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Hilbert's Theorem 90 for K_2, with Application to the Chow Groups of Rational Surfaces

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Merkur'ev and Suslin [16] have recently established some fundamental facts about the group K_2 of an arbitrary field. It is clear to anyone conversant with the work of Spencer Bloch [2-4] that these facts, when applied to the function fields of algebraic varieties, will greatly improve our knowledge of the Chow group $CH^2(X)$ of codimension 2 cycles modulo rational equivalence on a smooth variety X.

For X a projective surface over a field k, let $A_0(X) \subset CH^2(X)$ be the group of classes of degree zero 0-cycles. A smooth, projective, geometrically integral k-surface X is called *rational* if there exists an algebraic extension K/k such that the function field $K(X)$ of $X \times_k K$ is purely transcendental over K. In [4], Bloch applied the methods of algebraic K-theory to the study of $A_0(X)$ for X a rational k-surface; Sansuc and the author obtained some refinements in [9]. In both these papers, finiteness results for $A_0(X)$ when k is of an arithmetic nature were only obtained for special rational surfaces, namely conic bundles over the projective line, for which appeal could be made to the algebraic theory of quadratic forms (most notably to the Arason-Pfister theorem). In the present paper, I show that Merkur'ev and Suslin's results are enough to extend most of the results of [4] and [9] to *all* rational surfaces. For several reasons, I shall restrict myself to ground fields of characteristic zero.

Theorem A. *Let X be a rational k-surface, char. $k=0$.*

(i) *If k is finitely generated over* $\mathbf{Q}$, *and if there is a rational k-point on X* (*or a* 0-*cycle of degree* 1), *then $A_0(X)$ is finite;*

(ii) *If k is a local field, and if there is a rational k-point on X, then $A_0(X)$ is finite;*

(iii) *If k is a $\mathfrak{p}$-adic, field, and if X has good reduction, then $A_0(X)=0$;*

(iv) *If the cohomological dimension* $\operatorname{cd} k$ *of k is at most one, then $A_0(X)=0$.*

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Let k and X be as above, and let K/k be a Galois splitting field for X (cf. [4], Introduction). Let $G=\mathrm{Gal}(K/k)$, and let S be the k-torus dual to the G-module $\mathrm{Pic}\,X_K$ (here $X_K=X\times_k K$). Bloch has produced an exact sequence:

$$H^1(G, K_2K(X)/K_2K) \to A_0(X) \xrightarrow{\Phi} H^1(G, S(K))$$

and one knows ([4], [9]) that the image of Φ is finite if k is finitely generated over $\mathbf{Q}$, or (trivially) if k is a local field, and that it is zero in case (iii) of Theorem A and (trivially) in case (iv). In case (iii), X is known to have a k-point (the reduction of X is a rational surface over a finite field, which has a rational point according to a theorem of Weil). Granted these facts, the missing half of the proof of Theorem A is provided by the following Theorem B, which is the main result of the present paper, and which should also be of use in the study of rational equivalence on varieties other than rational surfaces.

Theorem B. *Let K/k be a Galois extension of fields, char. $k=0$, $G=\mathrm{Gal}(K/k)$. Let X be a geometrically integral k-variety. If there is a smooth rational k-point on X, or if* $\mathrm{cd}\,k\leqq 1$, *then*

$$H^1(G, K_2K(X)/K_2K)=0$$

and the natural map

$$j\colon K_2k(X)/K_2k \to (K_2K(X)/K_2K)^G$$

is an isomorphism.

The analogy with classical results on $K_1k(X)/K_1k=k(X)^*/k^*$ is striking. Theorem B also yields the computation of the (Zariski) $\mathcal{K}_2$-cohomology of rational surfaces with a k-point.

Theorem C. *Let X be a rational k-surface with a k-point, char. $k=0$. Then the natural map*

$$K_2k \to H^0(X, \mathcal{K}_2)$$

and the natural "reciprocity" map (cf. [4] (A.11))

$$H^1(X, \mathcal{K}_2) \to S(k)$$

are both isomorphisms.

Of course $H^2(X,\mathcal{K}_2)\cong CH^2(X)$ (Bloch, Quillen). That the reciprocity map should be an isomorphism was conjectured by Bloch, who proved it for conic bundles over the projective line ([5]).

Theorem B is proved in § 1. The applications to rational surfaces (Theorem A and C) are dealt with in § 2.

When k is a *local* or a *global* field (char. $k=0$) and no (smooth) k-point is at hand, the paper I originally submitted had a rather involved proof of the following facts:

- *For K/k finite, Theorem* B *holds up to finite groups*
- *For X a rational k-surface, $A_0(X)$ is always finite*
- *Theorem* C *holds up to finite groups.*

However, partially building upon an earlier version of this paper, Suslin [18] has managed to show that the map j in Theorem B and the map $K_2 k \to H^0(X, \mathscr{K}_2)$ in Theorem C are always isomorphisms (without any condition on k or on $X(k)$). This leads to a simplified proof of an improved version of the three facts above. At the referee's request, it is this better way of handling the case k local or global, $X(k)=\emptyset$, which is presented - apart - in § 3.

The papers [9] and [10] together with the present paper yield complete answers to the questions raised by Bloch in the introduction of [4], when char. $k=0$. The main problem now is to "determine" (the order of) the finite group $A_0(X)$, for X a rational surface over a number field. I refer the reader to [9], § 4, for the precise conjectures which Sansuc and I have made.

Here is a list of the results of [16] which will be used in the proof of Theorem B. Let k be a field, p a prime with $p \neq$ char. k, and assume k contains a primitive p-th root of unity ζ.

MS a *The Galois symbol $K_2 k/pK_2 k \to \mathrm{Br}\, k$, which to the symbol $\{a,b\}$ associates the class of the cyclic algebra $(a,b)_\zeta$, is an isomorphism onto the p-torsion subgroup $\mathrm{Br}_p k$ of the Brauer group $\mathrm{Br}\, k$* ([16], 9.4, 11.4, 11.5).

MS b (*Hilbert's Theorem* 90 *for K_2*) *Let K/k be a cyclic extension of degree n prime to char. k and let σ be a generator of* $\mathrm{Gal}(K/k)$. *Then* $\mathrm{Ker}(N_{K/k}: K_2 K \to K_2 k) = (1-\sigma) K_2 K$. ([16] 14.1)

MS c *Any p-torsion element of $K_2 k$ may be written as a symbol $\{a, \zeta\}$ for some $a \in k^*$* ([16] 10.4).

Another key ingredient in the proof of Theorem B for arbitrary varieties will be the following theorem of Bloch, proved by transcendental methods:

B6 *Let $k(X)$ be the function field of a curve X over an algebraically closed field k of characteristic zero, and let ζ be a primitive n-*th *root of unity in k. For $f \in k(X)^*$, the symbol $\{f, \zeta\}$ is zero in $K_2 k(X)$ (if and) only if $f \in k(X)^{*n}$* ([6], 1.24).

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§ 1. On the theme of Hilbert's Theorem 90 for K_2

Proposition 1. *Let k be a field, char. $k=0$, $\bar{k}$ an algebraic closure of k, X a geometrically integral k-variety. Let ζ be a primitive n-th root of unity in $\bar{k}$ (n integer), and let $f \in k(X)^*$ be such that $\{f, \zeta) = 0 \in K_2 \bar{k}(X)$. There then exists $\alpha \in k^*$ and $g \in k(X)^*$ such that $f = \alpha \cdot g^n$.*

Proof. For k algebraically closed and $\dim X = 1$, this is Bloch's result **B6.** Let k be arbitrary but $\dim X = 1$. By **B6**, we may write $f = g^n$, with $g \in \bar{k}(X)^*$. For σ

varying in $\mathfrak{g}=\mathrm{Gal}(\bar{k}/k)$, the elements ${}^{\sigma}g/g$ define a (continuous) 1-cocycle of $\mathfrak{g}$ with values in the group μ_n of n-th roots of unity in $\bar{k}$. By the Kummer sequence

$$1 \to \mu_n \to \bar{k}^* \xrightarrow{x \mapsto x^n} \bar{k}^* \to 1$$

and the usual Theorem 90, any such cocycle is of the form ${}^{\sigma}\beta/\beta$, for a fixed $\beta \in \bar{k}^*$ with $\beta^n = \alpha \in k^*$. Hence (g/β) is fixed under $\mathfrak{g}$, i.e. $(g/\beta) \in k(X)^*$, and $f = \alpha \cdot (g/\beta)^n$.

Lemma 1. *Let k be an algebraically closed field of characteristic zero. Any function field $k(X)$, for X an integral k-variety of dimension d, may be written as $L(C)$, where $L=k(T_1, \ldots, T_{d-1})$ is purely transcendental, and C is a geometrically integral L-curve with a smooth rational L-point.*

Proof of Lemma. We may assume $d \geqq 2$, and realize X as a hypersurface in $\mathbf{P}_k^{d+1}$. In view of the Bertini theorems ([13] 6.10, 6.11) it is possible to find a line $D \subset \mathbf{P}_k^{d+1}$ which cuts X transversally (hence avoids the singular locus) and which is such that the general $\mathbf{P}_k^2$ through this line cuts out on X an integral curve with no singularity at any of the finitely many points of $D \cap X$. The family of $\mathbf{P}_k^2$ going through D defines a dominating rational map $X \overset{\pi}{\dashrightarrow} \mathbf{P}_k^{d-1}$, the generic fibre of which is a geometrically integral L-curve – here $L = k(\mathbf{P}_k^{d-1})$. Any point $P \in D \cap X$ defines a smooth rational L-point of this curve (the rational section of π being given by the projective space on the tangent space to X at P; to make all this more "morphic", one may blow up $D \cap X$ by looking at the incidence correspondance in $X \times \mathbf{P}_k^{d-1}$). ∎

End of Proof of Proposition 1. The proof will be by induction on $\dim X$. Assume it has been proved for $\dim X < d$. Assume first that k is algebraically closed, and let $\dim X = d$, and $f \in k(X)^*$ be such that $\{f, \zeta\} = 0$ in $K_2 k(X)$. Write $k(X) = L(C)$ as in the Lemma. By the one-dimensional case, there exist $g \in L^*$ and $h \in L(C)^*$ such that $f = g \cdot h^n$. Hence

$$\{g, \zeta\} = \{g \cdot h^n, \zeta\} = \{f, \zeta\} = 0 \in K_2 L(C).$$

But $\{g, \zeta\}$ belongs to $K_2 L$, and the natural map $K_2 L \to K_2 L(C)$ is injective, in view of the existence of specialization maps corresponding to some smooth L-rational point (cf. e.g. [16] §2). Therefore $\{g, \zeta\} = 0 \in K_2 L$. By induction (or, since $L = k(\mathbf{P}_k^{d-1})$, by the following Remark 1), $g \in L^{*n}$ (k is algebraically closed). Hence $f \in k(X)^{*n}$. The proposition for $\dim X = d$ and k arbitrary follows from the special case k algebraically closed exactly as it did for $\dim X = 1$. ∎

Remark 1. Let k and X be as in the proposition. Assume moreover that $\bar{X} = X \times_k \bar{k}$ is locally factorial (e.g. X/k is smooth), that X/k is proper and that $\mathrm{Pic}\,\bar{X}$ has no torsion (If X/k is smooth and proper, this amounts to: $H^1(X, O_X) = 0$ and the Néron-Severi group of $\bar{X}$ has no torsion). The proof of the proposition now becomes very easy, and does not use **B6**. By the Galois cohomology argument in the above proof, it is enough to assume $k = \bar{k}$. If $\{f, \zeta\} = 0 \in K_2 k(X)$, then for any prime divisor $\Delta \subset X$, with associated valuation v, the tame symbol $T_v(\{f, \zeta\}) = \zeta^{v(f)}$ vanishes, hence n divides $v(f)$. Since the divisor of f is divisible by n, the assumptions imply that f is an n-th power in $k(X)^*$. ∎

Proposition 2. *Let X be a geometrically integral k-variety (char. $k=0$), and assume that either X has a smooth k-point or* $\operatorname{cd} k \leqq 1$. *For any algebraic extension K/k the natural map*

$$K_2 k(X)/K_2 k \to K_2 K(X)/K_2 K$$

is injective.

Proof. Since any element of the group K_2 of a field is the sum of finitely many symbols, we may restrict ourselves to the case K/k finite. Now, we may assume K/k is finite and Galois. The existence of a transfer map with the usual properties ([1] I§5, [17] §14, [14] §1.7, §2.4 Cor. 2), Bézout's lemma and the existence of Sylow subgroups of a finite group enable us to consider just the case where $\mathrm{Gal}(K/k)$ is a p-group. The structure of p-groups reduces us to the case where K/k is cyclic of prime degree p. Finally, since for such K/k, the extensions K/k and $k(\sqrt[p]{1})$ are linearly independent, a last transfer argument and Bézout's lemma allow us to assume that k contains a primitive p-th root of unity ζ.

Let now $z \in K_2 k(X)$ be such that $z_{|K(X)} = \beta_{|K(X)}$, with $\beta \in K_2 K$. Taking transfers, we get:

$$p z = \gamma \in K_2 k(X)$$

with $\gamma \in K_2 k$. If X has a smooth k-point, a specialization argument (cf. e.g. [16] §2) then shows the above equality to imply $\gamma = p\,\delta \in K_2 k$, for some $\delta \in K_2 k$. Such an equality always holds when $\operatorname{cd} k \leqq 1$, since in that case $K_2 k/pK_2 k = 0$, ([1] I.5.12; also, **MS a**). Combining both equalities, we see that $(z-\delta)$ belongs to the p-torsion of $K_2 k(X)$. By **MS c**, there exists $f \in k(X)^*$ such that

$$z - \delta = \{f, \zeta\} \in K_2 k(X) \tag{1}$$

Since $z_{|K(X)}$ comes from $K_2 K$, so does $\{f, \zeta\}$. Going over to $\bar{k}(X)$, we see that $\{f, \zeta\} \in K_2 \bar{k}(X)$ comes from $K_2 \bar{k}$, hence is zero (by a specialization argument, $\{f, \zeta\} = \{f(M), \zeta\}$ for some smooth $\bar{k}$-point M, and $\{f(M), \zeta\} = \{\sqrt[p]{f(M)}, \zeta^p\} = 0$). By Proposition 1, there exist $\alpha \in k^*$ and $g \in k(X)^*$ such that $f = \alpha \cdot g^p$. Hence $\{f, \zeta\} = \{\alpha, \zeta\}$ belongs to the image of $K_2 k$ in $K_2 k(X)$, and so does z by (1). ■

Theorem 1. *Let k be a field, char. $k=0$, and let X be a geometrically integral k-variety. Assume that either X has a smooth k-point or* $\operatorname{cd} k \leqq 1$. *For any Galois extension K/k with group G:*

(i) *The natural map*

$$j\colon K_2 k(X)/K_2 k \to (K_2 K(X)/K_2 K)^G$$

is an isomorphism;

(ii) $$H^1(G, K_2 K(X)/K_2 K) = 0.$$

The proof will be in several instalments. A crucial technical point is that it seems necessary to prove (i) to be able to embark on the proof of (ii) (see Steps I and IV) – although (ii) was our initial point of interest (see the introduction).

Step I. *Assume G is finite cyclic. Then* (ii) *holds.*

We have an obvious commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_2K \arrow[r] \arrow[d, "N_1"] & K_2K(X) \arrow[r] \arrow[d, "N_2"] & K_2K(X)/K_2K \arrow[r] \arrow[d, "N_3"] & 0 \\
0 \arrow[r] & K_2k \arrow[r] \arrow[d] & K_2k(X) \arrow[r] \arrow[d] & K_2k(X)/K_2k \arrow[r] & 0 \\
& K_2k/NK_2K \arrow[r, "\theta"] & K_2k(X)/NK_2K(X) & &
\end{tikzcd} \quad (2)$$

If X has a smooth k-point, a specialization argument shows θ to be injective, and $K_2k/NK_2K=0$ if $\operatorname{cd} k \leqq 1$ (as follows immediately from the projection formula). The snake lemma therefore gives a surjection

$$\operatorname{Ker} N_2 \twoheadrightarrow \operatorname{Ker} N_3.$$

The commutative diagram ([7], proof of lemma 3.5.3)

$$\begin{tikzcd}
K_2K(X) \arrow[rr, "N_4"] \arrow[dr, "N_2"'] & & K_2K(X) \\
& K_2k(X) \arrow[ur, "i"'] &
\end{tikzcd} \quad (3)$$

where $N_4 = \sum_{g\in G} g$, and i is induced by $k(X) \subset K(X)$, induces the following diagram (since we may write the same diagram as above for K/k instead of $K(X)/k(X)$)

$$\begin{tikzcd}
K_2K(X)/K_2K \arrow[rr, "N_4"] \arrow[dr, "N_3"'] & & K_2K(X)/K_2K \\
& K_2k(X)/K_2k \arrow[ur, "j"'] &
\end{tikzcd} \quad (4)$$

and under the assumptions of Theorem 1, Proposition 1 says that j is injective. Hence $\operatorname{Ker} N_4 = \operatorname{Ker} N_3$ in the above triangle. By **MS b**, $\operatorname{Ker} N_2$ consists of elements of the shape $({}^{\sigma}y - y)$ for σ a generator of G and $y \in K_2K(X)$. Since $\operatorname{Ker} N_2$ maps onto $\operatorname{Ker} N_3$ which coincides with $\operatorname{Ker} N_4$, we conclude $\hat{H}^{-1}(G, K_2K(X)/K_2K)=0$, hence $H^1(G, K_2K(X)/K_2K)=0$ since G is cyclic.

Step II. *Assume that k contains a primitive p-th root of unity ζ and that K/k is cyclic of prime degree p. Then* (i) *holds.*

By Proposition 2, the map is injective. Let $\alpha \in K_2K(X)$ be such that its class in $K_2K(X)/K_2K$ is G-invariant. To prove that the class of α comes from $K_2k(X)$, we shall be free to modify α by (the image of) an element of $K_2k(X)$, or of K_2K. Using triangle (4), we see:

$$p\alpha = \beta + \delta \in K_2K(X) \quad (5)$$

with $\beta \in K_2 k(X)$ and $\delta \in K_2 K$. By specialization to a smooth rational k-point M, and with an obvious (great) abuse of notation, we get $p\,\alpha(M) = \beta(M) + \delta \in K_2 K$, so that by changing α to $\alpha - \alpha(M)$ we may assume

$$p\,\alpha = \beta \in K_2 K(X) \tag{6}$$

with $\beta \in K_2 k(X)$. We may also assume (6) in the $\operatorname{cd} k \leq 1$ case, since $K_2 K/pK_2 K = 0$ in that case, hence δ in (5) is a p-th power.

Let us write $K = k(b)$, with $b^p = a \in k^*$. It is a direct consequence of [16] that any element in the kernel of the obvious map

$$K_2 k(X)/pK_2 k(X) \to K_2 K(X)/pK_2 K(X)$$

may be represented by a symbol $\{a, f\}$ for some $f \in k(X)^*$ (One only needs to use the classical exact sequence

$$0 \to k(X)^*/N_{K/k} K(X)^* \xrightarrow{\varphi} \operatorname{Br}_p k(X) \to \operatorname{Br}_p K(X),$$

where φ associates to $f \in k(X)^*$ the class of the cyclic algebra $(a, f)_\zeta$, and to replace Br_p by K_2/p, thanks to **MS a**). From (6) we therefore deduce:

$$\beta = p\,\gamma + \{a, f\} \in K_2 k(X) \tag{7}$$

with $f \in k(X)^*$ and $\gamma \in K_2 k(X)$.

Putting (6) and (7) together, and changing α to $(\alpha - \gamma)$, we may assume:

$$p\,\alpha = \{a, f\} \in K_2 K(X) \tag{8}$$

that is

$$p(\alpha - \{b, f\}) = 0 \in K_2 K(X)$$

(since $b^p = a$). The description of torsion in K_2 given by **MS c** then provides us with some $g \in K(X)^*$ such that

$$\alpha = \{b, f\} + \{\zeta, g\} \in K_2 K(X) \tag{9}$$

(note that $\{g^{-1}, \zeta\} = \{\zeta, g\}$).

Fix a generator σ of G by the condition

$${}^\sigma b/b = \zeta.$$

Since the class of α in $K_2 K(X)/K_2 K$ is G-invariant, applying σ to (9) and taking differences yields

$$\{\zeta, f \cdot {}^\sigma g/g\} = 0 \in K_2 K(X)/K_2 K.$$

Going over to $\bar{K} = \bar{k}$, we conclude by a specialization argument at some smooth $\bar{k}$-point that $\{\zeta, f \cdot {}^\sigma g/g\} = 0$ in $K_2 \bar{k}(X)$ (cf. end of proof of Proposition 2). Hence, by Proposition 1, there exist $\lambda \in K^*$ and $h \in K(X)^*$ with:

$$f \cdot {}^\sigma g/g = \lambda \cdot h^p \in K(X)^*. \tag{10}$$

Taking norms for K/k, we get $(f/N h)^p \in k^*$, hence there exists $\mu \in k^*$ with

$$f = \mu \cdot N(h). \tag{11}$$

Completion of the proof – first method (suggested by Bloch). First fix a smooth k-point M, and a regular systems of parameters at M, defining (compatible) specialization maps $k(X)^* \to k^*$, $K_2 k(X) \to K_2 k$, $K(X)^* \to K^*$, $K_2 K(X) \to K_2 K$ (the last two being G-linear). Plugging (11) in (8), we get (projection formula):

$$\begin{aligned} p\alpha = \{a,\mu\} + \{a, Nh\} &= \{a,\mu\} + N_{K/k}\{a,h\} \\ &= \{a,\mu\} + N_{K/k}\{b^p, h\} \\ &= \{a,\mu\} + p\,N_{K/k}\{b,h\} \in K_2 K(X). \end{aligned}$$

We are free to modify α by $N_{K/k}\{b,h\} \in K_2 k(X)$. Now:

$$p\alpha = \varepsilon \in K_2 K(X)$$

with $\varepsilon \in K_2 K$. A specialization argument shows $\varepsilon \in p K_2 K$, so we may modify α by an element of $K_2 K$ and assume

$$p\alpha = 0 \in K_2 K(X).$$

By **MS c**, there exists $g \in K(X)^*$ such that

$$\alpha = \{\zeta, g\} \in K_2 K(X).$$

Proceeding as above with (9), we here get

$${}^{\sigma}g/g = \lambda \cdot h^p$$

with $\lambda \in K^*$ and $h \in K(X)^*$. Changing g to $g/g(M)$, h to $h/h(M)$, and α to $\alpha - \{\zeta, g(M)\}$, we now have

$$\alpha = \{\zeta, g\} \in K_2 K(X)$$

with

$${}^{\sigma}g/g = h^p;$$

hence $N(h) \in k^*$ and in fact $N(h) = 1$ since $h(M) = 1$ (recall M is k-rational). By the usual Theorem 90 for $K(X)/k(X)$, we find $u \in K(X)^*$ such that $h = {}^{\sigma}u/u$; hence by the above equality, ${}^{\sigma}(g u^{-p})/(g u^{-p}) = 1$, that is $g u^{-p} \in k(X)^*$. Now

$$\alpha = \{\zeta, g\} = \{\zeta, g u^{-p}\} \in K_2 K(X)$$

comes from $K_2 k(X)$.

This method is easily adapted in the case where $\operatorname{cd} k \leqq 1$, but no smooth k-point is at hand. However, because of its use when extending part of Theorem 1 to the case when k is arbitrary and no smooth k-point is available, I shall describe an alternative approach. This approach, which was actually the initial one, and which evolved in conversations with Sansuc, has since then been used by Suslin ([18]) in his proof that Theorem 1 (i) holds without any restriction.

Completion of the proof – second method (suggested by Sansuc). We start again at (11). From $N(\lambda) = \mu^p$, that is $N(\lambda/\mu) = 1 \in k^*$, and from the usual Theorem 90 for K/k, we find $\rho\in K^*$ with $\lambda/\mu=\rho/^\sigma\rho$. Summarizing, we get $\mu\in k^*$, $\rho\in K^*$ and $h\in K(X)^*$ such that:

$$\mu^{-1}f=N(h)$$
$$\mu^{-1}f\frac{^\sigma(\rho\, g)}{\rho\, g}=h^p.$$

We are free to change f to $\mu^{-1}f$ (this modifies α by $\{b,\mu^{-1}\}\in K_2K$) and g to $\rho\, g$ (analogous reason). We now have:

$$^\sigma b/b=\zeta$$
$$f=N(h) \tag{12}$$
$$^\sigma g/g=h^p/N(h) \tag{13}$$

with $h\in K(X)^*$.

As a direct computation shows, (13) is equivalent to saying that the function

$$t=g/h^{\sigma+2\sigma^2+\ldots+(p-1)\sigma^{p-1}}$$

is invariant under σ, that is $t\in k(X)^*$. We have

$$\{\zeta,g\}=\{\zeta,t\}+\{\zeta,h^\sigma\}+\{\zeta^2,h^{\sigma^2}\}+\ldots+\{\zeta^{p-1},h^{\sigma^{p-1}}\}\in K_2K(X)$$

and since $\{\zeta,t\}$ comes from $K_2k(X)$, we may change α to $\alpha-\{\zeta,t\}$, and (9) and (12) now give

$$\begin{aligned}\alpha&=\{b,h\}+\{b,h^\sigma\}+\ldots+\{b,h^{\sigma^{p-1}}\}\\&\quad+\{\zeta,h^\sigma\}+\ldots+\{\zeta^{p-1},h^{\sigma^{p-1}}\}\\&=\{b,h\}+\{b\,\zeta,h^\sigma\}+\{b\,\zeta^2,h^{\sigma^2}\}+\ldots+\{b\,\zeta^{p-1},h^{\sigma^{p-1}}\}\\&=(\sum_{g\in G}g)\{b,h\}\end{aligned}$$

(since $\sigma^i b=\zeta^i b$). In view of triangle (3), this shows that α comes from $K_2k(X)$.

This step was really the core of the proof (once the results of [16] are known). The remainder is a sequence of easy reductions.

Step III. (i) *holds for any cyclic extension of prime degree.*

Let $L=k(\sqrt[p]{1})$ and let M be the composite field of K and L, which is Galois over L, with group $G=\mathrm{Gal}(K/k)$. Note that $[K:k]$ and $[L:k]$ are coprime. Consider the commutative diagram:

$$\begin{tikzcd}(K_2K(X)/K_2K)^G \arrow[r] & (K_2M(X)/K_2M)^G \arrow[r, "N_{M/K}"] & (K_2K(X)/K_2K)^G \\ K_2k(X)/K_2k \arrow[u] \arrow[r] & K_2L(X)/K_2L \arrow[u] \arrow[r, "N_{L/k}"] & K_2k(X)/K_2k. \arrow[u]\end{tikzcd}$$

Let α be in $(K_2K(X)/K_2K)^G$. Using the fact that the middle vertical arrow is an isomorphism (Step II) and that the composition of the horizontal arrows at each level is multiplication by $[L:k]$, we find that $[L:k]\alpha$ comes from $K_2 k(X)/K_2 k$. On the other hand, the commutativity of triangle (4) and the assumption that α is fixed under G show that $[K:k]\alpha$ comes from $K_2 k(X)/K_2 k$; as $[L:k]$ and $[K:k]$ are coprime, we are done.

Step IV. (i) *and* (ii) *hold for* $G=\mathrm{Gal}(K/k)$ *a finite* p*-group* (p *prime*).

From the previous steps, we know this for $G=\mathbf{Z}/p$. The proof will be by induction on n, where p^n is the order of G. Let us assume we know the statement for any p-group of order $\leqq p^{n-1}$, and let G be of order p^n. From the structure of p-groups, we know there exists a normal subgroup H of G with $H\cong\mathbf{Z}/p$. Let $T=G/H$, let $L=K^H$, so that $T=\mathrm{Gal}(L/k)$. We have the restriction-inflation sequence:

$$0\to H^1(T,(K_2K(X)/K_2K)^H)\to H^1(G,K_2K(X)/K_2K)\to H^1(H,K_2K(X)/K_2K)$$

where the last term vanishes by Step I and $(K_2K(X)/K_2K)^H$ may be identified with $K_2L(X)/K_2L$ by Step III. By the induction hypothesis, $H^1(T,K_2L(X)/K_2L)=0$, and we are done as far as (ii) is concerned. But (i) is immediate: one need only consider the injections (Prop. 2):

$$K_2k(X)/K_2k\hookrightarrow K_2L(X)/K_2L\hookrightarrow K_2K(X)/K_2K$$

and apply the induction hypothesis.

Step V. (ii) *holds for any* $G=\mathrm{Gal}(K/k)$ *finite.*

Indeed, for G_p a Sylow p-subgroup of G, the restriction map

$$H^k(G,M)\to H^k(G_p,M),$$

for M any G-module, is an injection on the p-primary subgroup of the torsion group $H^k(G,M)$ $(k\geqq 1)$.

Step VI. (i) *holds for any* $G=\mathrm{Gal}(K/k)$ *finite.*

Let $\alpha\in(K_2K(X)/K_2K)^G$; using diagram (4), we see that $[G]\alpha$ comes from $K_2k(X)/K_2k$. Let p be a prime, and $G_p=\mathrm{Gal}(K/L)$ be a Sylow p-subgroup of G. The proof that (3) commutes given in [7], 3.5.3, extends to give a commutative diagram:

$$\begin{tikzcd}
K_2L(X)/K_2L \arrow[rr, "\sum\limits_{g\in G/G_p} g"] \arrow[dr, "N_{L/k}"'] & & K_2K(X)/K_2K \\
& K_2k(X)/K_2k. \arrow[ur, "j"'] &
\end{tikzcd}$$

According to step IV (applied to K/L), there exists $\beta\in K_2L(X)/K_2L$ with image α in $K_2K(X)/K_2K$. From the above triangle, we deduce that $n_p\alpha$ comes from $K_2k(X)/K_2k$, where $n_p=[L:k]$ is prime to p. We can repeat the same argument for each $p|[G]$. As noted, $[G]\alpha$ also comes from $K_2k(X)/K_2k$. There is no prime number which divides $[G]$ and all n_p, for $p|[G]$, and we conclude with Bézout's lemma.

Step VII. *The theorem holds for any Galois extension K/k.*

Any element of $K_2K(X)$ may be expressed as a finite sum of symbols and therefore comes from some $K_2L(X)$ for $L \subset K$ and L/k finite. Part (i) of the theorem therefore extends from the finite case to the profinite one. Part (ii) therefore extends as well: by the initial remark, $K_2K(X)/K_2K$ is a continuous discrete G-module in the profinite case, and $H^1(G, K_2K(X)/K_2K)$ is the direct limit of all $H^1(G/H, (K_2K(X)/K_2K)^H)$ for H open normal subgroup of G. ∎

Corollary 1. *Let k be a field, char. $k=0$, and $\bar{k}$ an algebraic closure of k. Let X be a geometrically integral k-variety, and assume that either X has a smooth rational k-point or* $\operatorname{cd} k \leqq 1$. *Then $H^1(\operatorname{Gal}(\bar{k}/k), K_2\bar{k}(X)=0$.*

Indeed, there is an exact sequence

$$0 \to K_2\bar{k} \to K_2\bar{k}(X) \to K_2\bar{k}(X)/K_2\bar{k} \to 0$$

(the injection $K_2\bar{k} \hookrightarrow K_2\bar{k}(X)$ follows from a specialization argument at some smooth $\bar{k}$-rational point), and $H^1(\operatorname{Gal}(\bar{k}/k), K_2\bar{k})=0$ because $K_2\bar{k}$ is uniquely divisible ([1] I.1.3). ∎

Remarks 2

2.1. As easy transfer arguments show, Proposition 2 and Theorem 1 hold under the mere assumption that X has a "smooth 0-cycle of degree one", that is that the g.c.d. of the degrees of all finite extensions L/k, for which X has a smooth L-point, is one.

2.2. By Remark 1, the proofs of all results in this section need no reference to **B6**, when one restricts attention to smooth proper geometrically integral k-varieties X such that $\operatorname{Pic}\bar{X}$ has no torsion, for instance rational varieties.

§ 2. The Chow Groups of Rational Surfaces

Let k be a field of characteristic zero, let X be a rational k-surface (by definition, X is smooth and projective). I first recall a few facts from Bloch's paper [4]. For any extension K/k, taking sections of the Quillen resolution of the Zariski-sheaf $\mathscr{K}_{2,X_K}$ gives exact sequences:

$$0 \to \mathscr{Z}_K \to \bigoplus_{\gamma \in X_K^{(1)}} K(\gamma)^* \to \overset{0}{\bigoplus_{P \in X_K^{(2)}}} \mathbf{Z} \tag{15}$$

$$0 \to K_2K(X)/H^0(X_K, \mathscr{K}_2) \to \mathscr{Z}_K \to H^1(X_K, \mathscr{K}_2) \to 0. \tag{16}$$

The first sequence may be completed by a zero on the right as soon as X_K is K-birationally equivalent to $\mathbf{P}_K^2$ (since the cokernel of the last map is $A_0(X_K)$).

In [4], Bloch shows that the natural maps

$$K_2 K \to H^0(X_K, \mathscr{K}_2) \tag{17}$$

$$\operatorname{Pic} X_K \otimes_{\mathbf{Z}} K^* = H^1(X_K, \mathscr{K}_1) \otimes_{\mathbf{Z}} H^0(X_K, \mathscr{K}_1) \to H^1(X_K, \mathscr{K}_2) \tag{18}$$

are isomorphisms as soon as K is a splitting field for the rational k-surface X. In view of the Appendix of [4], (18) may be reinterpreted as follows: for K/k a Galois splitting field of X with Galois group G, the natural "reciprocity" map (which is a G-homomorphism):

$$H^1(X_K, \mathscr{K}_2) \to \operatorname{Hom}_{\mathbf{Z}}(\operatorname{Pic} X_K, K^*) \tag{18'}$$

is an isomorphism (of G-modules). We shall denote by S the k-torus, the group of characters of which is $\operatorname{Pic} X_K$ (for any Galois splitting field K, e.g. $K = \bar{k}$, the algebraic closure of k).

Proposition 3. *For X a rational k-surface* (*char.* $k=0$), *the natural map*

$$K_2 k \to H^0(X, \mathscr{K}_2)$$

(i) *is an isomorphism if X has a rational point*;
(ii) *is surjective if* $\operatorname{cd} k \leqq 1$.

Proof. (i) That the map is injective follows, readily from the existence of a k-point. Let K/k be a splitting field for X. Let z be in

$$H^0(X, \mathscr{K}_2) = \operatorname{Ker}[K_2 k(X) \xrightarrow{\text{Tame}} \bigoplus_{\gamma \in X^{(1)}} k(\gamma)^*].$$

Since all tame symbols of $z \in K_2 k(X)$ are trivial, so a fortiori are those of $z_{|K(X)} \in K_2 K(X)$, so that z belongs to $H^0(X_K, \mathscr{K}_2)$, which is none but $K_2 K$ by Bloch's result. We conclude that the class of z belongs to the kernel of

$$K_2 k(X)/K_2 k \to K_2 K(X)/K_2 K,$$

known to be zero by Proposition 2.

(ii) Same proof. ■

Proposition 4. *Let X be a rational k-surface* (*char.* $k=0$). *There are natural maps:*

$$\Phi\colon A_0(X) \to H^1_{\text{ét}}(k, S)$$
$$r\colon H^1(X, \mathscr{K}_2) \to S(k)$$

If $X(k)$ is not empty, or if $\operatorname{cd} k \leqq 1$, *$\Phi$ is an injection and r an isomorphism.*

Proof. Choose a Galois extension K/k which splits X, let $G = \operatorname{Gal}(K/k)$, and consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & K_2 k(X)/H^0(X, \mathscr{K}_2) \arrow[r] \arrow[d] & \mathscr{Z}_k \arrow[r] \arrow[d] & H^1(X, \mathscr{K}_2) \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & (K_2 K(X)/K_2 K)^G \arrow[r] & (\mathscr{Z}_K)^G \arrow[r] & S(k) \arrow[r] & {}
\end{tikzcd} \tag{19}$$

$$\to H^1(G, K_2 K(X)/K_2 K) \to A_0(X) \xrightarrow{\Phi} H^1(G, S(K))$$

which is obtained as follows. The second line is deduced from (16) by taking G-cohomology; the isomorphism (18′) has been used to identify $(H^1(X_K, \mathcal{K}_2))^G$ with $S(k) = \mathrm{Hom}_G(\mathrm{Pic}\, X_K, K^*)$, and the exact sequence (15) – completed by a zero – has been used to identify $H^1(G, \mathcal{Z}_K)$ with $A_0(X)$, as in [4]. The map $\mathcal{Z}_k \to (\mathcal{Z}_K)^G$ is obviously an isomorphism. The proposition is now an immediate consequence of Theorem 1. ∎

In view of the finiteness results of [4] and [9], as recalled in the Introduction, the proof of Theorems A and C is complete.

Proposition 5. *Let X be a rational k-surface over a local field k (char. $k=0$) and assume $X(k) \neq \emptyset$. Then the natural pairing*

$$A_0(X) \times \mathrm{Br}\, X \to \mathrm{Br}\, k \subset \mathbf{Q}/\mathbf{Z}$$

is non-degenerate on the left hand side.

Here $\mathrm{Br}\, X$ denotes the Brauer group of X, and the pairing is described in [4], Appendix.

Proof. This follows from [4], A.1 (compatibility of certain pairings), local duality (cf. [4], 2.1) together with Proposition 4. ∎

Remarks 3

3.1. In view of Remark 2.1, Propositions 3 and 4 hold under the mere assumption that X has a 0-cycle of degree one.

3.2. By Remark 2.2, the proofs of the results in this section, hence of Theorems A and C, need no reference to **B6**.

3.3. In contrast with Proposition 3, the natural map

$$K_2 k/pK_2 k \to H^0(X, \mathcal{K}_2/p\mathcal{K}_2),$$

with p a prime number, need not be surjective even if $X(k)$ is not empty. Let K/k by cyclic of degree p, assume k contains a primitive p-th root of unity, and let $K = k(\sqrt[p]{a})$ for $a \in k^*$. Let X be a rational k-surface with a k-point. We shall see that the above map is not surjective as soon as X has a non-trivial rational function $f \in k(X)^*$ the divisor of which is a norm for the extension K/k (non-trivial $= f$ is not of the shape $\alpha \cdot N_{K/k}(g)$, with $\alpha \in k^*$ and $g \in K(X)^*$), in other words as soon as $\hat{H}^{-1}(\mathrm{Gal}(K/k), \mathrm{Pic}\, X_K) \neq 0$. Many such examples are known, for instance the Châtelet surfaces ($p=2$). I claim that for f as above, all tame symbols mod. p of $\{a, f\}$ are equal to 1. Indeed, for $\gamma \in X^{(1)}$, the value of the tame symbol (mod. p) of $\{a, f\}$ at γ is $a^{v_\gamma(f)} \in k(\gamma)^*/k(\gamma)^{*p}$. Now, either γ_K is irreducible, in which case p divides $v_\gamma(f)$, or γ_K is reducible, and $K \subset k(\gamma)$, so a is a p-th power in $k(\gamma)$. If $\{a, f\}$ came from $K_2 k$, a specialization argument at a (smooth) rational k-point would imply

$$\{a, f\} = \{a, \alpha\} \in K_2 k(X)/pK_2 k(X)$$

for some $\alpha \in k^*$, hence $\{a, \alpha^{-1} f\} = 0$ in $K_2 k(X)/pK_2 k(X)$. But then $\alpha^{-1} f$ would lie in $N_{K/k} K(X)$ (apply the Galois symbol and use the sequence preceding (7) in the proof of Theorem 1): this we had excluded.

3.4. The finiteness of $A_0(X)$ for any rational surface over a local field of zero characteristic also follows from the conjunction of **MSa** and of [2], Corollary 3.6: Indeed, $A_0(X)$ is killed by $[K:k]$ for any extension K/k with X_K K-rational, and the quoted result of Bloch holds for k $\mathfrak{p}$-adic or real, along with k finite.

3.5. For X a rational variety over a finite field, Bloch [12] proved the vanishing of $A_0(X)$. He also gave (letter to the author) a very geometric proof of the vanishing of $A_0(X)$ for X a rational surface over a $\mathfrak{p}$-adic field, with good reduction.

3.6. Let X be a rational **R**-surface. In this case $A_0(X)=0$ if $X(\mathbf{R})=\emptyset$ (easy, since $A_0(X_{\mathbf{C}})=0$) and $A_0(X)=(\mathbf{Z}/2)^{s-1}$ if $X(\mathbf{R})$ is not empty and s denotes the number of connected components of $X(\mathbf{R})$; also, two points of $X(\mathbf{R})$ are rationally equivalent over X if and only if they lie in the same component of $X(\mathbf{R})$ ([8]; these results extend to higher dimension). Proposition 5 therefore implies that there are enough elements in the Brauer group of X to separate the real connected components by means of the obvious pairing

$$X(\mathbf{R}) \times \operatorname{Br} X \to \mathbf{Z}/2.$$

This may be reinterpreted as follows: on a rational **R**-surface, there are enough rational functions with divisor a norm (for **C**/**R**) to tell apart different connected components of $X(\mathbf{R})$.

§ 3. The Case where no (Smooth) Rational Point is Available

As mentioned in the Introduction, Suslin has recently extended some of the results of Sections 1 and 2:

Sa ([18], 5.8). *Let K/k be a Galois extension of fields, $G=\operatorname{Gal}(K/k)$. Let X be a geometrically integral k-variety. The natural map*

$$j\colon K_2\, k(X)/K_2\, k \to (K_2 K(X)/K_2 K)^G$$

is an isomorphism.

In other words, Theorem 1 (i) holds without any restriction.

Sb ([18], 5.6). *Let X/k be a complete smooth rational k-variety. Then the natural map*

$$K_2\, k \to H^0(X, \mathscr{K}_2)$$

is an isomorphism.

In other words, Proposition 3 holds without any restriction. For char. $k \neq 0$, the proof of these results uses recent work of Bloch-Gabber-Kato.

Proposition 6. *Let K/k be a finite cyclic extension of fields, $G=\operatorname{Gal}(K/k)$, and let X be a geometrically integral k-variety. There then exist isomorphisms:*

$$\begin{aligned} H^1(G, K_2K(X)/K_2K) &\simeq \hat{H}^{-1}(G, K_2K(X)/K_2K) \\ &\xrightarrow{\sim} \operatorname{Ker}[K_2\, k/NK_2K \to K_2\, k(X)/NK_2K(X)] \end{aligned}$$

the first one relying on the choice of a generator of G, the second one being canonical.

Proof. Inject **S a** into the proof of Theorem 1, Step I. ■

Lemma 2.

a) $K_2\mathbf{R}/NK_2\mathbf{C}\cong\mathbf{Z}/2$;

b) *For K/k a finite extension of $\mathfrak{p}$-adic fields, $K_2k/NK_2K=0$;*

c) *For K/k a finite extension of number fields, K_2k/NK_2K is finite; in fact, there is a natural isomorphism*

$$K_2k/NK_2K \xrightarrow{\sim} \bigoplus_{v\in S_\infty} K_2k_v/NK_2(K\otimes_k k_v)$$

where S_∞ denotes the set of real places of k. For K/k Galois, this simply reads

$$K_2k/NK_2K \xrightarrow{\sim} \bigoplus_{v\in S_\infty} K_2k_v/NK_2K^v,$$

where K^v denotes the completion of K at an arbitrary place above v.

Proof. a) and b) are well known (cf. [17], Appendix, A.15) or may be proved in the same manner as c). For K/k as in c), let $n=[K:k]$, and consider the commutative diagram ([14], §1.2, Lemma 3):

$$\begin{array}{ccc} K_2K/nK_2K & \longrightarrow & H^2(K,\mu_n^{\otimes 2}) \\ \Big\downarrow{\scriptstyle N_{K/k}} & & \Big\downarrow{\scriptstyle \mathrm{Cor}_{K/k}} \\ K_2k/nK_2k & \longrightarrow & H^2(k,\mu_n^{\otimes 2}) \end{array}$$

where $\mathrm{Cor}_{K/k}$ denotes the cohomological trace (corestriction) map. According to Tate's theorem, both horizontal arrows are isomorphisms. The cokernel of the left vertical map is none but K_2k/NK_2K. Consideration of the analogous diagrams at the places v (here $K\otimes_k k_v$ need not be a field, but this does not matter) enables us to identify the restriction map

$$K_2k/NK_2K \to \bigoplus_{v\in S_\infty} K_2k_v/NK_2(K\otimes_k k_v)$$

with the restriction map:

$$H^2(k,\mu_n^{\otimes 2})/\mathrm{Cor}\,H^2(K,\mu_n^{\otimes 2}) \to \bigoplus_{v\in S_\infty} H^2(k_v,\mu_n^{\otimes 2})/\mathrm{Cor}\,H^2(K\otimes_k k_v,\mu_n^{\otimes 2}).$$

Let $R_{K/k}\mu_n^{\otimes 2}$ be the Galois module (over k) obtained by pushing down the Galois module $\mu_n^{\otimes 2}$ from K to k. There is a natural surjective map $R_{K/k}\mu_n^{\otimes 2}\to\mu_n^{\otimes 2}$ of Galois modules over k, which induces the corestriction map:

$$H^2(K,\mu_n^{\otimes 2})\cong H^2(k,R_{K/k}\mu_n^{\otimes 2})\to H^2(k,\mu_n^{\otimes 2})$$

(the isomorphism being Shapiro's lemma). Define a finite Galois module M over k by the exact sequence:

$$0\to M\to R_{K/k}\mu_n^{\otimes 2}\to\mu_n^{\otimes 2}\to 0$$

and look at the commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^2(k,\mu_n^{\otimes 2})/H^2(k,R_{K/k}\,\mu_n^{\otimes 2}) & \longrightarrow & H^3(k,M) & \longrightarrow & H^3(k,R_{K/k}\,\mu_n^{\otimes 2}) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \bigoplus_{v\in S_\infty} [H^2(k_v,\mu_n^{\otimes 2})/H^2(k_v,R_{K/k}\,\mu_n^{\otimes 2})] & \longrightarrow & \bigoplus_{v\in S_\infty} H^3(k_v,M) & \longrightarrow & \bigoplus_{v\in S_\infty} H^3(k_v,R_{K/k}\,\mu_n^{\otimes 2}).
\end{array}$$

The two right vertical arrows are isomorphisms (Poitou, Tate), hence also the left one. ∎

Theorem 2. *Let K/k be a Galois extension of fields, $G=\mathrm{Gal}(K/k)$. Let X be a geometrically integral k-variety.*

a) *If k is a $\mathfrak{p}$-adic or a totally imaginary number field, then*

$$H^1(G, K_2K(X)/K_2K)=0;$$

b) *For $K/k=\mathbf{C}/\mathbf{R}$, there is a natural embedding*

$$H^1(G, K_2\,\mathbf{C}(X)/K_2\,\mathbf{C})\hookrightarrow \mathbf{Z}/2$$

which is zero if X has a smooth $\mathbf{R}$-point, and which is an isomorphism if and only if (-1) is a sum of 4 squares in $\mathbf{R}(X)$, as is always the case when $\dim X \leqq 2$ and X has no smooth $\mathbf{R}$-point.

c) *For k a number field, there is a natural embedding*

$$H^1(G, K_2K(X)/K_2K)\hookrightarrow (\mathbf{Z}/2)^T,$$

where T denotes the set of real places v of k such that X has no smooth k_v-point.

Proof. a) Note first that any finite extension of k is of the same type. For k as in a), the previous lemma shows

$$K_2\,k/NK_2K=0$$

for any finite extension K/k. We only have to go through the analogues of Steps I, IV, V and VII in the proof of Theorem 1. Proposition 6 and the previous remarks take care of Step I. In Step IV, one uses **Sa**, Steps V and VII are unchanged.

b) The natural embedding is given by Lemma 2 and Proposition 6. Since $K_2\mathbf{R}/NK_2\mathbf{C}$ is generated by the class of the symbol $\{-1,-1\}$, the embedding is an isomorphism if and only if $\{-1,-1\}$ belongs to $NK_2\mathbf{C}(X)\subset K_2\mathbf{R}(X)$. That this is the case when (-1) is a sum of 4 squares in $\mathbf{R}(X)$ follows from an easy computation with symbols. Conversely, from $\{-1,-1\}\in NK_2\mathbf{C}(X)$ follows $\{-1,-1,-1\}\in 2K_3^M\mathbf{R}(X)$, where K^M denotes Milnor's K-group. Going over to the Witt ring, one finds

$$\langle\!\langle -1,-1,-1\rangle\!\rangle = 0 \in I^3\mathbf{R}(X)/I^4\mathbf{R}(X),$$

hence $\langle\langle -1,-1,-1\rangle\rangle = 0 \in W(\mathbf{R}(X))$ thanks to the Arason-Pfister theorem ([15], X, 3.1).

(Let k be any field, char. $k \neq 2$, and let α, β, γ be in k^*. The same proof shows that γ is a reduced norm of the quaternion algebra $\left(\frac{\alpha, \beta}{k}\right)$ if and only if $\{\alpha, \beta\} \in NK_2 k(\sqrt{\gamma}) \subset K_2 k$. Such an equivalence holds more generally for cyclic algebras of prime degree, as follows from [16], 12.1, 15.6, **MS a**.)

The last assertion is a special case of a celebrated theorem of Pfister ([15], XI, 1.8).

c) Let $G_w \subset G$ be the decomposition group associated to a place w of K, $G_w = \mathrm{Gal}(K_w/k_w)$. In view of b) and of the fact that G_w and $G_{w'}$ are conjugate in G as soon as w and w' induce the same place on k, it is enough to show that the restriction map

$$\rho\colon H^1(G, K_2K(X)/K_2K) \to \prod_{w \in \Omega} H^1(G_w, K_2K_w(X)/K_2K_w)$$

is an injection; here Ω denotes the set of archimedean places of K including the complex ones. To prove this, we shall go through the same steps I, IV, V, VII as in Theorem 1.

Step I. *ρ is an injection when G is finite cyclic.*

Choosing a generator of G determines a generator for any subgroup of G. It is then an exercise in Galois cohomology to check that the following diagram is commutative:

$$\begin{array}{ccc}
H^1(G, K_2K(X)/K_2K) & \hookrightarrow & K_2k/NK_2K \\
\Big\downarrow{\scriptstyle\rho} & & \Big\downarrow \\
\prod\limits_{w \in \Omega} H^1(G_w, K_2K_w(X)/K_2K_w) & \hookrightarrow & \prod\limits_{w \in \Omega} K_2k_w/NK_2K_w;
\end{array}$$

here the vertical arrows are the obvious ones and the horizontal ones are the injections associated by Proposition 6 to the (compatible) choice of generators mentioned above. Since any place of k extends to a place of K, the right vertical arrow is an injection by Lemma 2, hence also ρ.

Step IV. *ρ is an injection if G is a (finite) p-group (p prime).*

As in the proof of Theorem 1, the proof will be by induction on the order of G. Let $H \subset G$ be as in that theorem, with $T = G/H$ and $L = K^H$. Any place w of K induces a place (also denoted by w) on L and k, and determines subgroups $G_w \subset G$, $H_w \subset H$, $T_w \subset T$, with $T_w = G_w/H_w$, and $G_w = \mathrm{Gal}(K_w/k_w)$, $H_w = \mathrm{Gal}(K_w/L_w)$, $T_w = \mathrm{Gal}(L_w/k_w)$. The restriction-inflation sequences at the global and local level are immediately seen to be compatible; using **S a**, we thus obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccc} 0 \longrightarrow & H^1(T, K_2L(X)/K_2L) & \longrightarrow & H^1(G, K_2K(X)/K_2K) & \\ & \Big\downarrow \rho_{L/k} & & \Big\downarrow \rho_{K/k} & \\ 0 \longrightarrow & \prod_{w\in\Omega} H^1(T_w, K_2L_w(X)/K_2L_w) & \longrightarrow & \prod_{w\in\Omega} H^1(G_w, K_2K_w(X)/K_2K_w) & \\ & & \longrightarrow & H^1(H, K_2K(X)/K_2K) & \\ & & & \Big\downarrow \rho_{K/L} & \\ & & \longrightarrow & \prod_{w\in\Omega} H^1(H_w, K_2K_w(X)/K_2K_w). & \end{array}$$

The two extreme vertical arrows are injective by the induction hypothesis (any place of L extends to a place of K), hence so is the middle one.

Step V. *ρ is an injection if G is finite.*

Follows from Step IV exactly as in Theorem 1.

Step VII. *ρ is an injection if G is profinite.*

Let $H \subset G$ be a normal open subgroup, $T = G/H$, $L = K^H$. In view of **Sa**, the inflation maps at the global and local levels yield a commutative diagram:

$$\begin{array}{ccc} H^1(T, K_2L(X)/K_2L) & \hookrightarrow & H^1(G, K_2K(X)/K_2K) \\ \Big\downarrow \rho_{L/k} & & \Big\downarrow \rho_{K/k} \\ \prod_{w\in\Omega} H^1(T_w, K_2L_w(X)/K_2L_w) & \hookrightarrow & \prod_{w\in\Omega} H^1(G_w, K_2K_w(X)/K_2K_w). \end{array}$$

Since any place of L extends to a place of K, the left vertical arrow is injective by Step V. As explained in the proof of Theorem 1, this is enough to conclude. ■

Remark 4. For X/k a rational surface over a number field and $K = \bar{k}$, the embedding in Theorem 2c) always is an isomorphism (Proposition 7). I do not know if this true for all surfaces.

Proposition 7. *Let X be a rational k-surface (char. $k = 0$).*

a) *The natural map $r: H^1(X, \mathcal{K}_2) \to S(k)$ is injective; for k a local or a global field,*

$$\operatorname{Coker}[H^1(X, \mathcal{K}_2) \xrightarrow{r} S(k)] \cong H^1(\operatorname{Gal}(\bar{k}/k), K_2\bar{k}(X)/K_2\bar{k}) \cong (\mathbf{Z}/2)^T$$

with $T = \emptyset$ if k is $\mathfrak{p}$-adic or $X(k) \neq \emptyset$, $T = \{1\}$ if $k = \mathbf{R}$ and $X(\mathbf{R}) = \emptyset$, and T the set of real places v of k with $X(k_v) = \emptyset$ if k is a number field.

b) *For k a local or a global field, the map*

$$\Phi: A_0(X) \to H^1(k, S)$$

is injective and $A_0(X)$ is finite.

(Notations are as in § 2)

Proof. That r is injective follows from **Sa** and diagram (19). The case $X(k) \neq \emptyset$ has been dealt with in §2. If k is a $\mathfrak{p}$-adic field, everything follows from Theorem 2 a) together with (19). If $k = \mathbf{R}$ and $X(\mathbf{R}) = \emptyset$, then $A_0(X) = 0$ (cf. 3.6), and everything follows from Theorem 2 b) together with (19). Note that $\bar{k}$ in a) could be replaced by K for K any Galois extension of k which splits X. Let now k be a number field, and K/k be such an extension. For each place $v \in T$, fix a place of K above v, denote by K^v the completion of K at this place and by G_v the decomposition group $G_v = \mathrm{Gal}(K^v/k_v)$. Since $A_0(X_{k_v}) = 0$ for each place $v \in T$, it follows from (19) that the map

$$S(k_v) \to H^1(G_v, K_2 K^v(X)/K_2 K^v) \qquad (\cong \mathbf{Z}/2)$$

is surjective. Restriction from k to each place $v \in T$ yields a commutative diagram:

$$\begin{array}{ccc} S(k) & \longrightarrow & H^1(G, K_2 K(X)/K_2 K) \\ \downarrow & & \downarrow \\ \prod_{v \in T} [S(k_v)/2S(k_v)] & \longrightarrow & \prod_{v \in T} H^1(G_v, K_2 K^v(X)/K_2 K^v). \end{array}$$

We have just seen that the lower horizontal arrow is surjective, and the right vertical map is injective (proof of Theorem 2 c)). Since each torus S_{k_v} is certainly split by a cyclic extension, $S(k)$ is dense in the product of the $S(k_v)$, and the left vertical map is surjective. Therefore the right vertical arrow is an isomorphism and the top map is surjective, which together with (19) implies that Φ is injective. That $A_0(X)$ is finite now follows from the fact that the image of Φ is finite. ∎

Remarks 5

5.1. For k an arbitrary field, the map Φ in Proposition 7 need not be injective when $X(k) = \emptyset$. Examples with $\mathrm{Ker}\,\Phi \neq 0$ and $k = \mathbf{Q}(T)$, $\mathbf{Q}_p((T))$, $\mathbf{C}(T_1, T_2, T_3)$ are given in [10].

5.2. Since **Sa** also holds for char. $k = p > 0$, so does Theorem 1. Proposition 4 then extends if X has a Galois splitting field. Once §1 of [9] is extended to the case where the ground field need not be perfect, it is an easy matter to extend the whole of Theorem A to char. $k > 0$, when X has a Galois splitting field. In the general case, one only obtains finiteness up to p-torsion. Such a result had already been proved in [11], starting from [16], but using ideas different from those of the present paper.

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