A. M. Vershik

THE ADIC REALIZATIONS OF THE ERGODIC ACTIONS WITH THE HOMEOMORPHISMS OF THE MARKOV COMPACT AND THE ORDERED BRATTELI DIAGRAMS

Abstract. For any ergodic transformation T of the Lebesgue space (X, μ) it is possible to introduce the topology τ into X such that

a) with provided topology X becomes the totally disconnected compact (Cantor set) with the structure of a markov compact and μ becomes a borel markov measure.
b) T becomes a minimal strictly ergodic homeomorphism of (X, τ);
c) orbit partition of T is the tail partition of the markov compact upto two classes of the partition.

The structure of markov compact is the same as a structure of the pathes in the Bratteli diagram of some AF-algebra. Bibliography–19.

1. The problem

Suppose α is an ergodic action of a countable group G on the Lebesgue space (X, μ). There is a number of the traditional realizations of such an action; the most well known is the symbolic (probabilistic) realization as the group of the shifts on the space of the function on the group G with an invariant measure.

Completely different realization (for the group Z) is defined in the papers of the author (see [1, 2, 3, 4]) – s.c. *adic transformation.* This realization is similar to the construction of the special set of the consecutive Rokhlin towers – see [1]; or in another words, this is a special transformation in the space of the infinite paths of the apropriate Bratteli diagram. The last space is a markov compactum (nonstationary) in the sense [1, 2] and the transformation is some generalization of the odometer. The word "realization" in this context means simply a metric isomorphism between the initial space and the markov compactum provided with a central measure; the adic transformation is defined almost everywhere and therefore is not a homeomorphism of the markov compact in general. In the series of the papers [5, 6, 7, 8] the homeomorphism of adic type on the markov compactum was studied. In particulary the application of K-theory had been

Partially supported Kapitza Fund (UK, *Warwick*) and RFFI (Russia) – grant No. 94-01-00921.

considered. We suggested the term "*Ergodic K-theory*" as a theory of the invariants of dynamical systems like K-functor.

In this paper we formulate much stronger realization theorem which gives a *homeomorphic* representation of an arbitrary ergodic transformation as the transformations of the adic type of a markov compact. The main point is the use of the techniques of the theory of decreasing sequences of the measurable partitions which had been developed 25 years ago (see [9, 10, 11, 13]) for another reason but which is extremelly helpful in this case. This gives us a new and independent proof of the previous theorem about adic realization as well as the strengthening of Jewitt–Krieger theorem about strictly ergodic and minimal realization of the ergodic transformation and Dye's theorem about orbit partitions. The orbit partition in our realization for group Z is a tail partition of the markov compact up to two orbits (strongly orbit equivalence in the sense of [6]) – theorem 2. For locally finite group the orbit partition and the tail partition coincide (theorem 1). We can get a diadic compact as markov for our goal but in this case we can guarantee the coincidence of the partitions only on the subset of the measure one (theorem 3).
Our techniques can be applied to the similar problems.

2. The formulation of the results

We will start with locally finite groups because the formulation is slightly simpler for this case but the proof is almost the same. It is not important what locally finite group will be chosen, so for the definiteness we are dealing with the group $G = \sum_{i=1}^{\infty}(Z/2)$ – infinite sum of $Z/2$.

Theorem 1. *Suppose that α is an ergodic action of the group $G = \sum_{i=1}^{\infty}(Z/2)$ on a Lebesgue space (X, μ), and let $Y = \prod_{i=1}^{\infty}(Z/2)$ be Cantor compactum – infinite product of the groups $Z/2$ provided with Haar (Bernoulli) measure m.*
There exists a measure preserving isomorphism $S : X \to Y$ between (X, μ) and (Y, m) such that
a) *for any $g \in G$ $\beta(g) \equiv S\alpha(g)S^{-1}$ is a homeomorphism of Y which preserves the measure m;*
b) *the orbit partition of the action β is the tail partition in Y (= two sequences $u = (u_n)$ and $v = (v_n)$ from Y are in the same element of the tail partition if $u_n = v_n$ for all $n > n(u, v)$).*

Corollary. 1. *Any ergodic action of the group G has a strictly ergodic, minimal realization;*

2. *Any two ergodic actions of the group G have an metrically isomorphic orbit partition.*

Indeed, the action β has both properties claimed in the first corollary and the theorem asserts that the orbit partitions is isomorphic to the standard one – the tail partition in (Y, m).

The first assertion is the generalization of Jewitt–Krieger theorem for locally finite group, the second one is just Dye's theorem.

Now we pass to the action of the group Z or of one automorphism. It is easy to prove that homeomorphism of an arbitrary markov compact can not have as the orbit partition the tail partition of the markov compact. So we have to change (glue) some blocks of the tail partition. The second theorem asserts that it can be done with a minimal modification of tail partition.

We will retain the definition of the Markov compact, tail partition and adic transformation especially because there are close to the theory of AF-algebras and prefer to use this terminology.

Let F – is locally finite infinite $Z_{(+)}$ – graded graph (Bratteli diagram) with the finite levels $F_n \in Z_{(+)}$ with one root $\varnothing \in F_0$ without end points, Vert $F = \cup F_n$, $0 \leqslant n < \infty$. Let $Y(F)$ is the space of all connected (infinite) paths in F started from $\varnothing$. The space $Y(F)$ is Markov compacts in a ordinary sense (nonstationary) – see Ref. [1, 2]. The *tail* partition is the partition in $Y(F)$ any element of which is the class of the paths which have the same "tail" – e.g. the paths are coincide from some level.

Now we will define the order in the space $Y(F)$. Suppose for any vertex $f \in F \subset$ Vert F of the level n the linear order $\vartheta(f)$ on the set F_{n-1} of the vertices of the level $n-1$ is defined. Because any path is a sequence of the vertices (we exclude the multiplicity of the edges for simplicity – in the case of multigraph instead the order on vertices we have to define the order on the incoming edges of the vertices) we can introduce the lexicographic order "from the end" on the space of paths $Y(F)$. It means that two paths are compatible with respect to this order if they belong to the same tail class and inside the class the order is lixicographic one with respect to the orders of $\vartheta(\cdot)$ on the each levels. This data define the *ordered* Bratteli diagram.

We can define this notion in terms of AF-algebras: this is the AF-algebra with the fixed Borel subalgebras in each simple subalgebra (which correspond to the vertices of F). We will not detail on this.

An *uniformly ordered* diagram is the ordered diagram in which the order on any given level F_{n-1} does not depend on the vertices of F_n, $n = 1, 2, \ldots$. An ordered diagram called *cyclic* if for any n and $f \in F_n$ the initial and final elements in F_{n-1} with respect to the orders $\varnothing(f)$ do not depend of the vertices f. The same name we will preserve for the corresponding ordered markov compact. The name "cyclic" will be explained in below.

The *adic* transformation on the space $Y(F)$ is a partial transformation of $Y(F)$ which sends the path to the next path in the sense of the above defined lexeographic order. The transformation is not defined for the paths which have no follower. Let us call this paths as *maximal (blind).* If the path has no predecessor we will call it *minimal (deaf)* . For the cyclic ordered diagram there are only one maximal and only one minimal path. Let us supplement the definition of the adic transformation for this case in the following way: the image of the (unique) maximal path is the (unique) minimal path. This definition turned adic transformation into the well-defined homeomorphism of the cyclic ordered Markov compact. We will call this homeomorphism as the *modified adic transformation.*

The orbit partition of the modified adic transformation in the cyclic Markov compact slightly differs from tail partition: two classes of the tail partition – the classes of the maximal and minimal paths join into one class of the orbit partition. On the complement to these classes the tail and the orbit partition are coincides.

Theorem 2. *For any ergodic automorphism T of a Lebesgue space (X, μ) there exists the cyclic ordered markov compact $Y(F)$, in which the tail partition is minimal and strictly ergodic with measure m, and the isomorphism $S : X \to Y(F)$ which sends measure μ to measure m such that STS^{-1} is the modified adic transformation. Consequently, we have the realization of T as strictly ergodic, minimal homeomorphism whose orbit partition is the tail partition of $Y(F)$ upto two classes.*

This theorem is also the generalization of the three theorems: Dye's theorem, Jewitt–Krieger theorem, and the theorem about adic realization. The proof is similar to the proof of the theorem 1 and uses the ideas of lacunary isomorphism. We can vary the theorem 2, f.e.

Theorem 3. *For any ergodic automorphism T of Lebesgue space (X, μ) there exists the measure preserving isomorphism $S : X \to Y$ where $Y = \prod_{i=1}^{\infty}(Z/2)$, and STS^{-1} is a homeomorphism of Y which has the orbit partition which is differ from the tail partition of Y on the set of the measure zero.*

In this theorem the compact is the same for all T in opposition to the theorem 2 but we can not say how big is a difference between the tail and orbit partitions – perhaps it much more bigger than in the theorem 2 where the difference was the smallest possible.

3. The main ideas of the proofs

We will give here only a short draft of the proof. The first step.

Lemma 1. ([9, 12]) *For any ergodic automorphism T its orbit partition $\tau(T)$ is a tame partition: it is the intersection of the decreasing sequence of the measurable partitions $\tau(T) = \cap\xi_n$ and each of ξ_n has the finite elements. Moreover, we can choose this sequence $\{\xi_n\}$ as ergodic diadic one: it means $\{\xi_n\}$ is a decreasing sequence with the trivial measurable intersection $\Lambda\xi_n = \nu$ and ξ_n is partition with the elements with the 2^n points of the equal measure there are no reasons to use this step for the locally finite group, of course.*

The second step.

Lemma 2. (*Lacunary isomorphism* [9, 13]). *Any decreasing diadic sequence of the measurable partition has a standard subsequence which can be realized as tail sequence in the compact $X = \prod_{i=1}^{\infty} X_n \quad \#X_n < \infty$ with Bernoulli measure.*

The final step which concludes the proof of the theorems 1–3 consists in the *special change of the topology* in order to realize the action of the group as a homeomorphic one (or in the case of the group Z – the automorphism as a homeomorphism). The corresponding construction is the sharpening of the general method which was used for lacunary isomorphism.

4. The examples

Any Bratteli diagram (and so any locally semisimple algebra with the given structure of the inductive limit) can be ordered and consequently gives the examples of adic transformations. Of course the various orders on the same algebra can give nonisomorphic adic transformations. The new idea is to use this link in the opposite direction: from AF to dynamical systems. We obtain new examples of the automorphism and new realizations of th famous one.

1. Pascal automorphism. Consider infinite Pascal triangle and define the natural order on the paths. The compact of the paths is not cyclic: we have countable many of the maximal and the minimal paths. The corresponding adic transformation – *Pascal* automorphism can be realized also in the compact $\Pi(Z/2)$ as following:

$$T(1^n 0^m 1abc\ldots) = 0^{m-1}1^{n+1}0abc\ldots.$$
$$n \geqslant 0,\quad 1 \leqslant m \leqslant \infty,\quad a,b,c\ldots = 0,1.$$
$$T(1^\infty) = 0^\infty,\quad T(1^n 0^\infty) = 0^\infty,\quad n \geqslant 0.$$

The transformation T is not a homeomorphism but with respect to Bernoulli measure it is a well defined automorphism of the measure space. It is not known whether this automorphism is a weak mixing with respect to Haar measure. (See [14]). It is also important to consider many dimensional Pascal graph in the same spirit.

2. Young automorphism. Consider the Young graph ([15, 16]) and the natural order on the Young diagrams – we have again a countable number of maximal and minimal paths. We obtain a *Young* automorphism which acts on the space of infinite Young tableau (=paths) with the central measures. Nothing is known about the properties of Young automorphism – it is important in the theory of representations of the infinite symmetric group.

3. Dynkin automorphisms. Consider the Dynkin diagrams of series A_n and corresponding AF-algebras (see [17, 4]). It gives us the family of the stationary adic transformations. It is the examples of substitutions in the sense of Morse–Hedlund. The connections with the theory of substitutions was discussed in [4, 18].

4. Bernoulli-like monster. Now we will give an example on the theorem 2, – the precise realization of an automorphism with the positive entropy as modified adic transformation in the cyclic minimal strictly ergodic Markov compact. The definition makes sense for a very general situation – for the general ordered Bratteli diagrams. This kind of the example was proposed by author in 80-th; see also [7] and [19].

Suppose the finite set $F = \{1, 2, \dots, r\}$, $r > 4$ which provided with the natural order is the first level of the diagram $-F = F_1$. Suppose we define the levels $F_0 = \varnothing$, $F_1 = F$, $F_2, \dots, F_{n-1}$ and the level F_{n-1} has the linear order with the first element f' and the last element f''.

Now we define the level F_n and the orders $\vartheta(f)$ on the F_{n-1} for all $f \in F_n$. The elements $f \in F_n$ are all the possible *orders* the F_{n-1} with the first element f' and the last element $-f''$. We can say the element of F_n is just a permutation of $F_{n-1} \setminus \{f', f''\}$. It means that for the number of the elements of F_n the following formula is true:

$$\#(F_n) = (\#(F_{n-1} - 2))!$$

(a terrible growth!).

Let us define the orders $\vartheta(f)$ on the F_{n-1} for each elements $f \in F_n$: but these order is included to the definition of F_n, because any $f \in F_n$ is by definition the order on the set F_{n-1} with the fixed first and last elements $-f', f''$. It remains to define the last and the first element on F_n, but the order on the F_{n-1}, so all the permutations can be arranged in the lexicographic order – this gives the order on the set F_n.

We have defined by induction the ordered Bratteli diagram which is cyclic in the sense of our definition – the unique maximal (blind) element is the path which consists of the last elements of all the levels, the unique minimal (deaf) path is the path which consists of the first elements on each level. Now we have the cyclic markov compact, modified adic transformation. It is not difficult to proof that the automorphism has positive entropy and even is a skew product over rational spectra and Bernoulli automorphisms as a leaves. It is not difficult to change the construction and to obtain the realization of the Bernoulli automorphism itself in similar form.

In a sense this is the universal construction of the ordered Bratteli diagrams – instead of the set all permutations on the F_{n-1} we can get as the set F_n some subset of it. (cf. [10]).

This paper was started when the author spent two months in Warwick University as Kapitza Fellow. Author express his gratitude to the Mathematical Department of Warwick University and to the Royal Society. Author express his gratitude to Prof. C. Skau, and T. Giordano for fruitful discussions.

References

1. A. M. Vershik, *Sov. Dokl.* **259** No. 3, 526–529 (1981).
2. A. M. Vershik, *J. Sov. Math.*, **28**, 667–673 (1985).
3. A. M. Vershik, A. N. Livshic, *Adv. Sov. Math.* **9** (1992).
4. A. M. Vershik, *St. Petersburg Math. J.*, **6** No. 3 (1994).
5. R. Herman, I. Putnam, and K. Skau, *Int. Math. J.*, **3** No. 6, 882–864 (1992).
6. T. Giordano, I. Putnam, and K. Skau, Preprint No. 11, Trondheim (1993).
7. M. Boyle and D. Handelman, *Pacif. Math. J.*, **164** No. 1, 1–13 (1994).
8. M. Boyle and D. Handelman, Preprint (1994).
9. A. M. Vershik, *Funct. Anal.*, **2** No. 3, 17–21 (1968).
10. A. M. Vershik, *Funct. Anal.*, **7** No. 3, 1–17 (1973).
11. A. M. Vershik, *Sov. Dokl.*, **193** No. 4, 748–751 (1970).
12. R. M. Belinskaya, *Funct. Anal.*, **2** No. 3, 4–20 (1968).
13. A. M. Vershik, *St. Petersburg Math. Journ.*, **6** No. 4, 1–68 (1994).
14. K. Petersen, Preprint Warwick Univ. 1994.
15. A. M. Vershik and S. V. Kerov, *J. Sov. Math.* **38**, 1701–1733 (1987).
16. A. M. Vershik, *Selecta Math. Sov.*, **12** (1992).
17. F. Goodman, P. De la Harpe, V. Jones, *Coxeter Graphs and Towers of Algebras* Publ. MSRI 62, Springer 1989.
18. A. N. Livshic, *St. Petersburg Math. J.*, **3** No. 1, 165–176 (1991).
19. C. Grllinberger, *Zeit. Wahr.*, **25**, 323–334 (1973).

С.-Петербургское отделение
Математического института
им. В. А. Стеклова РАН

Поступило 1 марта 1995 г.