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NILPOTENT ELEMENTS IN GROUP RINGS

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The main theorem gives necessary and sufficient conditions for the rational group algebra QG to be without (nonzero) nilpotent elements if G is a nilpotent or F•C group. For finite groups G , a characterisation of group rings RG over a commutative ring with the same property is given. As an application those nilpotent or F•C groups are characterised which have the group of units U(KG) solvable for certain fields K .

1. Introduction

Let RG be the group ring of a group G over a commutative ring R with identity. We investigate the conditions under which RG has no (nonzero) nilpotent elements. First we give a complete answer for finite groups G when R is the rational number field Q or a commutative ring of characteristic $n > 0$. Our proof depends on the following result of Claude Moser.

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THEOREM 1.1 [6]. *Let $Q_m = Q(\xi_m)$ be a cyclotomic field where ξ_m is a primitive m-th root of unity, m is odd and $m \geq 3$. Then the equation, $-1 = x^2 + y^2$, has a solution $x,y \in Q_m$ if and only if the multiplicative order of 2 modulo m is even.*

Next we describe (Theorem 4.8) nilpotent or F•C groups G such that QG has no nilpotent elements. The finite case of this theorem was announced by Berman [3] but he had a different proof in mind. It should be remarked that the question of absence of nilpotent elements in QG is related with the well known conjecture about the absence of zero divisors in QG if G is torsion free. As a matter of fact, if G is torsion free, QG has no nilpotent elements if and only if QG has no zero divisors [8].

We use our results to give a characterisation of these nilpotent or F•C groups G which have the property that the unit group, U(KG), is solvable when R = Q or $Z_p = Z/pZ$ if G has no element of order p. The case of finite groups has been settled by Bateman [2] and related results are given in [5]. We should mention that a complete characterisation of groups G such that U(KG) is solvable is dependent on the resolution of the well known conjecture that every unit of KG is trivial if G is torsion free.

We are indebted to I.B.S. Passi for giving us the reference [7].

2. Elementary facts and notations

2.1. Every idempotent e of a ring S without nilpotent elements is central.

Proof. Since $(eS(1-e))^2 = ((1-e)Se)^2 = 0$, we have

$$es = ese = se \quad \text{for all} \quad s \in S .$$

2.2. $$(\sum^{\oplus} R_i)(G) = \sum^{\oplus} R_i G .$$

2.3. $$R(G_1 \times G_2) = (RG_1)G_2 \approx RG_1 \otimes RG_2 .$$

2.4. If a commutative ring R has characteristic $n > 0$ and RG has no nilpotent elements then $n = p_1 p_2 \cdots p_k$, a product of distinct primes. We can write $R = \sum^{\oplus} R_i$ where R_i has characteristic p_i. Then RG has no nilpotent elements if and only if each $R_i G$ has no nilpotent elements.

2.5. Let G be a finite group. Suppose that $\gamma = \sum \gamma(g) g \in RG$ is nilpotent. Then $(G:1)\gamma(1)$ is a nilpotent element of R.

Proof. Let P be a prime ideal of R. Look at γ in $(R/P)G$. Considering the regular representation matrix of γ and taking trace we get that

$$(G:1)\gamma(1) = 0 \pmod{P}$$

and hence $(G:1)\gamma(1) \in \bigcap_P P$, which is nilpotent.

2.6. Let K be a field. If KG has no nilpotent elements then every finite subgroup of G is normal in G and hence the torsion elements form a subgroup $T(G)$ of G.

<u>Proof</u>. If $\mathrm{Char}\, K = p > 0$, then G has no element of order p as $(g-1)^p = 0$. Let H be a finite subgroup of G, then

$$\Big(\frac{1}{(H:1)} \sum_{h\in H} h\Big)^2 = \frac{1}{(H:1)} \sum_{h\in H} h$$

is a central idempotent by (2.1) and hence H is normal in G. As a corollary we have,

2.7. Let K be a field and G be finite. If KG has no nilpotent elements then G is abelian or Hamiltonian. We recall,

2.8. A group G is Hamiltonian if and only if

$$G = A \times E \times K_8$$

where A is an abelian group in which every element has odd order, E is an elementary abelian 2-group and K_8 is the quaternion group of order 8.

By a crossed product $K(G,\rho_{g,h},\alpha_g)$ we understand the set of finite sums, $\{\sum k_i\bar{g}_i \mid k_i \in K,\, g_i \in G\}$ where $\bar{g}_i$ is a symbol corresponding to g_i and $\rho\colon G \times G \to \dot{K}$, where $\dot{K}$ is the multiplicative group of the division ring K, is a factor system and α_g is an automorphism of K for each $g \in G$. Equality and addition are defined component wise. And, for $g,h \in G$, $k \in \dot{K}$

$$\bar{g}\cdot\bar{h} = \rho_{g,h}\,\overline{gh}$$

$$gk = k^{\alpha_g}g$$

where ρ and α are required to satisfy the necessary conditions for $K(G,\rho_{g,h},\alpha_g)$ to be a ring. For details we refer you to [4].

As a special case, if we have $\alpha_g = I$ for all $g \in G$, we call $K(G,\rho_{g,h},I) = K^t(G)$ the twisted group ring (See [9]). If $\rho_{g,h} = 1$ for all $g,h \in G$, we call $K(G,1,\alpha_g) = K_\alpha(G)$ the skew group ring. The following is easy to prove

2.9. If G is ordered and K is a division ring then the crossed product $K(G,\rho,\alpha)$ has no nilpotent elements.

For a normal subgroup N of G we shall denote by $\Delta_R(G,N)$ the kernel of the natural homomorphism $RG \to R(G/N)$. We shall write Char R for the characteristic of R.

3. Finite groups.

We first consider the case when R has characteristic $n > 0$.

PROPOSITION 3.1. *The group algebra Z_pG of a finite group G over the field Z_p of p-elements is without nilpotent elements if and only if G is abelian and p does not divide $(G:1)$.*

Proof. Suppose that Z_pG has no nilpotent elements. Since $g^p = 1$, $g \in G$ implies $(g-1) = 0$ in Z_pG, G has no elements of order p. Thus by Wedderburn theorem

$$Z_pG \simeq \sum^{\oplus}(D_i)_{n_i} ,$$

a direct sum of full matrix rings over division rings D_i. Since each D_i is a finite dimensional algebra over Z_p, it is cummutative. Also each $n_i = 1$, due to (2.1). Hence Z_pG is commutative. The converse also follows from the Wedderburn theorem.

PROPOSITION 3.2. Let G be finite and let R have characteristic $n > 0$. Then RG has no nilpotent elements if and only if R has no nilpotent elements, $(G:1)$ is not a zero divisor in R and G is abelian.

Proof. We first prove the necessity of the conditions. Suppose that $(G:1)r = 0$ for some $r \in R$. Then $(r \sum_{g\in G} g)^2 = 0$ and therefore $r = 0$. Obviously since $R \supset Z_n$, the ring of integers modulo n, it follows by (2.4) and Proposition 3.1 that G is abelian.

In order to prove the converse, suppose $0 \neq \gamma = \sum_g \gamma(g)g \in RG$ is nilpotent. We may suppose, by considering γg^{-1} if necessary for some g, that $\gamma(1) \neq 0$. It follows by (2.5) that $(G:1)\gamma(1)$ is nilpotent and therefore by hypothesis $(G:1)\gamma(1) = 0$. Hence $\gamma(1) = 0$ which is a contradiction, proving the proposition.

COROLLARY 3.3. *Let G be an arbitrary abelian group. Suppose R has characteristic $n > 0$. Then, RG has no nilpotent elements $\iff$ R has no nilpotent elements and $O(g)$ is not a zero divisor for any $g \in T(G)$, the torsion subgroup of G.*

Proof. We only need to prove the sufficiency of the conditions. We may suppose that G is finitely generated and so

$$G = T \times \langle x_1 \rangle \times \dots \times \langle x_m \rangle, \quad (T:1) < \infty, \quad O(x_i) = \infty,$$

and $RG = (RT)\langle x_1, \dots, x_m \rangle$ has no nilpotent elements as RT has none.

Let $K_8 = \langle i,j / i^2 = j^2 = t,\ t^2 = 1,\ ji = ijt \rangle$ be the quaternion group of order 8. We need to know when $R(K_8)$ can have nilpotent elements.

PROPOSITION 3.4. *Let R be a commutative ring which has no nilpotent elements. Suppose 2 is not a zero divisor. Then $R(K_8)$ has no nilpotent elements if and only if the equation*

$$x^2 + y^2 + z^2 = 0, \quad x,y,z \in R$$

has no nonzero solution.

Proof. Let $\gamma = \sum \gamma(g)g \in R(K_8)$ be such that $\gamma^2 = 0$. Since $G/\langle t \rangle$ is abelian, by the last result, we have $\gamma \in \Delta_R(K_8, \langle t \rangle)$. Also by 2.5 we can conclude that $\gamma(1) = 0$. Thus γ is of the form

$$\gamma = (ai+bj+cij)(1-t), \quad a,b,c \in R .$$

Then, $\gamma^2 = 2(a^2+b^2+c^2)(t-1) = 0$ if and only if $a^2 + b^2 + c^2 = 0$, completing the proof.

<u>THEOREM 3.5.</u> <u>The rational group algebra</u> QG <u>of a finite group</u> G <u>has no nilpotent elements if and only if one of the following is satisfied</u>

3.6. G is abelian

3.7. G is Hamiltonian of order $2^m t$, t odd, such that the multiplicative order of 2 modulo t is odd.

<u>Proof</u>. Suppose that G is a non abelian group without nilpotent elements in QG . Then by (2.7)

$$G = A \times E \times K_8$$

$$\begin{aligned} QG &= (QE)(A\times K_8) = (\overset{\oplus}{\sum} Q)(A\times K_8) \\ &= \overset{\oplus}{\sum} Q(A\times K_8) = \overset{\oplus}{\sum}(QA)K_8 . \end{aligned}$$

It is enough to consider $(QA)K_8$. But we know [1] that

$$QA = \overset{\oplus}{\sum_{d|e}} n_d Q_d ,$$

where $n_d Q_d$ denotes n_d copies of $Q(\xi_d)$ and e is the exponent of A .

$$(QA)(K_8) = \sum_{d|e} n_d Q_d (K_8) .$$

Now (3.7) follows by Proposition 3.4 and Theorem 1.1. The same argument together with Theorem 1.1 and Proposition 3.4 gives the converse.

4. Nilpotent or F•C groups.

In this section unless otherwise stated G will be assumed to be nilpotent or an F•C group. We have

PROPOSITION 4.1. *Let G be a nilpotent or F•C group. Suppose K is a prime field of Char $K = p \geq 0$. Then KG has no nilpotent elements if and only*

4.1. $T(G)$ has no element of order p (if $p > 0$)

4.2. Every idempotent of KG is central in KG and one of the following holds

4.3. $T(G)$ is abelian

4.4. $T(G)$ is Hamiltonian, $K = Q$ and modulo every odd n such that $T(G)$ has an element of order n, the multiplicative order of 2 is odd.

Proof (a). Let us suppose that KG has no nilpotent elements. Then (4.1) is obvious and (4.2) follows from (2.1). And if $T(G)$ is not abelian then it is Hamiltonian by (2.6). Now, (4.4) follows by (3.1) and (3.5).

(b). We assume (4.1), (4.2) and (4.3) or (4.4). Let $\gamma \in KG$ be an element such that $\gamma^2 = 0$. We may suppose that G is finitely generated. Then $T(G)$ is finite abelian or Hamiltonian satisfying (4.4). Consequently,

$$K(T(G)) = \sum_i^{\oplus} D_i$$

a direct sum of division rings. Since $G/T(G)$ is torsion free nilpotent (abelian if G is $F\cdot C$ [10]), it is ordered. We have

$$KG \simeq K(T(G))(G/T(G),\rho,\alpha)$$

a crossed product of $G/T(G)$ over $KT(G)$. Due to (4.2) we have

$$KG \simeq \sum^{\oplus} D_i(G/T(G),\rho_i,\alpha_i)$$

which is free of nilpotent elements by (2.9).

PROPOSITION 4.5. *Let G be a nilpotent or $F\cdot C$ group. Suppose K is a field of characteristic $p \geq 0$, $p \neq 2,3$. If G has no element of order p (if $p > 0$), then*

(4.2) and (4.3) $\Longleftrightarrow$ $U(KG)$ is solvable.

Proof. Assume (4.2) and (4.3). Suppose H is finitely generated subgroup of G. Then $T(H)$ is finite and

$$KT(H) = \sum_i^{\oplus} F_i$$

a direct sum of fields and due to (4.2)

$$KH = \sum^{\oplus} F_i(H/T(H),\rho_i,\alpha_i)$$

$$U(KH) = \prod^{\otimes} F_i H/T(H) \quad \text{(as } H/T(H) \text{ is ordered)}$$

which is solvable of degree $\leq m+1$ where G is solvable of degree m. Conversely, assume that $U(KG)$ is solvable then (4.2) follows from lemma 5 of Lanski [6] and commutativity of $T(G)$ follows from the finite case using the fact that a full matrix ring over a division ring has a solvable unit group if and only if it is a field or a 2×2 matrix ring over a field of 2 or 3 elements.

We investigate the absence of nilpotent elements further and replace condition (4.2) by a group theoretical condition.

In QG it follows from (4.2) that every finite subgroup is normal in G. And $T(G)$ is abelian or

$$T(G) = A \times E \times K_8$$

where A is an abelian group in which every element has odd order, E is elementary abelian 2-group and K_8 is the quaternion group of order 8. The formulae in [1] indicate that if every finite subgroup of an abelian group B is normal in G then all idempotents of QB are central in QG.

4.6. Let B be a finite abelian subgroup of G. Then every subgroup of B is normal in G if and only if every $x \in G$, induces the automorphism

$$B \ni b \to xbx^{-1} = b^i \quad \text{where} \quad i = i(x) .$$

We claim now

4.7. Every subgroup of K_8 is normal in G if and only if conjugation by $x \in G$ induces the identity automorphism on the group $K_8/\langle t\rangle \simeq Z_2 \oplus Z_2$.

Proof. Considering the subgroups $\{1,i,t,it\}$, $\{1,j,t,jt\}$ and $\{1,ij,t,ijt\}$ it is clear that the only automorphisms keeping these groups invariant are

0) $\theta_0: i \to i, \quad j \to j, \quad ij \to ij$

1) $\theta_1: i \to i, \quad j \to jt, \quad ij \to ijt$

2) $\theta_2: i \to it, \quad j \to jt, \quad ij \to ij$

3) $\theta_3: i \to it, \quad j \to j, \quad ij \to ijt$.

THEOREM 4.8. *Let G be a nilpotent or $F \cdot C$ group. Then QG has no nilpotent elements if and only if one of the following conditions is satisfied*

4.9. $T(G)$ is an abelian group, and for $x \in G$, we have locally on $T(G)$

$$xtx^{-1} = t^i \quad \text{for all} \quad t \in T(G), \quad i = i(x)$$

i.e. for a finite subgroup B of $T(G)$ we have $xyx^{-1} = y^i$ for all $y \in B$ and i depends on B and x .

4.10. $T(G) = A \times E \times K_8$, where A is an abelian group in which every element has odd order, E is an elementary abelian 2-group and K_8 is the quaternion group of eight elements such that (4.4) holds. Moreover,

1) K_8 is normal in G

2) conjugation by $x \in G$ induces the identity on $K_8/\langle t\rangle$ and

3) conjugation by $x \in G$ acts as in (4.9) on $A \times E$.

Proof. Because of (4.2), (4.6) and (4.7) we have only to prove the sufficiency of (4.9) and (4.10) separately. We may suppose that G is finitely generated. Then (4.9) implies that every subgroup of $T(G) = T$ is normal in G and hence by the formulae in [1], every idempotent of QT is central in QG . We have

$$QT = \overset{\oplus}{\sum} F_i ,$$

a direct sum of fields; and

$$\begin{aligned} QG &= (QT)(G/T,\rho,\alpha) \\ &= (\overset{\oplus}{\sum} F_i)(G/T,\rho,\alpha) \\ &= \overset{\oplus}{\sum} F_i(G/T,\rho_i,\alpha_i) \end{aligned}$$

which is by (2.9) free of nilpotent elements as G/T is ordered.

In order to prove the sufficiency of (4.10) we write $A_1 = A \times E$ and $T(G) = T = A_1 \times K_8$. Then

$$QA_1 = \overset{\oplus}{\sum_i} F_i, \quad \text{a direct sum of fields .}$$

Also every idempotent of QA_1 is central in QG due to (3) of (4.10). It is known that

$$Q(K_8) \simeq Q \oplus Q \oplus Q \oplus Q \oplus S ,$$

where S is the skew field of rational quaternions. If we pick an $x \in G$, $x \notin T$ then it can be checked directly that $Q\langle K_8, x\rangle$ has no nilpotent elements by using the fact that x induces one of the automorphisms $\theta_0 = I, \theta_1, \theta_2, \theta_3$ on K_8. Thus for example if we have a $\gamma \in Q\langle K_8, x\rangle$ such that $\gamma^2 = 0$. Then since $H = \langle K_8, x\rangle \supset K_8 \supset \langle t\rangle$ and $T(H/\langle t\rangle)$ is abelian, $\gamma \in \Delta(H, \langle t\rangle)$. Thus γ is of the form

$$\gamma = x^n(a_0+a_1i+a_2j+a_3ij)(1-t) + \dots ,$$

where x^n is the highest degree nonzero term in x, appearing in the support of γ. Let us suppose, for instance, that x^n acts like θ_1 on K_8. Then

$$\begin{aligned}\gamma^2 &= 2x^{2n}(a_0+a_1i+a_2jt+a_3ijt)(a_0+c_1i+a_2j+a_3i)(1-t) + \dots \\ &= 2x^{2n}(a_0^2+a_2^2+a_3^2+a_1^2t+2a_1a_0i+2a_1a_2j+a_2a_3ij+a_1a_3jt)(1-t) \\ &\quad + \dots \\ &= 0 .\end{aligned}$$

It is easy to see that $\gamma = 0$. It works the same way if x^n acts like θ_2 or θ_3.

We can conclude by (2.1) that every idempotent of $Q(K_8)$ commutes with x i.e. every idempotent of $Q(K_8)$ is central in QG, in particular $xSx^{-1} = S$ for all $x \in G$. Now we have

$$Q(A_1 \times K_8) = \sum_i F_i \otimes Q + \sum F_i \otimes Q + \sum F_i \otimes Q + \sum F_i \otimes Q + \sum F_i \otimes S$$

and each $F_i \otimes S$ being a simple ring, is a division ring S_i due to 4.4 and Theorem 3.5. Also $x^{-1}S_i x = S_i$. Thus we can write

$$Q(A \times K_8) = \sum_i^{\oplus} D_i ,$$

a direct sum of division rings with the corresponding idempotents central in QG. Now,

$$\begin{aligned} QG &\simeq (QT)(G/T,\rho,\alpha) \\ &\simeq (\sum_i^{\oplus} D_i)(G\ T,\rho,\alpha) \\ &\simeq \sum^{\oplus} D_i(G/T,\rho_i,\alpha_i) \end{aligned}$$

has no nilpotent elements due to (2.9). This completes the proof.

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