numeri A, B, C etc. aequationibus (12) satisfacientes eruantur, secundum, vt hi numeri in aequationibus (10) substituantur. Compensatio hoc modo prodiens dici poterit *perfecta* seu *completa*, vt distinguatur a compensatione *imperfecta* seu *manca*: hac scilicet denominatione designabimus, quae resultant ex iisdem quidem aequationibus (10), sed substratis valoribus quantitatum A, B, C etc., qui non satisfaciunt aequationibus (12), i. e. qui vel parti tantum satisfaciunt vel nullis. Quod vero attinet ad tales obseruationum mutationes, quae sub formulis (10) comprehendi nequeunt, a disquisitione praesente, nec non a denominatione compensationum exclusae sunto. Quum, quatenus aequationes (10) locum habent, aequationes (13) ipsis (12) omnino sint aequiualentes, illud discrimen ita quoque enunciari potest: Obseruationes complete compensatae omnibus aequationibus conditionalibus $X = 0$, $Y = 0$, $Z = 0$ etc. satisfaciunt, incomplete compensatae vero vel nullis vel saltem non omnibus; compensatio itaque, per quam omnibus aequationibus conditionalibus satisfit, necessario est ipsa completa.

19.

Iam quum ex ipsa notione compensationis sponte sequatur, aggregata duarum compensationum iterum constituere compensationem, facile perspicitur, nihil interesse, vtrum praecepta, per quae compensatio perfecta eruenda est, immediate ad obseruationes primitiuas applicentur, an ad obseruationes incomplete iam compensatas.

Reuera constituant $-\Theta$, $-\Theta'$, $-\Theta''$ etc. systema compensationis incompletae, quod prodierit e formulis (I)

$$\Theta p = A^\circ a + B^\circ b + C^\circ c + \text{etc.}$$
$$\Theta' p' = A^\circ a' + B^\circ b' + C^\circ c' + \text{etc.}$$
$$\Theta'' p'' = A^\circ a'' + B^\circ b'' + C^\circ c'' + \text{etc.}$$

etc.

Quum obseruationes his compensationibus mutatae omnibus aequationibus conditionalibus non satisfacere supponantur, sint $\mathfrak{A}^*$, $\mathfrak{B}^*$, $\mathfrak{C}^*$ etc.

etc. valores, quos X, Y, Z etc. ex illarum substitutione nanciscuntur. Quaerendi sunt numeri A^*, B^*, C^* etc. aequationibus (II) satisfacientes

$$\mathfrak{A}^* = A^*[aa] + B^*[ab] + C^*[ac] + \text{etc.}$$
$$\mathfrak{B}^* = A^*[ab] + B^*[bb] + C^*[bc] + \text{etc.}$$
$$\mathfrak{C}^* = A^*[ac] + B^*[bc] + C^*[cc] + \text{etc.}$$

etc., quo facto compensatio completa obseruationum isto modo mutatarum efficitur per mutationes nouas $-\varkappa$, $-\varkappa'$, $-\varkappa''$ etc., vbi $\varkappa$, $\varkappa'$, $\varkappa''$ etc. computandae sunt per formulas (III)

$$\varkappa p = A^* a + B^* b + C^* c + \text{etc.}$$
$$\varkappa' p' = A^* a' + B^* b' + C^* c' + \text{etc.}$$
$$\varkappa'' p'' = A^* a'' + B^* b'' + C^* c'' + \text{etc.}$$

etc. Iam inquiramus, quomodo hae correctiones cum compensatione completa obseruationum primitiuarum cohaereant. Primo manifestum est haberi

$$\mathfrak{A}^* = \mathfrak{A} - a\Theta - a'\Theta' - a''\Theta'' - \text{etc.}$$
$$\mathfrak{B}^* = \mathfrak{B} - b\Theta - b'\Theta' - b''\Theta'' - \text{etc.}$$
$$\mathfrak{C}^* = \mathfrak{C} - c\Theta - c'\Theta' - c''\Theta'' - \text{etc.}$$

etc. Substituendo in his aequationibus pro Θ, Θ', Θ'' etc. valores ex (I), nec non pro $\mathfrak{A}^*$, $\mathfrak{B}^*$, $\mathfrak{C}^*$ etc. valores ex II, inuenimus

$$\mathfrak{A} = (A^\circ + A^*)[aa] + (B^\circ + B^*)[ab] + (C^\circ + C^*)[ac] + \text{etc.}$$
$$\mathfrak{B} = (A^\circ + A^*)[ab] + (B^\circ + B^*)[bb] + (C^\circ + C^*)[bc] + \text{etc.}$$
$$\mathfrak{C} = (A^\circ + A^*)[ac] + (B^\circ + B^*)[bc] + (C^\circ + C^*)[cc] + \text{etc.}$$

etc., vnde patet, correlata aequationum conditionalium aequationibus (12) satisfacientia esse

$$A = A^\circ + A^*,\ B = B^\circ + B^*,\ C = C^\circ + C^* \text{ etc.}$$

Hinc vero aequationes (10), I et III docent, esse

$$\varepsilon = \Theta + \varkappa,\ \varepsilon' = \Theta' + \varkappa',\ \varepsilon'' = \Theta'' + \varkappa'' \text{ etc.}$$

i. e. compensatio obseruationum perfecta eadem prodit, siue immediate computetur, siue mediate proficiscendo a compensatione manca.

20.

Quoties multitudo aequationum conditionalium permagna est, determinatio correlatorum A, B, C etc. per eliminationem directam tam prolixa euadere potest, vt calculatoris patientia ei impar sit: tunc saepenumero commodum esse poterit, compensationem completam per approximationes successiuas adiumento theorematis art. praec. eruere. Distribuantur aequationes conditionales in duas pluresue classes, inuestigeturque primo compensatio, per quam aequationibus primae classis satisfit, neglectis reliquis. Dein tractentur obseruationes per hanc compensationem mutatae ita, vt solarum aequationum secundae classis ratio habeatur. Generaliter loquendo applicatio secundi compensationum systematis consensum cum aequationibus primae classis turbabit; quare, si duae tantummodo classes factae sunt, ad aequationes primae classis reuertemur, tertiumque systema quod huic satisfaciat eruemus; dein obseruationes ter correctas compensationi quartae subiiciemus, vbi solae aequationes secundae classis respiciuntur. Ita alternis vicibus, modo priorem classem modo posteriorem respicientes, compensationes continuo decrescentes obtinebimus, et si distributio scite adornata fuerat, post paucas iterationes ad numeros stabiles perueniemus. Si plures quam duae classes factae sunt, res simili modo se habebit: classes singulae deinceps in computum venient, post vltimam iterum prima et sic porro. Sed sufficiat hoc loco, hunc modum addigitauisse, cuius efficacia multum vtique a scita applicatione pendebit.

21.

Restat, vt suppleamus demonstrationem lemmatis in art. 8 suppositi, vbi tamen perspicuitatis caussa alias denotationes huic negotio magis adaptatas adhibebimus.

Sint itaque x°, x', x'', x''' etc. indeterminatae, supponamusque, ex aequationibus

$$n^{00}x^0 + n^{01}x' + n^{02}x'' + n^{03}x''' + \text{etc.} = X^0$$
$$n^{10}x^0 + n^{11}x' + n^{12}x'' + n^{13}x''' + \text{etc.} = X'$$
$$n^{20}x^0 + n^{21}x' + n^{22}x'' + n^{23}x''' + \text{etc.} = X''$$
$$n^{30}x^0 + n^{31}x' + n^{32}x'' + n^{33}x''' + \text{etc.} = X'''$$
etc.

sequi per eliminationem has

$$N^{00}X^0 + N^{01}X' + N^{02}X'' + N^{03}X''' + \text{etc.} = x^0$$
$$N^{10}X^0 + N^{11}X' + N^{12}X'' + N^{13}X''' + \text{etc.} = x'$$
$$N^{20}X^0 + N^{21}X' + N^{22}X'' + N^{23}X''' + \text{etc.} = x''$$
$$N^{30}X^0 + N^{31}X' + N^{32}X'' + N^{33}X''' + \text{etc.} = x'''$$
etc.

Substitutis itaque in aequatione prima et secunda secundi systematis valoribus quantitatum X, X', X'', X''' etc. e primo systemate, obtinemus

$$\begin{aligned} x^0 = {} & N^{00}(n^{00}x^0 + n^{01}x' + n^{02}x'' + n^{03}x''' + \text{etc.}) \\ & + N^{01}(n^{10}x^0 + n^{11}x' + n^{12}x'' + n^{13}x''' + \text{etc.}) \\ & + N^{02}(n^{20}x^0 + n^{21}x' + n^{22}x'' + n^{23}x''' + \text{etc.}) \\ & + N^{03}(n^{30}x^0 + n^{31}x' + n^{32}x'' + n^{33}x''' + \text{etc.}) \end{aligned}$$

etc., nec non

$$\begin{aligned} x' = {} & N^{10}(n^{00}x + n^{01}x' + n^{02}x'' + n^{03}x''' + \text{etc.}) \\ & + N^{11}(n^{10}x + n^{11}x' + n^{12}x'' + n^{13}x''' + \text{etc.}) \\ & + N^{12}(n^{20}x + n^{21}x' + n^{22}x'' + n^{23}x''' + \text{etc.}) \\ & + N^{13}(n^{30}x + n^{31}x' + n^{32}x'' + n^{33}x''' + \text{etc.}) \end{aligned}$$
etc.

Quum vtraque aequatio manifesto esse debeat aequatio identica, tum in priori tum in posteriori pro x^0, x', x'', x''' etc. valores quoslibet determinatos substituere licebit. Substituamus in priori

$$x^0 = N^{10},\ x' = N^{11},\ x'' = N^{12},\ x''' = N^{13} \text{ etc.}$$

in posteriori vero

$$x^0 = N^{00},\ x' = N^{01},\ x'' = N^{02},\ x''' = N^{03} \text{ etc.}$$

His ita factis subtractio producit

$$N^{10} - N^{01} = (N^{00}N^{11} - N^{10}N^{01})(n^{01} - n^{10})$$
$$+ (N^{00}N^{12} - N^{10}N^{02})(n^{02} - n^{20})$$
$$+ (N^{00}N^{13} - N^{10}N^{03})(n^{03} - n^{30})$$
$$+ \text{etc.}$$
$$+ (N^{01}N^{12} - N^{11}N^{02})(n^{12} - n^{21})$$
$$+ (N^{01}N^{13} - N^{11}N^{03})(n^{13} - n^{31})$$
$$+ \text{etc.}$$
$$+ (N^{02}N^{13} - N^{12}N^{03})(n^{23} - n^{32})$$
$$+ \text{etc. etc.}$$

quae aequatio ita quoque exhiberi potest

$$N^{10} - N^{01} = \Sigma\,(N^{0\alpha}N^{1\beta} - N^{1\alpha}N^{0\beta})(n^{\alpha\beta} - n^{\beta\alpha})$$

denotantibus α β omnes combinationes indicum inaequalium.

Hinc colligitur, si fuerit $n^{01} = n^{10}$, $n^{02} = n^{20}$, $n^{03} = n^{30}$, $n^{12} = n^{21}$, $n^{13} = {}^{31}$, $n^{23} = n^{32}$, etc., siue generaliter $n^{\alpha\beta} = n^{\beta\alpha}$, fore etiam

$$N^{10} = N^{01}$$

Et quum ordo indeterminatarum in aequationibus propositis sit arbitrarius, manifesto, in illa suppositione erit generaliter

$$N^{\alpha\beta} = N^{\beta\alpha}$$

22.

Quum methodus in hac commentatione exposita applicationem imprimis frequentem et commodam inueniat in calculis ad geodaesiam sublimiorem pertinentibus, lectoribus gratam fore speramus illustrationem praeceptorum per nonnulla exempla hinc desumta.

Aequationes conditionales inter angulos systematis triangulorum e triplici potissimum fonte sunt petendae.

I. Aggregatum angulorum horizontalium, qui circa eundem verticem gyrum integrum horizontis complent, aequare debet quatuor rectos.

II. Summa trium angulorum in quouis triangulo quantitati datae aequalis est, quum, quoties triangulum est in superficie curua, excessum illius summae supra duos rectos tam accurate computare liceat, vt pro absolute exacto haberi possit.

III. Fons tertius est ratio laterum in triangulis catenam clausam formantibus. Scilicet si series triangulorum ita nexa est, vt secundum triangulum habeat latus vnum a commune cum triangulo primo, aliud b cum tertio; perinde quartum triangulum cum tertio habeat latus commune c, cum quinto latus commune d, et sic porro vsque ad vltimum triangulum, cui cum praecedente latus commune sit k, et cum triangulo primo rursus latus l, valores quotientium $\frac{a}{l}$, $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$ .... $\frac{l}{k}$, innotescent resp. e binis angulis triangulorum successiuorum, lateribus communibus oppositis, per methodos notas, vnde quum productum illarum fractionum fieri debeat $= 1$, prodibit aequatio conditionalis inter sinus illorum angulorum, (parte tertia excessus sphaerici vel sphaeroidici, si triangula sunt in superficie curua, resp. diminutorum).

Ceterum in systematibus triangulorum complicatioribus saepissime accidit, vt aequationes conditionales tum secundi tum tertii generis plures se offerant, quam retinere fas est, quoniam pars earum in reliquis iam contenta est. Contra rarior erit casus, vbi aequationibus conditionalibus secundi generis adiungere oportet aequationes similes ad figuras plurium laterum spectantes, puta tunc tantum, vbi polygona formantur, in triangula per mensurationes non diuisa. Sed de his rebus ab instituto praesente nimis alienis, alia occasione fusius agemus. Silentio tamen praeterire non possumus monitum, quod theoria nostra, si applicatio pura atque rigorosa in votis est, supponit, quantitates per v, v', v'' etc. designatas reuera vel immediate obseruatas esse, vel ex obseruationibus ita deriuatas, vt inter se independentes maneant, vel saltem tales censeri possint. In praxi vulgari obseruantur anguli triangulorum ipsi, qui proin pro v, v', v'' etc. accipi possunt; sed memores esse debemus, si forte systema insuper contineat triangula talia, quorum anguli non sint immediate obseruati, sed prodeant tamquam summae vel differentiae angulorum reuera obseruatorum, illos non inter obseruatorum numerum referendos, sed in forma compositionis suae in calculis retinendos esse. Aliter vero res se habebit in modo obseruandi ei simili, quem sequutus est clar. Struve (Astronomische Nachrichten II, p. 431), vbi directiones singulorum laterum ab eodem vertice proficiscentium obtinentur per comparationem cum vna eademque directione arbitraria. Tunc scilicet hi ipsi anguli pro v, v', v'' etc. accipiendi sunt, quo pacto omnes anguli triangulorum in forma differentiarum se offerent, aequationesque conditionales primi generis, quibus per rei naturam sponte satisfit, tamquam superfluae cessabunt. Modus obseruationis, quem ipse sequutus sum in dimensione triangulorum annis praecedentibus perfecta, differt quidem tum a priori tam a posteriori modo, attamen respectu effectus posteriori aequiparari potest, ita vt in singulis stationibus directiones laterum inde proficiscentium ab initio quasi arbitrario numeratas pro quantitatibus v, v', v'' etc. accipere oporteat. Duo iam exempla elaborabimus, alterum ad modum priorem, alterum ad posteriorem pertinens.

23.

Exemplum primum nobis suppeditabit opus clar. de Krayenhof, *Précis historique des operations trigonometriques faites en Hollande*, et quidem compensationi subiiciemus partem eam systematis triangulorum, quae inter nouem puncta Harlingen, Sneek, Oldeholtpade, Ballum, Leeuwarden, Dockum, Drachten, Oosterwolde, Gröningen continentur. Formantur inter haec puncta nouem triangula in opera illo per numeros 121, 122, 123, 124, 125, 127, 128, 131, 132 denotata, quorum anguli (a nobis indicibus praescriptis distincta) secundum tabulam p. 77–81 ita sunt obseruati:

	Triangulum 121.			
0.	Harlingen	50°	58′	15″238
1.	Leeuwarden	82	47	15,351
2.	Ballum	46	14	27,202
	Triangulum 122.			
3.	Harlingen	51	5	39,717
4.	Sneek	70	48	33,445
5.	Leeuwarden	58	5	48,707
	Triangulum 123.			
6.	Sneek	49	30	40,051
7.	Drachten	42	52	59,382
8.	Leeuwarden	87	36	21,057
	Triangulum 124.			
9.	Sneek	45	36	7,492
10.	Oldeholtpade	67	52	0,048
11.	Drachten , ...	66	31	56,513
	Triangulum 125.			
12.	Drachten	53	55	24,745
13	Oldeholtpade . . .	47	48	52,580
14.	Oosterwolde	78	15	42,347
	Triangulum 127.			
15.	Leeuwarden	59	24	0,645
16.	Dockum	76	34	9,021
17.	Ballum	44	1	51,040
	Triangulum 128.			
18.	Leeuwarden	72	6	32,043
19.	Drachten	46	53	27,163
20.	Dockum	61	0	4,494

Triangulum 131.

21.	Dockum	57°	1′	55″292
22.	Drachten . . , . .	83	33	14,515
23.	Gröningen	39	24	52,397

Triangulum 132

24	Oosterwolde . . .	81	54	17,447
25.	Gröningen	31	52	46,094
26.	Drachten	66	12	57,246

Consideratio nexus inter haec triangula monstrat, inter 27 angulos, quorum valores approximati per obseruationem innotuerunt, 13 aequationes conditionales haberi, puta duas primi generis, nouem secundi, duas tertii. Sed haud opus erit, has aequationes omnes in forma sua finita hic adscribere, quum ad calculos tantummodo requirantur quantitates in theoria generali per $\mathfrak{A}$, a, a', a'' etc., $\mathfrak{B}$, b, b', b'' etc. etc. denotatae: quare illarum loco statim adscribimus aequationes supra per (13) denotatas, quae illas quantitates ob oculos ponunt: loco signorum ε, ε', ε'' etc. simpliciter hic scribemus (0), (1), (2) etc.

Hoc modo duabus aequationibus conditionalibus primi generis respondent sequentes:

$$(1) + (5) + (8) + (15) + (18) = -2''197$$
$$(7) + (11) + (12) + (19) + (22) + (26) = -0''436$$

Excessus sphaeroidicos nouem triangulorum inuenimus deinceps: 1″749; 1″147; 1″243; 1″698; 0″873; 1″167; 1″104; 2″161; 1″403. Oritur itaque aequatio conditionalis secundi generis prima haec *): $v^{(0)} + v^{(1)} + v^{(2)} - 180° \, 0' \, 1''749 = 0$, et perinde reliquae: hinc habemus nouem aequationes sequentes:

*) Indices in hoc exemplo per figuras arabicas exprimere praeferimus.

$$(0) + (1) + (2) = -3''958$$
$$(3) + (4) + (5) = +0,722$$
$$(6) + (7) + (8) = -0,753$$
$$(9) + (10) + (11) = +2,355$$
$$(12) + (13) + (14) = -1,201$$
$$(15) + (16) + (17) = -0,461$$
$$(18) + (19) + (20) = +2,596$$
$$(21) + (22) + (23) = +0,043$$
$$(24) + (25) + (26) = -0,616$$

Aequationes conditionales tertii generis commodius in forma logarithmica exhibentur: ita prior est

$$\log\sin(v^{(0)} - 0''583) - \log\sin(v^{(2)} - 0''583) - \log\sin(v^{(3)} - 0''382) + \log\sin(v^{(4)} - 0''382) - \log\sin(v^{(6)} - 0''414) + \log\sin(v^{(7)} - 0''414) - \log\sin(v^{(16)} - 0''389) + \log\sin(v^{(17)} - 0''389) - \log\sin(v^{(19)} - 0''368) + \log\sin(v^{(20)} - 0''368) = 0$$

Superfluum videtur, alteram in forma finita adscribere. His duabus aequationibus respondent sequentes, vbi singuli coëfficientes referuntur ad figuram septimam logarithmorum briggicorum:

$$17,068(0) - 20,174(2) - 16,993(3) + 7,328(4) - 17,976(6) + 22,672(7) - 5,028(16) + 21,780(17) - 19,710(19) + 11,671(20) = -371$$

$$17,976(6) - 0,880(8) - 20,617(9) + 8,564(10) - 19,082(13) + 4,375(14) + 6,798(18) - 11,671(20) + 13,657(21) - 25,620(23) - 2,995(24) + 33,854(25) = +370$$

Quum nulla ratio indicata sit, cur obseruationibus pondera inaequalia tribuamus, statuemus $p^{(0)} = p^{(1)} = p^{(2)}$ etc. $= 1$. Denotatis itaque correlatis aequationum conditionalium eo ordine, quo aequationes ipsis respondentes exhibuimus, per $A, B, C, D, E, F, G, H, I, K, L, M, N$, prodeunt ad illorum determinationem aequationes sequentes:

$$- 2''197 = 5\,A + C + D + E + H + I + 5{,}917\,N$$
$$- 0{,}436 = 6\,B + E + F + G + I + K + L + 2{,}962\,M$$
$$- 3{,}958 = A + 3\,C - 3{,}106\,M$$
$$+ 0{,}722 = A + 3\,D - 9{,}665\,M$$
$$- 0{,}753 = A + B + 3\,E + 4{,}696\,M + 17{,}096\,N$$
$$+ 2{,}355 = B + 3\,F - 12{,}053\,N$$
$$- 1{,}201 = B + 3\,G - 14{,}707\,N$$
$$- 0{,}461 = A + 3\,H + 16{,}752\,M$$
$$+ 2{,}596 = A + B + 3\,I - 8{,}039\,M - 4{,}874\,N$$
$$+ 0{,}043 = B + 3\,K - 11{,}963\,N$$
$$- 0{,}616 = B + 3\,L + 30{,}859\,N$$
$$- 371 = + 2{,}962\,B - 3{,}106\,C - 9{,}665\,D + 4{,}696\,E + 16{,}752\,H - 8{,}039\,I + 2902{,}27\,M - 459{,}33\,N$$
$$+ 370 = + 5{,}917\,A + 17{,}096\,E - 12{,}053\,F - 14{,}707\,G - 4{,}874\,I - 11{,}963\,K + 30{,}859\,L - 459{,}33\,M + 3385{,}96\,N$$

Hinc eruimus per eliminationem:

$A = -\ 0{,}598$	$H = +\ 0{,}659$
$B = -\ 0{,}255$	$I = +\ 1{,}050$
$C = -\ 1{,}234$	$K = +\ 0{,}577$
$D = +\ 0{,}086$	$L = -\ 1{,}351$
$E = -\ 0{,}447$	$M = -\ 0{,}109792$
$F = +\ 1{,}351$	$N = +\ 0{,}119681$
$G = +\ 0{,}271$	

Denique errores maxime plausibiles prodeunt per formulas

$$(0) = C + 17{,}068\,M$$
$$(1) = A + C$$
$$(2) = C - 20{,}174\,M$$
$$(3) = D - 16{,}993\,M$$

etc., vnde obtinemus valores numericos sequentes; in gratiam comparationis apponimus (mutatis signis) correctiones a clar. de Krayenhof obseruationibus applicatas:

		de Kr.			de Kr.
(0)	$= -\ 3''108$	$-\ 2''090$	(14)	$= +\ 0''795$	$+\ 2''400$
(1)	$= -\ 1,832$	$+\ 0,116$	(15)	$= +\ 0,061$	$+\ 1,273$
(2)	$= +\ 0,981$	$-\ 1,982$	(16)	$= +\ 1,211$	$+\ 5,945$
(3)	$= +\ 1,952$	$+\ 1,722$	(17)	$= -\ 1,732$	$-\ 7,674$
(4)	$= -\ 0,719$	$+\ 2,848$	(18)	$= +\ 1,265$	$+\ 1,876$
(5)	$= -\ 0,512$	$-\ 3,848$	(19)	$= +\ 2,959$	$+\ 6,251$
(6)	$= +\ 3,648$	$-\ 0,137$	(20)	$= -\ 1,628$	$-\ 5,530$
(7)	$= -\ 3,221$	$+\ 1,000$	(21)	$= +\ 2,211$	$+\ 3,486$
(8)	$= -\ 1,180$	$-\ 1,614$	(22)	$= +\ 0,322$	$-\ 3,454$
(9)	$= -\ 1,116$	0	(23)	$= -\ 2,489$	0
(10)	$= +\ 2,376$	$+\ 5,928$	(24)	$= -\ 1,709$	$+\ 0,400$
(11)	$= +\ 1,096$	$-\ 3,570$	(25)	$= +\ 2,701$	$+\ 2,054$
(12)	$= +\ 0,016$	$+\ 2,414$	(26)	$= -\ 1,606$	$-\ 3,077$
(13)	$= -\ 2,013$	$-\ 6,014$			

Aggregatum quadratorum nostrarum compensationum inuenitur $= 97,8845$. Hinc error medius, quatenus ex 27 angulis obseruatis colligi potest,

$$= \sqrt{\frac{97,8845}{13}} = 2''7440$$

Aggregatum quadratorum mutationum, quas clar. de Krayenhof ipse angulis obseruatis applicauit, inuenitur $= 341,4201$.

24.

Exemplum alterum suppeditabunt triangula inter quinque puncta triangulationis Hannoveranae, Falkenberg, Breithorn, Hauselberg, Wulfsode, Wilsede. Obseruatae sunt directiones *):

*) Initia, ad quae singulae directiones referuntur, hic tamquam arbitraria considerantur, quamquam reuera cum lineis meridianis stationum coincidant. Obseruationes in posterum complete publici iuris

In statione Falkenberg

0.	Wilsede	187°	47′	30″311
1.	Wulfsode	225	9	39,676
2.	Hauselberg	266	13	56,239
3.	Breithorn	274	14	43,634

In statione Breithorn

4.	Falkenberg	94	33	40,755
5.	Hauselberg	122	51	23,054
6.	Wilsede	150	18	35,100

In statione Hauselberg

7.	Falkenberg	86	29	6,872
8.	Wilsede	154	37	9,624
9.	Wulfsode	189	2	56,376
10.	Breithorn	302	47	37,732

In statione Wulfsode

11.	Hauselberg	9	5	36,593
12.	Falkenberg	45	27	33,556
13.	Wilsede	118	44	13,159

In statione Wilsede

14.	Falkenberg	7	51	1,027
15.	Wulfsode	298	29	49,519
16.	Breithorn	330	3	7,392
17.	Hauselberg	334	25	26,746

Ex his obseruationibus septem triangula formare licet.

Triangulum I.

Falkenberg	8°	0′	47″395
Breithorn	28	17	42,299
Hauselberg	143	41	29,140

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fient; interim figura inuenitur in Astronomische Nachrichten Vol. I. p. 441.

Triangulum II.

Falkenberg	86°	27′	13″323
Breithorn	55	44	54,345
Wilsede	37	47	53,635

Triangulum III.

Falkenberg	41	4	16,563
Hauselberg	102	33	49,504
Wulfsode	36	21	56,963

Triangulum IV.

Falkenberg	78	26	25,928
Hauselberg	68	8	2,752
Wilsede	35	25	34,281

Triangulum V.

Falkenberg	37	22	9,365
Wulfsode	73	16	39,603
Wilsede	69	21	11,508

Triangulum VI.

Breithorn	27	27	12,046
Hauselberg	148	10	28,108
Wilsede	4	22	19,354

Triangulum VII.

Hauselberg	34	25	46,752
Wulfsode	109	38	36,566
Wilsede	35	55	37,227

Aderunt itaque septem aequationes conditionales secundi generis (aequationes primi generis manifesto cessant), quas vt eruamus, computandi sunt ante omnia excessus sphaeroidici septem triangulorum. Ad hunc finem requiritur cognitio magnitudinis absolutae saltem vnius lateris: latus inter puncta Wilsede et Wulfsode est 22877,94 metrorum. Hinc prodeunt excessus sphaeroidici triangulorum I . . . 0″202; II . . . 2″442; III . . . 1″257; IV . . . . 1″919; V . . . . 1″957; VI . . . . 0″321; VII . . . . 1″295.

Iam si directionis eo ordine, quo supra allatae indicibusque distinctae sunt, per $\nu^{(0)}$, $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$ etc. designantur, trianguli I anguli fiunt $\nu^{(3)} - \nu^{(2)}$, $\nu^{(5)} - \nu^{(4)}$, $360^\circ + \nu^{(7)} - \nu^{(10)}$, adeoque aequatio conditionalis prima

$$-\nu^{(2)} + \nu^{(3)} - \nu^{(4)} + \nu^{(5)} + \nu^{(7)} - \nu^{(10)} + 179^\circ\, 59'\, 59''798 = 0$$

Perinde triangula reliqua sex alias suppeditant; sed leuis attentio docebit, has septem aequationes non esse independentes, sed secundam identicam cum summa primae, quartae et sextae; nec non summam tertiae et quintae identicam cum summa quartae et septimae: quapropter secundam et quintam negligemus. Loco remanentium aequationum conditionalium in forma finita, adscribimus aequationes correspondentes e complexu (13), dum pro characteribus ε, ε' etc. his (0), (1), (2) etc. vtimur:

$$-1''368 = -(2) + (3) - (4) + (5) + (7) - (10)$$
$$+1,773 = -(1) + (2) - (7) + (9) - (11) + (12)$$
$$+1,042 = -(0) + (2) - (7) + (8) + (14) - (17)$$
$$-0,813 = -(5) + (6) - (8) + (10) - (16) + (17)$$
$$-0,750 = -(8) + (9) - (11) + (13) - (15) + (17)$$

Aequationes conditionales tertii generis *octo* e triangulorum systemate peti possent, quum tum terna quatuor triangulorum I, II, IV, VI, tum terna ex his III, IV, V, VII ad hunc finem combinare liceat; attamen leuis attentio docet, *duas* sufficere, alteram ex illis, alteram ex his, quum reliquae in his atque prioribus aequationibus conditionalibus iam contentae esse debeant. Aequatio itaque conditionalis sexta nobis erit

$$\log\sin(\nu^{(3)} - \nu^{(2)} - 0''067) - \log\sin(\nu^{(5)} - \nu^{(4)} - 0''067)$$
$$+ \log\sin(\nu^{(14)} - \nu^{(17)} - 0''640) - \log\sin(\nu^{(2)} - \nu^{(0)} - 0''640)$$
$$+ \log\sin(\nu^{(6)} - \nu^{(5)} - 0''107) - \log\sin(\nu^{(17)} - \nu^{(16)} - 0''107) = 0$$

atque septima

$$\log\sin(\nu^{(2)} - \nu^{(1)} - 0''419) - \log\sin(\nu^{(12)} - \nu^{(11)} - 0''419)$$
$$+ \log\sin(\nu^{(14)} - \nu^{(17)} - 0''640) - \log\sin(\nu^{(2)} - \nu^{(0)} - 0''640)$$
$$+ \log\sin(\nu^{(13)} - \nu^{(11)} - 0''432) - \log\sin(\nu^{(17)} - \nu^{(15)} - 0''432)$$
$$= 0$$

quibus respondent aequationes complexus (13)

$$+25 = +4{,}31(0) - 153{,}88(2) + 149{,}57(3) + 39{,}11(4) - 79{,}64(5)$$
$$+ 40{,}53(6) + 31{,}90(14) + 275{,}39(16) - 307{,}29(17)$$
$$-3 = +4{,}31(0) - 24{,}16(1) + 19{,}85(2) + 36{,}11(11) - 28{,}59(12)$$
$$- 7{,}52(13) + 31{,}90(14) + 29{,}06(15) - 60{,}96(17)$$

Quodsi iam singulis directionibus eandem certitudinem tribuimus, statuendo $p^{(0)} = p^{(1)} = p^{(2)}$ etc. $= 1$, correlataque septem aequationum conditionalium, eo ordine, quem hic sequuti sumus, per A, B, C, D, E, F, G denotamus, horum determinatio petenda erit ex aequationibus sequentibus:

$$-1{,}368 = +6A - 2B - 2C - 2D + 184{,}72F - 19{,}85G$$
$$+1{,}773 = -2A + 6B + 2C + 2E - 153{,}88F - 20{,}69G$$
$$+1{,}042 = -2A + 2B + 6C - 2D - 2E + 181{,}00F + 108{,}40G$$
$$-0{,}813 = -2A - 2C + 6D + 2E - 462{,}51F - 60{,}96G$$
$$-0{,}750 = +2B - 2C + 2D + 6E - 307{,}29F - 133{,}65G$$
$$+25 = +184{,}72A - 153{,}88B + 181{,}00C - 462{,}51D$$
$$- 307{,}29E + 224868F + 16694{,}1G$$
$$-3 = -19{,}85A - 20{,}69B + 108{,}40C - 60{,}96D$$
$$- 133{,}65E + 16694{,}1F + 8752{,}39G$$

Hinc deducimus per eliminationem

$$A = -0{,}225$$
$$B = +0{,}344$$
$$C = -0{,}088$$
$$D = -0{,}171$$

$E = -\ 0,323$
$F = +\ 0,000215915$
$G = -\ 0,00547462$

Iam errores maxime plausibiles habentur per formulas:

$(0) = -\ C + 4,31\,F + 4,31\,G$
$(1) = -\ B - 24,16\,G$
$(2) = -\ A + B + C - 153,88\,F + 19,85\,G$

etc., vnde prodeunt valores numerici

$(0) = +\ 0''065$	$(9) = +\ 0''021$
$(1) = -\ 0,212$	$(10) = +\ 0,054$
$(2) = +\ 0,339$	$(11) = -\ 0,219$
$(3) = -\ 0,193$	$(12) = +\ 0,501$
$(4) = +\ 0,233$	$(13) = -\ 0,282$
$(5) = -\ 0,071$	$(14) = -\ 0,256$
$(6) = -\ 0,162$	$(15) = +\ 0,164$
$(7) = -\ 0,481$	$(16) = +\ 0,230$
$(8) = +\ 0,406$	$(17) = -\ 0,139$

Summa quadratorum horum errorum inuenitur $= 1,2288$; hinc error medius vnius directionis, quatenus e 18 directionibus obseruatis erui potest,

$$= \sqrt{\frac{1,2288}{7}} = 0''4190$$

25.

Vt etiam pars altera theoriae nostrae exemplo illustretur, indagamus praecisionem, qua latus Falkenberg-Breithorn e latere Wilsede-Wulfsode adiumento obseruationum compensatarum determinatur. Functio u, per quam illud in hoc casu exprimitur, est

$$u = 22877^{\mathrm{m}}94 \times \frac{\sin(v^{(13)} - v^{(12)} - 0''652) \cdot \sin(v^{(14)} - v^{(16)} - 0''814)}{\sin(v^{(1)} - v^{(0)} - 0''652) \cdot \sin(v^{(6)} - v^{(4)} - 0''814)}$$

Huius valor, e valoribus correctis directionum $\rho^{(0)}$, $\rho^{(1)}$ etc. inuenitur

$$= 26766^{m}\,68$$

Differentiatio autem illius expressionis suppeditat, si differentialia $d\rho^{(0)}$, $d\rho^{(1)}$ etc. minutis secundis expressa concipiuntur,

$$du = 0^{m}\,16991\,(d\rho^{(0)} - d\rho^{(1)}) + 0^{m}\,08836\,(d\rho^{(4)} - d\rho^{(6)})$$
$$- 0^{m}\,03899\,(d\rho^{(12)} - d\rho^{(13)}) + 0^{m}\,16731\,(d\rho^{(14)} - d\rho^{(16)})$$

Hinc porro inuenitur

$$[al] = -\ 0{,}08836$$
$$[bl] = +\ 0{,}13092$$
$$[cl] = -\ 0{,}00260$$
$$[dl] = +\ 0{,}07895$$
$$[el] = +\ 0{,}03899$$
$$[fl] = -\ 40{,}1315$$
$$[gl] = +\ 10{,}9957$$
$$[ll] = +\ 0{,}13238$$

Hinc denique per methodos supra traditas inuenitur, quatenus metrum pro vnitate dimensionum linearium accipimus,

$$\frac{1}{P} = 0{,}08329, \text{ siue } P = 12{,}006$$

vnde error medius in valore lateris Falkenberg-Breithorn metuendus $= 0{,}2886\,m$ metris, (vbi m error medius in directionibus obseruatis metuendus, et quidem in minutis secundis expressus), adeoque, si valorem ipsius m supra erutum adoptamus,

$$= 0^{m}\,1209$$

Ceterum inspectio systematis triangulorum sponte docet, punctum Hauselberg omnino ex illo elidi potuisse, incolumi manente nexu inter latera Wilsede-Wulfsode atque Falkenberg-Breithorn. Sed a bona methodo abhorreret, *supprimere* idcirco obseruationes, quae ad punctum Hauselberg referuntur *), quum certe ad praecisionem augendam conferre valeant. Vt clarius appareret, quantum praecisionis augmentum inde redundet, calculum denuo fecimus excludendo omnia, quae ad punctum Hauselberg referuntur, quo pacto e 18 directionibus supra traditis octo excidunt, atque reliquarum errores maxime plausibilis ita inueniuntur:

(0) = + 0″327	(12) = + 0″206
(1) = — 0,206	(13) = — 0,206
(3) = — 0,121	(14) = + 0,327
(4) = + 0,121	(15) = + 0,206
(6) = — 0,121	(16) = + 0,121

Valor lateris Falkenberg-Breithorn tunc prodit = $26766^m 63$, parum quidem a valore supra eruto discrepans, sed calculus ponderis producit

$$\frac{1}{P} = 0,13082 \text{ siue } P = 7,644$$

adeoque error medius metuendus = $0,36169\, m$ metris = $0^m 1515$. Patet itaque, per accessionem obseruationum, quae ad punctum Hauselberg referuntur, pondus determinationis lateris Falkenberg-Breithorn auctum esse in ratione numeri 7,644 ad 12,006, siue vnitatis ad 1,571.

*) Maior pars harum obseruationum iam facta erat, antequam punctum Breithorn repertum, atque in systema receptum esset.