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On the Finiteness of the Number of Unipotent Classes

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Let G be a reductive connected algebraic group over an algebraically closed field k; we identify G with the set of its k-rational points. Let $n(G)$ be the number of G-conjugacy classes of unipotent elements in G. In his talk at the International Congress in Moscow [7, Problem 10], Steinberg asked whether $n(G)$ is always finite. In this paper it will be shown that $n(G)$ is indeed finite. When k has characteristic 0, this follows from work of Dynkin and Kostant (see [3]). More generally, Richardson [5], has shown that $n(G)$ is finite if the characteristic of k is either 0 or $p>1$ with p a good prime for G (i.e. p does not divide any coefficient of the highest root of any simple component of G). In the general case, a simple argument, which was shown to us by Deligne, shows that $n(G)$ remains unchanged when the scalars are extended from k to an algebraically closed field containing k. Therefore, in order to prove that $n(G)$ is finite when the characteristic of k is a prime number $p>1$, one can assume, without loss of generality, that k is an algebraic closure of the finite field with p elements. In the rest of this paper, we shall adhere to this assumption.

Let q be a power of p. We shall always assume that G has a given $\mathbf{F}_q$-rational structure, where $\mathbf{F}_q$ is the subfield of k with q elements. We denote by F the corresponding Frobenius endomorphism of G.

The idea of the proof is roughly as follows. Using results from the representation theory (over fields of characteristic 0) of the finite groups $G^{F^s}=G(\mathbf{F}_{q^s})$, $(s\geqq 1)$, one shows that the unipotent G^{F^s}-conjugacy classes in G^{F^s} can be separated by the characters of a relatively small number of irreducible representations of G^{F^s}; the number of representations needed is bounded by a number depending only on the type of G, and in particular is independent of s. This gives a bound independent of s for the number of G^{F^s}-conjugacy classes of unipotent elements in G^{F^s}. As G is the union of the finite groups G^{F^s}, the same bound will give an estimate for $n(G)$. For example, when G is simple, adjoint, of rank $r\geqq 1$, we get the explicit estimate $n(G)\leqq |W|^2(r+1)^2$; here W is the Weyl group of G. (We have not attempted to get the best possible estimate for $n(G)$).

In this paper we make an extensive use of the methods and results of [2]. We also need a generalization of the basic construction in [2, 1.17, 1.20]. Let L be an

F-stable subgroup of G such that there exists a (not necessarily F-stable) parabolic subgroup P of G such that L is a Levi subgroup of P (equivalently, L is the centralizer of some F-stable torus in G). Using l-adic cohomology, we define a homomorphism $R^G_{L\subset P}$: $\mathscr{R}(L^F)\to\mathscr{R}(G^F)$ (here $\mathscr{R}(\ )$ denotes the Grothendieck group of representations of a finite group over an algebraic closure $\overline{\mathbf{Q}}_l$ of the l-adic numbers, $l\neq p$), see Section 1. When L is an F-stable maximal torus, this reduces to the construction in [*loc. cit.*]. On the other hand, in the special case where P itself is F-stable this is just inflation from L^F to P^F followed by induction from P^F to G^F. In the case where all simple factors of L are of type A, the above homomorphism $\mathscr{R}(L^F)\to\mathscr{R}(G^F)$ was constructed independently by B. Srinivasan [6], by a different method which makes use of [2] and which is valid only for q sufficiently large; she also proves results similar to our Propositions 9 and 10 under the same restrictions on L and q.

A key result on $R^G_{L\subset P}$ is the orthogonality Theorem 8, which is a partial generalization of [2, Theorem 6.8]. Using this result one can prove that, for G of type B or D (possibly twisted) and $q=2^e$, the number of G^F-conjugacy classes of unipotent elements in G^F equals the number of isomorphism classes of irreducible unipotent representations (in the sense of [2, 7.8]) of G^F. (This fact will be proved elsewhere). For example, $SO_8(\mathbf{F}_{2^e})$, corresponding to a split quadratic form over $\mathbf{F}_{2^e}$, has 14 unipotent representations; exactly one of them is cuspidal.

It is likely that the homomorphism $R^G_{L\subset P}$ is actually independent of the choice of the parabolic subgroup P; Deligne has an argument to show that this is true provided that q is large. (For the case where L is an F-stable maximal torus, this is again true, cf. [2, 4.3]).

The work on this paper was started during a visit at the IHES in December 1974; the paper was completed during another visit at the IHES in December 1975. On these and other occasions, the author has had valuable conversations with Deligne and now wishes to thak him.

Notations. Let H be a finite group. All representations of H will be assumed to be over $\overline{\mathbf{Q}}_l$ and of finite dimension; we shall also call them H-modules. We put $H^\vee=\mathrm{Hom}(H,\overline{\mathbf{Q}}_l^*)$. If ρ,ρ' are two H-modules we put

$$\langle\rho,\rho'\rangle_H=|H|^{-1}\sum_{h\in H}Tr(h,\rho)\,Tr(h^{-1},\rho').$$

If X is a scheme over k, we put $H^i_c(X)=H^i_c(X,\overline{\mathbf{Q}}_l)$ (l-adic cohomology with compact support). Unless otherwise stated, the notations will be consistent with those of [2].

1. Consider a pair $L\subset P$ where P is a parabolic subgroup of G and L is a Levi subgroup of P. We assume that L is F-stable, but make no assumption on P. Let U_P be the unipotent radical of P. The scheme over k

$$S=S_{L\subset P,G}=\{g\in G\,|\,g^{-1}F(g)\in F\,U_P\}$$

is acted on by the finite group $G^F\times L^F$ as follows:

$$(g_0,l)\colon g\mapsto g_0\,g\,l^{-1}\qquad ((g_0,l)\in G^F\times L^F, g\in S).$$

By transport of structure, $G^F \times L^F$ acts on the vector space $H_c^i(S)$ (for any $i \geqq 0$) by the maps $(g_0, l)^{*-1}$.

Let π be an L^F-module. We may regard π as a $G^F \times L^F$-module, with G^F acting trivially. The tensor product (over $\bar{\mathbf{Q}}_l$) $H_c^i(S) \otimes \pi$ is again a $G^F \times L^F$ module in a natural way. Clearly, the L^F-invariant part $(H_c^i(S) \otimes \pi)^{L^F}$ is G^F-stable. Taking alternating sum over i, we get a virtual G^F-module

$$R_{L \subset P}^G(\pi) = \sum_i (-1)^i (H_c^i(S) \otimes \pi)^{L^F}.$$

Its character at $g_0 \in G$ is given by

$$(1.1) \quad Tr(g_0, R_{L \subset P}^G(\pi)) = |L^F|^{-1} \sum_{l \in L^F} \sum_i (-1)^i Tr((g_0, l)^{*-1}, H_c^i(S)) Tr(l, \pi).$$

Clearly, $\pi \mapsto R_{L \subset P}^G(\pi)$ can be regarded as a homomorphism $R_{L \subset P}^G \colon \mathscr{R}(L^F) \to \mathscr{R}(G^F)$.

2. **Lemma.** *If $g_0 \in G^F$ is unipotent, we have*

$$Tr(g_0, R_{L \subset P}^G(\pi)) = |L^F|^{-1} \sum_{\substack{l \in L^F \\ \text{unipotent}}} \sum_i (-1)^i Tr((g_0, l)^{*-1}, H_c^i(S)) Tr(l, \pi).$$

Let $l \in L$; we can write $l = su = us$ with $s \in L^F$ semisimple, $u \in L^F$ unipotent. Using [2, 3.2] we see that

$$(2.1) \quad \sum_i (-1)^i Tr((g_0, l)^{*-1}, H_c^i(S)) = \sum_i (-1)^i Tr((g_0, l)^{*-1}, H_c^i(S^s))$$

where S^s is the fixed point scheme of s on S. Clearly, S^s is empty when $s \neq 1$, i.e. when l is non unipotent; in this case, the expression (2.1) is zero and the lemma follows.

3. **Lemma.** *Let T be an F-stable maximal torus in L and let B be a Borel subgroup of G such that $T \subset B \subset P$. Let $B_1 = B \cap L$. The map which sends $(g, g') \in S_{L \subset P, G} \times S_{T \subset B_1, L}$ to $g'' = gg'$ defines an isomorphism*

$$L^F | (S_{L \subset P, G} \times S_{T \subset B_1, L}) \xrightarrow{\sim} S_{T \subset B, G}.$$

(L^F acts on $S_{L \subset P, G} \times S_{T \subset B_1, L}$ by l: $(g, g') \mapsto (gl^{-1}, lg')$.)

If $g \in G$, $g' \in L$ satisfy $g^{-1}F(g) \in FU_P$, $g'^{-1}F(g') \in FU_{B_1}$, we have

$$\begin{aligned}(gg')^{-1}F(gg') &= g'^{-1}g^{-1}F(g)F(g') \in g'^{-1}FU_P F(g') = g'^{-1}F(g')FU_P \subset FU_{B_1}FU_P \\ &= FU_B\end{aligned}$$

so that $gg' \in S_{T \subset B, G}$. Now let $g'' \in S_{T \subset B, G}$, so that $g''^{-1}F(g'') \in FU_B$. We may write $g''^{-1}F(g'') = u_1 u_2$ with $u_1 \in FU_{B_1}$, $u_2 \in FU_P$ uniquely determined. By Lang's theorem, we can find $g' \in L$ such that $g'^{-1}F(g') = u_1$; if we put $g = g''g'^{-1}$, we have $g^{-1}F(g) = F(g')u_2 F(g')^{-1} \in FU_P$. Any other choice for $g' \in L$ is of the form lg' ($l \in L^F$); this changes g into $l^{-1}g$. Thus we have defined a morphism

$$S_{T \subset B, G} \to L^F | (S_{L \subset P, G} \times S_{T \subset B_1, L})$$

which is clearly the inverse of the morphism in the lemma.

More generally, let Q be a parabolic subgroup of G such that $Q \subset P$ and let M be a Levi subgroup of Q such that $M \subset L$. We assume that $FM = M$. As Deligne has observed, in this situation one has the transitivity formula $R^G_{L \subset P} \circ R^L_{M \subset L \cap Q} = R^G_{M \subset Q}$ as homomorphisms $\mathscr{R}(M^F) \to \mathscr{R}(G^F)$; the proof is identical to the one given above, using the appropriate generalization of Lemma 3.

4. Now let $\theta \in (T^F)^\vee$ and let π_θ be a one dimensional T^F-module with character θ. Clearly, $R^G_{T \subset B}(\pi_\theta)$ is the same as R^θ_T of [2, 4.3] (at this point, the reader is warned that the action of T^F on $S_{T \subset B, G}$ considered in 1 is the inverse of that in [2, 1.17]). In particular, $R^G_{T \subset B}(\pi_\theta)$ is independent of B. We shall denote it here as $R^G_T(\theta)$. Similarly, $R^L_{T \subset B_1}(\pi_\theta) = R^L_T(\theta)$ is independent of B.

5. **Corollary.** *If θ is as above, we have $R^G_{L \subset P}(R^L_T(\theta)) = R^G_T(\theta)$.*

Using Lemma 3 and the Künneth formula we see that for any integer $t \geqq 0$:

$$\bigoplus_{i+j=t} (H^i_c(S_{L \subset P, G}) \otimes H^j_c(S_{T \subset B_1, L}))^{L^F} \cong H^t_c(S_{T \subset B, G}),$$

(the isomorphism being compatible with the natural action of $G^F \times T^F$). For any T^F-module M, we denote by M_θ the part of M on which T^F acts by θ^{-1}. We then have an isomorphism of G^F-modules

$$(5.1)\qquad \bigoplus_{i+j=t} (H^i_c(S_{L \subset P, G}) \otimes H^j_c(S_{T \subset B_1, L})_\theta)^{L^F} \cong H^t_c(S_{T \subset B, G})_\theta .$$

Taking alternating sum over t, we get

$$\begin{aligned} R^G_T(\theta) &= \sum_t (-1)^t H^t_c(S_{T \subset B, G})_\theta = \sum_{i,j} (-1)^{i+j} (H^i_c(S_{L \subset P, G}) \otimes H^j_c(S_{T \subset B_1, L})_\theta)^{L^F} \\ &= \sum_j (-1)^j R^G_{L \subset P}(H^j_c(S_{T \subset B_1, L})_\theta) = R^G_{L \subset P}(R^L_T(\theta)). \end{aligned}$$

6. **Corollary.** *Let θ be as above and let π be an L^F-submodule of $H^j_c(S_{T \subset B_1, L})_\theta$. Then $R^G_{L \subset P}(\pi)$ can be regarded as a $\mathbf{Z}$-linear combination of irreducible G^F-submodules of $H^t_c(S_{T \subset B, G})_\theta$ (for t variable).*

By (5.1), $(H^i_c(S_{L \subset P, G}) \otimes \pi)^{L^F}$ is isomorphic to a G^F-submodule of $H^{i+j}_c(S_{T \subset B, G})_\theta$ and the result follows.

7. Let (T, θ) be as above. Using a fixed isomorphism of k^* with the part of order prime to p of $\mathbf{Q}/\mathbf{Z}$, we may identify T^F with $Y(T)/(F-1)Y(T)$ where $Y(T)$ is the lattice of one parameter subgroups of T. (See [2, 5.2]). Thus we may regard θ as a homomorphism $Y(T) \to \overline{\mathbf{Q}}^*_l$. We now consider another F-stable maximal torus $T' \subset L$ and $\theta' \in (T'^F)^\vee$. Let $n \in G$ be such that $n T' n^{-1} = T$. The map $t' \mapsto n t' n^{-1}$ ($t' \in T'$) induces a homomorphism $ad(n)$: $Y(T') \to Y(T)$. Let

$$N_{\theta, \theta'} = \{n \in G \mid n T' n^{-1} = T,\ \theta \circ ad(n) = \theta'\};$$

here $\theta \circ ad(n)$ and θ' are regarded as homomorphisms $Y(T') \to \overline{\mathbf{Q}}^*_l$.

8. **Theorem.** *Let T, T', θ, θ' be as above. Let B, B' be Borel subgroups in G such that $T \subset B \subset P$, $T' \subset B' \subset P$; we put $B_1 = B \cap L$, $B'_1 = B' \cap L$. Let π, π' be irreducible L^F-modules such that $\langle \pi, R^L_T(\theta) \rangle_{L^F} \neq 0$, $\langle \pi', R^L_T(\theta') \rangle_{L^F} \neq 0$. Assume that $N_{\theta, \theta'} \subset L$.*

Then

(8.1) $\langle R^G_{L\subset P}(\pi), R^G_{L\subset P}(\pi')\rangle_{G^F} = \langle \pi, \pi'\rangle_{L^F}$.

Let $\bar\pi'$ be the L^F-module dual to π', so that $Tr(l, \bar\pi') = Tr(l^{-1}, \pi')$, $l \in L^F$. Then $R^G_{L\subset P}(\bar\pi')$ is the dual of $R^G_{L\subset P}(\pi')$; this follows from (1.1) and from the fact that $\sum_i (-1)^i Tr((g_0, l)^{*-1}, H^i_c(S_{L\subset P, G}))$ is an integer for any $(g_0, l)\in G^F$ (see [2, 3.3]). Similarly, $\sum_j (-1)^j H^j_c(S_{T'\subset B'_1, L})_{\theta'^{-1}}$ is the dual of $\sum_j (-1)^j H^j_c(S_{T'\subset B'_1, L})_{\theta'}$ hence $\langle \bar\pi', \sum_j (-1)^j H^j_c(S_{T'\subset B'_1, L})_{\theta'^{-1}}\rangle_{L^F} \neq 0$. We may identify $\bar\pi'$ with an L^F-submodule of $H^{j'}_c(S_{T'\subset B'_1, L})_{\theta'^{-1}}$ for some j'. Similarly, we may identify π with an L^F-submodule of $H^j_c(S_{T\subset B_1, L})_\theta$. We have

$$\begin{aligned}\langle R^G_{L\subset P}(\pi), R^G_{L\subset P}(\pi')\rangle_{G^F} &= |G^F|^{-1} \sum_{g_0\in G^F} Tr(g_0, R^G_{L\subset P}(\pi))\, Tr(g_0, R^G_{L\subset P}(\bar\pi'))\\ &= |G^F|^{-1} |L^F|^{-2} \sum_{i,i'} (-1)^{i+i'} \sum_{\substack{g\in G^F\\ l, l'\in L^F}} Tr((g_0, l)^{*-1}, H^i_c(S))\, Tr((g_0, l)^{*-1}, H^{i'}_c(S))\\ &\quad\cdot Tr(l, \pi)\, Tr(l', \bar\pi')\\ &= \dim\Big(\sum_i (-1)^i H^i_c(G^F | S\times S)\otimes \pi\otimes\bar\pi')^{L^F\times L^F}\Big),\end{aligned}$$

where the action of G^F on $S\times S$ is g_0: $(g, g')\mapsto (g_0 g, g_0 g')$, $g_0\in G^F$, $(g, g')\in S\times S$ and the action of $L^F\times L^F$ on $H^i_c(G^F | S\times S)\otimes\pi\otimes\bar\pi'$ is

$$(l, l'):\ \alpha\otimes\beta\otimes\gamma \mapsto (l, l')^{*-1}\alpha\otimes l\beta\otimes l'\gamma.$$

The map $(g, g')\mapsto(x, x', y)$, $x = g^{-1}F(g)$, $x' = g'^{-1}F(g')$, $y = g^{-1}g'$, defines an isomorphism of $G^F | S\times S$ with

$$\mathfrak{S} = \{(x, x', y)\in FU_P\times FU_P\times G \mid xF(y) = yx'\}.$$

Under this isomorphism, the action of $L^F\times L^F$ on $G^F | S\times S$ becomes the following action of $L^F\times L^F$ on $\mathfrak{S}$:

$$(x, x', y)\mapsto (lxl^{-1}, l'x'l'^{-1}, l\,y\,l'^{-1}),\qquad (l, l')\in L^F\times L^F.$$

We must show that $\dim(\sum_i (-1)^i (H^i_c(\mathfrak{S})\otimes\pi\otimes\bar\pi')^{L^F\times L^F}) = \langle\pi, \pi'\rangle_{L^F}$. This will clearly follow from the result below:

Lemma. *With the assumptions of the theorem,* $\dim(H^i_c(\mathfrak{S})\otimes\pi\otimes\bar\pi')^{L^F\times L^F}$ *is equal to* $\langle\pi, \pi'\rangle_{L^F}$ *if* $i = 2d$ ($d = \dim U_P + \dim(U_P\cap FU_P)$) *and is zero if* $i\neq 2d$.

The partition $\mathfrak{S} = \mathfrak{S}'\cup\mathfrak{S}''$ with $\mathfrak{S}' = \{(x, x', y)\in\mathfrak{S} \mid y\in P\}$, $\mathfrak{S}'' = \{(x, x', y)\in\mathfrak{S} \mid y\in G-P\}$ is stable under the action of $L^F\times L^F$ and gives rise to a natural long exact sequence

$$\cdots\to H^{i-1}_c(\mathfrak{S}')\to H^i_c(\mathfrak{S}'')\to H^i_c(\mathfrak{S})\to H^i_c(\mathfrak{S}')\to\cdots$$

This, in turn, gives rise to an exact sequence

$$\begin{aligned}(8.2)\quad \ldots &\to (H^{i-1}_c(\mathfrak{S}')\otimes\pi\otimes\bar\pi')^{L^F\times L^F}\to (H^i_c(\mathfrak{S}'')\otimes\pi\otimes\bar\pi')^{L^F\times L^F}\to\\ &\to (H^i_c(\mathfrak{S})\otimes\pi\otimes\bar\pi')^{L^F\times L^F}\to (H^i_c(\mathfrak{S}')\otimes\pi\otimes\bar\pi')^{L^F\times L^F}\to\cdots\end{aligned}$$

If $(x, x', y) \in \mathfrak{S}'$, we have $y \in P$; we may write $y = lu$ with $l \in L$, $u \in U_P$ uniquely determined. We show that we must have $l \in L^F$ and $u \in U_P \cap FU_P$. First note that P, FP have L as a common Levi subgroup; it follows that $P \cap U_{FP} = U_P \cap FP = U_P \cap U_{FP}$ (they all coincide with the unipotent radical of $P \cap FP$). From $xF(y) = yx'$ we get $xF(l)F(u) = lux'$ hence $u = l^{-1}F(l)u'$ with $u' = F(l)^{-1}xF(l)F(u)x' \in FU_P$. Thus $u \in U_P \cap FP = U_P \cap FU_P$, and $l^{-1}F(l) = uu' \in L \cap FU_P = \{1\}$, so that $l \in L^F$. Conversely, if $x \in FU_P$ and $y \in L^F(U_P \cap FU_P)$ are given, there is a unique $x' \in FU_P$ such that $(x, x', y) \in \mathfrak{S}'$; in fact, $x' = y^{-1}xF(y)$ clearly belongs to FU_P. Thus the map $(x, x', y) \mapsto (x, y)$ is an isomorphism of $\mathfrak{S}'$ with $FU_P \times L^F(U_P \cap FU_P)$. As FU_P and $U_P \cap FU_P$ are affine spaces and L^F is a finite set, we see that $H_c^i(\mathfrak{S}') = 0$ for $i \neq 2d$ and $H_c^{2d}(\mathfrak{S}') \cong H_c^0(L^F)(-d)$. This last isomorphism is compatible with the action of $L^F \times L^F$, where the action of $L^F \times L^F$ on L^F is (l, l'): $y \mapsto l y l'^{-1}$, $(l, l') \in L^F \times L^F$, $y \in L^F$. It follows that

$$\begin{aligned}
&\dim((H_c^{2d}(\mathfrak{S}') \otimes \pi \otimes \bar{\pi}')^{L^F \times L^F}) = \dim((H_c^0(L^F) \otimes \pi \otimes \bar{\pi}')^{L^F \times L^F}) \\
&\quad = |L^F|^{-2} \sum_{l, l' \in L^F} \# \{y \in L^F \mid y = l y l'^{-1}\}\, Tr(l, \pi)\, Tr(l', \bar{\pi}') \\
&\quad = |L^F|^{-2} \sum_{l, y \in L^F} Tr(l, \pi)\, Tr(y^{-1} l y, \bar{\pi}') = |L^F|^{-1} \sum_{l \in L^F} Tr(l, \pi)\, Tr(l, \bar{\pi}') = \langle \pi, \pi' \rangle_{L^F}.
\end{aligned}$$

Using (8.2) we see that it is enough to prove that $(H_c^i(\mathfrak{S}'') \otimes \pi \otimes \bar{\pi}')^{L^F \times L^F} = 0$ for all i. As $\pi \subset H_c^j(S_{T \subset B_1, L})_\theta$, $\bar{\pi}' \subset H_c^{j'}(S_{T' \subset B_1', L})_{\theta'^{-1}}$, it is also enough to prove that, for all i, j, j', we have

(8.3) $\quad (H_c^i(\mathfrak{S}'') \otimes H_c^j(S_{T \subset B_1, L})_\theta \otimes H_c^{j'}(S_{T' \subset B_1', L})_{\theta'^{-1}})^{L^F \times L^F} = 0.$

Let $\tilde{\mathfrak{S}}'' = \{(\tilde{x}, \tilde{x}', \tilde{y}) \in FU_B \times FU_{B'} \times (G - P) \mid \tilde{x}F(\tilde{y}) = \tilde{y}\tilde{x}'\}$. The map which sends $((x, x', y), g, g') \in \mathfrak{S}'' \times S_{T \subset B_1, L} \times S_{T' \subset B_1', L}$ to $(g^{-1}xF(g), g'^{-1}x'F(g'), g^{-1}yg')$ defines an isomorphism

$$(L^F \times L^F) \backslash (\mathfrak{S}'' \times S_{T \subset B_1, L} \times S_{T' \subset B_1', L}) \xrightarrow{\sim} \tilde{\mathfrak{S}}''.$$

This can be seen just as in the proof of Lemma 3. It follows that (8.3) is equivalent to

(8.4) $\quad H_c^i(\tilde{\mathfrak{S}}'')_{\theta, \theta'^{-1}} = 0, \qquad$ for all i,

where, for any $T^F \times T'^F$-module M, $M_{\theta, \theta'^{-1}}$ denotes the part of M on which $T^F \times T'^F$ acts by (θ^{-1}, θ'); the action of $T^F \times T'^F$ on $\tilde{\mathfrak{S}}''$ is:

$$(t, t'): (\tilde{x}, \tilde{x}', \tilde{y}) \mapsto (t\tilde{x}t^{-1}, t'\tilde{x}'t'^{-1}, t\tilde{y}t'^{-1}), \qquad (x, x', y) \in \tilde{\mathfrak{S}}'', \ (t, t') \in T^F \times T'^F.$$

The rest of the argument is based on [2, 6.6 and 6.7]. Let $N(T, T') = \{n \in G \mid Tn = nT'\}$ and let $W(T, T') = T \backslash N(T, T') = N(T, T')/T'$. For any $w \in W(T, T')$ we put $G_w = B\dot{w}B'$ where $\dot{w}$ is a representative of w in $N(T, T')$. Then the G_w, for $w \in W(T, T')$ such that $\dot{w} \notin P$ form a finite partition of $G - P$ into locally closed subschemes; for any such w, let $\tilde{\mathfrak{S}}''_w = \{(\tilde{x}, \tilde{x}', \tilde{y}) \in \tilde{\mathfrak{S}}'' \mid \tilde{y} \in G_w\}$. The $\tilde{\mathfrak{S}}''_w$ for w such that $\dot{w} \notin P$ form a finite partition of $\tilde{\mathfrak{S}}''$ into locally closed subschemes, stable under $T^F \times T'^F$. As in [*loc. cit.*] we see that in order to prove (8.4) it is enough to show that

(8.5) $\quad H_c^i(\tilde{\mathfrak{S}}''_w)_{\theta, \theta'^{-1}} = 0, \qquad$ for all i, and all $w \in W(T, T')$ such that $\dot{w} \notin P$.

According to [*loc. cit.*] we have $H^i_c(\tilde{\mathfrak{S}}''_w)_{\theta,\theta'^{-1}}=0$ for all i, provided that $\dot w\notin N_{\theta,\theta'}$. If $w\in W(T,T')$ is such that $\dot w\notin P$, we have also $\dot w\notin L$ and our assumption $N_{\theta,\theta'}\subset L$ shows that $\dot w\notin N_{\theta,\theta'}$. This ends the proof of the lemma, hence that of the theorem.

9. **Proposition.** *Let $L\subset P$ be as in 1 and let $\theta\colon L^F\to\bar{\mathbf{Q}}_l^*$ be a homomorphism whose restriction to the image of $\tilde L^F\to L^F$ is trivial. (Here $\tilde L\to L$ is the simply-connected covering of the derived group of L). Then*

(9.1) $|L^F|^{-1}\sum_{T\subset L}|T^F|R^G_T(\theta|T^F)$ (summation over all F-stable maximal tori in L)

is a virtual representation of G^F.

According to [2, (7.14.1)], the unit representation of L^F can be expressed as a sum

$$1=|L^F|^{-1}\sum_{T\subset L}|T^F|R^L_T(1)$$

We now tensor both sides of the last equality by θ and use that $\theta\otimes R^L_T(1)=R^L_T(\theta|T^F)$, cf. [2, 1.27]. We get

$$\theta=|L^F|^{-1}\sum_{T\subset L}|T^F|R^L_T(\theta|T^F).$$

We now apply $R^G_{L\subset P}$ to this equality and use Corollary 5. We see that the expression (9.1) equals $R^G_{L\subset P}(\theta)$. This ends the proof.

10. **Proposition.** *Let $T\subset L\subset P$ be as before and let $\theta\in(T^F)^\vee$. We assume that $N_{\theta,\theta}\subset L$. Write $R^L_T(\theta)=\sum_{i=1}^n c_i\pi_i$ with c_i non-zero integers and π_i mutually non-isomorphic, irreducible L^F-modules. Then there exist well-defined signs $\varepsilon_i=\pm 1$ such that*

$$R^G_T(\theta)=\sum_{i=1}^n(\varepsilon_i c_i)(\varepsilon_i R^G_{L\subset P}(\pi_i))$$

with $\varepsilon_i R^G_{L\subset P}(\pi_i)$ mutually non-isomorphic, irreducible G^F-modules. In particular, any irreducible G^F-module ρ such that $\langle\rho,R^G_T(\theta)\rangle_{G^F}\neq 0$ is of the form $\rho=\pm R^G_{L\subset P}(\pi)$, for some well-defined irreducible L^F-module π with $\langle\pi,R^L_T(\theta)\rangle_{L^F}\neq 0$.

By (8.1) we have

$$\langle R^G_{L\subset P}(\pi_i),R^G_{L\subset P}(\pi_j)\rangle_{G^F}=\langle\pi_i,\pi_j\rangle_{L^F}$$

for all i,j. It follows that there exist well defined signs $\varepsilon_i=\pm 1$ $(1\leqq i\leqq n)$ such that $\varepsilon_i R^G_{L\subset P}(\pi_i)$ are mutually non-isomorphic, irreducible G^F-modules. By Corollary 5, we have $R^G_{L\subset P}(R^L_T(\theta))=R^G_T(\theta)$. Applying $R^G_{L\subset P}$ to the equality $R^L_T(\theta)=\sum_{i=1}^n c_i\pi_i$ we get $R^G_T(\theta)=\sum_{i=1}^n c_i R^G_{L\subset P}(\pi_i)$. This ends the proof.

11. *Remark.* The signs ε_i in Proposition 10 are given by

(11.1) $\varepsilon_i=(-1)^{\sigma(G)-\sigma(L)}$

where $\sigma(G)$ is the dimension of a maximal $\mathbf{F}_q$-split torus of G and $\sigma(L)$ is defined analogously; we shall not make use of this result in this paper. Clearly, (11.1) is a consequence of the following result.

12. **Proposition.** *For any irreducible L^F-module π, we have*

(12.1) $\quad (-1)^{\sigma(G)-\sigma(L)} R^G_{L\subset P}(\pi)\otimes St_G = \mathrm{Ind}_{L^F\uparrow G^F}(\pi\otimes St_L)$

where St_G (resp. St_L) is the Steinberg representation of G^F (resp. L^F). (Compare [4, Cor. 1.3]).

Under the assumption that $\langle\pi, R^L_T(\theta)\rangle_{L^F}\neq 0$ for some pair (T,θ) as in 4, with $N_{\theta,\theta}\subset L$, the identity (12.1) can be deduced from (8.1) and results of [2] (especially [2, 6.3], [2, 7.6]); this is already sufficient for the proof of (11.1). Deligne has shown that the assumption on π is actually unnecessary. We ommit further details.

13. **Theorem.** *There are only finitely many G-conjugacy classes of unipotent elements in G. In fact, if G is simple, adjoint of rank $r\geqq 1$, there are at most $|W|^2(r+1)^2$ such classes.*

In the rest of this paper (except in section 20), G will be assumed to have a smmoth and connected centre; it is clearly sufficient to prove the theorem under this assumption.

14. Let G^* be the group dual to G as in [2, 5.21]. The derived group $G^{*\prime}$ of G^* is simply connected ([2, 5.23]) hence the centralizer of any semisimple element in G^* is connected.

We shall say that a semisimple element $x\in G^*$ is exceptional if x is in the derived group of G^* and if its centralizer $Z_{G^*}(x)$ in G^* has semisimple rank equal to that of G^*. (We shall also use the term "G^*-exceptional"). Of course, for this definition, the $\mathbf{F}_q$ structure on G is not needed.

Now, any pair (T,θ) with T an F-stable maximal torus in G and $\theta\in(T^F)^\vee$ gives rise to a semisimple element $x\in G^{*F}$ well defined up to G^{*F}-conjugacy, cf. [2, (5.21.6)]; this correspondence depends on the choice of an embedding $k^*\subset\overline{\mathbf{Q}}_l^*$. If (T,θ), (T',θ') give rise to $x\in G^{*F}$, $x'\in G^{*F}$ then (T,θ), (T',θ') are geometrically conjugate (in the sense of [2, 5.5], i.e. $N_{\theta,\theta'}$ is non-empty) if and only if x, x' are G^{*F}-conjugate (cf. [2, 5.24]); moreover, any semisimple class in G^{*F} comes from some pair (T,θ).

We shall say that a pair (T,θ) is exceptional (or G-exceptional) if it gives rise to an element $x\in G^{*F}$ which is exceptional.

Now, a semisimple element $x\in G^{*\prime F}$ is exceptional if and only if $Z_{G^*}(x)$ is not contained in any F-stable Levi subgroup L^* of a proper parabolic subgroup of G^*. In fact, if $Z_{G^*}(x)$ has semisimple rank strictly less than that of G^*, then the centralizer in G^* of the identity component of the centre of $Z_{G^*}(x)$ would be such an L^*.

It follows that a pair (T,θ) in G is exceptional if and only if θ is trivial on the intersection of T^F with the centre of G and if $N_{\theta,\theta}$ is not contained in any F-stable Levi subgroup of a proper parabolic subgroup of G.

15. The adjoint group of G (G modulo its centre) will be denoted $\bar{G}$. It has a canonical direct product decomposition $\bar{G}=\bar{G}_A\times\bar{G}_B$ with $\bar{G}_A$ a product of simple

groups of type $A_r (r \geqq 1)$ and $\bar{G}_B$ without simple factors of type $A_r (r \geqq 1)$. This decomposition is over k but it is automatically F-stable. Similarly, the derived group $G^{*'F}$ of G^{*F} has a canonical, F-stable direct product decomposition $G^{*'} = G_A^{*'} \times G_B^{*'}$ with $G_A^{*'}$ a product of groups isomorphic to $SL(n)$, $n \geqq 2$ and $G_B^{*'}$ with no such factors.

Let $\mathscr{E}(G)$ be the set of all pairs (T, θ) with T an F-stable maximal torus in G and $\theta \in (T^F)^\vee$ such that there exists an F-stable Levi subgroup L of a parabolic subgroup in G with the following properties:

(a) $T \subset L$ and (T, θ) is L-exceptional.

(b) If $\bar{T}$ is the image of T in $\bar{L}$, then the character of $\bar{T}^F$ defined by θ is trivial on $\bar{T}^F \cap \bar{L}_A$.

We also define $\mathscr{E}^*(G)$ to be the set of all semisimple elements $x \in G^{*F}$ for which there exists an F-stable Levi subgroup L^* of a parabolic subgroup in G^* with the following properties:

(a*) $x \in L^*$ and x is L^*-exceptional.

(b*) $x \in L_B^{*'}$.

16. **Lemma.** *Let ρ be a virtual G^F-module. There exist irreducible G^F-modules $\rho_j (1 \leqq j \leqq n)$ and numbers $c_j \in \bar{\mathbf{Q}}_l (1 \leqq j \leqq n)$ such that*

(16.1) *for any j, $(1 \leqq j \leqq n)$ there exists $(T, \theta) \in \mathscr{E}(G)$ such that $\langle \rho_j, R_T^G(\theta) \rangle_{G^F} \neq 0$*

(16.2) *$Tr(g_0, \rho) = \sum_i c_j \, Tr(g_0, \rho_j)$ for any unipotent element $g_0 \in G^F$.*

We may assume that $\dim G > 0$ and that the lemma is proved for groups with smooth and connected centre and whose dimension is $< \dim G$.

The natural map $G \to \bar{G}$ induces a bijection from the set of G^F-conjugacy classes of unipotents in G^F to the set of $\bar{G}^F$-conjugacy classes of unipotents in $\bar{G}^F$ (since the centre of G is connected.) It follows that any class function on G^F will agree, on the unipotents in G^F, with a class function on G^F which is constant on the fibres of $G^F \to \bar{G}^F$. Thus, we may assume that ρ comes from a virtual $\bar{G}^F$ module (under the map $\mathscr{R}(\bar{G}^F) \to \mathscr{R}(G^F)$). If $\dim \bar{G} < \dim G$, the lemma is true for $\bar{G}$ by assumption; using the previous argument, we deduce that it is also true for G. Thus, we may assume that $G = \bar{G}$.

Write $G = G_A \times G_B$ as in 15. If G_A and G_B have both dimension > 0, the lemma will be true for G_A and G_B; it follows easily that it is also true for their product G. Thus, we may assume that G equals G_A or G_B. To prove the lemma, we may now also assume that ρ is irreducible. According to [2, 7.7], there exists an F-stable maximal torus $T \subset G$ and $\theta \in (T^F)^\vee$ such that $\langle \rho, R_T^G(\theta) \rangle_{G^F} \neq 0$. Assume first that for such a pair (T, θ) there exists an F-stable Levi subgroup L of a proper parabolic subgroup P of G such that $N_{\theta, \theta} \subset L$ (hence $T \subset L$). By Proposition 10, there exists an irreducible L^F-module π such that $\rho = \varepsilon R_{L \subset P}^G(\pi)$, $\varepsilon = \pm 1$. As L has a smooth and connected centre and $\dim L < \dim G$, we may apply the induction hypothesis and conclude that there exist irreducible L^F-modules $\pi_j (1 \leqq j \leqq m)$ and numbers $b_j \in \bar{\mathbf{Q}}_l$ such that $\langle \pi_j, R_{T_j}^L(\theta_j) \rangle_{L^F} \neq 0$ for some $(T_j, \theta_j) \in \mathscr{E}(L)$, $(1 \leqq j \leqq m)$ and $Tr(l, \pi) = \sum_j b_j \, Tr(l, \pi_j)$ for any unipotent element $l \in L^F$. Using Lemma 2, we see that for any unipotent element $g_0 \in G^F$ we have

$$Tr(g_0, \rho) = \varepsilon\, Tr(g_0, R^G_{L\subset P}(\pi)) = |L^F|^{-1}\, \varepsilon \sum_{\substack{l\in L^F\\ \text{unipotent}}} \sum_i (-1)^i\, Tr((g_0, l)^{*-1}, H^i_c(S))\, Tr(l, \pi)$$

$$= |L^F|^{-1}\, \varepsilon \sum_{\substack{l\in L^F\\ \text{unipotent}}} \sum_i (-1)^i\, Tr((g_0, l)^{*-1}, H^i_c(S)) \sum_j b_j\, Tr(l, \pi_j)$$

$$= \sum_j b_j\, |L^F|^{-1}\, \varepsilon \sum_{\substack{l\in L^F\\ \text{unipotent}}} \sum_i (-1)^i\, Tr((g_0, l)^{*-1}, H^i_c(S))\, Tr(l, \pi_j)$$

$$= \sum_j \varepsilon b_j\, Tr(g_0, R^G_{L\subset P}(\pi_j)).$$

(Here $S = S_{L\subset P, G}$). By Corollary 6, $R^G_{L\subset P}(\pi_j)$ is a $\mathbf{Z}$-linear combination $\sum_h a_{jh}\, \rho_{jh}$ of irreducible G^F-submodules ρ_{jh} of $\bigoplus_t H^t_c(S_{T_j\subset B_j, G})_{\theta_j}$ for a suitable Borel subgroup B_j, $T_j \subset B_j \subset P$. For each ρ_{jh} there exists a pair (T', θ') such that

$$\langle \rho_{jh}, R^G_{T'}(\theta') \rangle_{G^F} \neq 0.$$

As $\rho_{jh} \subset H^t_c(S_{T_j\subset B_j, G})_{\theta_j}$ for some t, we see from [2, 6.2] that (T_j, θ_j) and (T', θ') are geometrically conjugate. As $(T_j, \theta_j) \in \mathscr{E}(L)$ and $\mathscr{E}(L) \subset \mathscr{E}(G)$, we have $(T_j, \theta_j) \in \mathscr{E}(G)$. As (T_j, θ_j) and (T', θ') are geometrically conjugate, we must also have $(T', \theta') \in \mathscr{E}(G)$. We have

$$Tr(g_0, \rho) = \sum_j \varepsilon b_j \sum_h a_{jh}\, Tr(g_0, \rho_{jh})$$

for any unipotent element $g_0 \in G^F$, as required.

It remains to consider the case where there is no L (an F-stable Levi subgroup of a proper parabolic subgroup) with $N_{\theta,\theta} \subset L$. As G is adjoint, (T, θ) must be G-exceptional. If $G = G_B$ then clearly $(T, \theta) \in \mathscr{E}(G)$ and there is nothing to be proved.

Assume now that $G = G_A$. The semisimple conjugacy class in G^{*F} determined by (T, θ) (cf. 14) is represented by a semisimple element $x \in G^{*F}$ whose centralizer has the same semisimple rank as G^*. As G^* is isomorphic over k to a product of $SL(n)$'s, we see that x must be in the centre of G^*. It follows that θ must be the the restriction to T^F of a homomorphism $\theta': G^F \to \bar{\mathbf{Q}}_l^*$ such that θ' is trivial on $\pi(\tilde{G}^F)$, where $\pi: \tilde{G} \to G$ is the simply connected covering of G. According to [2, 1.27], we have $R^G_T(\theta) = \theta' \otimes R^G_T(1)$. Hence $\langle \theta'^{-1} \otimes \rho, R^G_T(1) \rangle_{G^F} = \langle \rho, R^G_T(\theta) \rangle_{G^F} \neq 0$. Clearly, $(T, 1) \in \mathscr{E}(G)$ and $Tr(g_0, \rho) = Tr(g_0, \theta'^{-1} \otimes \rho)$ for all unipotent elements $g_0 \in G^F$ (indeed, $\theta'(g_0) = 1$ for such g_0). This ends the proof.

17. **Lemma.** *Let $e(G)$ be the number of G^{*F}-conjugacy classes of elements in $\mathscr{E}^*(G)$. The number of G^F-conjugacy classes of unipotent elements in G^F is at most equal to $|W|^2 e(G)$.*

Lemma 16 shows that the number of G^F-conjugacy classes of unipotent elements in G^F is at most equal to the number of distinct irreducible G^F-modules ρ such that $\langle \rho, R^G_T(\theta) \rangle_{G^F} \neq 0$ for some $(T, \theta) \in \mathscr{E}(G)$, hence it is at most equal to

$$\sum_{\substack{(T,\theta)\in\mathscr{E}(G)\\ \text{mod } G^F}} \langle R^G_T(\theta), R^G_T(\theta) \rangle_{G^F}$$

According to the orthogonality formula [2, 6.8], we have $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{G^F} \leqq |W|$ for any (T,θ) hence the previous sum is at most equal to

$$|W|\,\#((T,\theta)\in\mathscr{E}(G) \bmod G^F).$$

The correspondence $(T,\theta)\mapsto x\in G^{*F}$ in 14 defines a surjective map $\mathscr{E}(G)/G^F \to \mathscr{E}^*(G)/G^F$; the fibres of this map can be described as the sets of G^F-conjugacy classes of pairs (T,θ) in a fixed geometric conjugacy class (cf. 14) or, alternatively as the sets of $Z_{G^*}(x)^F$-conjugacy classes of F-stable maximal tori in G^* containing x, for various $x\in\mathscr{E}^*(G)$. In particular, these fibres have at most $|W|$ elements. Thus

$$\#((T,\theta)\in\mathscr{E}(G) \bmod G^F) \leqq |W|\,\#(x\in\mathscr{E}^*(G) \bmod G^{*F}) = |W|\,e(G).$$

and the lemma is proved.

18. **Lemma.** *Assume that G is simple, adjoint, of rank $r\leqq 1$. Then $e(G)\leqq(r+1)^2$.*

Let $\tilde{\mathscr{E}}^*(G)$ be the set of semisimple elements $x\in G^*$ such that there exists a Levi subgroup L^* of a parabolic subgroup of G^* such that $L^{*\prime}$ is almost simple of type $\neq A_n (n\geqq 1)$ and has the property that $x\in L^*$ and x is L^*-exceptional (cf. 14). Here we regard the group with just one element as being almost simple of type A_0.

Let $\tilde{e}(G)$ be the number of G^*-conjugacy classes of elements in $\tilde{\mathscr{E}}^*(G)$. We now note that there are at most $(r+1)$ G^*-conjugacy classes of G^*-exceptional elements in G^*. This fact is implicit in the work of Borel-de Siebenthal [1] (see also [8, § 1]). It can be seen as follows. Consider a maximal torus T^* in G^* and let Y be the lattice of its one parameter subgroups so that $T^*\cong Y\otimes k^*$. By identifying k^* with the part of order prime to p in $\mathbf{Q}/\mathbf{Z}$, we get an exact sequence $0\to Y\to Y\otimes\mathbf{Q}_{(p)}\to T^*\to 0$, where $\mathbf{Q}_{(p)}$ denotes the ring of rational numbers with no p in denominator. We regard $Y\otimes\mathbf{Q}_{(p)}$ as a subgroup of $Y\otimes\mathbf{R}$. The roots of G^* with respect to T^* may be regarded as homomorphisms $Y\to\mathbf{Z}$ or $Y\otimes\mathbf{R}\to\mathbf{R}$. The set of points in $Y\otimes\mathbf{R}$ where all roots take non-integral values is a union of (open) simplices. Let us choose one of these simplices C. Let $\bar{C}$ be the closure of C and let $C'=\bar{C}\cap(Y\otimes\mathbf{Q}_{(p)})$. The map $Y\otimes\mathbf{Q}_{(p)}\to T^*$ defines a bijection of C' with a subset of T^* which contains precisely one representative in each G^*-conjugacy class of semisimple elements in G^*. Moreover, a point in C' corresponds to an exceptional class if and only if it is a vertex of the closed simplex $\bar{C}$. It remains to observe that C has exactly $(r+1)$ vertices; note that in general not all vertices of C are in $Y\otimes\mathbf{Q}_{(p)}$.

Next we note that there are at most $(r+1)$ G^*-conjugacy classes of subgroups of G^* which are Levi subgroups of parabolic subgroups and whose derived group is simple of type $\neq A_n (n\geqq 1)$. In fact, there are at most $(r+1)$ connected (possibly empty) subgraphs of type $\neq A_n (n\geqq 1)$, of the Coxeter graph of G^*; this can be easily checked using the classification of Coxeter graphs. It follows that $\tilde{e}(G)\leqq (r+1)^2$.

It is easy to see that $\mathscr{E}^*(G)\subset\tilde{\mathscr{E}}^*(G)$. Two elements in $\mathscr{E}^*(G)$ which are conjugate under G^* are actually conjugate under G^{*F} (since the centralizer of any semisimple element in G^* is connected). It follows that $e(G)\leqq\tilde{e}(G)$ and the lemma is proved.

Combining Lemmas 17 and 18, we deduce

19. **Lemma.** *With the assumptions of Lemma* 18, *the number of G^F-conjugacy classes of unipotent elements in G^F is at most equal to* $|W|^2(r+1)^2$.

We now prove Theorem 13. We may clearly assume, without loss of generality, that G is simple, adjoint of rank $r \geqq 1$. Let $n=|W|^2(r+1)^2$. Assume that there exist unipotent elements $u_1, u_2, \ldots, u_{n+1}$ in G such that no two of them are conjugate under G. We shall derive a contradiction as follows. We can find integers $s_i \geqq 1$ $(1 \leqq i \leqq n+1)$ such that u_i is fixed by F^{s_i} for all i, $1 \leqq i \leqq n+1$. Let $s=\prod_i s_i$. Then $u_i \in G^{F^s}$ for all i, $1 \leqq i \leqq n+1$. Applying Lemma 19 with F replaced by F^s, we see that there must exist $i \neq j$, $1 \leqq i, j \leqq n+1$ such that u_i, u_j are conjugate under G^{F^s} hence also under G. This contradiction completes the proof of the theorem.

We now state our final theorem. (We no longer make assumptions on the centre of G).

20. **Theorem.** *There exists a number N depending only on the Dynkin diagram of G, such that the set of restrictions to the unipotent elements in G^F of the irreducible characters of G^F has at most N elements.*

It is easy to see that there exists a number m depending only on the Dynkin diagram of G, such that the number of G^F-conjugacy classes of F-stable Levi subgroups of parabolic subgroups of G is at most m.

Let L be such a Levi subgroup. Let v_L be the number of distinct functions on the unipotent elements in L^F which are restrictions of irreducible characters of L^F. When π runs through the irreducible representations of L^F, the restrictions of the characters of $R^G_{L \subset P}(\pi)$ (with P a fixed parabolic subgroup for which L is a Levi subgroup) to the unipotent elements in G^F form a set with at most v_L elements; this is a consequence of Lemma 2. It follows that the restrictions of the characters of $\pm R^G_{L \subset P}(\pi)$ to the unipotent elements in G^F form a set with at most $2v_L$ elements. Let $\mathscr{S}$ be the the set of isomorphism classes of irreducible representations of G^F which are not of the form $\pm R^G_{L \subset P}(\pi)$ with $L \neq G$ and π an irreducible L^F-module. Let α be the the number of distinct functions on the unipotent elements in G^F which are restrictions of characters of representations in $\mathscr{S}$. We have clearly

$$v_G \leqq \alpha + 2\sum v_L$$

(sum over a set of representatives of the G^F-conjugacy classes of subgroups L as above, with $L \neq G$). It follows that $v_G \leqq \alpha + 2m \sup_{L \neq G}(v_L)$.

We may assume that the theorem is true for G replaced by $L(L \neq G)$. Thus, it is enough to show that α is bounded above by a number depending only on the Dynkin diagram of G.

Let $\rho \in \mathscr{S}$ and let (T, θ) be a pair as in 4 such that $\langle \rho, R^G_T(\theta)\rangle_{G^F} \neq 0$, cf. [2, 7.7]. Let $\mathscr{M}(T)$ be the set of all $\theta' \in T^F$ such that there is no L as above $(L \neq G)$ with $N_{\theta', \theta'} \subset L$. Using Proposition 10, we see that $\theta \in \mathscr{M}(T)$. Next, we observe that for any homomorphism $\chi: G^F \to \overline{\mathbf{Q}}_l^*$ such that χ is trivial on G'^F (G' is the derived group of G), the characters of π, $\pi \otimes \chi$ have the same restriction to the unipotent elements in G^F. Moreover, $R^G_T(\theta) \otimes \chi = R^G_T(\theta \otimes \chi)$. Let $\overline{\mathscr{M}}(T)$ be the the quotient set of $\mathscr{M}(T)$ obtained by identifying $\theta', \theta'' \in \mathscr{M}(T)$ if and only if $\theta'' = \theta' \otimes \chi$, for some χ as above. Now, there exists a number m' depending only on the Dynkin diagram of G such that $\bar{\mathcal{M}}(T)$ has at most m' elements. (This can be seen using the method of proof of Lemma 18.) Note also that, for any pair (T, θ), there are at most $|W|$ distinct irreducible G^F-modules ρ such that $\langle \rho, R_T^G(\theta)\rangle_{G^F} \neq 0$ (see the proof of Lemma 17). The previous discussion shows that

$$\alpha \leqq \#\{T \bmod G^F\}\,|W|\,m' \leqq |W|^2\,m'$$

and the theorem follows.

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