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On conjugating representations and adjoint representations of semisimple groups

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0. Introduction

Let G be a semisimple, affine algebraic group over an algebraically closed field of characteristic 0 and $A(G)$ the algebra of regular functions on G. In [18], Richardson considered $A(G)$ as a $C(G)G$-module, where $C(G)$ is the algebra of regular class functions and G is acting via the conjugating representation. Richardson proved that, when $C(G)$ is a polynomial algebra, the G-homogeneous component of $A(G)$ indexed by weight λ breaks up as a tensor product $C(G)\otimes E_\lambda$, where E_λ is a direct sum of irreducible G-modules of highest weight λ. Here we consider the conjugating representation in arbitrary characteristic and prove the appropriate version of Richardson's Theorem (under small, and almost certainly unnecessary, characteristic/root-system restrictions). One can no longer decompose $A(G)$ into homogeneous components but instead we prove the existence of an ascending $C(G)G$-module filtration, indexed by the set X^+ of dominant weights and successive quotients of the form $C(G)\otimes E_\lambda(\lambda\in X^+)$, where E_λ is a direct sum of induced modules of highest weight λ.

Richardson's Theorem is a rather precise analogue of a theorem of Kostant, [16], concerning the action of the adjoint group of a semisimple complex Lie algebra $\mathfrak{g}$ on the algebra $A(\mathfrak{g})$ of polynomial functions of $\mathfrak{g}$. We prove the appropriate version of this result in characteristic p (with small, and not entirely unnecessary, characteristic/root system restrictions). The filtrations of both $A(G)$ and $A(\mathfrak{g})$ are obtained as special cases of the corollary to the theorem which we prove in § 1.5.

The results in this paper are obtained by combining the geometric algebra in [18] (which is characteristic free) with methods and results developed over the last few years in characteristic-free representation theory of reductive groups, [9].

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1. (C, G)-modules

1.1. Let k be an algebraically closed field. All modules for an affine algebraic group over k will be supposed rational (see [6], 1.1) but not necessarily finite dimensional. Let G be a reductive group over k, B a Borel subgroup, $T \subset B$ a maximal torus and U the unipotent radical of B. Let X be the character group of T. For a T-module V and $\lambda \in X$, V^λ denotes the λ-weight space of V. We choose the system of positive roots Φ^+ in the root system of G so that B is the negative Borel subgroup. Let $W = N_G(T)/T$ be the Weyl group and $w_0 \in W$ be the longest element. For $\lambda \in X$, let $\lambda^* = -w_0 \lambda$.

For $\lambda \in X$, k_λ denotes the one dimensional B-module on which T acts with weight λ. We denote by $Y(\lambda)$ the induced module $\operatorname{Ind}_B^G k_\lambda$. Then $Y(\lambda) \neq 0$ precisely when λ belongs to the set X^+ of dominant weights and, for $\lambda \in X^+$, $Y(\lambda)$ has formal character given by Weyl's Character Formula (see [6], Ch. 1).

1.2. Let C be a commutative k-algebra. Given a C-module V and a k-space M we write $|M| \otimes V$ for the vector space $M \otimes V$ viewed as a C-module with C acting via $c(m \otimes v) = m \otimes cv$, for $c \in C$, $m \in M$, $v \in V$. By a (C, G)-module we mean a k-vector space V which has the structure of a C-module and a rational G-module in such a way that $c(gv) = g(cv)$ for all $c \in C$, $g \in G$, $v \in V$. Morphisms of (C, G)-modules, (C, G)-submodules etc. are defined in the usual way. Given a (C, G)-module V and a G-module M we regard the C-module $|M| \otimes V$ as a (C, G)-module with G acting diagonally. We occasionally regard C as a (C, G)-module on which C acts via the regular action and G acts trivially.

By a good filtration of a G-module V we mean an ascending filtration $0 = V_0, V_1, \ldots$ of V such that, for each $i > 0$, V_i/V_{i-1} is either 0 or isomorphic to $Y(\lambda_i)$ for some $\lambda_i \in X^+$. For a fixed $\lambda \in X^+$, the number of successive quotients isomorphic to $Y(\lambda)$ in such a filtration is independent of the choice of good filtration ([6], (12.1.1)) and denoted $(V : Y(\lambda))$.

Let π be a finite subset of X^+ which is saturated in the sense that $\mu \in \pi$ whenever $\lambda \in \pi$, $\mu \in X^+$ and μ is less than λ in the natural partial order on X. We say that a G-module V belongs to π if every dominant weight of V belongs to π. Among all submodules belonging to π, of an arbitrary rational G-module V, there is a unique maximal one, denoted $O_\pi(V)$. Notice that a G-module homomorphism $V_1 \to V_2$ induces a homomorphism $O_\pi(V_1) \to O_\pi(V_2)$ by restriction, making O_π a left exact functor. Notice also that if V is a (C, G)-module, $O_\pi(V)$ is a (C, G)-submodule of V and O_π determines a left exact functor on (C, G)-modules.

Proposition. *Let V be a (C, G)-module which (as a G-module) has a good filtration. Let π be a finite, saturated subset of X^+, λ a maximal element of π and $\pi' = \pi \backslash \{\lambda\}$. Then $O_\pi(V)/O_{\pi'}(V)$ and $|Y(\lambda)| \otimes O_\pi(V)^\lambda$ are isomorphic C-modules.*

Proof. Let $M = O_\pi(V)/O_{\pi'}(V)$. Then M, as a G-module, is isomorphic to a direct sum of copies of $Y(\lambda)$, by [6], (12.1.2) and (12.1.2). Moreover, $\operatorname{End}_G(Y(\lambda)) \cong k$, e.g. by [6], (1.5.3) and therefore the map φ: $Y(\lambda) \otimes \operatorname{Hom}_G(Y(\lambda), M) \to M$ defined by $\varphi(y \otimes \theta) = \theta(y)$, $y \in Y(\lambda)$, $\theta \in \operatorname{Hom}_G(Y(\lambda), M)$, is a k-space isomorphism. Regarding $\mathrm{Hom}_G(Y(\lambda), M)$ as a (C, G)-module on which C acts via $(c\theta)(y) = c\theta(y)$ $(c \in C$, $\theta \in \mathrm{Hom}_G(Y(\lambda), M)$, $y \in Y(\lambda))$ and on which G acts trivially, φ is a (C, G)-module isomorphism $|Y(\lambda)| \otimes \mathrm{Hom}_G(Y(\lambda), M) \to M$. Again, since M is a direct sum of copies of $Y(\lambda)$, restriction to the λ-weight space shows that $\mathrm{Hom}_G(Y(\lambda), M) \cong M^\lambda$, as (C, G)-modules (with trivial G-action). Hence we have $M \cong |Y(\lambda)| \otimes M^\lambda$. But the natural map $O_\pi(V) \to M$ induces an isomorphism $O_\pi(V)^\lambda \to M^\lambda$ so that $O_\pi(V)/O_{\pi'}(V)$ is isomorphic to $|Y(\lambda)| \otimes O_\pi(V)^\lambda$, as required.

1.3. We regard the following as the characteristic-free analogue of [18], Proposition 3.1.

Proposition. *Let A be a finitely generated, commutative, k-algebra on which G acts rationally as k-algebra automorphisms. Suppose that A has a good filtration and let $C = A^G$, the algebra of invariants. Then, for every finite, saturated subset π of X^+, $O_\pi(A)$ is a finitely generated C-module.*

Proof. Let λ be a maximal element of π and $\pi' = \pi \setminus \{\lambda\}$. Then $O_\pi(A)/O_{\pi'}(A) \cong |Y(\lambda)| \otimes O_\pi(A)^\lambda$ by 1.2 Proposition, so by induction on $|\pi|$ it suffices to show that $O_\pi(A)^\lambda$ is a finitely generated C-module. Moreover, multiplication by a coset representative of w_0 in $N_G(T)$ induces an isomorphism $O_\pi(A)^\lambda \to O_\pi(A)^{w_0\lambda}$ so it suffices to show that $O_\pi(A)^{w_0\lambda}$ is finitely generated.

Let $A_0 = A^U$. Since $w_0\lambda$ is a lowest weight of $O_\pi(A)$, we have $O_\pi(A)^{w_0\lambda} \leqq A_0^{w_0\lambda}$. On the other hand, we have $\bar{A}^{w_0\lambda} = 0$ by [6], (12.1.6) and (1.5.2), where $\bar{A} = (A/O_\pi(A))^U$ so that $O_\pi(A)^{w_0\lambda} = A_0^{w_0\lambda}$. Furthermore A_0 is a T-module and $A_0^T = A^B = C$, by [4], (2.1) Theorem and A_0 is finitely generated, e.g., by [8], Corollary, § 3. Therefore we may (and do) replace A by A^U and G by T. Let $\chi = -w_0\lambda$. Then $A \otimes k[\chi]$ is a finitely generated k-algebra on which T-acts and so $(A \otimes k[\chi])^T$ is finitely generated (e.g. by [12], 14.3 Theorem and exercise 1, Ch. V), by $a_i \otimes \chi^{d_i}$ say, $1 \leqq i \leqq n$, $d_i \geqq 0$. Then $A^{w_0\lambda}$ is generated as C-module by $\{a_i\colon 1 \leqq i \leqq n \text{ and } d_i = 1\}$.

1.4. For a C-module or (C, G)-module V we denote by w.h.d.(V) the weak homological dimension of V(as a C-module).

Proposition. *Suppose C has finite global dimension and V is a (C, G)-module with a good filtration. Then w.h.d.$(V) = \max\{$w.h.d. $O_\sigma(V)$: $\sigma \in \mathscr{S}\}$, where $\mathscr{S}$ is the set of finite, saturated subsets of X^+.*

Proof. Let $d = \max\{$w.h.d. $O_\sigma(V)$: $\sigma \in \mathscr{S}\}$ and let π be such that $d =$ w.h.d. $O_\pi(V)$. We claim that

$$\text{w.h.d. } O_\sigma(V)/O_\tau(V) \leqq d$$

for every $\sigma, \tau \in \mathscr{S}$ with $\sigma \supset \tau$. By induction on $|\sigma| - |\tau|$ it suffices to consider the case $\sigma = \tau \cup \{\lambda\}$ for some $\lambda \notin \tau$. Then we have $O_\sigma(V)/O_\tau(V) \cong |Y(\lambda)| \otimes O_\sigma(V)^{w_0\lambda}$ by (1.2b) Proposition. Now $O_\sigma(V)^{w_0\lambda}$ is a C-module summand of $O_\sigma(V)$ so that w.h.d. $O_\sigma(V)^{w_0\lambda} \leqq d$ and hence w.h.d. $O_\sigma(V)/O_\tau(V) \leqq d$, proving the claim.

Let M be a C-module. By the claim and the long exact sequence we have that

$$\mathrm{Tor}_e^C(O_\tau(V), M) \to \mathrm{Tor}_e^C(O_\sigma(V), M) \tag{1}$$

is injective for all $\tau, \sigma \in \mathscr{S}$, $\tau \subset \sigma$ and $e \geqq d$. Label the elements of X^+ in sequence $\lambda_1, \lambda_2, \ldots$ such that $i<j$ whenever $\lambda_i<\lambda_j$ and $\pi=\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for some n. We set $\pi(r)=\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $V_r=O_{\pi(r)}(V)$ for $r \geqq 1$. We have that

$$\operatorname{Tor}_e^C(V, M)=\varinjlim_r \operatorname{Tor}_e^C(V_r, M)$$

and that $\operatorname{Tor}_e^C(V_r, M) \to \operatorname{Tor}_e^C(V_{r+1}, M)$ is injective, for $r \geqq 1$ and $e \geqq d$, by (1) and [3], Ch. VI, Proposition 1.3. Hence $\operatorname{Tor}_e^C(V, -)=0$ for $e>d$. Choosing M so that $\operatorname{Tor}_d^C(O_\pi(V), M) \neq 0$, i.e., $\operatorname{Tor}_d^C(V_n, M) \neq 0$ we get $\operatorname{Tor}_d^C(V, M) \neq 0$ and so $\text{w.h.d.}(V)=d$.

1.5. Our main result is the corollary given in this section. In the rest of paper we apply this in the important special cases $A=A(G)$ and $A(\mathfrak{g})$, the coordinate rings of G and its Lie algebra $\mathfrak{g}$.

Theorem. *Let A be a finitely generated commutative k-algebra on which G acts rationally as k-algebra automorphisms and put $C=A^G$. Suppose that A has a good filtration (as a G-module), that C is a free polynomial k-algebra and A is a flat C-module. Let π be a finite saturated subset of X^+, λ a maximal element of π and $\pi'=\pi \backslash \{\lambda\}$. Then $O_\pi(A)/O_{\pi'}(A)$ is isomorphic to $|E| \otimes C$ as a (C, G)-module, where E is isomorphic to a direct sum of a finite number of copies of $Y(\lambda)$.*

Proof. By 1.4 Proposition, $O_\pi(A)$ is flat over C, and therefore $O_\pi(A)^\lambda$, a C-module summand of $O_\pi(A)$, is also flat. Now by 1.2 Proposition, $M=O_\pi(A)/O_{\pi'}(A)$ is also a flat C-module. Moreover $O_\pi(A)$, and therefore M, is a finitely generated C-module by 1.3 Proposition. Hence M is a finitely generated projective C-module by [3], Ch. VI, Ex. 3. Now M, as a (C, G)-module is isomorphic to $|Y(\lambda)| \otimes M^\lambda$ (where C acts trivially on M^λ) by the proof of 1.2 Proposition. Now M^λ is a C-module direct summand of M and hence also a finitely generated projective C-module, and therefore free by the Serre Conjecture [17], [21]. Hence $M^\lambda=|V| \otimes C$ for some finite dimensional, trivial G-module V and so $M \cong |Y(\lambda)| \otimes (|V| \otimes C) \cong |E| \otimes C$, where $E=Y(\lambda) \otimes V$, a direct sum of $\dim V$ copies of $Y(\lambda)$.

Corollary. *Under the hypotheses of the Theorem, A has an ascending (C, G)-module filtration $0=A_0, A_1, \ldots$ where $A_i/A_{i-1} \cong |E_i| \otimes C$, E_i a finite direct sum of copies of $Y(\lambda_i)$ $(i \geqq 1)$ and $\lambda_1, \lambda_2, \ldots$ is a labelling of the elements of X^+ such that $i<j$ whenever $\lambda_i<\lambda_j$. For a given labelling, the multiplicity $(E_i: Y(\lambda_i))$ is independent of the choice of such a filtration.*

In particular, A is a free C-module.

Proof. Let $\pi(i)=\{\lambda_1, \lambda_2, \ldots, \lambda_i\}$ and $A_i=O_{\pi(i)}(A)$ for $i \geqq 1$. Put $A_0=0$. Then $A_i/A_{i-1} \cong |E_i| \otimes C$, for $i \geqq 1$, with E_i a direct sum of finitely many copies of $Y(\lambda_i)$, by the Theorem. let $\mathscr{M}$ be a maximal ideal of C. Then $A/A\mathscr{M}$ has a G-modules filtration with quotients isomorphic to the E_i $(i \geqq 1)$. Hence $(E_i: Y(\lambda_i))=(A/A\mathscr{M}: Y(\lambda_1))$, and is therefore independent of the filtration.

2. Conjugating representations and adjoint representations

2.1. We adopt the following conventions. For a k-space V, we denote the dual space by V^*. If V is a finite dimensional G-module, V^* is viewed as a G-module in the usual way. The coordinate ring of an affine k-variety Z is denoted by $A(Z)$; if G acts on Z then we put $C(Z)=A(Z)^G$, the algebra of invariants. For a finitely generated, reduced commutative k-algebra A, we denote the corresponding "classical" variety by $\mathrm{Spm}(A)$ (the maximum spectrum).

Let p be the characteristic of k. Recall that, for an indecomposable root system Ψ, p is called good unless one of the following holds: Ψ has type B, C or D and $p=2$; Ψ has type E_6, E_7, F_4 or G_2 and $p=2$ or 3; Ψ has type E_8 and $p=2$, 3 or 5. We call p very good for Ψ if p is good and, in addition, if Ψ has type A_{l+1} then $p \nmid l+1$. We call p good (resp. very good) for an arbitrary root system if it is good (resp. very good) for each indecoomposable component. We call p good for G (resp. very good for G), or simply good (resp. very good), if it is good (resp. very good) for the root system of G.

Let $\mathfrak{g}=\mathrm{Lie}(G)$ and $\mathfrak{t}=\mathrm{Lie}(T)$. For the rest of the paper $r=\dim T$, the rank of G. We call an element $x\in\mathfrak{g}$ strongly regular if the centralizer $Z_{\mathfrak{g}}(x)$ of x in $\mathfrak{g}$ has dimension r. In applying 1.5 Corollary to $A(\mathfrak{g})$ we shall use the following collection of more or less known results.

Proposition. *Suppose that either G is almost simple, simply connected and p is very good or that $G=GL_n(k)$ for some $n\geqq 1$.*

(i) *The restriction map θ: $C(\mathfrak{g})\to A(\mathfrak{t})^W$ is an isomorphism.*
(ii) *$C(\mathfrak{g})$ is a free polynomial ring in r indeterminates.*
(iii) *$\mathfrak{g}$ contains a non-empty open set of strongly regular, semisimple elements.*
(iv) *$A(\mathfrak{g})$ is a flat $C(\mathfrak{g})$-module.*
(v) *$\mathrm{Spm}(A(\mathfrak{g}))\to\mathrm{Spm}(C(\mathfrak{g}))$ is separable.*
(vi) *$A(\mathfrak{g})$ has a good filtration.*

Proof. First consider the case in which G is semisimple. If $p=0$, (i) and (ii) are well known results of Chevalley; for (iii) see [11], p. 133/134. Also, (iv) is true since $A(\mathfrak{g})$ is free over $C(\mathfrak{g})$ by Kostant, [16]. Separability is automatic in characteristic 0 and every G-module of countable dimension has a good filtration (the $Y(\lambda)$'s are the irreducible G-modules). We therefore suppose $p\neq 0$. Let $\mathfrak{u}=\mathrm{Lie}(U)$ and $\mathfrak{n}=\mathrm{Lie}(U^+)$, where U^+ is the unipotent radical of B^+, the Borel subgroup opposite to B. As in [14], we identify $\mathfrak{t}^*$ with $\{l\in\mathfrak{g}^*: l(\mathfrak{n})=l(\mathfrak{u})=0\}$ and $\mathfrak{n}^*$ with $\{l\in\mathfrak{g}^*: l(\mathfrak{t})=l(\mathfrak{u})=0\}$.

There is a G-invariant, non-degenerate, bilinear form on $\mathfrak{g}$ (see [19], I, 5.3. Lemma) inducing a G-isomorphism $\mathfrak{g}^*\to\mathfrak{g}$ taking $\mathfrak{t}^*$ to $\mathfrak{t}$. In proving (i), (ii) and (iv) we may therefore replace $\mathfrak{g}$ by $\mathfrak{g}^*$ and $\mathfrak{t}$ by $\mathfrak{t}^*$. Hence (i) is true by [14], Theorem 4(i), and (ii) is true by (i) and [5], Corollary of Theorem 3. To prove that $A(\mathfrak{g}^*)$ is flat over $C(\mathfrak{g}^*)$ we use the criterion of Lemma 2.2 of [18]. We must show that φ: $\mathfrak{g}^*\to Y=\mathrm{Spm}(C(\mathfrak{g}))$ is surjective and the fibres have all irreducible components of dimension $d=\dim\mathfrak{g}^*-\dim Y=\dim\mathfrak{g}-r$. Surjectivity follows from (i) and the fibers are irreducible by [14], Theorem 4 (vii). For $y\in Y$ we have $\dim\varphi^{-1}(y)\geqq d$ on general grounds ([12], 4.1 Theorem) so we only need check that $\dim\varphi^{-1}(y)\leqq d$. By [14], Theorem 4 (iv), $\varphi^{-1}(y)$ contains an element $l=l_1+l_2$, with $l_1\in\mathfrak{t}^*$, $l_2\in\mathfrak{n}^*$. By [14], 3.10, $\varphi^{-1}(y)$

$=G.(l_1+\mathfrak{n}^*)$. Now $l_1+\mathfrak{n}^*$ is stable under the action of B so we get $\varphi^{-1}(y) = \bigcup_{w\in W} Uw(l_1+\mathfrak{n}^*)$, by the Bruhat decomposition, and therefore $\dim\varphi^{-1}(y) \leqq \dim U+\dim\mathfrak{n}^*=d$, as required.

Note that the centraliser $Z_W(\mathfrak{t})=\{1\}$, by [14], 2.3 Proposition. Hence there exists $x\in\mathfrak{t}$ such that $s_\alpha(x)\neq x$, for every reflection $s_\alpha\in W$, i.e., $d\alpha(x)\neq 0$, for every $\alpha\in\Phi$. Now x is strongly regular with centraliser $\mathfrak{t}$ in $\mathfrak{g}$ and (iii) follows as in [20], 6.8. We have (v) by [2], AG, (2.4) Proposition and the argument of [19], p. 200 (3). For (vi), see [1], 4.4 Proposition.

Now let $G=\mathrm{GL}_n(k)$. Then by the fundamental theorem of symmetric functions $C(\mathfrak{t})^W$ is freely generated by $e_1,\ldots,e_n$, where $e_i=\theta(d\chi_i)$ $(1\leqq i\leqq n)$ and $\chi_i\in C(G)$ is the trace function on the i^{th} exterior power of the natural representation. Hence θ is surjective. Now (iii) follows as in the semisimple case above and the injectivity of θ follows as in [11], p. 133/134. Moreover, (ii) follows from (i). It follows, also from (i), that $\varphi\colon\mathfrak{g}\to\mathrm{Spm}(C(\mathfrak{g}))$ is surjective. One obtains that the fibres are connected, as in [14], 3.10, and have dimension $\dim\mathfrak{g}-r$, as above. Hence (iv) holds by [18], Lemma 2.2. We have $C(\mathfrak{g})=A(\mathfrak{g})^{G_0}$, where $G_0=\mathrm{SL}_n(k)$, since the centre of G acts trivially on $A(\mathfrak{g})$, so (v) follows as in the semisimple case above. We have (vi) by [1], 4.3 (and [6], Proposition 3.2.7 (iii)).

2.2. **Theorem.** *Let $Z=G$ (resp. $Z=\mathfrak{g}$) with G acting on Z via conjugation (resp. the adjoint action). Assume the either $G=\mathrm{GL}_n(k)$ for some $n\geqq 1$ or that G is almost simple, simply connected and $p\neq 2$ or G does not have type E_7 or E_8 (resp. G is almost simple, simply connected and p is good). Let $A=A(Z)$ and $C=C(Z)$. Then A has a (C,G)-module filtration $0=A_0,A_1,A_2,\ldots$ where $A_i/A_{i-1}=|E_i|\otimes C$, for $i\geqq 1$, E_i is a finite direct sum of copies of $Y(\lambda_i)$ and $\lambda_1,\lambda_2,\ldots$ is a labelling of X^+ such that $i<j$ whenever $\lambda_i<\lambda_j$. For a fixed labelling, the multiplicity $(E_i\colon Y(\lambda_i))$ is $\dim Y(\lambda_i)^T$ $(i\geqq 1)$.*

In particular A is a free C-module.

Proof. First suppose that Z is not isomorphic to $sl_n(k)$.

If G is semisimple and $Z=G$ then C is a polynomial algebra in r indeterminates by [20], 6.1 Theorem and A is flat over C by [18], Proposition 2.3. The same holds for $Z=\mathrm{GL}_n(k)$ as one may see, e.g., from the proof of 2.1 Proposition. Also A has a G-module filtration with sections $Y(\lambda)\otimes Y(\lambda^*)$ $(\lambda\in X^+)$, [7], 1.4(17) (or [15], Theorem 1 or [13], II, 4.20 Proposition). Hence A has a good filtration by [6], (10.8.5) Theorem and [6], Proposition 3.1.1. If $Z=\mathfrak{g}$ then C is a polynomial algebra in r indeterminates, A is flat over C and A has a good filtration, by 2.1 Proposition, (ii), (iv) and (vi). Hence in all these cases, A satisfies the hypothese of 1.5 Corollary and hence has a filtration of the required form. For $Z=G$ semisimple, Richardson shows [18], Lemma 8.3, that there is a maximal ideal $\mathcal{M}$ of C such that $A/A\mathcal{M}$ is isomorphic, as a G module, to $A(G/T)$, and the argument works to for $G=\mathrm{GL}_n(k)$. (Actually, in [18], Lemma 8.3, k has characteristic 0 but the proof works generally, it is based on the separability of $G\to\mathrm{Spm}(C(G))$, which is true in arbitrary characteristic by [20], 6.9 Theorem and [2], *AG*, (2.4 Proposition.)

We shall now show that the same holds for $Z=\mathfrak{g}$ (G not isomorphic to $\mathrm{SL}_n(k)$). By 2.1 Proposition, (iii) and (v) we can pick $x\in\mathfrak{t}$ such that x is strongly regular and $d\varphi_x$ is surjective, where $\varphi\colon\mathfrak{g}\to\mathrm{Spm}(C(\mathfrak{g}))$ is the natural map. Let

O_x be the G-orbit of x. Then O_x is closed, [2], Theorem 9.2 and it is not difficult to see that $O_x=\varphi^{-1}(y)$, where $y=\varphi(x)$. By [18], Lemma 8.2, $A\mathcal{M}$ is a prime ideal, where $\mathcal{M}\leqq C$ is the ideal of y. Hence $A(O_x)$ is isomorphic to $A/A\mathcal{M}$ as a G-module and k-algebra. Consider the map $\pi\colon G\to O_x$, $\pi(g)=Ad(g)\,x$ for $g\in G$. Then $Z_G(x)=T$ and the kernel of $d\pi_1$ is $Z_{\mathfrak{g}}(x)=\mathfrak{t}$. Hence, by dimensions, $d\pi_1$ is surjective, and π in separable. Hence by [2], (6.7 Proposition, π induces an isomorphism $G/T\to O_x$. Where therefore have that $A/A\mathcal{M}$ is G-isomorphic to $A(G/T)$, as required. Hence, for Z not isomorphic to $sl_n(k)$, we have $(E_i\colon Y(\lambda_i))=(A/A\mathcal{M}\colon Y(\lambda_i))$ (see the proof of 1.5 Corollary) and so $(E_i\colon Y(\lambda_i))=(A(G/T)\colon\ Y(\lambda_i))$ which is $\dim(A(G/T)\otimes Y(\lambda_i^*))^G$, by [6], (12.1.1). However, $A(G/T)$ is the induced module $\mathrm{Ind}_T^G\,k$ and so by reciprocity and the tensor identity, [6], (1.1.2), (1.1.7) we obtain $(E_i\colon Y(\lambda_i))=\dim Y(\lambda_i^*)^T$. However, the formal character of $Y(\lambda_i^*)$ is equal to the formal character of $Y(\lambda_i)^*$ (e.g., by Weyl's Character Formula, [6], (2.2.6)) and so $\dim Y(\lambda_i^*)^T=\dim Y(\lambda_i)^T$, giving the desired multipicity assertion.

It remains to deal with the Lie algebra of a special linear group. Let $G=\mathrm{GL}_n(k)$, $G_0=\mathrm{SL}_n(k)$, $\mathfrak{g}_0=\mathrm{Lie}(G_0)$. Let $f\in A(\mathfrak{g})$ be the trace function. Then we have a short exact sequence

$$0\to f.A(\mathfrak{g})\to A(\mathfrak{g})\to A(\mathfrak{g}_0)\to 0 \tag{1}$$

where the first map is inclusion and the second is restriction. By 2.1 Proposition (vi), $A(\mathfrak{g})$ has a good filtration as a G-module, and hence by [6], Proposition 3.2.7 (iii), as a G_0-module. Moreover, $f.A(\mathfrak{g})\cong A(\mathfrak{g})$ as a G_0-module so that $H^1(G_0,f.A(\mathfrak{g}))=0$, by [10], Corollary 6. Hence we get a short exact sequence $0\to f.A(\mathfrak{g})^{G_0}\to A(\mathfrak{g})^{G_0}\to A(\mathfrak{g}_0)^{G_0}\to 0$. But G is the product of G_0 and the centre so $A(\mathfrak{g})^{G_0}=A(\mathfrak{g})^G$ and we get a short exact sequence

$$0\to f.C(\mathfrak{g})\to C(\mathfrak{g})\to C(\mathfrak{g}_0)\to 0. \tag{2}$$

Now $C(\mathfrak{g})$ is a free polynomial ring in n variables, one of which is f (see the proof of 2.1 Proposition (ii)) and so, by (2), $C(\mathfrak{g}_0)$ is free on $n-1$ variables. By 2.1 Proposition (iv), $A(\mathfrak{g})$ is a flat $C(\mathfrak{g})$-module. Hence $C(\mathfrak{g}_0)\otimes_{C(\mathfrak{g})}A(\mathfrak{g})$ is a flat $C(\mathfrak{g}_0)$-module, i.e., $A(\mathfrak{g}_0)$ is a flat $C(\mathfrak{g}_0)$-module. Hence by 1.5 Corollary, $A(\mathfrak{g}_0)$ has a filtration of the required form and it only remains to calculated the multiplicities.

Let X^+ be the set of dominant weights of the diagonal torus T of G and X_0^+ the set of dominant weights of the diagonal torus T_0 of G_0. The kernel of restriction $X(T)\to X(T_0)$ is $\mathbb{Z}\omega$, where ω is the determinant function on T. Let $\mathscr{I}$ be the set of $\lambda\in X^+$ such that $Y(\lambda)^T\neq 0$ and $\mathscr{I}_0$ the set of $\lambda\in X_0^+$ such that $Y_0(\lambda)^{T_0}\neq 0$ ($Y_0(\lambda)$ the module induced from the module k_λ for the Borel subgroup B_0 of lower triangular matrices). If λ, $\mu\in\mathscr{I}$ then $\lambda-\mu\in\mathbb{Z}\,\Phi$ and, since $\mathbb{Z}\Phi\cap\mathbb{Z}\omega=(0)$, the restriction $\varphi\colon\mathscr{I}\to\mathscr{I}_0$ is injective. It is also easy to check that φ is surjective. From what has already been proved, it follows that $A(\mathfrak{g})$ has a $(C(\mathfrak{g}),G)$-filtration $0=A(\mathfrak{g})_0, A(\mathfrak{g})_1,\dots$ where $A(\mathfrak{g})_i/A(\mathfrak{g})_{i-1}\cong|E(\mathfrak{g})_i|\otimes C(\mathfrak{g})$, for $i\geqq 1$, $E(\mathfrak{g})_i$ isomorphic to $\dim Y(\lambda_i)^T$ copies of $Y(\lambda_i)$ and $\lambda_1,\lambda_2,\dots$ a labelling of $\mathscr{I}$ such that $i<j$ whenever $\lambda_i<\lambda_j$. By tensoring with $C(\mathfrak{g}_0)$ over $C(\mathfrak{g})$, and using (1), (2), we obtain a $(C(\mathfrak{g}_0),G_0)$-filtration $0=A(\mathfrak{g}_0)_0$, $A(\mathfrak{g}_0)_1,\dots$ of $A(\mathfrak{g}_0)_i/A(\mathfrak{g})_{i-1}\cong|E(\mathfrak{g}_0)_i|\otimes C(\mathfrak{g}_0)$ for $i>0$, $E(\mathfrak{g}_0)_i\cong E(\mathfrak{g})_i|_{G_0}$. Hence, by [6], Proposition 3.2.7 (i), $E(\mathfrak{g}_0)_i$ is isomorphic to $\dim Y(\lambda_i)^T$ copies of $Y_0(\varphi(\lambda_i))$. It follows, from the injectivity of restriction $\mathbb{Z}\Phi \to X(T_0)$ and [6], Proposition 3.2.7 (i), that $\dim Y(\lambda_i)^T = \dim Y_0(\varphi(\lambda_i))^{T_0}$. Hence the multiplicities in the filtration $A(\mathfrak{g}_0)_i$ $(i \geqq 0)$ of $A(\mathfrak{g}_0)$ are as stated in the theorem. One has the same multiplicities in any such filtration by 1.5 Corollary.

Remarks. 1. In the course of the proof we obtained that $C(\mathfrak{g}_0)$ is a free polynomial algebra so that one only requires p to be good for 2.1 Proposition (ii) to hold. For $p=2$, $G_0=\mathrm{SL}_2(k)$, $A(\mathfrak{t}_o)^W$ has a non-zero, degree one element and $C(\mathfrak{g}_0)$ does not so 2.1 Proposition (i) fails in that case.

2. The characteristic/root-system restrictions in the case $Z=G$ come entirely from [6], (10.8.5) Theorem and are almost certainly unnecessary. In any case, by Richardson, [18], Theorem C, $A(G)$ is always free over $C(G)$ (G semisimple, simply connected). On the other hand, for $Z=\mathfrak{g}$, some restriction is definitely necessary. It follows from the conclusion of the theorem that $A(\mathfrak{g})$ has a good filtration and this is not always the case for G semisimple, simply connected (see [9], for a counterexample).

3. One quickly obtains Richardson's result. [18], Theorem A (for G semisimple, simply connected) from the above theorem. Suppose k has characteristic 0. Let $\lambda_1, \lambda_2, \ldots$ be as in our theorem. The $\lambda=\lambda_i$ homogeneous component $A(G)_\lambda$ of $A(G)$ is the G-submodule of $A(G)_i$ generated by $A(G)_i^\lambda$. We have $A(G)_i = A(G)_\lambda \oplus A(G)_{i-1}$, as (C, G)-modules. Hence $A(G)_\lambda$ is isomorphic to $|E_i| \otimes C(G)$ (by the above theorem) as asserted by [18], Theorem A.

4. Assume the hypotheses of 2.2 Theorem and let P be a parabolic subgroup of G with unipotent radical V. Then one obtains a filtration of A^V as a $(P/V, C)$-module, of the kind given in 2.2 Theorem, since the V-fixed point functor is exact on G-modules with a good filtration (see [8], 1.4 Proposition and § 2, Proposition). In particular, A^V is a free C-module.

5. As in [18], Theorem A, our result is not constructive. However, in the case $A=A(G)$, $G=\mathrm{SL}_2(k)$, we can give a more concrete description, as follows. We regard A as a $G\times G$-module in the usual way. By [7], 1.4 (16), (17), there is a uniquely determined $G\times G$-submodule $A(m)$ say $(m\geqq 0)$ with sections $Y(0)\otimes Y(0)$, $Y(1)\otimes Y(1), \ldots, Y(m)\otimes Y(m)$ (identifying weights with integers in the usual way). Now regard $A(m)$ as a G-module via the diagonal action. By [6], (10.8.5) Theorem, $A(m)$ has a good filtration and so by [6], (12.1.6) there is a uniquely determined G-submodule $A(m,n)$, for $n \leqq m$, such that $A(m,n)$ has a good filtration and $(A(m,n): Y(j))=0$ for $j>2n$ and $(A(m)/A(m,n): Y(j))=0$ for $j \leqq 2n$. Then $C(G)=\bigcup_{m\geqq 0} A(m,0)$ and it is possible to show that $A(m,n).A(r,s) = A(m+r, n+s)$ (for $m\geqq n$, $r \geqq s$). We put $A_0=0$ and $A_{i+1}=\bigcup_{m\geqq i} A(m,i)$, for $i \geqq 0$. One may deduce that $A_{i+1}/A_i \cong |Y(2i)\otimes C(G)$ $(i\geqq 0)$ and the transversal $Y(2i)$ may be realised in A_{i+1}/A_i as $(A(i)+A_i)/A_i$.

References

1. Andersen, H.H., Jantzen, J.C.: Cohomology of induced representations for algebraic groups. Math. Ann. 269, 487–525 (1984)
2. Borel, A.: Linear Algebraic Groups. New York: Benjamin 1969

3. Cartan, H., Eilenberg, S.: Homological Algebra. Oxford: University Press 1956
4. Cline, E., Parshall, B., Scott, L., Kallen, W. van der: Rational and generic cohomology. Invent. Math. **39**, 143–163 (1977)
5. Demazure, M.: Invariants symétriques entires des groupes de Weyl et torison. Invent. Math. **21**, 287–301 (1973)
6. Donkin, S.: Rational Representations of Algebraic Groups: Tensor Products and Filtrations (Lect. Notes Math., vol. 1140) Berlin Heidelberg New York: Springer 1985
7. Donkin, S.: Skew modules for reductive groups. J. Algebra (in press)
8. Donkin, S.: Invariants of unipotent radicals. Math. Z. (in press)
9. Donkin, S.: Good filtrations of rational modules for reductive groups, Representations of Finite Groups and Related Topics, Proc. Symp. Pure Math.: Am. Math. Soc. **47**, (1987) (in press)
10. Friedlander, E.: A canonical filtration for certain rational modules. Math. Z. **188**, 433–438 (1984/85)
11. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics 9, Berlin Heidelberg New York: Springer 1972
12. Humphreys, J.E.: Linear Algebraic Groups. Graduate Texts in Mathematics 21, Berlin Heidelberg New York: Springer 1975
13. Jantzen, J.C.: Representations of Algebraic Groups, Pure and Applied Mathematics, vol. 131, Academic Press 1987
14. Kac, V., Weisfeiler, B.: Coadjoint action of a semisimple algebraic group and the center of the enveloping algebra in characteristic *p*. Indagationes Math. **38**, 136–151 (1976)
15. Koppinen, M.: Good bimodule filtrations for coordinate rings. J. Lond. Math. Soc. (2) **30**, 244–250 (1984)
16. Kostant, B.: Lie group representations on polynomial rings. Am. J. Math. **85**, 327–404 (1963)
17. Quillen, D.: Projective modules over polynomial rings, Invent. Math. **36**, 167–171 (1976)
18. Richardson, R.W.: The conjugating representation of a semisimple group, Invent. Math. **54**, 229–245 (1979)
19. Springer, T., Steinberg, R.: Conjugacy Classes. In: Seminar on Algebraic Groups and Related Finite Groups (Lect. Notes Math., vol. 131, pp. 167–266) Berlin Heidelberg New York: Springer 1970
20. Steinberg, R.: Regular elements of semisimple groups. Publ. I.H.E.S. **25**, 49–80 (1965)
21. Suslin, A.A.: Projective modules over polynomial rings are free. Dokl. Nauk. S.S.S.R. 229 (1976) (=Soviet Math. Dokl.) **17**, 1160–1164 (1976)

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