Inventiones math. 53, 165–184 (1979)

Inventiones mathematicae

Representations of Coxeter Groups and Hecke Algebras

David Kazhdan[1] and George Lusztig[2★]

[1] Harvard University, Department of Mathematics, Cambridge, MA02138, USA
[2] MIT, Department of Mathematics, Cambridge, MA02139, USA

§ 1. Introduction

Let W be a Coxeter group and let S be the corresponding set of simple reflections. Following [2, Ch. IV, § 2, Ex. 34], we define an algebra $\tilde{\mathscr{H}}$ over the polynomial ring $\mathbb{Z}[q]$ as follows. $\tilde{\mathscr{H}}$ has basis elements T_w, one for each $w \in W$. The multiplication is defined by the rules

$$T_w T_{w'} = T_{ww'}, \quad \text{if } l(ww') = l(w) + l(w')$$
$$(T_s + 1)(T_s - q) = 0, \quad \text{if } s \in S;$$

here $l(w)$ is the length of w.

In the case where W is a Weyl group and q is specialized to a fixed prime power, $\tilde{\mathscr{H}} \otimes_{\mathbb{Z}[q]} \mathbb{C}$ can be interpreted as the algebra of intertwining operators of the space of functions on the flag manifold of the corresponding finite Chevalley group $G(F_q)$ (see [loc. cit., Ex. 24]). Therefore, the problem of decomposing this space of functions into irreducible representations of $G(F_q)$ is equivalent to the problem of decomposing the regular representation of $\tilde{\mathscr{H}} \otimes_{\mathbb{Z}[q]} \mathbb{C}$. It is known that, in this case, $\tilde{\mathscr{H}} \otimes_{\mathbb{Z}[q]} \mathbb{C}$ is isomorphic to the group algebra of W; however, in general, this isomorphism cannot be defined without introducing a square root of q (see [1]).

It is therefore, natural to extend the ground ring of $\tilde{\mathscr{H}}$ as follows. For any Coxeter group (W, S) we define the Hecke algebra $\mathscr{H}$ to be $\tilde{\mathscr{H}} \otimes_{\mathbb{Z}[q]} A$, where A is the ring of Laurent polynomials with integral coefficients in the indeterminate $q^{1/2}$.

Our purpose is to construct representations of $\mathscr{H}$ endowed with a special basis. They will be defined in terms of certain graphs. We define a W-graph to be a set of vertices X, with a set Y of edges (an edge is a subset of X consisting of two elements) together with two additional data: for each vertex $x \in X$, we are given a subset I_x of S and, for each ordered pair of vertices y, x such that $\{y, x\} \in Y$, we are given an integer $\mu(y, x) \neq 0$. These data are subject to the requirements (1.0.a), (1.0.b) below. Let E be

★ Both authors were supported in part by the National Science Foundation

0020-9910/79/0053/0165/$04.00

the free A-module with basis X. Then

for any $s \in S$

$$\tau_s(x) = \begin{cases} -x, & \text{if } s \in I_x \\ qx + q^{1/2} \sum\limits_{\substack{y \in X \\ s \in I_y \\ \{y,x\} \in Y}} \mu(y,x)\, y, & \text{if } s \notin I_x \end{cases} \tag{1.0.a}$$

defines an endomorphism of E (i.e. the sum over y is assumed to be always finite), and

for any $s \neq t$ in S such that st has finite order m, we require that

$$\underbrace{\tau_s \tau_t \tau_s \ldots}_{m \text{ factors}} = \underbrace{\tau_t \tau_s \tau_t \ldots}_{m \text{ factors}}. \tag{1.0.b}$$

In other words, there is a unique representation $\varphi\colon \mathscr{H} \to \mathrm{End}(E)$ such that $\varphi(T_s) = \tau_s$, for each $s \in S$.

We shall construct, for any W, such a graph. First, we give some definitions. Let $a \to \bar{a}$ be the involution of the ring A defined by $\overline{q^{1/2}} = q^{-1/2}$. This extends to an involution $h \to \bar{h}$ of the ring $\mathscr{H}$, defined by

$$\overline{\sum a_w T_w} = \sum \bar{a}_w T_{w^{-1}}^{-1}.$$

(Note that T_w is an invertible element of $\mathscr{H}$, for any $w \in W$; for example, if $s \in S$, we have $T_s^{-1} = q^{-1} T_s + (q^{-1} - 1)$.) For any $w \in W$, we define $q_w = q^{l(w)}$, $\varepsilon_w = (-1)^{l(w)}$. Let $\leqq$ be the usual order relation on W (defined, for example, in [11]). We can now state

Theorem 1.1. *For any $w \in W$, there is a unique element $C_w \in \mathscr{H}$ such that*

$$\bar{C}_w = C_w \tag{1.1.a}$$

$$C_w = \sum_{y \leqq w} \varepsilon_y \varepsilon_w q_w^{1/2} q_y^{-1} \overline{P_{y,w}}\, T_y \tag{1.1.b}$$

where $P_{y,w} \in A$ is a polynomial in q of degree $\leqq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$, and $P_{w,w} = 1$.

The following statement is equivalent to Theorem 1.1:

For any $w \in W$, there is a unique element $C'_w \in \mathscr{H}$ such that $\bar{C}'_w = C'_w$ and $C'_w = q_w^{-1/2} \sum\limits_{y \leqq w} P_{y,w} T_y$, where $P_{y,w} \in A$ is a polynomial in q of degree $\leqq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$. (1.1.c)

The elements C_w and C'_w are related by the identity $C' = \varepsilon_w j(C_w)$, where j is the involution of the ring $\mathscr{H}$ given by $j(\sum a_w T_w) = \sum \bar{a}_w \varepsilon_w q_w^{-1} T_w$.

It may be conjectured that all coefficients of the polynomial $P_{y,w}$ are non-negative integers.

Definition 1.2 *Given $y, w \in W$ we say that $y \prec w$ if the following conditions are satisfied: $y < w$, $\varepsilon_y = -\varepsilon_w$ and $P_{y,w}$ (given by Theorem 1.1) is a polynomial in q of degree exactly $\frac{1}{2}(l(w) - l(y) - 1)$; in this case, the coefficient of the highest power of q in $P_{y,w}$ is denoted $\mu(y,w)$. It is a non-zero integer. If $w \prec y$, we set $\mu(w,y) = \mu(y,w)$.*

Let W^0 be the group opposed to W. Then $(W \times W^0, S \amalg S^0)$ is a Coxeter group. Let Γ_W be the graph whose vertices are the elements of W and whose edges are the subsets of W of the form $\{y, w\}$ with $y \prec w$. For each $w \in W$, let $I_w = \mathcal{L}(w) \amalg \mathcal{R}(w)^0 \subset S \amalg S^0$, where $\mathcal{L}(w) = \{s \in S \mid sw < w\}$, $\mathcal{R}(w) = \{s \in S \mid ws < w\}$.

Theorem 1.3. *Γ_W, together with the assignment $w \to I_w$ and with the function μ defined above, is a $W \times W^0$-graph.*

Now, given any W-graph Γ, and a subset S' of S, we can regard Γ as W'-graph (where W' is the subgroup of W generated by S') by replacing the set $I_x \subset S$, for each vertex x of Γ, by the set $I_x \cap S'$. In particular, Γ_W can be regarded as a W-graph and as a W^0-graph.

Given any W-graph, Γ, we define a preorder relation $\underset{\Gamma}{\leqq}$ on the set of vertices Γ as follows: we say that the vertices x, x' satisfy $x \underset{\Gamma}{\leqq} x'$, if there exists a sequence of vertices $x = x_0, x_1, \ldots, x_n = x'$ such that for each i, $(1 \leqq i \leqq n)$, $\{x_{i-1}, x_i\}$ is an edge of Γ and $I_{x_{i-1}} \not\subset I_{x_i}$. The equivalence relation on the set of vertices, corresponding to this preorder is denoted $\underset{\Gamma}{\sim}$. (Thus, $x \underset{\Gamma}{\sim} x'$ means that $x \underset{\Gamma}{\leqq} x' \underset{\Gamma}{\leqq} x$.) Each equivalence class, regarded as a full subgraph of Γ (with the same sets I_x and the same function μ) is itself a W-graph. The set of equivalence classes is an ordered set with respect to $\underset{\Gamma}{\leqq}$. In the case of the $W \times W^0$-graph Γ_W, the equivalence classes for $\underset{\Gamma_W}{\sim}$ are called the 2-sided cells of W. When Γ_W is regarded as a W-graph, we shall use the notation $\underset{L}{\leqq}$, $\underset{L}{\sim}$ instead of $\underset{\Gamma_W}{\leqq}$, $\underset{\Gamma_W}{\sim}$; the corresponding equivalence classes are called the left cells of W. When Γ_W is regarded as a W^0-graph, we shall use the notation $\underset{R}{\leqq}$, $\underset{R}{\sim}$ instead of $\underset{\Gamma_W}{\leqq}$, $\underset{\Gamma_W}{\sim}$; the corresponding equivalence classes are called the right cells of W.

In the case where W is the symmetric group s_n, we have

Theorem 1.4. *Let X be a left cell of $W = s_n$, let Γ be the W-graph associated to X and let ρ be the representation of $\mathcal{H}$ (over the quotient field of A) corresponding to Γ. Then ρ is irreducible and the isomorphism class of the W-graph Γ depends only on the isomorphism class of ρ and not on X.*

This gives, in particular, a distinguished basis (defined uniquely up to simultaneous homotety) for any complex irreducible representation of σ_n, with respect to which s_n acts through integral matrices.

Our investigation has started from trying to understand Springer's work connecting unipotent classes and representations of Weyl groups. This had led us to the following question on singularities of Schubert varieties. Let G be a semisimple group over an algebraically closed field, and let $\mathcal{B}$ be the variety of Borel subgroups of G. We fix $B_0 \in \mathcal{B}$, and for each w in the Weyl group W, let $\mathcal{B}_w$ be the set of all $B \in \mathcal{B}$ such that B_0 and B are in relative position w (a Bruhat cell of dimension $l(w)$.) Let $\overline{\mathcal{B}_w}$ be the closure of $\mathcal{B}_w$ (a Schubert variety). Let $T^*(\mathcal{B})$ be the cotangent bundle of $\mathcal{B}$ and let $\mathcal{N}_w \subset T^*(\mathcal{B})$ be the conormal bundle of $\mathcal{B}_w$. Its closure $\overline{\mathcal{N}_w}$ in $T^*(\mathcal{B})$ is an irreducible variety of dimension equal to $\dim(\mathcal{B})$. There is a natural projection $\pi_w: \overline{\mathcal{N}_w} \to \overline{\mathcal{B}_w}$. Now let $y \in W$ be such that $y < w$. Then $\overline{\mathcal{B}_y} \subset \overline{\mathcal{B}_w}$. The question is: for which pairs $y < w$ is it true that $\dim \pi_w^{-1}(\mathcal{B}_y) = \dim(\mathcal{B}) - 1$? It seems likely that when

$G=GL_n$, the condition is precisely that $y \prec w$ in the sense of Definition 1.2. (When $G \neq GL_n$, this is not, in general, true.)

Our polynomials $P_{y,w}$ appear to be very closely connected with the structure of singularities of Schubert varieties. More precisely, $P_{y,w}$ can be regarded as a measure for the failure of local Poincaré duality on the Schubert cell $\overline{\mathscr{B}_w}$ in a neighborhood of a point in $\mathscr{B}_y$. Some results in this direction are formulated in the Appendix.

Another starting point of our investigation was trying to understand the work of Jantzen [6] and Joseph [7, 8] relating primitive ideals in enveloping algebras with representations of Weyl groups.

Let $\underline{g}$ be a semisimple complex Lie algebra. We wish to state a conjecture relating our results with the theory of infinite dimensional representations of $\underline{g}$. We shall need some notations. Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$ and let $\underline{b}$ be a Borel subalgebra containing $\underline{h}$. Let $\rho\colon \underline{h} \to \mathbb{C}$ be the linear function on $\underline{h}$ which takes the value 1 on each simple coroot vector. Let W be the Weyl group of $\underline{g}$ with respect to $\underline{h}$ and let S be its set of simple reflections determined by $\underline{b}$. For each $w \in W$, let M_w be the Verma module with highest weight $-w(\rho)-\rho$ and let L_w be its unique irreducible quotient. We can now state

Conjecture 1.5

$$\operatorname{ch} L_w = \sum_{y \leqq w} \varepsilon_y \varepsilon_w P_{y,w}(1) \operatorname{ch} M_y \tag{1.5.a}$$

$$\operatorname{ch} M_w = \sum_{y \leqq w} P_{w_0 w, w_0 y}(1) \operatorname{ch} L_y \tag{1.5.b}$$

for all $w \in W$, where $P_{y,w}$ is the polynomial in q given by Theorem 1.1, *and $P_{y,w}(1)$ denotes its value for $q=1$.*

1.6. *Remarks.* a) The identities (1.5.a) and (1.5.b) are equivalent (see Theorem 3.1).
b) It is known and easy to prove that

$$\operatorname{ch} L_w = \sum_{y \leqq w} \sum_j (-1)^j \dim \operatorname{Ext}^j(M_y, L_w) \operatorname{ch} M_y$$

where Ext is taken in the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand. (See, for example, [4].) It is also known that $\operatorname{Ext}^j(M_y, L_w)=0$ if $j > l(w)-l(y)$. (Casselman and Schmid; see also Delorme [4].)

David Vogan has proved [14] that our conjecture 1.5 is equivalent to the formula

$$P_{y,w} = \sum_{i \geqq 0} q^i \dim \operatorname{Ext}^{l(w)-l(y)-2i}(M_y, L_w) \qquad (y \leqq w)$$

and it is also equivalent to the vanishing of $\operatorname{Ext}^j(M_y, L_w)$ for $j \not\equiv l(w)-l(y)$ (mod 2).

c) Conjecture 1.5, together with the results of Joseph [8] and Vogan [13] would imply that the ideal $\operatorname{Ann}(L_w)$ of the universal enveloping algebra of g, annihilating L_w, contains the ideal $\operatorname{Ann}(L_{w'})$ if and only if $w \underset{L}{\leqq} w'$.

1.7. In [6] a distinguished class $\mathcal{S}_W$ of irreducible representations of a Weyl group W was introduced. (Its definition, which will not be reproduced here, was suggested by the representation theory of finite Chevalley groups.) Let X be a left cell of W; it gives rise to a W-graph hence to a representation of $\mathcal{H}$. Specializing $q^{1/2}$ to 1, we get an integral representation of W. The corresponding representation over $\mathbb{Q}$ is not, in general, irreducible. However, it seems likely that it contains a unique irreducible component in the class $\mathcal{S}_W$. We expect that all representations in $\mathcal{S}_W$ are obtained in this way and that two left cells give rise to the same representation in $\mathcal{S}_W$ if and only if they are contained in the same 2-sided cell.

§ 2. The Proofs of Theorems 1.1 and 1.3

Let us define for each $x, y \in W$, an element $R_{x,y} \in A$ by the formula

$$T_{y^{-1}}^{-1} = \sum_x \overline{R_{x,y}}\, q_x^{-1}\, T_x. \tag{2.0.a}$$

The following formulae provide an inductive procedure for computing $R_{x,y}$:

$$R_{x,y} = \begin{cases} R_{sx,sy}, & \text{if } sx<x \text{ and } sy<y \\ R_{xs,ys}, & \text{if } xs<x \text{ and } ys<y \end{cases} \tag{2.0.b}$$

$$R_{x,y} = (q-1)\, R_{sx,y} + q R_{sx,sy}, \quad \text{if } sx>x \text{ and } sy<y. \tag{2.0.c}$$

It follows easily that $R_{x,y} \neq 0$ if and only if $x \leqq y$; when $x \leqq y$, $R_{x,y}$ is a polynomial in q of degree $l(y)-l(x)$. Here are some further properties of $R_{x,y}$.

Lemma 2.1

(i) $\overline{R_{x,y}} = \varepsilon_x \varepsilon_y q_x q_y^{-1} R_{x,y}$.

(ii) $\sum_{x \leqq t \leqq y} \varepsilon_t \varepsilon_x R_{x,t} R_{t,y} = \delta_{x,y}$, *for all* $x \leqq y$ *in* W.

(iii) $R_{x,y} = (q-1)^{l(y)-l(x)}$ *for all* $x \leqq y$ *such that* $l(x) \geqq l(y)-2$.

(iv) *If* W *is finite and* w_0 *is its longest element, we have* $R_{w_0 y, w_0 x} = R_{x,y}$ *for all* $x, y \in W$.

Proof. (i) follows easily from (2.0.b), (2.0.c). Applying the involution $h \to \bar{h}$ to (2.0.a), we get

$$T_y = \sum_x R_{x,y}\, q_x\, T_{x^{-1}}^{-1}$$

hence the matrices $(R_{x,y}\, q_x)$, $(\overline{R_{x,y}}\, q_x^{-1})$ are inverse to each other. By (i), the matrices $(R_{x,y}\, q_x)$, $(\varepsilon_x \varepsilon_y R_{x,y}\, q_y^{-1})$ are inverse to each other, hence (ii). The formula (iii) is obvious for $x=y$. Assume now that $x \leqq y$ and $l(x)=l(y)-1$. There is a reduced expression $y = s_1 \dots s_i \dots s_n$ such that $x = s_1 \dots \hat{s}_i \dots s_n$. Using (2.0.b), the computation of $R_{x,y}$ is reduced to the case where $1=i=n$, in which case (iii) is obvious. Assume now that $x \leqq y$ and $l(x)=l(y)-2$. There is a reduced expression $y = s_1 \dots s_i \dots s_j \dots s_n$ such that $x = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_n$. Using (2.0.b), the computation of $R_{x,y}$ is reduced to the case where $i=1, j=n$. Using then (2.0.c) with $s=s_i$, we see that

$$R_{x,y}=(q-1)\,R_{s_1\ldots\hat{s}_n,\,\hat{s}_1\ldots\hat{s}_n}+qR_{s_1\ldots\hat{s}_n,\,\hat{s}_1\ldots s_n}.$$

By the previous case, $R_{s_1\ldots\hat{s}_n,\,\hat{s}_1\ldots\hat{s}_n}=q-1$; moreover, $R_{s_1\ldots\hat{s}_n,\,\hat{s}_1\ldots s_n}=0$, since $\hat{s}_1\ldots s_n \nleqq s_1\ldots\hat{s}_n$. Thus, (iii) is proved. (iv) follows easily by induction from (2.0.b), (2.0.c).

2.2. *Proof of Theorem 1.1*

Uniqueness. The equality $\bar{C}_w=C_w$ can be written in the form

$$\sum_{x\leqq w}\varepsilon_x\varepsilon_w q_w^{1/2}q_x^{-1}\bar{P}_{x,w}T_x=\sum_{y\leqq w}\varepsilon_y\varepsilon_w q_w^{-1/2}q_y P_{y,w}\sum_{x\leqq y}\overline{R_{x,y}}\,q_x^{-1}T_x$$

or, equivalently, in the form

$$\varepsilon_x\varepsilon_w q_w^{1/2}q_x^{-1}\bar{P}_{x,w}=\sum_{x\leqq y\leqq w}\varepsilon_y\varepsilon_w q_w^{-1/2}q_y q_x^{-1}\bar{R}_{x,y}P_{y,w}\qquad(\text{for all } x\leqq w).\tag{2.2.a}$$

This is also equivalent to

$$\begin{aligned}&q_w^{1/2}q_x^{-1/2}\bar{P}_{x,w}-q_w^{-1/2}q_x^{1/2}P_{x,w}\\&\quad=\sum_{x<y\leqq w}\varepsilon_x\varepsilon_y q_w^{-1/2}q_y q_x^{-1/2}\bar{R}_{x,y}P_{y,w}\qquad(\text{for all } x<w).\end{aligned}\tag{2.2.b}$$

If the $P_{y,w}$ are known for all y, $x<y\leqq w$ (where $x<w$ is fixed), the Eq. (2.2.b) cannot have more than one solution $P_{x,w}$. Indeed, our assumptions on $P_{x,w}$ imply that $q_w^{-1/2}q_x^{1/2}P_{x,w}$ is a polynomial in $q^{-1/2}$ without constant term, while $q_w^{1/2}q_x^{-1/2}\bar{P}_{x,w}$ is a polynomial in $q^{1/2}$ without constant term. Thus, there cannot be cancellations between these two expressions.

Existence. Clearly, $C_e=T_e$. Assume now that $w\neq e$ and that the existence of C_w satisfying (1.1.a) and (1.1.b) has already been proved for elements w' of length $<l(w)$. We can write $w=sv$, where $s\in S$ and $l(w)=l(v)+1$. Thus C_v is already constructed; the Definition 1.2 can be applied to C_v, so that the relation $z\prec v$ and the corresponding integer $\mu(z,v)$ have a meaning. We now define

$$C_w=(q^{-1/2}T_s-q^{1/2})\,C_v-\sum_{\substack{z\prec v\\ sz<z}}\mu(z,v)\,C_z.$$

To check that C_w satisfies (1.1.a) it is enough to observe that

$$\overline{q^{-1/2}T_s-q^{1/2}}=q^{-1/2}T_s-q^{1/2}.$$

A straightforward computation shows that

$$C_w=\sum_{y\leqq w}\varepsilon_y\varepsilon_w q_w^{1/2}q_y^{-1}\bar{P}_{y,w}T_y$$

where

$$P_{y,w}=q^{1-c}P_{sy,v}+q^cP_{y,v}-\sum_{\substack{z\\ y\leqq z\prec v\\ sz<z}}\mu(z,v)\,q_z^{-1/2}q_v^{1/2}q^{1/2}P_{y,z}\qquad(y\leqq w)\tag{2.2.c}$$

and $c=1$ if $sy<y$, $c=0$ if $sy>y$. (We shall make the convention that $P_{x,v}=0$ when $x\nleq v$.)

(2.2.c) shows that $P_{y,w}$ is a polynomial in q of degree $\leqq\frac{1}{2}(l(v)-l(y))$ if $y<w$ and that $P_{w,w}=1$. Thus, C_w satisfies (1.1.b) and Theorem 1.1 is proved.

2.3. *Proof of Theorem* 1.3. In the process of proving Theorem 1.1, we have seen that

$$T_s C_v = q\,C_v + q^{1/2} C_{sv} + q^{1/2} \sum_{\substack{z\prec v\\ sz<z}} \mu(z,v)\,C_z, \qquad \text{if } s\in S \text{ and } sv>v. \tag{2.3.a}$$

A similar proof (interchanging left and right) shows that

$$C_v T_s = q\,C_v + q^{1/2} C_{vs} + q^{1/2} \sum_{\substack{z\prec v\\ zs<z}} \mu(z,v)\,C_z, \qquad \text{if } s\in S \text{ and } vs>v. \tag{2.3.b}$$

We now show that

$$T_s C_v = -C_v \qquad \text{if } s\in S \text{ and } sv<v. \tag{2.3.c}$$

We may assume that (2.3.c) is known for elements v' satisfying $sv'<v'$, $l(v')<l(v)$. Using (2.3.a) with v replaced by sv, we see that

$$\begin{aligned}
T_s C_v &= T_s\Big(q^{-1/2} T_s C_{sv} - q^{1/2} C_{sv} - \sum_{\substack{z\prec sv\\ sz<z}} \mu(z,sv)\,C_z\Big)\\
&= q^{-1/2}((q-1)\,T_s + q)\,C_{sv} - q^{1/2} T_s C_{sv} + \sum_{\substack{z\prec sv\\ sz<z}} \mu(z,sv)\,C_z\\
&= q^{1/2} C_{sv} - q^{-1/2} T_s C_{sv} + \sum_{\substack{z\prec sv\\ sz<z}} \mu(z,sv)\,C_z\\
&= -C_v
\end{aligned}$$

as required. An entirely similar proof shows hat

$$C_v T_s = -C_v \qquad \text{if } s\in S \text{ and } vs<v. \tag{2.3.d}$$

To complete the proof of Theorem 1.3 it is now enough to verify the following two statements.

Let $x,y\in W$, $s\in S$ be such that $x<y$, $sy<y$, $sx>x$.
Then $x\prec y$ if and only if $y=sx$.
Moreover, this implies that $\mu(x,y)=1$. (2.3.e)

Let $x,y\in W$, $s\in S$ be such that $x<y$, $ys<y$, $xs<x$.
Then $x\prec y$ if and only if $y=xs$.
Moreover, this implies that $\mu(x,y)=1$. (2.3.f)

Comparing the coefficients of T_{sx} in the two sides of (2.3.c) with $v=y$, we see that,

$$P_{x,y} = P_{sx,y}, \qquad \text{if } x<y,\ sy<y,\ sx>x. \tag{2.3.g}$$

If $sx\neq y$, it follows that $\deg P_{x,y}=\deg P_{sx,y}\leqq\frac{1}{2}(l(y)-l(x))<\frac{1}{2}(l(y)-l(x)-1)$ hence the relation $x\prec y$ is not satisfied. If $sx=y$, it follows that $P_{x,y}=P_{y,y}=1$, hence $x\prec y$ and

$\mu(x, y)=1$. This proves (2.3.e). The proof of (2.3.f) is entirely similar. This completes the proof of Theorem 1.3.

We now state a property of the preorders $\underset{L}{\leqq}$ and $\underset{R}{\leqq}$ on W.

Proposition 2.4. (i) *If* $x \underset{L}{\leqq} y$, *then* $\mathscr{R}(x) \supset \mathscr{R}(y)$. *Hence, if* $x \underset{L}{\sim} y$, *then* $\mathscr{R}(x)=\mathscr{R}(y)$.

(ii) *If* $x \underset{R}{\leqq} y$, *then* $\mathscr{L}(x) \supset \mathscr{L}(y)$. *Hence, if* $x \underset{R}{\sim} y$, *then* $\mathscr{L}(x)=\mathscr{L}(y)$.

Proof. It is easy to check that, given $s \in S$, we have

$$sy>y \Rightarrow \mathscr{R}(sy) \supset \mathscr{R}(y) \tag{2.4.a}$$

$$ys>y \Rightarrow \mathscr{L}(ys) \supset \mathscr{L}(y). \tag{2.4.b}$$

Assume now that $x \prec y$ and $\mathscr{L}(x) \not\subset \mathscr{L}(y)$. From (2.4.b) we see that $x^{-1} y \notin S$. Using (2.3.f), we see that $\mathscr{R}(x) \supset \mathscr{R}(y)$. This, together with (2.4.a), show that $x \underset{L}{\leqq} y \Rightarrow \mathscr{R}(x) \supset \mathscr{R}(y)$. The proof of (ii) is entirely similar.

2.5. For each $y \leqq w$ in W we define

$$N_{y,w}=q_y \sum_{y \leqq z \leqq w} R_{y,z}. \tag{2.5.a}$$

The following result is stated for future reference.

Lemma 2.6. (i) *For each* $x \leqq y$ *in* W, $P_{x,y}$ *is a polynomial in* q *with constant term* 1.

(ii) *Given* $y<w$ *in* W, *the following two conditions are equivalent:*

$$P_{y',w}=1, \quad \textit{for all } y \leqq y' \leqq w \tag{2.6.a}$$

and

$$N_{y',w}=q_w, \quad \textit{for all } y \leqq y' \leqq w. \tag{2.6.b}$$

(iii) *For each* $y<w$ *such that* $l(w)=l(y)+1$, *we have* $N_{y,w}=q_w$ *and* $P_{y,w}=1$. *In particular, we have* $y \prec w$ *and* $\mu(y,w)=1$.

(iv) *For each* $y<w$ *such that* $l(w)=l(y)+2$, *we have* $N_{y,w}=q_w$ *and* $P_{y,w}=1$.

(v) *For each* $w \in W$, *we have*

$$q_w^{-1} \sum_{y \leqq w} q_y P_{y,w} = \overline{\sum_{y \leqq w} q_y P_{y,w}}.$$

(vi) *If* W *is finite and* w_0 *is its longest element, then* $P_{y,w_0}=1$ *for all* $y \in W$.

Proof. (i) follows immediately from the inductive formula (2.2.c). To prove (ii), we may assume, by induction on $l(w)-l(y)$, that $P_{y',w}=1$ for all y' such that $y<y' \leqq w$. Then, the identity

$$P_{y,w}=\sum_{y \leqq y' \leqq w} \varepsilon_y \varepsilon_{y'} R_{y,y'} \overline{P_{y',w}}\, q_{y'}^{-1} q_w$$

(see 2.2.a) becomes

$$P_{y,w}=q_y^{-1}q_w\overline{P_{y,w}}+\sum_{y<y'\leqq w}\varepsilon_y\varepsilon_{y'}R_{y,y'}q_{y'}^{-1}q_w.$$

Using Lemma 2.1(i) this can be also written as

$$\begin{aligned}P_{y,w}&=q_y^{-1}q_w\overline{P_{y,w}}+\sum_{y<y'\leqq w}\overline{R_{y,y'}q_y^{-1}}q_w\\&=q_y^{-1}q_w\overline{P_{y,w}}-q_y^{-1}q_w+q_w\sum_{y\leqq y'\leqq w}\overline{R_{y,y'}q_y},\end{aligned}$$

hence

$$P_{y,w}-q_y^{-1}q_w\overline{P_{y,w}}=q_w\overline{N_{y,w}}-q_y^{-1}q_w. \tag{2.6.c}$$

If $P_{y,w}=1$, it follows that $N_{y,w}=q_w$. Conversely, if $N_{y,w}=q_w$, it follows that

$$q_y^{1/2}q_w^{-1/2}(P_{y,w}-1)=\overline{q_y^{1/2}q_w^{-1/2}(P_{y,w}-1)}.$$

But $q_y^{1/2}q_w^{-1/2}(P_{y,w}-1)$ is a polynomial in $q^{-1/2}$ without constant term; therefore it can be fixed by the involution $a\to\bar{a}$ only if it is zero. It follows that $P_{y,w}=1$ and (ii) is proved. Using Lemma 2.1(iii), we see that with the assumptions of (iii) we have $N_{y,w}=q_yR_{y,y}+q_yR_{y,w}=q_y+q_y(q-1)=q_w$. Using (ii), we deduce that $P_{y,w}=1$, hence (iii). Under the assumptions of (iv), it is known that there are exactly two elements z_1, z_2 such that $y<z_1<w$, $y<z_2<w$. Using Lemma 2.1(iii), we see that

$$N_{y,w}=q_y(R_{y,y}+R_{y,z_1}+R_{y,z_2}+R_{y,w})=q_y(1+(q-1)+(q-1)+(q-1)^2)=q_w.$$

Using (ii) and the fact that $N_{z_1,w}=N_{z_2,w}=q_w$ (given by (iii)) it follows that $P_{y,w}=1$. The identity (v) is just the identity $\mathscr{X}(C_w)=\mathscr{X}(\bar{C}_w)$, where $\mathscr{X}\colon\mathscr{H}\to A$ is the algebra homomorphism defined by $\mathscr{X}(T_y)=\varepsilon_y$ for all y. (vi) follows by applying repeatedly (2.3.g).

§3. An Inversion Formula

Our next result describes, in the case where W is finite, the inverse of the triangular matrix $(P_{x,y})$, where $P_{x,y}$ is defined to be zero if $x\not\leqq y$.

Theorem 3.1. *Assume that W is finite and let w_0 be its longest element. We have*

$$\sum_{x\leqq z\leqq y}\varepsilon_x\varepsilon_zP_{x,z}P_{w_0y,w_0z}=\delta_{x,y},\quad \textit{for all } x\leqq y \textit{ in } W. \tag{3.1.a}$$

Proof. Let $M_{x,y}$ be the left hand side of (3.1.a). We may assume that $x<y$ and that $M_{t,s}=0$ for all $t<s$ such that $l(s)-l(t)<l(y)-l(x)$. We start with the identity (2.2.a):

$$P_{x,z}=\sum_{x\leqq t\leqq z}\varepsilon_x\varepsilon_tR_{x,t}\overline{P_{t,z}}\,q_t^{-1}q_z\quad (x\leqq z \text{ in } W).$$

It follows that

$$M_{x,y}=\sum_{x\leqq z\leqq y}\varepsilon_x\varepsilon_z\sum_{\substack{x\leqq t\leqq z\\ z\leqq s\leqq y}}\varepsilon_x\varepsilon_t\varepsilon_y\varepsilon_s R_{x,t}\overline{P_{t,z}}R_{w_0y,\,w_0s}\overline{P_{w_0s,\,w_0z}}\,q_t^{-1}q_zq_z^{-1}q_s$$

$$=\sum_{\substack{t,s\\ x\leqq t\leqq s\leqq y}}\varepsilon_y\varepsilon_s q_t^{-1}q_s R_{x,t}R_{w_0y,\,w_0s}\overline{M_{t,s}}.$$

The only t,s which can contribute to this sum satisfy $t=s$ or $t=x$, $s=y$. Thus,

$$M_{x,y}=q_x^{-1}q_y\overline{M_{x,y}}+\sum_{x\leqq t\leqq y}\varepsilon_y\varepsilon_t R_{x,t}R_{w_0y,\,w_0t}.$$

Using Lemma 2.1(iv) and (ii), we see that the last sum (over t) equals

$$\sum_{x\leqq t\leqq y}\varepsilon_y\varepsilon_t R_{x,t}R_{t,y}=0.$$

Thus $M_{x,y}=q_x^{-1}q_y\overline{M_{x,y}}$ hence $q_x^{1/2}q_y^{-1/2}M_{x,y}=q_x^{-1/2}q_y^{1/2}\overline{M_{x,y}}$. The bounds on the degree of the polynomials $P_{y,w}$ described in Theorem 1.1 imply that $\overline{q_x^{-1/2}q_y^{1/2}M_{x,y}}$ is a polynomial in $q^{1/2}$ without constant term. Hence it cannot be fixed by the involution $a\to\bar{a}$, unless it is zero. Thus, $M_{x,y}=0$, as required.

Corollary 3.2. *Let $x<y$ be two elements of W (assumed to be finite). The following conditions are equivalent: $x\prec y$ and $w_0y\prec w_0x$. If these conditions are satisfied, we have $\mu(x,y)=\mu(w_0y,w_0x)$.*

Proof. We can assume that $\varepsilon_x=-\varepsilon_y$. The difference $P_{w_0y,\,w_0x}-P_{x,y}$ is equal to $\sum_{x<z<y}\varepsilon_x\varepsilon_z P_{x,z}P_{w_0y,w_0z}$ and one checks easily that the last expression is a polynomial in q of degree $<\frac{1}{2}(l(y)-l(x)-1)$. Therefore, the $\frac{1}{2}(l(y)-l(x)-1)$-th power of q appears in $P_{x,y}$ with the same coefficients as in P_{w_0y,w_0x}.

3.3. *Remarks.* a) The map $x\to w_0x$ reverses each of the preorders $\underset{L}{\leqq},\underset{R}{\leqq},\underset{LR}{\leqq}$ on W. Hence it induces an order reversing involution on the set of left cells of W, on the set of right cells of W and on the set of 2-sided cells of W.

b) Setting $q=0$ in the identity (3.1.a) and using Lemma 2.6(i) we get the following known identity [11]:

$$\sum_{x\leqq z\leqq y}\varepsilon_x\varepsilon_z=\delta_{x,y},\qquad\text{for all } x\leqq y \text{ in } W.$$

§ 4. Some Preliminaries to the Proof of Theorem 1.4

4.1. Let us fix two reflections s,t in S such that st has order 3. Let

$\mathscr{D}_L(s,t)=\{w\in W\mid \mathscr{L}(w)\cap\{s,t\}$ has exactly one element$\}$

$\mathscr{D}_R(s,t)=\{w\in W\mid \mathscr{R}(w)\cap\{s,t\}$ has exactly one element$\}$.

If $w\in\mathscr{D}_L(s,t)$, then exactly one of the elements sw,tw is in $\mathscr{D}_L(s,t)$; we denote it $*w$. The map $w\to *w$ is an involution of $\mathscr{D}_L(s,t)$. Similarly, we have an involution

$w \to w^*$ of $\mathscr{D}_R(s,t)$: w^* is the unique element of $\mathscr{D}_R(s,t) \cap \{ws, wt\}$. Let $\langle s,t \rangle$ be the group of order 6 generated by s,t. We shall prove

Theorem 4.2. *Let y, w be two elements in $\mathscr{D}_L(s,t)$.*

(i) *If $yw^{-1} \notin \langle s,t \rangle$, then we have $y \prec w$ if and only if $^*y \prec {}^*w$, and then $\mu(y,w) = \mu(^*y, {}^*w)$.*

(ii) *If $yw^{-1} \in \langle s,t \rangle$, then we have $y \prec w$ if and only if $^*w \prec {}^*y$, and then $\mu(y,w) = \mu(^*w, {}^*y) = 1$.*

Let y, w be two elements in $\mathscr{D}_R(s,t)$.

(iii) *If $y^{-1}w \notin \langle s,t \rangle$, then we have $y \prec w$ if and only if $y^* \prec w^*$, and then $\mu(y,w) = \mu(y^*, w^*)$.*

(iv) *If $y^{-1}w \in \langle s,t \rangle$, then we have $y \prec w$ if and only if $w^* \prec y^*$, and then $\mu(y,w) = \mu(w^*, y^*) = 1$.*

Proof. Throughout this proof, we shall use the following notations. For any $x < x'$ in W such that $\varepsilon_x = -\varepsilon_{x'}$, we set $d(x,x') = \frac{1}{2}(l(x') - l(x) - 1)$ and let $\mu(x,x')$ be the coefficient of $g^{d(x,x')}$ in $P_{x,x'}$. Thus $x \prec x'$ if and only if $\mu(x,x') \neq 0$. If P' is a polynomial in q, we say that $P_{x,x'} \sim P'$ if $P_{x,x'} - P'$ is of degree $< d(x,x')$. In particular, $P_{x,x'} \sim \mu(x,x')\, q^{d(x,x')}$.

It is enough to prove statements (i) and (ii). With the assumptions of (ii), we have $y \prec w$ if and only if $y < w$ and $l(w) = l(y) + 1$ and then $\mu(y,w) = 1$. (See Lemma 2.6(iii).) The conclusion of (ii) follows immediately. In the remainder of the proof we shall assume that $y, w \in \mathscr{D}_L(s,t)$ and $yw^{-1} \notin \langle s,t \rangle$. We may assume that $\varepsilon_y = -\varepsilon_w$. (This is equivalent to $\varepsilon_{^*y} = -\varepsilon_{^*w}$.) There are two cases to consider.

Case 1. $\quad {}^*y \cdot y^{-1} = {}^*w \cdot w^{-1}$.

In this case, we may assume without loss of generality that $tsy < sy < y < ty$ and $tsw < sw < w < tw$, so that $^*y = sy$, $^*w = sw$. It is clear that the conditions $y < w$ and $sy < sw$ are equivalent. Thus, we may assume that $y < w$. From (2.2.c), it follows that $P_{y,w} = P_{sy,sw}$ if $y \nleqq sw$ and

$$P_{y,w} \sim P_{sy,sw} + q P_{y,sw} - \sum_{\substack{y \prec z \prec sw \\ sz < z}} \mu(y,z)\, \mu(z,sw)\, q^{d(y,w)} \tag{4.2.a}$$

if $y \leqq sw$. Thus, we can assume that $y \leqq sw$. This implies that $ty \leqq sw$, since $t \in \mathscr{L}(sw)$. From (2.3.e), we see that for any z in the last sum, such that $z \neq ty$, $z \neq tws$, we have $t \in \mathscr{L}(sw) \Rightarrow t \in \mathscr{L}(z) \Rightarrow t \in \mathscr{L}(y)$, a contradiction. On the other hand, $z = ty$ satisfies $sz < z$, while $z = tws$ doesn't satisfy $sz < z$. Thus the sum over z has exactly one term: $z = ty$. We have $\mu(y, ty) = 1$, hence (4.2.a) becomes

$$P_{y,w} \sim P_{sy,sw} + q P_{y,sw} - \mu(ty, sw)\, q^{d(y,w)}.$$

By (2.3.g), we have $P_{y,sw} = P_{ty,sw}$ so that $q P_{y,sw} - \mu(ty,sw)\, q^{d(y,w)}$ is a polynomial in q of degree $< d(y,w)$. It follows that $P_{y,w} \sim P_{sy,sw}$ as required.

Case 2. $\quad {}^*y \cdot y^{-1} \neq {}^*w \cdot w^{-1}$.

In this case, we may assume without loss of generality that $tsy < sy < y < ty$, $sw < w < tw < stw$, so that $^*y = sy$, $^*w = tw$. We can clearly assume that $sy < tw$:

This implies that $tsy < w$ and $y < stw$. From (2.2.c), it follows that

$$P_{sy,tw} = P_{tsy,w} \quad \text{if } sy \nleqq w$$

and

$$P_{sy,tw} \sim P_{tsy,w} + q P_{sy,w} - \sum_{\substack{sy \prec z \prec w \\ tz < z}} \mu(sy,z)\,\mu(z,w)\, q^{d(sy,tw)}$$

if $sy \leqq w$. We have $s \in \mathscr{L}(w)$, $s \notin \mathscr{L}(tsy)$ and $w \neq stsy$, hence, by (2.3.e), the relation $tsy \prec w$ cannot hold. Thus, if $sy \nleqq w$, we have $P_{sy,tw} \sim 0$, hence $sy \prec tw$ fails to be true. On the other hand, if $sy \nleqq w$, we must have also $y \nleqq w$ (since $s \in \mathscr{L}(w)$), hence $y \prec w$ also fails to be true. Thus, we may assume that $sy \leqq w$, so that

$$P_{sy,tw} \sim q P_{sy,w} - \sum_{\substack{sy \prec z \prec w \\ tz < z}} \mu(sy,z)\,\mu(z,w)\, q^{d(sy,tw)}.$$

From (2.3.e) we see that for any z in the last sum, such that $z \neq y$, $z \neq sw$, we have $s \in \mathscr{L}(w) \Rightarrow s \in \mathscr{L}(z) \Rightarrow s \in \mathscr{L}(sy)$, a contradiction. On the other hand, neither $z = y$ nor $z = sw$ satisfy $tz < z$. It follows that $P_{sy,tw} \sim q P_{sy,w}$. By (2.3.g), we have $P_{sy,w} = P_{y,w}$ (we must have $y \leqq w$, since $sy \leqq w$ and $s \in \mathscr{L}(w)$). Thus, $P_{sy,tw} \sim q P_{y,w}$, hence $\mu(sy,tw) = \mu(y,w)$, as required.

Corollary 4.3. (i) *Let y, w be two elements in $\mathscr{D}_L(s,t)$. If $y \underset{R}{\sim} w$, then $^*y \underset{R}{\sim} w^*$.*

(ii) *Let y, w be two elements in $\mathscr{D}_R(s,t)$. If $y \underset{L}{\sim} w$, then $y^* \underset{L}{\sim} w^*$.*

Proof. We first note that, if $x \in \mathscr{D}_L(s,t)$, then $^*x \underset{L}{\sim} x$, hence, by Proposition 2.4(i), we have $\mathscr{R}(^*x) = \mathscr{R}(x)$. Now let y, w be two elements in $\mathscr{D}_L(s,t)$ such that $y \underset{R}{\sim} w$. Then there exists a sequence $y = y_1, y_2, \dots, y_n = w$ such that $\{y_i, y_{i+1}\}$ is an edge of Γ_W and $\mathscr{R}(y_i) \not\subset \mathscr{R}(y_{i+1})$ for $i = 1, \dots, n-1$, and there exists a sequence $w = w_1, w_2, \dots, w_m = y$ such that $\{w_j, w_{j+1}\}$ is an edge of Γ_W and $\mathscr{R}(w_j) \not\subset \mathscr{R}(w_{j+1})$ for $j = 1, \dots, m-1$. Clearly, all elements y_i, w_j are in the same right cell, hence, by Proposition 2.4(ii), we have $\mathscr{L}(y_i) = \mathscr{L}(y)$ for all i, $\mathscr{L}(w_j) = \mathscr{L}(y)$ for all j. Since $y \in \mathscr{D}_L(s,t)$, it follows that $y_i \in \mathscr{D}_L(s,t)$ for all i and $w_j \in \mathscr{D}_L(s,t)$ for all j. Hence $^*y_i$ and $^*w_j$ are well defined. Theorem 4.2 shows that $\{^*y_i, {}^*y_{i+1}\}$ is an edge of Γ_W for $i = 1, \dots, n-1$ and that $\{^*w_j, {}^*w_{j+1}\}$ is an edge of Γ_W for $j = 1, \dots, m-1$. By the remark at the beginning of the proof, we have $\mathscr{R}(y_i) = \mathscr{R}(^*y_i)$ for all i and $\mathscr{R}(w_j) = \mathscr{R}(^*w_j)$ for all j. It follows that $\mathscr{R}(^*y_i) \not\subset \mathscr{R}(^*y_{i+1})$ for $i = 1, \dots, n-1$ and $\mathscr{R}^*(w_j) \not\subset \mathscr{R}(^*w_{j+1})$ for $j = 1, \dots, m-1$. This shows that $^*y = {}^*y_1 \underset{R}{\leqq} {}^*y_2 \underset{R}{\leqq} \dots \underset{R}{\leqq} {}^*y_n = w = {}^*w_1 \underset{R}{\leqq} {}^*w_2 \underset{R}{\leqq} \dots \underset{R}{\leqq} {}^*w_m = {}^*y$ hence $^*y \underset{R}{\sim} {}^*w$ and (i) is proved. The proof of (ii) is entirely similar.

§ 5. Proof of Theorem 1.4

In [12], Vogan defines for any Weyl group W, an equivalence relation on W by means of a "generalized τ-invariant". In his language, Corollary 4.3 can be reformulated to say that two elements $y, w \in W$ such that $y \underset{L}{\sim} w$, must have the same generalized τ-invariant (provided that the Coxeter graph of W is simply laced). Moreover, in the case where W is the symmetric group s_n, Jantzen and Vogan have shown [loc. cit., Thm. 6.5] that if $y, w \in W$ have the same generalized τ-invariant, then $y \approx w$, where $\approx$ is the equivalence relation generated by the relations $x \approx sx$ where $s \in S$, $x < sx$, $\mathscr{L}(x) \not\subset \mathscr{L}(sx)$. On the other hand, it is clear that two elements equivalent under $\approx$ are equivalent under $\underset{L}{\sim}$. Thus, for $W = s_n$, the equivalence relations $\underset{L}{\sim}$ and $\approx$ coincide. The equivalence relation $\approx$ on s_n has been studied by combinatorists (see, for example [9, 5.1.4 and Ex. 5]). The following result is known: If X is an equivalence class for $\approx$ (i.e. a left cell) and if y, y' are distinct elements of X^{-1}, then the $\approx$ equivalence classes $X_y, X_{y'}$ containing y, y' respectively, are disjoint; moreover $X = X_y$ for some $y \in X^{-1}$. We now show that the W-graphs $\Gamma_y, \Gamma_{y'}$ associated to the left cells $X_y, X_{y'}$ $(y, y' \in X^{-1})$ are isomorphic. We have $y^{-1} \approx y'^{-1}$, hence, by the definition of $\approx$, we are reduced to the case where there exist $s, t \in S$ such that $(st)^3 = 1$, $y \in \mathscr{D}_R(s, t)$ and $y^* = y'$ (* defined with respect to s, t). It follows that all elements of X_y and of $X_{y'}$ are in $\mathscr{D}_R(s, t)$ (cf. Proposition 2.4(i)) and that $w \to w^*$ is a bijection of X_y onto $X_{y'}$ (cf. Corollary 4.3(ii)). It defines an isomorphism between the W-graphs $\Gamma_y, \Gamma_{y'}$ (cf. Theorem 4.2). In particular, for any $y, y' \in X^{-1}$, the representations $\rho_y, \rho_{y'}$ of $\mathscr{H}$ associated to $\Gamma_y, \Gamma_{y'}$ are isomorphic. The sum of the representations of $\mathscr{H}$ associated to the various left cells is equal to the regular representation (over some field containing A). If ρ is the representation corresponding to X, then $\sum_{y \in X^{-1}} \rho_y = (\dim \rho)\, \rho$ is a subrepresentation of the regular representation. It follows that ρ is irreducible, and that the left cells which give rise to a representation isomorphic to ρ are exactly the left cells X_y $(y \in X^{-1})$. This completes the proof of Theorem 1.4.

§ 6. Examples

6.1. Let W be a Weyl group of type A_3 with $S = \{s_1, s_2, s_3\}$ such that $s_1 s_3 = s_3 s_1$. There are exactly two pairs of elements $y < w$ in W such that $y \prec w$, $l(w) - l(y) > 1$. These are $s_2 < s_2 s_1 s_3 s_2$ and $s_1 s_3 < s_1 s_3 s_2 s_3 s_1$. For both pairs we have $P_{y,w} = 1 + q$.

6.2. Let (W, S) be a Coxeter group such that for any $s \neq t$ in S, the order $m_{s,t}$ of st is $2, 3, 4, 6$ or ∞. There is a standard graph Γ associated to (W, S): its set of vertices is S and $\{s, t\}$ is an edge precisely when $m_{s,t} \geqq 3$. We associate to each $s \in S$ the set $I_s = \{s\}$ and we consider a function μ on the set of ordered pairs, s, t which are joined in Γ such that $\mu(s,t)\, \mu(t,s) = 4 \cos^2 \pi / m_{s,t}$. This is a W-graph (see the work of Kilmoyer [3]).

We shall now give some examples of W-graphs associated to left cells in the Coxeter group (W, S). In all these examples, the function μ is identically 1, hence it will be omitted. The vertices will be represented by circles, inside which we describe the corresponding subset of S.

If W is of type A_2 with Coxeter graph $\overset{1}{\vdash}\!\!-\!\!\overset{2}{\dashv}$, the W-graphs arising from the left cells of W are:

$\ominus$, ①—②, (1, 2).

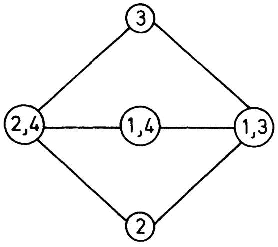
If W is of type B_2 with Coxeter graph $\overset{1}{\vdash}\!=\!\overset{2}{\dashv}$, the W-graphs arising from the left cells of W are:

$\ominus$, ①—②—①, ②—①—②, (1, 2).

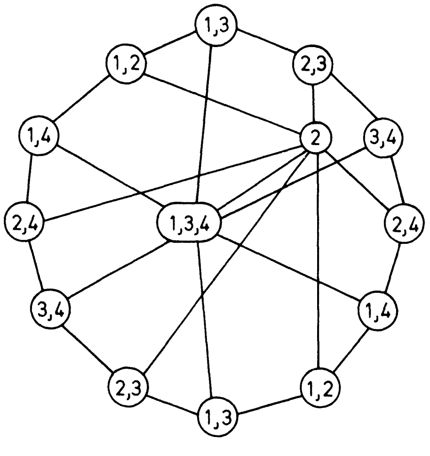
If W is of type A_3, with Coxeter graph $\overset{1}{\vdash}\!-\!\overset{2}{+}\!-\!\overset{3}{\dashv}$, the W-graphs arising from the left cells of W are:

$\ominus$, ①—②—③, (1, 2)—③, (2, 3)—(1, 3)—(1, 2), (1, 2, 3).

An example of W-graph associated to a left cell of W of type A_4 (with Coxeter graph $\overset{1}{\vdash}\!-\!\overset{2}{+}\!-\!\overset{3}{+}\!-\!\overset{4}{\dashv}$):

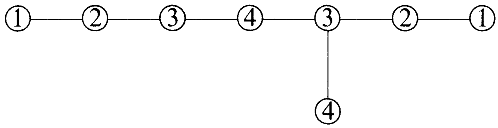


An example of W-graph associated to a left cell of W of type D_4 (with Coxeter graph 1—2<3, 4):

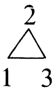


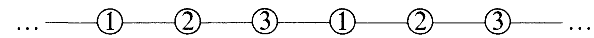
An example of W-graph associated to a left cell of the non-crystalographic finite Coxeter group W of type H_4 with simple reflections s_1, s_2, s_3, s_4 such that

$(s_1 s_2)^3=(s_2 s_3)^3=(s_3 s_4) \ =1$:



(6.2.c)

An example of W-graph associated to a left cell of the affine Weyl group W with Coxeter graph  :



(6.2.d)

6.3. Let u be a unipotent element in $GL_n(\mathbb{C})$ and let $\mathscr{B}_u$ be the variety of Borel subgroups containing u. Let X be the set of irreducible components of $\mathscr{B}_u$. We associate to u a graph Γ_u as follows: the set of vertices of Γ_u is X; two vertices are joined precisely when the corresponding components of $\mathscr{B}_u$ have an intersection of dimension equal to $\dim(\mathscr{B}_u)-1$. To each component C of $\mathscr{B}_u$, we associate a set I_C of simple reflections in the Weyl group W as follows. We identify the set of simple reflections in W with the set of conjugacy classes of rank 1 parabolic subgroups. Let $\mathscr{P}_s$ be the class of parabolic subgroups corresponding to s and let π_s: $\mathscr{B}\rightarrow\mathscr{P}_s$ be the natural projection. We say that $s\in I_C$, if C is a union of fibres of the map π_s. We have verified that, for $n\leqq 6$, Γ_u, together with the assignment $C\rightarrow I_C$ and with the function $\mu\equiv 1$ is a W-graph, and that the W-graphs obtained in this way from the various unipotent classes in $GL_n(\mathbb{C})$ are the same as the W-graphs associated to the left cells of W. We have also shown that the graph (6.2.b) has an analogous geometric interpretation in terms of a unipotent class in $SO_8(\mathbb{C})$.

Appendix

We will discuss here some algebraic geometry related to the polynomials $R_{y,w}$ and $P_{y,w}$.

The lemmas in this Appendix are not difficult to prove and their proofs will be generally omitted.

Let k be an algebraic closure of the prime field F_p with p elements. We will consider algebraic varieties over k. For any such variety we denote by $H^*(X)$ the étale cohomology of X with values in the constant sheaf $\mathbb{Q}_l$, where l is a fixed prime $\neq p$. If $x\in X$, we denote by $H^*_{\langle x\rangle}(X)$ the cohomology of X with support in x (see [5, Éxp. 13, p. 2]). There is an exact sequence

$$\ldots\rightarrow H^i(X)\rightarrow H^i(X-x)\rightarrow H^{i+1}_{\langle x\rangle}(X)\rightarrow H^{i+1}(X)\rightarrow\ldots$$

If X is non-singular, of dimension d at x, then $H^i_{\langle x\rangle}(X)=0$ if $i\neq 2d$ and $H^{2d}_{\langle x\rangle}(X)=\mathbb{Q}_l(-d)$.

Definition A1. *Let X be an irreducible variety of dimension d.*

(**a**) *X is rationally smooth if for all $x \in X$, we have $H^i_{\langle x\rangle}(X)=0$ if $i \neq 2d$ and $H^{2d}_{\langle x\rangle}(X)=\mathbb{Q}_l(-d)$.*

(**b**) *We say that $x \in X$ is a rationally smooth point of X if there exists an open, rationally smooth neighborhood of x in X.*

(**c**) *We denote by $\mathscr{S}(X) \subset X$ the set of all points of X which are not rationally smooth.*

$\mathscr{S}(X)$ is a closed subset of X, contained in the set $\operatorname{Sing}(X)$ of singular points of X.

Let G be a semisimple adjoint algebraic group over k. Let B_0 be a Borel subgroup, $T_0 \subset B_0$ a maximal torus and let W be the corresponding Weyl group.

The set $\mathscr{B}$ of Borel subgroups of G has a natural structure of projective G-variety: $(g,B) \to B^g$. The set $\mathscr{B}^{T_0} \subset \mathscr{B}$ of T_0-invariant points is in 1-1 correspondence with W: $w \leftrightarrow B_0^w$. Given two points B_1, B_2 in $\mathscr{B}$, we say that B_1, B_2 are in relative position w ($w \in W$) if, for some $g \in G$, we have $B_1^g = B_0$, $B_2^g = B_0^w$ (we then write $B_1 \xrightarrow{w} B_2$). For any $w \in W$, we denote by $\mathscr{B}_w$ the set of all $B \in \mathscr{B}$ such that $B_0 \xrightarrow{w} B$. Its closure $\overline{\mathscr{B}_w}$ is called a Schubert variety. It is known that, given two elements $y, w \in W$, we have $y \leqq w$ if and only if $\overline{\mathscr{B}_y} \subset \overline{\mathscr{B}_w}$

Theorem A2. *Given $y<w$ in W, the following conditions are equivalent:*

(a) $\mathscr{B}_y \cap \mathscr{S}(\overline{\mathscr{B}_w}) = \emptyset$

(b) $N_{y',w} = q_w$, *for all y', $y \leqq y' \leqq w$*

(c) $P_{y',w} = 1$, *for all y', $y \leqq y' \leqq w$.*

We have seen already (Lemma 2.6(ii)) that (b), (c) are equivalent. By induction on $l(w)-l(y)$, we can restrict ourselves to the case where, for all y', $y<y' \leqq w$, we have $\mathscr{B}_{y'} \cap \mathscr{S}(\overline{\mathscr{B}_w}) = \emptyset$ and $P_{y',w}=1$. In the rest of the proof, y and w are fixed.

We now fix an F_p-rational structure on G such that G is F_p-split and T_0, B_0 are defined over F_p. Then $\mathscr{B}, \mathscr{B}_W$ and, more generally, all algebraic varieties X we will deal with will be F_p-varieties. For such a variety, we denote by $|X|_r$ the number of F_{p^r}-rational points of X.

The Hecke algebra $\mathscr{H}$ will enter in the proof by means of the following Lemma. For any tripe $w_1, w_2, w_3 \in W$, let $\mathscr{N}(w_1, w_2, w_3) \subset \mathscr{B}$ be the set of all points $B \in \mathscr{B}$ such that $B_0 \xrightarrow{w_2} B \xrightarrow{w_3} B_0^{w_1}$.

Lemma A3. *There is a unique polynomial $c(W_1, w_2, w_3; q)$ such that $c(w_1, w_2, w_3; p^r) = |\mathscr{N}(w_1, w_2, w_3)|_r$ for all $r \geqq 1$. We have*

$$T_{w_2} T_{w_3} = \sum_{w_1 \in W} c(w_1, w_2, w_3; q)\, T_{w_1}.$$

For any $w', w'' \in W$, we denote by $\mathscr{B}_{w'}(w'')$ the set of points $B \in \mathscr{B}$ such that $B_0^{w''} \xrightarrow{w'} B$. We define

$$U = \overline{\mathscr{B}_w} \cap \mathscr{B}_{w_0}(y w_0), \qquad V = \overline{\mathscr{B}_w} \cap \mathscr{B}_{w_0 y}(w_0).$$

Then U is an open neighborhood of $\mathscr{B}_y$ in $\overline{\mathscr{B}_w}$. We have $U^{T_0} = V^{T_0} = \{b\}$, where $b = B_0^y$.

Lemma A4. (a) $|U|_r = N_{y,w}(p^r)$.

(b) *There is a canonical T_0-invariant isomorphism*

$$U \simeq \mathscr{B}_y \times V.$$

Part (a) follows from Lemma A3 and the following elementary statement: the polynomials $R_{w',w}$ (see (2.0.a)) satisfy the identity

$$T_w T_{w_0} = \sum_{w' \leqq w} R_{w',w} q_{w'} T_{w' w_0}.$$

The isomorphism in (b) is the restriction of an isomorphism

$$\mathscr{B}_{w_0}(y w_0) \cong \mathscr{B}_y \times \mathscr{B}_{w_0 y}(w_0).$$

Remark A5. Our assumptions imply that $V-0$ is rationally smooth.

Lemma A6. *There exists an F_p-isomorphism of algebraic varieties φ: $L \simeq k^n$ (where $n = l(w_0 y)$) such that*

(a) $\varphi(b) = 0$,

(b) *the induced action of T_0 on k^n is given by*

$$t: (e_1, \ldots, e_n) \to (\chi_1(t) e_1, \ldots, \chi_n(t) e_n)$$

where $\chi_1, \ldots, \chi_n$ are characters of T_0.

(c) *There exists an imbedding j: $\mathbb{G}_m \to T_0$ such that for all i, $1 \leqq i \leqq n$, the composition $\chi_i \cdot j$: $\mathbb{G}_m \to \mathbb{G}_m$ is given by $\lambda \to \lambda^{a_i}$, $a_i > 0$.*

We will identify L with k^n via φ and V with the corresponding subvariety of k^n. We will regard L and V as G_m-varieties.

Lemma A7. *Let Z be an algebraic variety with an action ψ: $\mathbb{G}_m \times Z \to Z$ of $\mathbb{G}_m$ and let $z_0 \in Z$ be a $\mathbb{G}_m$-invariant point. Suppose that $\mathbb{G}_m$ "contracts Z to z_0" i.e. ψ can be extended to a morphism*

$$\begin{array}{rcl} \tilde{\psi}\colon \mathbb{A}^1 \times Z & \longrightarrow & Z \\ \uparrow & & \| \\ \psi\colon \mathbb{G}_m \times Z & \longrightarrow & Z \end{array}$$

such that $\tilde{\psi}(0 \times Z) = z_0$. Then $H^i_{\langle z_0 \rangle}(Z) \cong H^{i-1}(Z - z_0)$ for $i \neq 1$ and $H^1_{\langle z_0 \rangle}(Z) = 0$.

Definition A8. (a) *Let Y be an affine algebraic variety with a $\mathbb{G}_m$-action. We say that this action is standard if there exists a finite group $\Gamma \subset \mathbb{G}_m$, a variety Y_0 and an action of Γ on Y_0 such that Y is isomorphic as a $\mathbb{G}_m$-variety to $\Gamma \backslash (\mathbb{G}_m \times Y_0)$ where Γ acts diagonally on $\mathbb{G}_m \times Y_0$.*

(b) *We say that an action of $\mathbb{G}_m$ on an algebraic variety X is locally standard if there is a covering $X = Y_1 \cup Y_2 \cup \ldots \cup Y_m$, where Y_i are open, $\mathbb{G}_m$-invariant affine subsets of X such that the action of $\mathbb{G}_m$ on each Y_i is standard.*

Lemma A9. *Let X be an algebraic variety with a locally standard action of $\mathbb{G}_m$. Then*

(a) *the geometric quotient $\pi: X \to \hat{X}$ (=set of $\mathbb{G}_m$-orbits on X) exists.*

(b) *$R^i\pi_*(\mathbb{Q}_l)$ is zero if $i \neq 0, 1$ and is isomorphic to $\mathbb{Q}_l$ (resp. $\mathbb{Q}_l(-1)$) for $i=0$ (resp. $i=1$).*

(c) *X is rationally smooth if and only if $\hat{X}$ is rationally smooth.*

(d) *$|X|_r = (p^r - 1)|\hat{X}|_r$, for all $r \geqq 1$.*

(e) *If X' is a $\mathbb{G}_m$-invariant closed subset of X, then the $\mathbb{G}_m$ action on X' is locally standard.*

Lemma A10. *The action of $\mathbb{G}_m$ on $L-0$ (and, hence on $V-0$) is locally standard.*

Consider the geometric quotients (for the $\mathbb{G}_m$-action) $\pi: L-0 \to \hat{L}$ and $\pi: V-0 \to \hat{V}$. It follows from Lemmas A6, A9, A10 and Remark A5 that $\bar{L}$ and $\bar{V} \subset \bar{L}$ are projective, rationally smooth varieties and that

$$|\hat{V}|_r = \frac{N_{y,w}(p^r)\cdot p^{-rl(y)} - 1}{p^r - 1},$$

for all $r \geqq 1$. It is easy to see that the function

$$\alpha(q) \overset{\text{def}}{=} \frac{N_{y,w}\cdot q_y^{-1} - 1}{q-1}$$

is a polynomial in q. By the Lefschetz fixed point formula, we have

$$\alpha(p^r) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(F^r, H^i(\hat{V}))$$

where $d = l(w) - l(y) - 1 = \dim(\hat{V})$ and F is the Frobenius map relative to the F_p-structure. $\hat{V}$ is rationally smooth, projective, hence it satisfies the Weil conjecture. (P. Deligne, La conjecture de Weil, II.) It follows that $H^{2i+1}(\hat{V}) = 0$ for all i and that all eigenvalues of F on $H^{2i}(\hat{V})$ are equal to p^i.

Using the Leray spectral sequence for $\pi: V-0 \to \hat{V}$, we get an exact sequence

$$\ldots \to H^{i+1}(V-0) \to H^i(\hat{V})(-1) \xrightarrow{\wedge\omega} H^{i+2}(\hat{V}) \xrightarrow{\pi^*} H^{i+2}(V-0) \to \ldots \tag{1}$$

where $\omega \in H^2(\hat{V})(1)$ is the restriction oof the corresponding class in $H^2(\hat{L})(1)$.

Let

$$\Pi_{2i} = \operatorname{coker}(H^{2i-2}(\hat{V})(-1) \xrightarrow{\wedge\omega} H^{2i}(\hat{V})),$$

$$\Pi'_{2i} = \ker(H^{2i}(\hat{V})(-1) \xrightarrow{\wedge\omega} H^{2i+2}(\hat{V})).$$

By Poincaré duality on $\hat{V}$, we have

$$\dim \Pi_{2i} = \dim \Pi'_{2d-2i}. \tag{2}$$

Since $H^{2i+1}(\hat{V}) = 0$, it follows from (1) and Lemma A7 that $H'_{2i} \cong H^{2i+1}(V-0) \cong H^{2i+2}_{\langle 0\rangle}(V)$ and that, for $i \neq 0$, $\Pi_{2i} \cong H^{2i}(V-0) \cong H^{2i+1}_{\langle 0\rangle}(V)$, for $i \neq 0$. Hence,

$$\begin{aligned}\dim \Pi'_{2i} &= \dim H^{2i+2}_{\langle 0\rangle}(V) = \dim H^{2l(y)+2i+2}_{\langle b\rangle}(U)\\ &= \dim H^{2l(y)+2i+2}_{\langle b\rangle}(\overline{\mathscr{B}_w}) \end{aligned}\tag{3}$$

$$\begin{aligned}\dim \Pi_{2i} &= \dim H^{2i+1}_{\langle 0\rangle}(V) = \dim H^{2l(y)+2i+1}_{\langle b\rangle}(U)\\ &= \dim H^{2l(y)+2i+1}_{\langle b\rangle}(\overline{\mathscr{B}_w}), \quad \text{for } i \neq 0. \end{aligned}\tag{4}$$

We have:

$$\begin{aligned} N_{y,w}(p^r)\cdot p^{-rl(y)}-1 &= |\hat{V}|_r(p^r-1)\\ &= \sum_i \operatorname{Tr}(F^r, H^{2i}(\hat{V}))(p^r-1)\\ &= \sum_i (\operatorname{Tr}(F^r, H^{2i}(\hat{V})(-1)) - \operatorname{Tr}(F^r, H^{2i+2}(\hat{V})))\\ &= \sum_i (\operatorname{Tr}(F^r, \Pi'_{2i}) - \operatorname{Tr}(F^r, \Pi_{2i}))\\ &= \sum_i p^{r(i+1)}\dim \Pi'_{2i} - \sum_i p^{ri}\dim \Pi_{2i}\\ &= \sum_i p^{r(d-i+1)}\dim \Pi_{2i} - \sum_i p^{ri}\dim \Pi_{2i}, \quad \text{(by (2)).} \end{aligned}$$

Since this is true for all r, we have an identity of polynomials in q:

$$N_{y,w}\cdot q_y^{-1} - 1 = \sum_i q_w q_y^{-1} q^{-i}\dim \Pi_{2i} - \sum_i q^i \dim \Pi_{2i}.$$

On the other hand, in the proof of Lemma 2.6(ii), we have seen that

$$N_{y,w}\cdot q_y^{-1} - 1 = \overline{P_{y,w}}\cdot q_w q_y^{-1} - P_{y,w}.$$

It follows that

$$q_w^{-1/2} q_y^{1/2}(P_{y,w} - \sum_i q^i\cdot \dim \Pi_{2i}) = \overline{q_w^{-1/2} q_y^{1/2}(P_{y,w} - \sum_i q^i\cdot \dim \Pi_{2i})}. \tag{5}$$

By the Lefschetz theorem [5, Éxp. 13] for V, we have $\dim \Pi_{2i}=0$ if $i>d/2 = 1/2(l(w)-l(y)-1)$. It follows that the left hand side of (5) is a polynomial in $q^{-1/2}$ without constant term, hence it cannot be fixed by the involution $a\to\bar{a}$ unless it is zero. Thus, we have

$$P_{y,w} = \sum_{i\geq 0} q^i \dim \Pi_{2i} = 1 + \sum_{i\geq 1} q^i \dim H^{2l(y)+2i+1}_{\langle b\rangle}(\overline{\mathscr{B}_w}) \tag{6}$$

and

$$P_{y,w} = \sum_{i\geq 0} q^i \dim \Pi'_{2d-2i} = \sum_{i\geq 0} q^i \dim H^{2l(w)-2i}_{\langle b\rangle}(\overline{\mathscr{B}_w}). \tag{7}$$

Moreover, it follows from Lemma A7 that $H^j_{\langle b\rangle}(\overline{\mathscr{B}_w})=0$ for $j\leqq 2l(y)+1$. Note also that $\dim H^i_{\langle x\rangle}(\overline{\mathscr{B}_w})$ is constant when x runs through $\mathscr{B}_y$. Using now (6) and (7) it follows directly that $P_{y,w}=1$ if and only if $\mathscr{B}_y \cap \mathscr{S}(\overline{\mathscr{B}_w})=0$, and Theorem A2 is proved.

Remark A 11. In the process of proving Theorem 2, we have also obtained the explicit formulae (6), (7) for $P_{y,w}$, valid for any $y<w$ in W such that $P_{y',w}=1$ for all $y<y'\leqq w$. From this, we see that for such pairs $y<w$, we have $y\prec w$ if and only if (with the notations of the previous proof) we have $d\equiv 0 \pmod 2$ and $\dim \Pi_d \neq 0$ (i.e. if $\hat{V}$ has non trivial "primitive cohomology" in the middle dimension).

Corollary A12. *For any $w\in W$, $\mathscr{S}(\overline{\mathscr{B}_w})$ has codimension $\geqq 3$ in $\overline{\mathscr{B}_w}$.*

(See Lemma 2.6(iii), (iv).)

This is in contrast with the behaviour of the singular set of $\overline{\mathscr{B}_w}$. For example, if $G=Sp_4$ and w, w' are the two elements of length 3 in W, then one of the Schubert cells $\mathscr{B}_w, \mathscr{B}_{w'}$ is non-singular, and the other one has a singular set of codimension 2.

References

1. Benson, C.T., Curtis, C.W.: On the degrees and rationality of certain characters of finite Chevalley groups, Trans. Amer. Math. Soc. **165**, 251–273 (1972) and **202**, 405–406 (1975)
2. Bourbaki, N.: Groupes et algèbres de Lie, Chapitres 4, 5, 6, Paris: Hermann 1968
3. Curtis, C.W., Iwahori, N., Kilmoyer, R.: Hecke algebras and characters of parabolic type of finite groups with (B, N) pair, I.H.E.S. Publ. Math. **40**, 81–116 (1972)
4. Delorme, P.: Extensions dans la categorie $\mathcal{O}$ de Bernstein-Gelfand-Gelfand. Applications, preprint, Palaiseau, 1978
5. Grothendieck, A., SGA2, Amsterdam: North Holland 1968
6. Jantzen, C.J.: Moduln mit einem höchsten Gewicht. Habilitationsschrift, Bonn, 1977
7. Joseph, A.: A characteristic variety for the primitive spectrum of a semisimple Lie algebra, preprint, Bonn, 1976
8. Joseph, A.: W-module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra, preprint, Tel-Aviv, 1978
9. Knuth, D.E.: The art of computer programming, Addison-Wesley, 1975
10. Lusztig, G.: A class of irreducible representations of a Weyl group, in press (1979)
11. Verma, D.N.: Möbius inversion for the Bruhat ordering on a Weyl group, Ann. Sci. E.N.S., 4e serie, t. **4**, 393–398 (1971)
12. Vogan, D.: A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra, preprint, Princeton, 1978
13. Vogan, D.: Ordering in the primitive spectrum of a semisimple Lie algebra, preprint, Princeton, 1978
14. Vogan, D.: Irreducible characters of semisimple Lie groups II. The Kazhdan-Lusztig conjectures, manuscript, 1979

Received March 11, 1979