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PURE AND APPLIED MATHEMATICS.

CONGRUENCES RELATING TO THE SUMS OF PRODUCTS OF THE FIRST *n* NUMBERS AND TO OTHER SUMS OF PRODUCTS.

By J. W. L. GLAISHER.

Theorems of Wilson, Lagrange, Wolstenholme, Ferrers.

§1. In Vol. v.* of the *Quarterly Journal* Wolstenholme has shown that, if n be any prime greater than 3,

(i) the numerator of the harmonic progression

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}$$

is divisible by n^2;

(ii) the numerator of

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}$$

is divisible by n;

(iii) the number of combinations of $2n-1$ things, taken $n-1$ together, diminished by 1, is divisible by n^3.

In Vol. XXII.† of the *Messenger of Mathematics* Mr. G. Osborn has given a simple proof of properties (i) and (ii); and in Vol XXIII.‡ of the same journal Dr. Ferrers has shown that

* "On certain properties of prime numbers," pp. 35–39.

† "Note on the numerator of a harmonical progression," pp. 51, 52.

‡ "Two theorems on prime numbers," pp. 56–58.

(iv) if n be an uneven prime, the sum of the products of the numbers $1, 2, 3, \ldots, n-1$, taken r together, r being less than $n-1$, is divisible by n;

(v) if $2n+1$ be a prime, the sum of the products of the squares of the numbers $1, 2, \ldots, n$, taken r together, r being less than $n-1$, is divisible by $2n+1$.

§ 2. In the first portion of the present paper I give proofs of these theorems. It will be seen that they are all closely connected with an investigation of Lagrange's, from which, however, he did not himself actually deduce any of them except (iv), and even this result he did not enunciate explicitly. The remainder of the paper contains investigations which lead to more general theorems of the same class, as well as to other formulæ and congruences relating to the divisibility of the sums of products of n quantities taken r together.

§ 3. Let

$$(x+1)(x+2)\ldots(x+n-1) = x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + \ldots + A_{n-2} x + A_{n-1},$$

so that A_r denotes the sum of the products of the numbers $1, 2, \ldots, n-1$ taken r together.

Putting $x+1$ for x, and multiplying by $x+1$, we have

$$(x+1)^n + A_1(x+1)^{n-1} + A_2(x+1)^{n-2} + \ldots + A_{n-2}(x+1)^2 + A_{n-1}(x+1)$$
$$= (x+n)(x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + \ldots + A_{n-2} x + A_{n-1}),$$

whence, using $(n)_r$ to denote the number of combinations of n things r together, we find by equating coefficients

$$(n)_2 = A_1,$$
$$(n)_3 + (n-1)_2 A_1 = 2A_2,$$
$$(n)_4 + (n-1)_3 A_1 + (n-2)_2 A_2 = 3A_3,$$
$$(n)_5 + (n-1)_4 A_1 + (n-2)_3 A_2 + (n-3)_2 A_3 = 4A_4,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(n)_{n-1} + (n-1)_{n-2} A_1 + (n-2)_{n-3} A_2 + \ldots + (3)_2 A_{n-3} = (n-2) A_{n-2},$$
$$1 + A_1 + A_2 + A_3 + \ldots + A_{n-2} = (n-1) A_{n-1}.$$

If n be an uneven prime, the first equation shows that A_1 is divisible by n, then the second shows that A_2 is divisible by n, the third that A_3 is divisible by n, and so on up to A_{n-2}, the last equation showing that $A_{n-1}+1$ is divisible by n, which is Wilson's theorem.

The fact that $A_1, A_2, \ldots, A_{n-2}$ are all divisible by n when n is an uneven prime constitutes the theorem (iv), which has thus been proved.

§ 4. In the original equation let $x=-n$, n being still supposed to be an uneven prime. Then, since the left-hand side $=A_{n-1}$, the equation becomes, after division by n,

$$0=n^{n-2}-A_1 n^{n-3}+A_2 n^{n-4}-\ldots+A_{n-3}n-A_{n-2}.$$

If $n>3$, A_{n-3} is divisible by n, so that all the terms except the last have n^2 as a factor, and therefore the last, A_{n-2}, must be divisible by n^2. This is the theorem (i).

§ 5. Since the numerator of

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}$$

is equal to $A^2_{n-2}-2A_{n-3}A_{n-1}$; and, since, if n be a prime greater than 3, A_{n-2} and A_{n-3} are divisible by n, it follows that this numerator is divisible by n, which is the theorem (ii).

§ 6. Let $x=n$ in the original equation, n being a prime greater than 3. The equation then becomes

$$(n+1)(n+2)\ldots(2n-1)=n^{n-1}+A_1 n^{n-2}+\ldots+A_{n-2}n+A_{n-1},$$

whence

$$(n+1)(n+2)\ldots(2n-1)-1.2.3\ldots(n-1)=\text{a multiple of } n^3,$$

i.e. $\qquad (2n-1)_{n-1}-1\equiv 0, \quad \text{mod. } n^3,$

which is the theorem (iii).

The theorems (i), ..., (iv) have thus been proved.

§ 7. The investigation in § 3 was given by Lagrange in the *Nouveaux Mémoires* of the Berlin Academy for 1771.* It forms his proof (perhaps the first published) of Wilson's theorem, which had been enunciated, without proof, the previous year in Waring's *Meditationes Algebraicæ.* Although Lagrange proves, as above, that if n is an uneven prime,

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* "Démonstration d'un Théoreme nouveau concernant les nombres premiers," pp. 125—137.

A_1, A_2, ..., A_{n-2} are divisible by n, he appears to have regarded these results merely as subsidiary to the proof of Wilson's theorem; and he does not draw special attention to them as possessing an interest of their own. Lagrange's investigation is referred to by H. J. S. Smith in Art. 10 of his report on the Theory of Numbers.†

Mr. Osborn's method was to show first that A_{n-2} and the numerator of $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}$ were divisible by n; and thence to deduce, by the formula used in § 5, that A_{n-3} was also divisible by n. From the identity

$$(n-1)(n-2)\ldots\{n-(n-1)\}=(n-1)!$$

he obtained the equation

$$n^{n-2}-A_1n^{n-3}+A_2n^{n-4}-\ldots+A_{n-3}n-A_{n-2}=0,$$

whence he inferred, as in § 4, that A_{n-2} must be divisible by n^2.

Wolstenholme's own method, which was somewhat lengthy, differed entirely from Mr. Osborn's and from that employed in this paper. He first proved independently that A_{n-2} was divisible by n^2, and A_{n-3} by n, whence he deduced, as in § 5, that the numerator of $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{(n-1)^2}$ was divisible by n.

Thence from the equation

$$\frac{\{n+n-1\}\{n+n-2\}\ldots(n+1)}{(n-1)!}$$
$$=\frac{n^{n-1}+A_1n^{n-2}+\ldots+A_{n-3}n^2+A_{n-2}n}{(n-1)!}+1$$

he showed that $(2n-1)_{n-1}-1$ must be divisible by n^3.

Starting from the equation

$$(x+1)(x+2)\ldots(x+n-1)=x^{n-1}+A_1x^{n-2}+\ldots+A_{n-2}x+A_{n-1},$$

Dr. Ferrers proved the theorem (iv), not by equating coefficients as in Lagrange's method, but by putting $x=1, 2, \ldots, n-2$, and considering the values of A_1, A_2, ..., A_{n-2} as determinants, which result from these equations. Dr. Ferrers proved (v) independently by a similar method.

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† "Collected Mathematical Papers," Vol. I., p. 46.

§ 8. The theorem (v) may be deduced from (iv) as follows: The quantity A_r is the sum of the products of the numbers $1, 2, 3, \ldots, n-1$ taken r together. Each term is therefore included in the general form

$$fg \ldots u(n-u)\, v(n-u) \ldots,$$

$f, g, \ldots$ being the factors (if any) whose complements $n-f$, $n-g, \ldots$ do not occur; and $u, v, \ldots$ the factors (if any) whose complements $n-u$, $n-v, \ldots$ do occur. Let n be uneven (so that a factor cannot be equal to its complement), and consider a term $fu(n-u)\,v(n-v) \ldots$, in which only a single factor f occurs without its complement; then A_r contains also the term $(n-f)\,u(n-u)\,v(n-v) \ldots$, and the sum of these two terms is divisible by n. Thus the sum of all the terms in A_r which contain only one factor without its complement $\equiv 0$, mod. n. Consider now a term containing exactly two factors f, g without their complements: combining this term with the three others in which f, g are replaced by $n-f, g$; $f, n-g$; $n-f, n-g$ respectively, we obtain a sum which is divisible by $fg + (n-f)\,g + f(n-g) + (n-f)(n-g) = n^2$. Thus the sum of all the terms in A_r which contain exactly two factors without their complements $\equiv 0$, mod. n^2. In the same way we see that the sum of all the terms in A_r which contain exactly three factors without their complements $\equiv 0$, mod. n^3; and so on. If r is uneven, there must be at least one factor in each term without its complement; and therefore, whether n be prime or not, so long as it is uneven,

$$A_{2t+1} \equiv 0, \quad \text{mod. } n.$$

If r is even we can group as above all the terms which contain at least one factor without its complement, so that their sum $\equiv 0$, mod. n: but the terms in which every factor has its complement cannot be so grouped. These terms are of the form $u(n-u)\,v(n-v) \ldots \equiv (-1)^{\frac{1}{2}r} u^2 v^2 \ldots$. Thus if $n = 2m+1$, and if B_t denote the sum of the products of $1^2, 2^2, \ldots, m^2$ taken t together, then, whether n be prime or not,

$$B_t \equiv (-1)^t A_{2t}, \quad \text{mod. } n = 2m+1;$$

for every term $u^2 v^2 \ldots$ in B_t gives rise to a term

$$u(n-u)\,v(n-v) \ldots$$

in A_{2t} and *vice versâ*.

Now $A_{2t} \equiv 0$, mod. n, if n be a prime greater than 3, so that in this case we have $B_t \equiv 0$, mod. n, which is the theorem (v).

Theorems relating to the values of $S_r(1, 2, ..., n)$.

§ 9. It will be noticed that the reasoning employed in the preceding section shows that in general $A_r \equiv 0$, mod. n, where r and n are any uneven numbers, and r is less than n; and it also distinguishes between the classes of terms in A_r, showing that the sum of those which contain s factors without their complements $\equiv 0$, mod. n^s.

This mode of classifying the terms will now be considered in detail, and from the resulting formulæ it will be shown that, if n is a prime greater than 3, not only is A_{n-2} divisible by n^2, but also A_r, r being any uneven number > 1 and $< n$ (§ 16). It will also be shown that, r and n being both uneven, and $n > 3$, $r > 1$,

$$\frac{n(n-r)}{2} A_{r-1} - A_r$$

is a multiple of n^3, and, when n is a prime > 5 and $r > 3$, a multiple of n^4 (§ 19).

§ 10. Let $S_r(a_1, a_2, ..., a_i)$ denote the sum of the products of the numbers $a_1, a_2, ..., a_i$ taken r together.

Let l be any number greater than any of the numbers $a_1, a_2, ..., a_i$, and such that no one of the numbers $a_1, a_2, ..., a_i$ is equal to any one of the complementary numbers

$$l - a_1,\ l - a_2,\ ...,\ l - a_i\,;$$

and let $\sigma_r(a_1, a_2, ..., a_i;\ l)$ denote the sum of the products obtained by multiplying each product in $S_r(a_1, a_2, ..., a_i)$ by the product formed of the complementary factors, *i.e.*, so that, if

$$S_r(a_1, a_2, ..., a_i) = \Sigma a_p a_q ... a_s,$$

then $\sigma_r(a_1, a_2, ..., a_i;\ l) = \Sigma a_p a_q ... a_s (l - a_p)(l - a_q)...(l - a_s)$.

For example,

$$S_2(1, 2, 3, 4) = 1.2 + 1.3 + 1.4 + 2.3 + 2.4 + 3.4 = 35,$$

$$\sigma_2(1, 2, 3, 4;\ 10) = 1.2.9.8 + 1.3.9.7 + 1.4.9.6$$
$$+ 2.3.8.7 + 2.4.8.6 + 3.4.7.6 = 1773.$$

§ 11. The following investigation enables us to express $S_{2t}(a_1, a_2, \ldots, a_i, l-a_i, \ldots, l-a_2, l-a_1)$ in terms of

$$\sigma_t(a_1, a_2, \ldots, a_i; l), \; \sigma_{t-1}(a_1, a_2, \ldots, a_i; l), \text{ \&c.}$$

The sum of the terms in $S_{2t}(a_1, \ldots, a_i, l-a_i, \ldots, l-a_1)$ in which no factor occurs without its complement is evidently $\sigma_t(a_1, a_2, \ldots, a_i; l)$. Consider next the terms in which exactly two factors occur without their complements. As in § 8, if f, g be two such factors (*i.e.* so that all the other factors have their complements), we have also terms in which the corresponding factors are $l-f$, g; f, $l-g$; $l-f$, $l-g$ respectively, the other factors remaining the same; the sum of these four terms therefore is $l^2 \times$ product of $t-1$ factors and their complements. In forming the pairs f, g we have to exclude the cases of $g = l-f$, so that, if we put $\mu = i - t$, the number of pairs is $(2\mu+2)\,2\mu$. Thus the sum of all the terms in which two factors occur without the complements is

$$\frac{(2\mu+2)\,2\mu}{2!}\,\frac{l^2}{4}\,\sigma_{t-1}(a_1, a_2, \ldots, a_i; l)$$

$$= \frac{(\mu+1)\,\mu}{2!}\,l^2\sigma_{t-1}(a_1, a_2, \ldots, a_i; l).$$

Now consider the terms in which exactly four factors occur without their complements. Each set of four factors f, g, h, k without their complements gives rise to a system of sixteen terms, the sum of which is $l^4 \times$ product of $t-2$ factors and their complements; and the number of products of four factors f, g, h, k which can be formed from $\mu+2$ quantities and their $\mu+2$ complements, subject to the condition that no factor occurs with its complement, is

$$\frac{(2\mu+4)(2\mu+2)\,2\mu\,(2\mu-2)}{4!}.*$$

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* It is easy to see that the number of combinations of the $n\ (=2i)$ quantities $a_1, a_2, \ldots, a_i, a_1', a_2', \ldots, a_i'$, taken r together, and in which no suffix occurs twice, is $\frac{n(n-2)\ldots(n-2r+2)}{r!}$; for, in forming the combinations two together, a_1 can combine with all except a_1, a_1', *i.e.* with $n-2$ quantities, so that the number of combinations is $\frac{n(n-2)}{2!}$; in forming the combinations three together the pair a_1a_2 can combine with all except a_1, a_1', a_2, a_2', *i.e.* with $n-4$ quantities, so that the number of combinations is $\frac{n(n-2)(n-4)}{3!}$; and so on.

Thus the sum of these terms

$$=\frac{(2\mu+4)(2\mu+2)2\mu(2\mu-2)}{4\,!}\,\frac{l^4}{16}\,\sigma_{t-2}(a_1, a_2, \ldots, a_i;\ l)$$

$$=\frac{(\mu+2)(\mu+1)\mu(\mu-1)}{4\,!}\,l^4\sigma_{t-2}(a_1, a_2, \ldots, a_i;\ l).$$

Similarly we may obtain the sum of the terms which contain six factors without their complements, and so on.

Thus we find that, if $\mu=i-t$,

$$S_{2t}(a_1, a_2, \ldots, a_i, l-a_i, \ldots, l-a_2, l-a_1)=\sigma_t(a_1, a_2, \ldots, a_i;\ l)$$
$$+\frac{(\mu+1)\mu}{2\,!}\,l^2\sigma_{t-1}(a_1, a_2, \ldots, a_i;\ l)$$
$$+\frac{(\mu+2)(\mu+1)\mu(\mu-1)}{4\,!}\,l^4\sigma_{t-2}(a_1, a_2, \ldots, a_i;\ l)$$
$$+\ \&c.$$

The series terminates either by the occurrence of a zero coefficient or when the suffix of σ is zero, the value of $\sigma_0(a_1, a_2, \ldots, a_i;\ l)$ being unity.

§ 12. In the same manner we obtain the corresponding formula for the case where the suffix of S is uneven, viz.

$$S_{2t+1}(a_1, a_2, \ldots, a_i, l-a_i, \ldots, l-a_2, l-a_1)=\mu l\sigma_t(a_1, a_2, \ldots, a_i;\ l)$$
$$+\frac{\mu(\mu^2-1^2)}{3\,!}\,l^3\sigma_{t-1}(a_1, a_2, \ldots, a_i;\ l)$$
$$+\frac{\mu(\mu^2-1^2)(\mu^2-2^2)}{5\,!}\,l^5\sigma_{t-2}(a_1, a_2, \ldots, a_i;\ l)$$
$$+\ \&c.$$

where $\mu=i-t$ as before.

§ 13. In these formulæ let $a_1, a_2, \ldots, a_i$ be the numbers $1, 2, \ldots, m$ and let $l=2m+1$. We then have

$$\text{(i)}\ \ S_{2t}(1, 2, \ldots, 2m)=\sigma_t(1, 2, \ldots, m;\ 2m+1)+\frac{(\mu+1)\mu}{2\,!}\,l^2\sigma_{t-1}\text{*}$$
$$+\frac{(\mu+2)(\mu+1)\mu(\mu-1)}{4\,!}\,l^4\sigma_{t-2}+\ \&c.,$$

* For the sake of brevity the arguments of the σ's are omitted in all the terms after the first in each series.

$$\text{(ii)}\ S_{2t+1}(1, 2, \ldots, 2m) = \mu l \sigma_t(1, 2, \ldots, m; 2m+1) + \frac{\mu(\mu^2 - 1^2)}{3!} l^3 \sigma_{t-1} + \frac{\mu(\mu^2 - 1^2)(\mu^2 - 2^2)}{5!} l^5 \sigma_{t-2} + \&c.,$$

where $\mu = m - t$, $l = 2m + 1$.

§ 14. Let $a_1, a_2, \ldots, a_i$ be the numbers $1, 2, \ldots, m-1$ and let $l = 2m$; the numbers $a_1, a_2, \ldots, a_i, l - a_i, \ldots, l - a_2, l - a_1$ are then the numbers $1, 2, \ldots, m-1, m+1, \ldots, 2m-1$, that is, all the numbers from 1 to $2m - 1$ with the exception of m.

Now, evidently,

$$S_r(1, 2, \ldots, 2m-1) = S_r(1, 2, \ldots, m-1, m+1, \ldots, 2m-1) + mS_{r-1}(1, 2, \ldots, m-1, m+1, \ldots, 2m-1),$$

for the first term on the right-hand side contains all the terms in which m does not occur, and the second term all the terms in which m does occur.

Now, putting $r = 2t$, we have, from § 11,

$$S_{2t}(1, 2, \ldots, m-1, m+1, \ldots, 2m-1)$$
$$= \sigma_t(1, 2, \ldots, m-1; 2m) + \frac{(\mu'+1)\mu'}{2!} l^2 \sigma_{t-1} + \&c.,$$

where $\mu' = m - 1 - t$, $l = 2m$;

and, from § 12,

$$S_{2t-1}(1, 2, \ldots, m-1, m+1, \ldots, 2m-1)$$
$$= \mu l \sigma_{t-1}(1, 2, \ldots, m-1; 2m) + \frac{\mu(\mu^2 - 1^2)}{3!} l^3 \sigma_{t-2} + \&c.,$$

where $\mu = m - t$, $l = 2m$.

Putting $\mu - 1$ for μ' in the first series, and adding it to the second series multiplied by $m = \frac{l}{2}$, we find

$$\text{(iii)}\ S_{2t}(1, 2, \ldots, 2m-1) = \sigma_t(1, 2, \ldots, m-1; 2m) + \frac{\mu^2}{2!} l^2 \sigma_{t-1} + \frac{\mu^2(\mu^2 - 1^2)}{4!} l^4 \sigma_{t-2} + \frac{\mu^2(\mu^2 - 1^2)(\mu^2 - 2^2)}{6!} l^6 \sigma_{t-3} + \&c.,$$

where $$\mu = m - t,\ l = 2m.$$

In the same manner, by putting $r = 2t + 1$, we find

$$\text{(iv)}\ S_{2t+1}(1, 2, \ldots, 2m-1) = (\mu - \tfrac{1}{2})\left\{l\sigma_t(1, 2, \ldots, m-1; 2m) + \frac{\mu(\mu-1)}{3!}l^3\sigma_{t-1} + \frac{(\mu+1)\mu(\mu-1)(\mu-2)}{5!}l^5\sigma_{t-2} + \&c.\right\},$$

where $$\mu = m - t, \quad l = 2m.$$

§ 15. The μ-coefficients in the series (iii) and (iv) are of the form, $\dfrac{\text{integer}}{2}$. This is evident from the manner in which they have been obtained, viz. in the case of (iii),

$$\frac{\mu^2}{2} = \frac{\mu(\mu-1)}{2!} + \tfrac{1}{2}\mu,$$

$$\frac{\mu^2(\mu^2-1^2)}{4!} = \frac{(\mu+1)\mu(\mu-1)(\mu-2)}{4!} + \tfrac{1}{2}\frac{(\mu+1)\mu(\mu-1)}{3!},$$

$$\frac{\mu^2(\mu^2-1^2)(\mu^2-2^2)}{6!} = \frac{(\mu+2)(\mu+1)\ldots(\mu-3)}{6!} + \tfrac{1}{2}\frac{(\mu+2)(\mu+1)\ldots(\mu-2)}{5!},$$

&c. &c.

Similarly, in the case of (iv),

$$\mu - \tfrac{1}{2} = \mu - 1 + \tfrac{1}{2},$$

$$(\mu - \tfrac{1}{2})\frac{\mu(\mu-1)}{3!} = \frac{\mu(\mu-1)(\mu-2)}{3!} + \tfrac{1}{2}\frac{\mu(\mu-1)}{2!},$$

$$(\mu - \tfrac{1}{2})\frac{(\mu+1)\mu(\mu-1)(\mu-2)}{5!} = \frac{(\mu+1)\mu\ldots(\mu-3)}{5!} + \tfrac{1}{2}\frac{(\mu+1)\mu\ldots(\mu-2)}{4!},$$

&c. &c.

§ 16. Formula (i), § 13, shows that

$$S_{2t}(1, 2, \ldots, 2m) \equiv \sigma_t(1, 2, \ldots, m; 2m+1), \quad \text{mod. } (2m+1)^2.$$

In § 3 it was shown that $S_{2t}(1, 2, \ldots, 2m)$ is divisible by

$2m+1$, if $2m+1$ is prime; therefore $\sigma_t(1, 2, ..., m; 2m+1)$ must be divisible by $2m+1$ if $2m+1$ is prime.

Formula (ii), § 13, shows that

$$S_{2t+1}(1, 2, ..., 2m) \equiv \mu(2m+1)\sigma_t(1, 2, ..., m; 2m+1),$$
$$\text{mod.}(2m+1)^3,$$

and since, as has just been shown, $\sigma_t(1, 2, ..., m; 2m+1)$ is divisible by $2m+1$ when $2m+1$ is prime, it follows that $S_{2t+1}(1, 2, ..., 2m)$ must be divisible by $(2m+1)^2$ when $2m+1$ is prime, except when $t=0$.

§ 17. It will be shown in § 43 that if m be prime $S_r(1, 2, ..., 2m-1)$ is divisible by m for the values 1, 2, ..., $m-2$ of r. Assuming this result we see from formula (iii), § 14, that $\sigma_r(1, 2, ..., m-1; 2m)$ is also so divisible for the values 1, 2, ..., $\frac{1}{2}(m-3)$ of r. It follows therefore from formula (iv), § 14, that, when m is an uneven prime greater than 3, $S_r(1, 2, ..., 2m-1)$ is divisible by m^2 for the values, 3, 5, ..., $m-2$ of r.*

§ 18. It is evident from formula (ii), § 13, that $S_{2t+1}(1, 2, ..., 2m)$ is always divisible by $2m+1$; and, from formula (iv), § 14, that $S_{2t+1}(1, 2, ..., 2m-1)$ is always divisible by m. Thus $S_{2t+1}(1, 2, ..., n)$ is divisible by $n+1$, if n is even, and by $\frac{1}{2}(n+1)$ if n is uneven.

§ 19. By multiplying formula (i), § 13, by μl and subtracting (ii), we find

$$\mu l S_{2t}(1, 2, ..., 2m) - S_{2t+1}(1, 2, ..., 2m)$$
$$= (2\mu+1)\left\{\frac{\mu(\mu+1)}{3!} l^3 \sigma_{t-1}(1, 2, ..., m; 2m+1)\right.$$
$$\left. + \frac{2(\mu+2)(\mu+1)\mu(\mu-1)}{5!} l^5 \sigma_{t-2} + \frac{3(\mu+3)\ldots(\mu-2)}{7!} l^7 \sigma_{t-3} + \&\text{c.}\right\},$$

where, as before

$$\mu = m-t, \quad l = 2m+1.$$

When $2m+1$ is a prime, $\sigma_{t-1}(1, 2, ..., m; 2m+1)$ is divisible by $2m+1$ if $t>1$ (§16). Thus in this case the right-hand side

* These results are extended to other values of r in §§ 51, 52.

is divisible by $(2m+1)^4$, so that we have, generally,

$$(m-t)(2m+1)\,S_{2t}(1, 2, \ldots, 2m) \equiv S_{2t+1}(1, 2, \ldots, 2m), \quad \text{mod. } (2m+1)^3,$$

and, when $2m+1$ is a prime, and $t>1$,

$$(m-t)(2m+1)\,S_{2t}(1, 2, \ldots, 2m) \equiv S_{2t+1}(1, 2, \ldots, 2m), \quad \text{mod. } (2m+1)^4.$$

If $t=1$, the modulus is $(2m+1)^5$.

§ 20. Similarly by multiplying formula (iii), § 14, by $(\mu-\frac{1}{2})\,l$ and subtracting formula (iv), we find after reduction

$$(\mu-\tfrac{1}{2})\,l\,S_{2t}(1, 2, \ldots, 2m-1) - S_{2t+1}(1, 2, \ldots, 2m-1)$$

$$= \tfrac{1}{2}(4\mu^2-1)\left\{\frac{\mu}{3\,!}\,l^3\sigma_{t-1}(1, 2, \ldots, m-1;\ 2m) + \frac{2\mu(\mu^2-1)}{5\,!}\,l^5\sigma_{t-2} + \frac{3\mu(\mu^2-1)(\mu^2-2^2)}{7\,!}\,l^7\sigma_{t-3} + \&\text{c.}\right\},$$

where $\mu = m-t, \quad l = 2m.$

Thus, generally,

$$(2m-2t-1)\,mS_{2t}(1, 2, \ldots, 2m-1) \equiv S_{2t+1}(1, 2, \ldots, 2m-1), \quad \text{mod. } m^3.$$

§ 21. When m is prime, $\sigma_r(1, 2, \ldots, m-1;\ 2m)$ is divisible by m for the values $1, 2, \ldots, \frac{1}{2}(m-3)$ of r (§ 17). It follows therefore that when m is prime

$$(2m-2t-1)\,mS_{2t}(1, 2, \ldots, 2m-1) \equiv S_{2t+1}(1, 2, \ldots, 2m-1), \quad \text{mod. } m^4,$$

for the values $4, 6, 8, \ldots, m-1$ of $2t$.*

If $2t=2$, the modulus is m^3.

§ 22. It is evident that, n being any number,

$$S_r(1, 2, \ldots, n) = S_r(1, 2, \ldots, n-1) + nS_{r-1}(1, 2, \ldots, n-1).$$

Putting $r=2t+1$, this formula becomes

$$S_{2t+1}(1, 2, \ldots, n) = S_{2t+1}(1, 2, \ldots, n-1) + nS_{2t}(1, 2, \ldots, n-1).$$

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* This result is extended in § 53.

Now we know from § 18 that, if n is uneven, $S_{2t+1}(1, 2, \ldots, n-1)$ is divisible by n, and, if n is even, by $\frac{1}{2}n$. Thus generally, if n is uneven, $S_{2t+1}(1, 2, \ldots, n)$ is divisible by n, and, if even, by $\frac{1}{2}n$.

If n be a prime greater than 3, then $S_{2t}(1, 2, \ldots, n-1)$ is divisible by n (§ 1), and $S_{2t+1}(1, 2, \ldots, n-1)$ by n^2 (§ 15), when $t>0$. Therefore $S_{2t+1}(1, 2, \ldots, n)$, $t>0$, is divisible by n^2, when n is a prime greater than 3.

§ 23. Putting $r=2t$, we have

$$S_{2t}(1, 2, \ldots, n) = S_{2t}(1, 2, \ldots, n-1) + nS_{2t-1}(1, 2, \ldots, n-1);$$

so that

$$S_{2t}(1, 2, \ldots, n) \equiv S_{2t}(1, 2, \ldots, n-1), \quad \text{mod. } n^2 \text{ or mod. } \tfrac{1}{2}n^2,$$

according as n is uneven or even.

If n is a prime >3, $S_{2t}(1, 2, \ldots, n-1)$, $2t<n-1$, is divisible by n, and $S_{2t-1}(1, 2, \ldots, n-1)$, $t>1$, by n^2 (§ 17); therefore, if $2t<n-1$, $S_{2t}(1, 2, \ldots, n)$ is divisible by n; and, if $t>1$,

$$\frac{S_{2t}(1, 2, \ldots, n)}{n} \equiv \frac{S_{2t}(1, 2, \ldots, n-1)}{n}, \quad \text{mod. } n^2.$$

If $t=1$, the modulus is n.

§ 24. It has thus been shown that, if n is a prime >3, $S_r(1, 2, \ldots, n)$ is always divisible by n, when $r<n-1$, and by n^2, when r is uneven and >1.

§ 25. I now proceed to consider the sum of products which has been denoted by $\sigma_r(a_1, a_2, \ldots, a_i;\ l)$ (§ 10).

Since

$$a_1(l-a_1)\,a_2(l-a_2)\ldots a_r(l-a_r)$$

$$=(-1)^r a_1^2 a_2^2 \ldots a_r^2 \left(1-\frac{l}{a_1}\right)\left(1-\frac{l}{a_2}\right)\left(1-\frac{l}{a_r}\right)$$

$$=(-1)^r a_1^2 a_2^2 \ldots a_r^2 \left\{1 - l\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_r}\right) + l^2\Sigma\frac{1}{a_1 a_2} - \&c.\right\},$$

we have

$$\sigma_r(a_1, a_2, \ldots, a_i;\ l) = (-1)^r\{S_r(a_1^2, a_2^2, \ldots, a_i^2)$$
$$- lS_r'(a_1^2, a_2^2, \ldots, a_i^2) + l^2 S_r''(a_1^2, a_2^2, \ldots, a_i^2) - \&c.\},$$

where $S_r'(a_1^2, a_2^2, \ldots, a_i^2)$ may be derived from $S_r(a_1^2, a_2^2, \ldots, a_i^2)$ by multiplying each term $a_1^2a_2^2\ldots a_r^2$ by $\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_r}$, or from $S_{r-1}(a_1^2, a_2^2, \ldots, a_i^2)$ by multiplying each term $a_1^2a_2^2\ldots a^2_{r-1}$ by the sum of the numbers not occurring in it, *i.e.* by $a_r+a_{r+1}+\ldots+a_i$. Similarly $S_r''(a_1^2, a_2^2, \ldots, a_i^2)$ may be derived from $S_r(a_1^2, a_2^2, \ldots, a_i^2)$ by multiplying each term $a_1^2a_2^2\ldots a_r^2$ by $\Sigma\frac{1}{a_1a_2}$, the products being formed from the numbers occurring in the term; or from $S_{r-2}(a_1^2, a_2^2, \ldots, a_i^2)$ by multiplying each term $a_1^2a_2^2\ldots a^2_{r-2}$ by the sum of the pairs of products formed from the other numbers $a_{r-1}, a_r, \ldots, a_i$; and so on.

§ 26. Substituting for the σ's in the formulæ (i), ..., (iv), §§ 13, 14, we find

$$\text{(i)}\quad S_{2t}(1, 2, \ldots, 2m) \equiv (-1)^t\{S_t(1^2, 2^2, \ldots, m^2) - (2m+1)S_t'(1^2, 2^2, \ldots, m^2)\},\quad \text{mod. } (2m+1)^2,$$

$$\text{(ii)}\quad S_{2t+1}(1, 2, \ldots, 2m) \equiv (-1)^t(m-t)(2m+1) \times\{S_t(1^2, 2^2, \ldots, m^2) - (2m+1)S_t'(1^2, 2^2, \ldots, m^2)\},\quad \text{mod. } (2m+1)^3,$$

$$\text{(iii)}\quad S_{2t}(1, 2, \ldots, 2m-1) \equiv (-)^t[S_t\{1^2, 2^2, \ldots, (m-1)^2\} - 2mS_t'\{1^2, 2^2, \ldots, (m-1)^2\}],\quad \text{mod. } 2m^2,$$

$$\text{(iv)}\quad S_{2t+1}(1, 2, \ldots, 2m-1) \equiv (-)^t m(2m-2t-1) \times[S_t\{1^2, 2^2, \ldots, (m-1)^2\} - 2mS_t'\{1^2, 2^2, \ldots, (m-1)^2\}],\quad \text{mod. } 4m^3.$$

If $2m+1$ is prime, $S_r(1, 2, \ldots, 2m)$ is divisible by $2m+1$ for all values of r, and by $(2m+1)^2$ if r is uneven (§ 16). Each of the congruences (i) and (ii) therefore shows that $S_t(1^2, 2^2, \ldots, m^2)$ is divisible by $2m+1$, when $2m+1$ is prime (Ferrers's theorem, § 1). If m is prime, $S_r(1, 2, \ldots, 2m-1)$ is divisible by m for all values of r from 1 to $m-2$, and by m^2 for all uneven values of r from 3 to $m-2$ (§ 17). Each of the congruences (iii) and (iv) therefore shows that $S_t\{1^2, 2^2, \ldots, (m-1)^2\}$ is divisible by m for the values $1, 2, \ldots, \frac{1}{2}(m-3)$ of t when m is prime and > 3.*

* This result is extended to other values of t in § 58.

§ 27. We may obtain also some results with respect to the numbers $S'_r(1^2, 2^2, \ldots, m^2)$, &c. For example, we see from (i) that, $2m+1$ being not necessarily prime,

$$S_{2t}(1, 2, \ldots, 2m) + (-1)^{t-1} S_t(1^2, 2^2, \ldots, m^2)$$

is divisible by $2m+1$, and that the quotient

$$\equiv (-1)^{t-1} S'_t(1^2, 2^2, \ldots, m^2), \quad \text{mod. } 2m+1.$$

§ 28. We have, identically, (§ 5)

$$S_{m-2}\{1^2, 2^2, \ldots, (m-1)^2\} = \{S_{m-2}(1, 2, \ldots, m-1)\}^2$$
$$- 2(m-1)!\, S_{m-3}(1, 2, \ldots, m-1).$$

First, suppose m uneven and not prime; then $S_{m-2}(1, 2, \ldots, m-1)$ is divisible by m and so is $(m-1)!$ Secondly, suppose m to be a prime > 3; then $S_{m-2}(1, 2, \ldots, m-1)$ is divisible by m^2 and $S_{m-3}(1, 2, \ldots, m-1)$ is divisible by m. In both cases therefore the right-hand side is divisible by m. Thus, when m is any uneven number > 3,

$$S_{m-2}\{1^2, 2^2, \ldots, (m-1)^2\} \equiv 0, \quad \text{mod. } m.$$

§ 29. It follows therefore from formulæ (iii) and (iv) of § 26 that when m is uneven and > 3,

$$S_{2m-4}(1, 2, \ldots, 2m-1) \equiv 0, \quad \text{mod. } m,$$

and

$$S_{2m-3}(1, 2, \ldots, 2m-1) \equiv 0, \quad \text{mod. } m^2.$$

Tables and Formulæ.

§ 30. For the purpose of verifying the preceding results in special cases I have calculated the values of $S_r(1, 2, \ldots, n)$, and some other sums of products for certain values of n. The table of $S_r(1, 2, \ldots, n)$ which is given at the end of the paper (§ 50, pp. 26–28) contains the values of $S_r(1, 2, \ldots, n)$ as far as $n = 22$. If it is used as a table of A_r, *i.e.* of $S_r(1, 2, \ldots, n-1)$, the heading of the column is $n-1$: thus, for example, to obtain the value of A_r for $n = 11$, we enter the table in the column headed 10.

§ 31. The values of $\sigma_r(1, 2, 3, 4, 5;\ 11)$ are:

$$\sigma_1 = 110;\quad \sigma_2 = 4708;\quad \sigma_3 = 97416;\quad \sigma_4 = 966240;\quad \sigma_5 = 3628800.$$

The values of $\sigma_r(1, 2, 3, 4;\ 10)$ are:

$$\sigma_1 = 70;\quad \sigma_2 = 1773;\quad \sigma_3 = 19080;\quad \sigma_4 = 72576.$$

The values of $S_r(1^2, 2^2, 3^2, 4^2, 5^2)$ are:

$S_1 = 55$; $S_2 = 1023$; $S_3 = 7645$; $S_4 = 21076$; $S_5 = 14400$.

The values of $S_r'(1^2, 2^2, 3^2, 4^2, 5^2)$ are:

$S_1' = 15$; $S_2' = 600$; $S_3' = 7395$; $S_4' = 31050$; $S_5' = 32880$.

§ 32. The formulæ (i), ..., (iv) of § 13, 14 form a system of results which are of some interest for their own sake; so that it seems worth while to notice that they may be expressed in a more symmetrical manner as follows:

$$\text{(i)}\quad S_{2t}(1, 2, \ldots, 2m) = \sigma_t(1, 2, \ldots, m;\ 2m+1) + \frac{\lambda^2 - 1^2}{2\,!}\left(\frac{l}{2}\right)^2 \sigma_{t-1}$$

$$+ \frac{(\lambda^2 - 1^2)(\lambda^2 - 3^2)}{4\,!}\left(\frac{l}{2}\right)^4 \sigma_{t-2} + \&c.\ (\lambda = 2m - 2t + 1,\ l = 2m + 1);$$

$$\text{(ii)}\quad S_{2t+1}(1, 2, \ldots, 2m) = \lambda \frac{l}{2} \sigma_t(1, 2, \ldots, m;\ 2m+1)$$

$$+ \frac{\lambda(\lambda^2 - 2^2)}{3\,!}\left(\frac{l}{2}\right)^3 \sigma_{t-1} + \frac{\lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)}{5\,!}\left(\frac{l}{2}\right)^5 \sigma_{t-2} + \&c.$$

$$(\lambda = 2m - 2t,\ l = 2m + 1);$$

$$\text{(iii)}\quad S_{2t}(1, 2, \ldots, 2m-1) = \sigma_t(1, 2, \ldots, m-1;\ 2m) + \frac{\lambda^2}{2!}\left(\frac{l}{2}\right)^2 \sigma_{t-1}$$

$$+ \frac{\lambda^2(\lambda^2 - 2^2)}{4\,!}\left(\frac{l}{2}\right)^4 \sigma_{t-2} + \&c.\ (\lambda = 2m - 2t,\ l = 2m);$$

$$\text{(iv)}\quad S_{2t+1}(1, 2, \ldots, 2m-1) = \lambda \frac{l}{2} \sigma_t(1, 2, \ldots, m-1;\ 2m)$$

$$+ \frac{\lambda(\lambda^2 - 1^2)}{3\,!}\left(\frac{l}{2}\right)^3 \sigma_{t-1} + \frac{\lambda(\lambda^2 - 1^2)(\lambda^2 - 3^2)}{5\,!}\left(\frac{l}{2}\right)^5 \sigma_{t-2} + \&c.$$

$$(\lambda = 2m - 2t - 1,\ l = 2m).$$

Thus in every equation, if the left-hand side be written $S_r(1, 2, \ldots, n-1)$, we have $\lambda = n - r$, $l = n$.

§ 33. In the earlier sections of this paper the results have been expressed by means of the notation A_r, which is naturally suggested by the usual form of Wilson's theorem. It is therefore interesting to give a complete list of the properties of A_r which have been obtained. The quantity denoted by A_r is the sum of the products of the numbers 1, 2, 3, ..., $n-1$ taken r together. Thus A_r is the same as $S_r(1, 2, ..., n-1)$.

The following is the list of properties. The suffix r is always supposed to be less than $n-1$.

I. If n is prime,

(i) $A_{n-1} \equiv -1$, mod. n (Wilson);

(ii) $A_{n-2} \equiv 0$, mod. n^2 (Wolstenholme);

(iii) $A_r \equiv 0$, mod. n (Ferrers);

(iv) A_r (r uneven and > 1) $\equiv 0$, mod. n^2 (§ 16);

(v) A_r (r uneven and > 3) $\equiv \dfrac{n(n-r)}{2} A_{r-1}$, mod. n^4 (§ 19).

In (i) and (iii) n may be any uneven prime, in (ii) and (iv) n must be > 3, and in (v) must be > 5.

II. If $n-1=m$ is an uneven prime,

(i) A_r (r even and $< m-1$) $\equiv 0$, mod. m (§ 23);

(ii) A_r (r uneven and > 1) $\equiv 0$, mod. m^2 (§ 22);

(ii′) $A_1 \equiv 0$, mod. m (§ 22).

III. If $n = 2m$, where m is prime and > 3,

(i) A_r ($r = 2, 4, ..., m-3$) $\equiv 0$*, mod. m (§ 17);

(ii) A_r ($r = 3, 5, ..., m-2$) $\equiv 0$*, mod. m^2 (§ 17);

(iii) A_r ($r = 5, 7, ..., m$) $\equiv \dfrac{n(n-r)}{2} A_{r-1}$*, mod. m^4 (§ 21);

(iv) $A_{n-3} \equiv 0$,† mod. m^2 (§ 29),

(v) $A_{n-4} \equiv 0$,† mod. m (§ 29).

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* In §§ 51–54 (pp. 29–31), these results are extended and it is shown that, if m be an uneven prime, and > 3 in (iii),

(i) A_r (r even, except $r = m-1$ and $2m-2$) $\equiv 0$, mod. m (§ 51).

(ii) A_r (r uneven and > 1) $\equiv 0$, mod. m^2 (§ 52).

(iii) A_r (r uneven and > 3, including $r = 2m-1$) $\equiv \dfrac{n(n-r)}{2} A_r$, mod. m^4 (§ 53),

(iv) $A_{m-1} \equiv -2$, $A_{2m-2} \equiv 1$, mod. m (§ 51).

† These formulæ are true also when m is any uneven number > 3 (§ 29).

IV. If n is uneven,

(i) A_r (r uneven) $\equiv 0$, mod. n (§ 18),

(ii) A_r (r uneven and >1) $\equiv \frac{n(n-r)}{2} A_{r-1}$, mod. n^3 (§ 19).

V. If $n-1$ is even $=2m$,

(i) A_r (r uneven) $\equiv 0$*, mod. m (§ 22).

VI. If $n-1$ is uneven $=m$,

(i) A_r (r uneven) $\equiv 0$, mod. m (§ 22).

VII. If n is even $=2m$,

(i) A_r (r uneven) $\equiv 0$, mod. m (§ 18),

(ii) A_r (r uneven and >1) $\equiv \frac{n(n-r)}{2} A_{r-1}$, mod. m^3 (§ 20).

In (ii) r may be equal to $2m-1$.

§ 34. We may also add the following results which are not expressible by the notation A_r.

(i) $S_{2t}(1, 2, \ldots, n) \equiv S_{2t}(1, 2, \ldots, n-1)$, mod. n^2 or mod. $\frac{n^2}{2}$,

according as n is uneven or even (§ 23).

(ii) $\frac{S_{2t}(1, 2, \ldots, n)}{n} \equiv \frac{S_{2t}(1, 2, \ldots, n-1)}{n}$, mod. n^2,

when n is a prime >3 (§ 23).

(iii) $S_r(1^2, 2^2, \ldots, m^2) \equiv 0$, mod. $2m+1$,

when $2m+1$ is prime (Ferrers, and § 26).

(iv) $S_r(1^2, 2^2, \ldots, m^2)\left(r=1, 2, \ldots, \frac{m-2}{2}\right) \equiv 0$,† mod. $m+1$,

when $m+1$ is a prime >3 (§ 26).

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* It is shown in § 53 that, if m is an uneven prime,

(i) A_r (r even, except $r=m-1$ and $2m-2$) $\equiv 0$, mod. m (§ 55),

(ii) A_r (r uneven and >1) $\equiv 0$, mod. m^2 (§ 56).

(iii) $A_{m-1} \equiv -2$, $A_{2m-2} \equiv 1$, mod. m (§ 55).

† It is shown in § 58 that r may have all values from 1 to $m-1$ except $\frac{1}{2}m$; and that $S_{\frac{1}{2}m}(1^2, 2^2, \ldots, m^2) \equiv (-1)^{\frac{1}{2}m+1}\, 2$, mod. $m+1$.

(v) $S_{m-1}(1^2, 2^2, \ldots, m^2) \equiv 0, \quad \text{mod. } m+1,$
when $m+1$ is any uneven number > 3 (§ 28).

(vi) $S_{2t}(1, 2, \ldots, 2m) + (-1)^{t-1} S_t(1^2, 2^2, \ldots, m^2)$

$$\equiv (-1)^{t-1} S'_t(1^2, 2^2, \ldots, m^2), \quad \text{mod. } 2m+1,$$

when $2m+1$ is prime (§ 27).

Congruences relating to A_r.

§ 35. A few other congruences relating to the A's may be derived from the fundamental equation in § 3, viz.,

$$(x+1)(x+2)\ldots(x+n-1) = x^{n-1} + A_1 x^{n-2} + A_2 x^{n-3} + \ldots + A_{n-2} x + A_{n-1}.$$

Putting $x = 1$ and -1, we have

$$n! = 1 + A_1 + A_2 + \ldots + A_{n-2} + A_{n-1},$$

$$0 = 1 - A_1 + A_2 - \ldots + (-1)^{n-2} A_{n-2} + (-1)^{n-1} A_{n-1},$$

whence, if n is uneven,

$$n! = 2A_1 + 2A_3 + \ldots + 2A_{n-2}.$$

Thus, when n is uneven and not a prime,

$$\frac{A_1}{n} + \frac{A_3}{n} + \ldots + \frac{A_{n-2}}{n} \equiv 0, \quad \text{mod. } n.$$

When n is prime and > 3, all these A's except A_1 are divisible by n^2, and in this case we find

$$\frac{A_3}{n^2} + \frac{A_5}{n^2} + \ldots + \frac{A_{n-2}}{n^2} \equiv \frac{\mu_1}{2}, \quad \text{mod. } n,$$

where μ_1 is the residue with respect to n of the quotient

$$\frac{(n-1)! - (n-1)}{n}.$$

§ 36. Putting $x = -n$, we have, if n is even,

$$2A_{n-1} = n^{n-1} - A_1 n^{n-2} + \ldots - A_{n-3} n^2 + A_{n-2} n.$$

Thus, when n is even,

$$2A_{n-1} \equiv nA_{n-2}, \quad \text{mod. } n^2;$$

that is $$\frac{2A_{n-1}}{n} \equiv A_{n-2}, \quad \text{mod. } n.$$

If n is uneven,

$$0 = n^{n-1} - A_1 n^{n-2} + \ldots + A_{n-3} n^2 - A_{n-2} n.$$

Thus in this case we have $A_{n-2} \equiv n A_{n-3}$, mod. n^3, and mod. n^4 when n is a prime > 3. These results are, however, included in IV. (ii) and I. (v) of § 33.

§ 37. Putting $x = n$,

$$(n+1)(n+2)\ldots(2n-1) = n^{n-1} + A_1 n^{n-2} + \ldots$$
$$+ A_{n-3} n^2 + A_{n-2} n + A_{n-1},$$

and we have, if n is uneven,

$$0 = n^{n-1} - A_1 n^{n-2} + \ldots + A_{n-3} n^2 - A_{n-2} n,$$

so that, n being uneven,

$$(n+1)(n+2)\ldots(2n-1) - (n-1)!$$
$$= 2A_1 n^{n-2} + 2A_3 n^{n-4} + \ldots + 2A_{n-2} n.$$

Thus

$$(n+1)(n+2)\ldots(2n-1) - (n-1)! \equiv 0, \text{ mod. } n, n^2, \text{ or } n^3,$$

according as n is even, uneven, or a prime > 3.

The third case gives Wolstenholme's theorem (§ 6).

§ 38. These equations show also that, if n is uneven,

$$\frac{(n+1)(n+2)\ldots(2n-1) - (n-1)!}{n^2} \equiv \frac{2A_{n-2}}{n}, \text{ mod. } n^2,$$

and, if n is a prime > 3,

$$\frac{(n+1)(n+2)\ldots(2n-1) - (n-1)!}{n^3} \equiv \frac{2A_{n-2}}{n^2}, \text{ mod. } n^2.$$

If n is prime, $(n-1)! \equiv -1$, mod. n; and therefore, when n is a prime > 3, we have

$$\frac{(2n-1)_{n-1} - 1}{n^3} \equiv \frac{2A_{n-2}}{n^2}, \text{ mod. } n.$$

§ 39. By putting $x = (k-1)n$, k being any positive integer, we obtain similar results in which

$$(2n-1)(2n-2)\ldots(2n-1)$$

is replaced by

$$(kn-1)(kn-2)\ldots(kn-n+1),$$

viz., we have

$$(kn-1)(kn-2)\ldots(kn-n+1)-(n-1)!\equiv 0, \text{ mod. } n, n^2, \text{ or } n^3,$$

according as n is even, uneven, or a prime >3. Thus in the third case $(kn-1)_{n-1}\equiv 1$, mod. n^3.

In the relations corresponding to those in § 38, the factor 2 on the right-hand side becomes $k(k-1)$; *ex. gr.* if n is a prime >3,

$$\frac{(kn-1)(kn-2)\ldots(kn-n+1)-(n-1)!}{n^3}\equiv k(k-1)\frac{A_{n-2}}{n^2}, \text{ mod. } n^2,$$

and
$$\frac{(kn-1)_{n-1}-1}{n^3}\equiv k(k-1)\frac{A_{n-2}}{n^2}, \text{ mod. } n.$$

It may be remarked that k and l being any positive integers, and n a prime >3,

$$l(l-1)(kn-1)_{n-1}-k(k-1)(ln-1)_{n-1}$$
$$\equiv l(l-1)-k(k-1), \text{ mod. } n^5.$$

§ 40. Proceeding as in § 37, we have generally, n being uneven and x unrestricted,

$$(x+1)(x+2)\ldots(x+n-1)-(x-1)(x-2)\ldots(x-n+1)$$
$$=2A_1x^{n-2}+2A_3x^{n-4}+\ldots+2A_{n-2}x\equiv 0, \text{ mod. } n \text{ and mod. } x.$$

When n is a prime >3,

$$(x+1)(x+2)\ldots(x+x-1)-(x-1)(x-2)\ldots(x-n+1)$$
$$\equiv n(n-1)x^{n-2}\equiv -nx^{n-2}, \text{ mod. } n^2,$$

and therefore, multiplying by x and dividing by $n!$,

$$(x+n-1)_n-(x)_n\equiv x^{n-1}, \text{ mod. } n$$
$$\equiv 1, \text{ mod. } n, \text{ if } x \text{ is any number prime to } n.$$

If x is any of the numbers 1, 2, ..., $n-1$, we have

$$x(x+1)(x+2)\ldots(x+n-1)\equiv -n, \text{ mod. } n^2.$$

By giving x its separate values, however, this congruence merely repeats Wilson's theorem.

§ 41. Since

$$S_{n-2}\{1^2, 2^2, \ldots, (n-1)^2\} = A^2_{n-2} - 2A_{n-3}A_{n-1} \quad (\S\ 28),$$

we have

$$S_{n-2}\{1^2, 2^2, \ldots, (n-1)^2\} \equiv -2A_{n-3}A_{n-1}, \text{ mod. } n^2 \text{ or } n^4,$$

according as n is uneven or a prime > 3.

In the latter case, therefore,

$$\frac{S_{n-2}\{1^2, 2^2, \ldots, (n-1)^2\}}{n} \equiv \frac{2A_{n-3}}{n} \equiv \frac{2A_{n-2}}{n^2}, \text{ mod. } n.$$

§ 42. The following theorems relate to the divisibility of the A's by p, where p is a prime not greater than n.

It was shown in Vol. XXX., p. 152 of the *Quarterly Journal* that, if q and s be less than any prime p, then $(kp+q)_{gp+s}$ is divisible by p if $s > q$, and $\equiv (k)_g \times (q)_s$, mod. p, if $s \leqq q$.

Thus the quantities

$$(kp+q)_1,\ (kp+q)_2,\ \ldots,\ (kp+q)_q$$

are not divisible by p, and the quantities

$$(kp+q)_{q+1},\ (kp+q)_{q+2},\ \ldots,\ (kp+q)_{p-1}$$

are divisible by p.

Now let $n = kp + q$, p being a prime and $q < p$. The relation (§ 3) connecting A_q and the preceding A's is

$$(kp+q)_{q+1} + (kp+q-1)_q A_1 + \ldots + (kp+1)_2 A_{q-1} = qA_q.$$

Every one of the coefficients on the left-hand side is divisible by p, and therefore A_q must be divisible by p (since q cannot be so divisible, being $< p$).

The next relation

$$(kp+q)_{q+2} + (kp+q-1)_{q+1} A_1 + \ldots + (kp)_2 A_q = (q+1) A_{q+1}$$

shows in the same way that A_{q+1} is divisible by p.

In the next relation

$$(kp+q)_{q+3} + (kp+q-1)_{q+2} A_1 + \ldots + (kp)_3 A_q$$
$$+ (kp-1)_2 A_{q+1} = (q+2) A_{q+2}$$

all the coefficients except the last are divisible by p. The last is not so, but A_{q+1} is; and therefore A_{q+2} is divisible by p.

Proceeding in this manner we see that A_{q+3}, ..., are divisible by p, the general relation being

$$(kp+q)_{q+t} + (kp+q-1)_{q+t-1} A_1 + \dots + (kp)_t A_q$$
$$+ (kp-1)_{t-1} A_{q+1} + \dots + (kp-t+2)_2 A_{q+t-2} = (q+t-1) A_{q+t-1}.$$

If $q+t<p$, the coefficients in the first line are divisible by p; those in the second line on the left-hand side are not so,* but the A's are, and therefore A_{q+t-1} is so. Thus A_q, A_{q+1}, ..., A_{p-2} are divisible by p.

§ 43. We have therefore shown that, if n be any number, and A_r denote the sum of the products of the numbers 1, 2, 3, ..., $n-1$ taken r together; and if p be any prime not exceeding n, and if q be the remainder when n is divided by p; then

$$A_q,\ A_{q+1},\ A_{q+2},\ \dots,\ A_{p-2}$$

are all divisible by p. If $q=0$, A_q is, of course, not so divisible.

In particular, if p be greater than $\frac{1}{2}n$, the remainder q is $n-p$, and the theorem shows that

$$A_{n-p},\ A_{n-p+1},\ \dots,\ A_{p-2}$$

are divisible by p.

§ 44. The relation involving A_{p-1} is

$$(kp+q)_p + (kp+q-1)_{p-1} A_1 + (kp+q-2)_{p-2} A_2 + \dots$$
$$+ \{(k-1)p+q+2\}_2 A_{p-2} = (p-1) A_{p-1}.$$

The first term is not divisible by p, but $\equiv k$, mod. p. All the other terms $\equiv 0$, mod. p, so that we find

$$A_{p-1} + k \equiv 0, \text{ mod. } p,$$

which is a generalisation of Wilson's theorem.

§ 45. As an example of the theorems in §§ 43 and 44, let $n=23$, so that A_r is the sum of the products of the numbers 1, 2, 3, ..., 22, taken r together.

* The coefficients, which may be written $\{(k-1)p+p-1\}_{t-1}$, ..., $\{(k-1)p+p-t+2\}_2$ are $\equiv (p-1)_{t-1}$, ..., $(p-t+2)_2$, mod. p.

Let $p=23$, then $k=1$, $q=0$. Thus $A_{22}+1$ is divisible by 23 (Wilson's theorem) and $A_1, A_2, \ldots, A_{21}$ are divisible by 23 (Lagrange's and Ferrers's theorem). From § 16 we know that $A_3, A_5, \ldots, A_{21}$ are divisible by 23^2.

Let $p=19$, then $k=1$, $q=4$, and $A_4, A_5, \ldots, A_{17}, A_{18}+1$ are divisible by 19.

Let $p=17$, then $k=1$, $q=6$, and $A_6, A_7, \ldots, A_{15}, A_{16}+1$ are divisible by 17.

Let $p=13$, then $k=1$, $q=10$, and $A_{10}, A_{11}, A_{12}+1$ are divisible by 13.

Let $p=11$, then $k=2$, $q=1$, and $A_1, A_2, \ldots, A_9, A_{10}+2$ are divisible by 11.*

Let $p=7$, then $k=3$, $q=2$, and $A_2, A_3, A_4, A_5, A_6+3$ are divisible by 7.

Let $p=5$, then $k=4$, $q=3$, and A_3, A_4+4 are divisible by 5.

Let $p=3$, then $k=7$, $q=2$, and A_2+7, that is A_2+1 is divisible by 3.

Expressions for $A_1, A_2, \ldots$

§ 46. I have calculated the values of $A_1, A_2, \ldots, A_7$ in terms of n by means of the formulæ in § 3. These expressions are as follows:

$$A_1=\tfrac{1}{2}n(n-1),$$

$$A_2=\tfrac{1}{24}n(n-1)(n-2)(3n-1),$$

$$A_3=\tfrac{1}{48}n^2(n-1)^2(n-2)(n-3),$$

$$A_4=\tfrac{1}{5760}n(n-1)(n-2)(n-3)(n-4)(15n^3-30n^2+5n+2),$$

$$A_5=\tfrac{1}{11520}n^2(n-1)^2(n-2)(n-3)(n-4)(n-5)(3n^2-7n-2),$$

$$A_6=\tfrac{1}{2903040}n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)\times(63n^5-315n^4+315n^3+91n^2-42n-16),$$

$$A_7=\tfrac{1}{5806080}n^2(n-1)^2(n-2)(n-3)(n-4)(n-5)(n-6)(n-7)\times(9n^4-54n^3+51n^2+58n+16).$$

---

* In the case of $p=11$, we know from § 57 that A_r is divisible by 11 for all even values of r except $r=10$ and 20, and by 11^2 for all uneven values except $r=11$ and 21; and that $S_{10}\equiv-2$, $S_{20}\equiv1$, mod. 11, and $S_{11}\equiv-4.11$, mod. 11^2, $S_{21}\equiv3.11$, mod. 11^2.

§ 47. For the actual calculation it was convenient to put

$$A_1 = n(n-1)\alpha_1, \quad A_2 = n(n-1)(n-2)\alpha_2,$$

$$A_3 = n(n-1)\dots(n-3)\alpha_3, \ \dots,$$

and to obtain the values of $\alpha_1, \alpha_2, \alpha_3, \dots$ from the formulæ

$$\alpha_1 = \frac{1}{2\,!},$$

$$2\alpha_2 = \frac{1}{3\,!} + \frac{n-1}{2\,!}\alpha_1,$$

$$3\alpha_3 = \frac{1}{4\,!} + \frac{n-1}{3\,!}\alpha_1 + \frac{n-2}{2\,!}\alpha_2,$$

$$4\alpha_4 = \frac{1}{5\,!} + \frac{n-1}{4\,!}\alpha_1 + \frac{n-2}{3\,!}\alpha_2 + \frac{n-3}{2\,!}\alpha_3,$$

&c. &c.

§ 48. It is perhaps worth while also to give the corresponding expressions for $S_r(1, 2, \dots, n)$. These expressions are of course deducible from § 46 by putting $n+1$ for n. Thus, writing for brevity S_r in place of $S_r(1, 2, \dots, n)$, we have

$$S_1 = \tfrac{1}{2}(n+1)n,$$

$$S_2 = \tfrac{1}{24}(n+1)n(n-1)(3n+2),$$

$$S_3 = \tfrac{1}{48}(n+1)^2n^2(n-1)(n-2),$$

$$S_4 = \tfrac{1}{5760}(n+1)n(n-1)(n-2)(n-3)(15n^3+15n^2-10n-8),$$

$$S_5 = \tfrac{1}{11520}(n+1)^2n^2(n-1)(n-2)(n-3)(n-4)(3n^2-n-6),$$

$$S_6 = \tfrac{1}{2903040}(n+1)n(n-1)(n-2)(n-3)(n-4)(n-5)$$
$$\times(63n^5-315n^3-224n^2+140n+96),$$

$$S_7 = \tfrac{1}{5806080}(n+1)^2n^2(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$$
$$\times(9n^4-18n^3-57n^2+34n+80).$$

§ 49. I may remark that for the calculation of the numerical values of the A's, when n is a given number, I found it more convenient to use the formulæ in the original form of § 3 than in the modified form of § 47. For the formation of a table such as that in § 50, the formula

$$S_r(1, 2, \ldots, n) = S_r(1, 2, \ldots, n-1) + nS_{r-1}(1, 2, \ldots, n-1) \quad (\S\, 22)$$

affords much the readiest process. As a fact, however, the table was calculated as far as $n=10$ by § 3 (*i.e.* each column independently): it was then verified, and extended to $n=20$, by this formula.

Table of the values of $S_n(1, 2, 3, \ldots, n)$.

§ 50. The following is the table referred to in § 30. It gives the values of $S_r(1, 2, 3, \ldots, n)$ for $n=1, 2, 3, \ldots, 22$. The lines correspond to the values of r, and the heading of the column is the value of n. Thus, for example, $S_4(1, 2, \ldots, 9)=63273$.

	1	2	3	4	5	6	7	8	9
1	1	3	6	10	15	21	28	36	45
2		2	11	35	85	175	322	546	870
3			6	50	225	735	1960	4536	9450
4				24	274	1624	6769	22449	63273
5					120	1764	13132	67284	269325
6						720	13068	118124	723680
7							5040	109584	1172700
8								40320	1026576
9									362880

	10	11	12	13
1	55	66	78	91
2	1320	1925	2717	3731
3	18150	32670	55770	91091
4	157773	357423	749463	1474473
5	902055	2637558	6926634	16669653
6	3416930	13339535	44990231	135036473
7	8409500	45995730	206070150	790943153
8	12753576	105258076	657206836	3336118786
9	10628640	150917976	1414014888	9957703756
10	3628800	120543840	1931559552	20313753096
11		39916800	1486442880	26596717056
12			479001600	19802759040
13				6227020800

	14	15	16
1	105	120	136
2	5005	6580	8500
3	143325	218400	323680
4	2749747	4899622	8394022
5	37312275	78558480	156952432
6	368411615	928095740	2185031420
7	2681453775	8207628000	23057159840
8	14409322928	54631129553	185953177553
9	56663366760	272803210680	1146901283528
10	159721605680	1009672107080	5374523477960
11	310989260400	2706813345600	18861567058880
12	392156797824	5056995703824	48366009233424
13	283465647360	6165817614720	87077748875904
14	87178291200	4339163001600	102992244837120
15		1307674368000	70734282393600
16			20922789888000

	17	18
1	153	171
2	10812	13566
3	468180	662796
4	13896582	22323822
5	299650806	549789282
6	4853222764	10246937272
7	60202693980	147560703732
8	577924894833	1661573386473
9	4308105301929	14710753408923
10	24871845297936	102417740732658
11	110228466184200	557921681547048
12	369012649234384	2353125040549984
13	909299905844112	7551527592063024
14	1583313975727488	17950712280921504
15	1821602444624640	30321254007719424
16	1223405590579200	34012249593822720
17	355687428096000	22376988058521600
18		6402373705728000

	19	20
1	190	210
2	16815	20615
3	920550	1256850
4	34916946	53327946
5	973941900	1672280820
6	20692933630	40171771630
7	342252511900	756111184500
8	4465226757381	11310276995381
9	46280647751910	135585182899530
10	381922055502195	1307535010540395
11	2503858755467550	10142299865511450
12	12953636989943896	63030812099294896
13	52260903362512720	311333643161390640
14	161429736530118960	1206647803780373360
15	371384787345228000	3599979517947607200
16	610116075740491776	8037811822645051776
17	668609730341153280	12870931245150988800
18	431565146817638400	13803759753640704000
19	121645100408832000	8752948036761600000
20		2432902008176640000

	21	22
1	231	253
2	25025	30107
3	1689765	2240315
4	79721796	116896626
5	2792167686	4546047198
6	75289668850	136717357942
7	1599718388730	3256091103430
8	27188611869881	62382416421941
9	373100999802531	971250460939913
10	4154823851430525	12363045847086207
11	37600535086859745	129006659818331295
12	276019109275035346	1103230881185949736
13	1634980697246583456	7707401101297361068
14	7744654310169576800	43714229649594412832
15	28939583397335447760	199321978221066137360
16	83637381699544802976	720308216440924653696
17	181664979520697076096	2021687376910682741568
18	284093315901811468800	4280722865357147142912
19	298631902863216384000	6548684852703068697600
20	186244810780170240000	6756146673770930688000
21	51090942171709440000	4148476779335454720000
22		1124000727777607680000

Addition to the paper.

§ 51. Since this paper has been in type I have found that the values of r for which the congruences (i), (ii), (iii) of III., § 33 (p. 17) hold good may be extended. For, putting $n=2p$, so that

$$A_r = S_r(1, 2, \ldots, 2p-1),$$

and supposing p an uneven prime, we have, as in § 44,

$$(2p)_p + (2p-1)_{p-1} A_1 + (2p-2)_{p-2} A_2 + \ldots + (p+2)_2 A_{p-2}$$
$$= (p-1) A_{p-1};$$

whence, since all the A's on the left-hand side are divisible by p,

$$A_{p-1} \equiv -2, \quad \text{mod. } p \text{ (§ 44)}.$$

The next equation, giving A_p, is

$$(2p)_{p+1} + (2p-1)_p A_1 + (2p-2)_{p-1} A_2 + \ldots + (p+1)_2 A_{p+1} = pA_p.$$

Now on the left-hand side all the terms except the first two and the last are divisible by p^2 (for in each case both the coefficient and the A is divisible by p), and we have

$$(2p)_{p+1} \equiv 2p, \quad \text{mod. } p^2,$$
$$(2p-1)_p A_1 \equiv -p, \quad \text{mod. } p^2,$$
$$(p+1)_2 A_{p-1} \equiv -p, \quad \text{mod. } p^2,$$

so that the equation gives

$$2p - p - p \equiv pA_p, \quad \text{mod. } p^2,$$

and therefore $A_p \equiv 0$, mod. p.

The next equation is

$$(2p)_{p+2} + (2p-1)_{p+1} A_1 + (2p-2)_p A_2 + \ldots + (p+1)_3 A_{p-1} + (p)_2 A_p$$
$$= (p+1) A_{p+1},$$

giving $$A_{p+1} \equiv 0, \quad \text{mod. } p.$$

Similarly in the next equation the only terms which need examination are the first term, in which no A occurs, and the term involving A_{p-1}, for all the other A's are divisible by p. The coefficient of A_{p-1} is $(p+1)_4$ which $\equiv 0$, mod. p, and we therefore find $A_{p+2} \equiv 0$, mod. p. This reasoning holds good as far as the equation giving A_{2p-3}, which $\equiv 0$, mod. p

But in the equation giving A_{2p-2} the coefficient of A_{p-1} is $(p+1)_p$ which $\equiv 1$, mod. p, so that in this case we have

$$-2 \equiv -2A_{2p-2}, \quad \text{mod. } p,$$

that is, $$A_{2p-2} \equiv 1, \quad \text{mod. } p.$$

Thus $A_r \equiv 0$, mod. p, for all values of r from $r=1$ to $r=2p-3$ inclusive, except $r=p-1$; and

$$A_{p-1} \equiv -2, \quad A_{2p-2} \equiv 1, \quad \text{mod. } p.$$

§ 52. Expressed in the S-form these results are: if p is an uneven prime, then

$$S_r(1, 2, \ldots, 2p-1) \equiv 0, \quad \text{mod. } p,$$

for all values of r from 1 to $2p-3$ inclusive, except $r=p-1$; and

$$S_{p-1}(1, 2, \ldots, 2p-1) \equiv -2, \quad \text{mod. } p,$$
$$S_{2p-2}(1, 2, \ldots, 2p-1) \equiv 1, \quad \text{mod. } p.$$

Now, reasoning as in § 17, we see from (iii), § 14, that

$$\sigma_r(1, 2, \ldots, p-1;\ 2p) \equiv 0, \quad \text{mod. } p,$$

for the values $r = 1, 2, \ldots, p-2$, excepting only $r=\tfrac{1}{2}(p-1)$, and therefore, from (iv), § 14,

$$S_r(1, 2, \ldots, 2p-1) \equiv 0, \quad \text{mod. } p^2,$$

for all uneven values of r from 3 to $2p-3$ inclusive, except $r=p$.

In the special case of $r=p$, we have

$$t=\tfrac{1}{2}(p-1) \text{ and therefore } \mu=\tfrac{1}{2}(p+1).$$

Thus formula (iv) gives

$$S_p(1, 2, \ldots, 2p-1) \equiv p^2\sigma_{\frac{1}{2}(p-1)}(1, 2, \ldots, p-1;\ 2p), \quad \text{mod. } p^3,$$

so that in this case also the theorem holds good, and we have

$$S_r(1, 2, \ldots, 2p-1) \equiv 0, \quad \text{mod. } p^2,$$

for all uneven values of r from 3 to $2p-3$ inclusive.

From (iii), § 14, we have

$$\sigma_{\frac{1}{2}(p-1)}(1, 2, \ldots, p-1;\ 2p) \equiv S_{p-1}(1, 2, \ldots, 2p-1), \quad \text{mod. } p^2$$
$$\equiv -2, \quad \text{mod. } p,$$

and therefore

$$S_p(1, 2, \ldots, 2p-1) \equiv -2p^2, \text{ mod. } p^3.$$

Also

$$S_{2p-1}(1, 2, \ldots, 2p-1) = (2p-1)!$$
$$= p \times (p-1)! \times (p+1)(p+2)\ldots(2p-1).$$

The two products $(p-1)!$ and $(p+1)(p+2)\ldots(2p-1)$ are each $\equiv -1$, mod. p, and therefore

$$S_{2p-1}(1, 2, \ldots, 2p-1) \equiv p, \text{ mod. } p^2.$$

§ 53. Proceeding as in § 21, we see that

$$(2p-r)pS_{r-1}(1, 2, \ldots, 2p-1) \equiv S_r(1, 2, \ldots, 2p-1), \text{ mod. } p^4,$$

for the values $r = 5, 7, \ldots, 2p-1$, excepting only $r = p+2$.

In this excepted case the formula of § 20 gives

$$(p-2)pS_{p-1}(1, 2, \ldots, 2p-1) - S_{p+2}(1, 2, \ldots, 2p-1)$$
$$\equiv \frac{p^4(p-1)(p-2)}{3} \sigma_{\frac{1}{2}(p-1)}(1, 2, \ldots, p-1;\ 2p), \text{ mod. } p^5,$$

so that in this case also, if $p > 3$, the theorem is true, and we have generally

$$(2p-r)pS_{r-1}(1, 2, \ldots, 2p-1) \equiv S_r(1, 2, \ldots, 2p-1), \text{ mod. } p^4,$$

for all uneven values of r from 5 to $2p-1$ inclusive.

We also see that

$$(p-2)pS_{p+1}(1, 2, \ldots, 2p-1) - S_p(1, 2, \ldots, 2p-1)$$
$$\equiv -\tfrac{4}{3}p^4, \text{ mod. } p^5.$$

§ 54. It has thus been shown that if $n = 2p$, so that

$$A_r = S_r(1, 2, \ldots, 2p-1),$$

where p is an uneven prime, then

(i) A_r(r even, except $r = p-1$, and $2p-2) \equiv 0$, mod. p.

(ii) A_r(r uneven and $> 1) \equiv 0$, mod. p^2.

(iii) A_r (r uneven and > 3, including $r = 2p-1$)

$$\equiv p(2p-r)A_{r-1}, \text{ mod. } p^4.$$

(iv) $A_{p-1} \equiv -2,\ A_{2p-2} \equiv 1,$ mod. p.

(v) $A_p \equiv -2p^2,$ mod. p^3.

(vi) $A_{2p-1} \equiv p,$ mod. p^2.

(vii) $(p-2)pA_{p+1} - A_{p+2} \equiv -\tfrac{4}{3}p^4,$ mod. p^5.

In (iii) and (vii) p must be >3. The results (i), ..., (iv) are stated in the note on p. 17.

§ 55. Consider now the corresponding theorems in which $n = 2p + 1$, so that

$$A_r = S_r(1, 2, \ldots, 2p),$$

p being an uneven prime as before.

We have, as in § 44,

$$(2p+1)_p + (2p)_{p-1}A_1 + (2p-1)_{p-2}A_2 + \ldots + (p+3)_2 A_{p-2}$$
$$= (p-1)A_{p-1},$$

giving $$A_{p-1} \equiv -2, \text{ mod. } p$$

as in the general theorem (§ 44).

The next equation is

$$(2p+1)_{p+1} + (2p)_p A_1 + (2p-1)_{p-1}A_2 + \ldots + (p+2)_2 A_{p-1} = pA_p.$$

None of the coefficients on the left-hand side are divisible by p, and we cannot therefore obtain the residue of A_p, mod. p from this equation by the simple procedure of § 51. We know, however, that $A_p \equiv 0$, mod. p by virtue of the general theorem (§ 22) that, if $n = 2m$, then

$$A_r (r \text{ uneven}) \equiv 0, \text{ mod. } m,$$

whether m be prime or not.

The next equation is

$$(2p+1)_{p+2} + (2p)_{p+1}A_1 + (2p-1)_p A_2 + \ldots + (p+1)_2 A_p$$
$$= (p+1)A_{p+1},$$

which gives, as in § 51, $A_{p+1} \equiv 0$, mod. p. We can thus show, as in § 51, that $A_{p+2} \equiv 0,\ A_{p+3} \equiv 0, \ldots, A_{2p-3} \equiv 0$, mod. p; and $A_{2p-2} \equiv 1$, mod. p.

Thus $A_r \equiv 0$, mod. p for all even values of r, except $r = p - 1$ and $2p - 2$;

and $$A_{p-1} \equiv -2,\ A_{2p-2} \equiv 1, \text{ mod. } p.$$

§ 56. We may, however, prove these results more simply, and also obtain fresh results similar to those in § 52 by means of the formula (§ 22)

$$S_r(1, 2, \ldots, 2p) = S_r(1, 2, \ldots, 2p-1) + 2pS_{r-1}(1, 2, \ldots, 2p-1).$$

This equation shows that $S_r(1, 2, \ldots, 2p)$ is divisible by p whenever $S_r(1, 2, \ldots, 2p-1)$ is so divisible; thus

$$S_r(1, 2, \ldots, 2p) \equiv 0, \text{ mod. } p,$$

for all even values of r, except $r = p-1$ and $2p-2$; and also

$$S_{p-1}(1, 2, \ldots, 2p) \equiv -2, \text{ mod. } p,$$
$$S_{2p-2}(1, 2, \ldots, 2p) \equiv 1, \text{ mod. } p.$$

The equation also shows that, when r is uneven and > 1,

$$S_r(1, 2, \ldots, 2p) \equiv 0, \text{ mod. } p^2,$$

except when $r-1 = p-1$ and $2p-2$, that is, when $r = p$ and $2p-1$.

Corresponding to these cases we find from the same equation

$$S_p(1, 2, \ldots, 2p) \equiv -4p, \text{ mod. } p^2;$$

and $$S_{2p-1}(1, 2, \ldots, 2p) \equiv 3p, \text{ mod. } p^2.$$

§ 57. It has thus been shown that, if $n = 2p+1$, so that

$$A_r = S_r(1, 2, \ldots, 2p),$$

where p is an uneven prime, then

(i) A_r (r even, except $r = p-1$ and $2p-2$) $\equiv 0$, mod. p,

(ii) A_r (r uneven and > 1, except $r = p$ and $2p-1$) $\equiv 0$, mod. p^2,

(iii) $A_{p-1} \equiv -2$, $A_{2p-2} \equiv 1$, mod. p,

(iv) $A_p \equiv -4p$, $A_{2p-1} \equiv 3p$, mod. p^2.

The results (i), (ii), (iii) have been stated in the note on p. 18.

§ 58. Since $S_{2t}(1, 2, \ldots, 2p-1)$ is divisible by p, if p is an uneven prime, for all values of $2t$, except $2t = p-1$ and $2p-2$, we see from (iii) of § 26 that $S_t\{1^2, 2^2, \ldots, (p-1)^2\}$ is divisible by p, except when $t=\frac{1}{2}(p-1)$ or $p-1$. For $t=\frac{1}{2}(p-1)$, we have

$$S_{\frac{1}{2}(p-1)}\{1^2, 2^2, \ldots, (p-1)^2\} \equiv (-1)^{\frac{1}{2}(p-1)} S_{p-1}(1, 2, \ldots, 2p-1), \text{ mod. } p$$
$$\equiv (-1)^{\frac{1}{2}(p+1)} 2, \quad \text{mod. } p.$$

Also $\quad S_{p-1}\{1^2, 2^2, \ldots, (p-1)^2\} \equiv 1, \quad \text{mod. } p.$

Thus, if $n+1$ is prime,

$$S_r(1^2, 2^2, \ldots, n^2) \equiv 0, \quad \text{mod. } n+1,$$

for all values of r from 1 to $n-1$ except $\frac{1}{2}n$, and

$$S_{\frac{1}{2}n}(1^2, 2^2, \ldots, n^2) \equiv (-1)^{\frac{1}{2}n+1} 2, \quad \text{mod. } n+1.$$

§ 59. In conclusion it will be convenient to place together in one list all the results which have been obtained relative to the divisibility of $S_r(1, 2, \ldots, n)$*. This list corresponds to the A-list in § 35.

For brevity S_r is written in place of $S_r(1, 2, \ldots, n)$; r is supposed to be $< n$, and by a prime is to be understood a prime greater than 2.

I. n even,

$$S_r \equiv 0, \quad \text{mod. } n+1, \quad (r \text{ uneven}).$$

II. $n+1$ a prime > 3,

$$S_r \equiv 0, \quad \text{mod. } n+1, \quad (r \text{ even}),$$
$$S_r \equiv 0, \quad \text{mod. } (n+1)^2, \quad (r \text{ uneven and } > 1).$$

III. n uneven,

$$S_r \equiv 0, \quad \text{mod. } n, \quad (r \text{ uneven}).$$

IV. n a prime,

$$S_r \equiv 0, \quad \text{mod. } n, \quad (r \text{ even and } < n-1),$$
$$S_r \equiv 0, \quad \text{mod. } n^2, \quad (r \text{ uneven and } > 1),$$
$$S_{n-1} \equiv -1, \quad \text{mod. } n.$$

V. n even $= 2m$,

$$S_r \equiv 0, \quad \text{mod. } m, \quad (r \text{ uneven}).$$

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* This list of results is also given in the *Messenger*, Vol. XXVIII., pp. 184–187.

VI. n uneven $= 2m - 1$,

$$S_r \equiv 0, \quad \text{mod. } m, \quad (r \text{ uneven}).$$

Also when m is uneven,

$$S_{n-3} \equiv 0, \quad \text{mod. } m; \quad S_{n-2} \equiv 0, \quad \text{mod. } m^2.$$

VII. $n = 2p$, where p is prime,

$$S_r \equiv 0, \quad \text{mod. } p, \quad (r \text{ even, except } r = p - 1 \text{ and } 2p - 2),$$

$$S_r \equiv 0, \quad \text{mod. } p^2, \quad (r \text{ uneven, except } r = p \text{ and } 2p - 1),$$

$$S_{p-1} \equiv -2 \;, \quad S_{2p-2} \equiv 1 \;, \quad \text{mod. } p,$$

$$S_p \equiv -4p, \quad S_{2p-1} \equiv 3p, \quad \text{mod. } p^2.$$

VIII. $n = 2p - 1$, where p is prime,

$$S_r \equiv 0, \quad \text{mod. } p, \quad (r \text{ even, except } r = p - 1 \text{ and } 2p - 2),$$

$$S_r \equiv 0, \quad \text{mod. } p^2, \quad (r \text{ uneven and } > 1),$$

$$S_{p-1} \equiv -2, \quad S_{2p-2} \equiv 1, \quad \text{mod. } p,$$

$$S_p \equiv -2p^2, \quad \text{mod. } p^3; \quad S_{2p-1} \equiv p, \quad \text{mod. } p^2.$$

IX. If p is a prime not exceeding $n+1$, and if k be the quotient and q the remainder when $n+1$ is divided by p, then

$$S_q, \; S_{q+1}, \; \ldots, \; S_{p-2}, \text{ and } S_{p-1} + k$$

are all divisible by p. If q is zero, S_q is to be omitted.

X. The congruence

$$S_r \equiv \frac{(n+1)(n+1-r)}{2} S_{r-1}$$

holds good for

(i) mod. $(n+1)^3$, if n is even and > 2, and r uneven and > 1,

(ii) mod. $(n+1)^4$, if $n+1$ is a prime > 5, and r uneven and > 3,

(iii) mod. m^3, if $n = 2m - 1$, and r is uneven and > 1 including $r = 2m - 1$,

(iv) mod. p^4, if $n = 2p - 1$, where p is a prime > 3, and r is uneven and > 3, including $r = 2p - 1$.

THE SCATTERING OF ELECTRO-MAGNETIC WAVES BY A SPHERE.

By G. W. Walker, A.R.C.Sc., B.A., Scholar of Trinity College, Cambridge; Isaac Newton Student in the University of Cambridge.

In the *Quarterly Journal of Mathematics*, Vol. xxx., p. 204, I investigated the scattering of electro-magnetic waves by small, slightly conducting spheres. The method of approximation adopted in that paper is valid when dealing with small values of conductivity, but fails entirely with large values. Professor Thomson, whom I consulted about the matter, suggested that it would be interesting to work out the general case, and to ascertain, if possible, the reason for the widely different laws of scattering in the two extreme cases (1) of a perfect conductor, and (2) of a perfect insulator.

Before proceeding with the mathematical analysis of the question, it will be advisable to consider the purely physical aspect of the matter.

It is usually assumed that K, the specific inductive capacity, is infinite for a conductor. The distinction between true and apparent specific inductive capacity is not, I think, sufficiently insisted on.

Let us consider this question carefully. Suppose that we place an imperfect conductor in an electrical field which is practically steady: at any point of the body there will be an electrical force. Now, inasmuch as the body conducts a little, we shall have a separation of electricity at any point. Positive electricity will flow in one direction, and negative electricity in the other. This electricity will be stopped at the surface of the body, and its effect will be to reduce the electrical force inside the body.

Separation of electricity will go on until there is no force in the interior of the body. When such a state of things is reached the body satisfies our conception of a perfect conductor.

Thus we see that, if sufficient time is allowed to elapse, badly conducting bodies will act in a steady field as if they were perfect conductors. It is otherwise when the external electrical field is rapidly changing.

The electricity in an imperfect conductor has not time to take up its equilibrium value.

In the case of optics we deal with waves of exceedingly short periods, and so we are not at liberty to make the apparent specific inductive capacity infinite; but we must retain the true specific inductive capacity and the true electrical conductivity.

We shall find from the analysis that the conductivity even of metals is not sufficiently great to be regarded as infinite when considering waves of optical frequency.

We now proceed to the analytical treatment of the problem of the scattering of plane curves by a sphere which is neither a perfect insulator nor a perfect conductor.

For the ether we have K_0 and μ_0,

$$\left.\begin{aligned} 4\pi\frac{\partial f}{\partial t} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \text{ \&c.,} \\ -\mu_0\frac{\partial \alpha}{\partial t} &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \text{ \&c.} \\ f = \frac{K_0 X}{4\pi},\ g &= \frac{K_0 Y}{4\pi},\ h = \frac{K_0 Z}{4\pi}. \end{aligned}\right\} \ldots (1),$$

Since we are to consider periodic values, we have

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 0 = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z},$$

K_0 and μ_0 being constants.

Again, for the sphere we have K_1, k_1, and μ_1 constants.

$$\left.\begin{aligned} 4\pi\left\{\frac{\partial f}{\partial t} + \frac{4\pi k_1}{K_1} f\right\} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}, \text{ \&c.,} \\ -\mu_1\frac{\partial \alpha}{\partial t} &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}, \text{ \&c.} \\ f = \frac{K_1 X}{4\pi},\ g &= \frac{K_1 Y}{4\pi},\ h = \frac{K_1 Z}{4\pi}, \\ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} &= 0 = \frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z}. \end{aligned}\right\} \ldots (2),$$

We shall suppose that all the quantities are proportional to e^{ipt}, and so write $\frac{\partial f}{\partial t} = ipf$, and omit the time factor.

It has been proved by Professor H. Lamb (*Proc. Lond. Math. Soc.*, XIII., p. 189, 1881) that the most general expressions which satisfy the conditions (1) are

$$\left.\begin{aligned}4\pi ipf &= \Sigma \frac{1}{2n+1}\left\{(n+1)f_{n-1}(\kappa r)\frac{\partial \omega_n'}{\partial \chi} - n\kappa^2 r^{2n+3} f_{n+1}(\kappa r)\frac{\partial}{\partial \chi}\frac{\omega_n'}{r^{2n+1}}\right\}\\ &\qquad + \Sigma (2n+1)\,\kappa^2 f_n(\kappa r)\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\omega_n\\ \alpha &= \Sigma\left\{(n+1)f_{n-1}(\kappa r)\frac{\partial \omega_n}{\partial \chi} - n\kappa^2 r^{2n+3} f_{n+1}(\kappa r)\frac{\partial}{\partial \chi}\frac{\omega_n}{r^{2n+1}}\right\}\\ &\qquad + \Sigma f_n(\kappa r)\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\omega_n',\end{aligned}\right\} \quad \ldots\ldots\ldots(3),$$

with symmetrical expressions for g, h, β, and γ, where

$$\kappa^2 = p^2 K_0 \mu_0, \quad K_0\mu_0 = \frac{1}{V_0^2},$$

$$f_n(\kappa r) = \left(\frac{1}{\kappa r}\frac{\partial}{\partial \kappa r}\right)^n \frac{e^{-i\kappa r}}{\kappa r},$$

and ω_n and ω_n' are arbitrary solid spherical harmonics of degree n.

These are the proper expressions for the scattered waves in the ether.

Let the incident wave be $f_0 e^{i(pt+\kappa z)}$, where $z = r\cos\vartheta = r\mu$.

It may readily be proved that

$$e^{i\kappa z} = e^{i\kappa r\mu} = \Sigma_0^\infty (2n+1)\frac{S_n(\kappa r)}{i^n} P_n(\mu),$$

where $$S_n(\kappa r) = (\kappa r)^n\left(\frac{1}{\kappa r}\frac{\partial}{\partial \kappa r}\right)^n \frac{\sin \kappa r}{\kappa r},$$

and $P_n(\mu)$ is the zonal harmonic of order n.

Hence for the incident wave we have

$$\left.\begin{aligned}&f = f_0 \Sigma_0^\infty (2n+1)\frac{S_n(\kappa r)}{i^n} P_n(\mu),\ g = 0,\ h = 0\\ &\alpha = 0,\ \beta = -4\pi V_0 f_0 \Sigma_0^{-\infty} (2n+1)\frac{S_n(\kappa r)}{i^n} P_n(\mu)\end{aligned}\right\}\ldots(4).$$

Similarly inside the sphere we may write

$$\left.\begin{aligned}\alpha &= \Sigma\left\{(n+1)f_{n-1}(\lambda r)\frac{\partial_1\omega_n}{\partial\chi} - n\lambda^2 r^{2n+3} f_{n+1}(\lambda r)\frac{\partial}{\partial\chi}\frac{{}_1\omega_n}{r^{2n+1}}\right\} \\ &\qquad + \Sigma f_n(\lambda r)\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}{}_1\omega_n' \\ 4\pi&\left\{ip + \frac{4\pi k_1}{K_1}\right\}f \\ &= \Sigma\frac{1}{(2n+1)}\left\{(n+1)f_{n-1}(\lambda r)\frac{\partial_1\omega_n'}{\partial\chi} - n\lambda^2 r^{2n+3} f_{n+1}(\lambda r)\frac{\partial}{\partial\chi}\frac{{}_1\omega_n'}{r^{2n+1}}\right\} \\ &\quad + \Sigma(2n+1)\lambda^2 f_n(\lambda r)\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}{}_1\omega_n\end{aligned}\right\} \quad \text{.........(5),}$$

with symmetrical expressions for g, h, β, and γ, where

$$\lambda^2 = p^2 K_1\mu_1 - 4\pi i p\mu_1 k_1.$$

Since the series is to be convergent at the origin, we must take

$$f_n(\lambda r) = \left(\frac{1}{\lambda r}\frac{\partial}{\partial\lambda r}\right)^n \frac{\sin\lambda r}{\lambda r}.$$

The expressions represent the waves inside the sphere.

In expressing the boundary conditions we shall require the normal components of electric and magnetic force.

(1) For the incident wave

$$f\frac{x}{r} = f_0 \sin\vartheta\cos\phi\,\Sigma_0(2n+1)\frac{S_n(\kappa r)}{i^n}P_n(\mu),$$

$$\beta\frac{y}{r} = -4\pi V_0 f_0 \sin\vartheta\sin\phi\,\Sigma_0(2n+1)\frac{S_n(\kappa r)}{i^n}P_n(\mu).$$

Now $$(2n+1)P_n(\mu) = \frac{\partial P_{n+1}}{\partial\mu} - \frac{\partial P_{n-1}}{\partial\mu}.$$

Thence

$$\begin{aligned}\Sigma_0(2n+1)\frac{S_n(\kappa r)P_n}{i^n} &= \Sigma_0\frac{S_n(\kappa r)}{i^n}\left(\frac{\partial P_{n+1}}{\partial\mu} - \frac{\partial P_{n-1}}{\partial\mu}\right) \\ &= \Sigma_1\frac{\partial P_n}{\partial\mu}\left\{\frac{S_{n-1}(\kappa r)}{i^{n-1}} - \frac{S_{n+1}(\kappa r)}{i^{n+1}}\right\} \\ &= i\Sigma_1(2n+1)\frac{S_n(\kappa r)}{i^n\kappa r}\frac{\partial P_n}{\partial\mu}.\end{aligned}$$

Thus

$$\left.\begin{aligned} f\frac{x}{r} &= if_0 \cos\phi\, \Sigma_1^\infty \frac{2n+1}{i^n} \frac{S_n(\kappa r)}{\kappa r} \sin\vartheta \frac{\partial P_n}{\partial \mu} \\ \beta\frac{y}{r} &= -4\pi i V_0 f_0 \sin\phi\, \Sigma_1^\infty \frac{2n+1}{i^n} \frac{S_n(\kappa r)}{\kappa r} \sin\vartheta \frac{\partial P_n}{\partial \mu} \end{aligned}\right\} \ldots(6).$$

(2) For the reflected waves

$$\left.\begin{aligned} 4\pi i p\left\{\frac{x}{r}f + \frac{y}{r}g + \frac{z}{r}h\right\} &= -\Sigma n(n+1)\frac{f_n(\kappa r)}{r}\omega_n' \\ \left(\frac{x}{r}\alpha + \frac{y}{r}\beta + \frac{z}{r}\gamma\right) &= -\Sigma n(n+1)(2n+1)\frac{f_n(\kappa r)}{r}\omega_n \end{aligned}\right\} \ldots(7).$$

(3) For the waves in the sphere

$$\left.\begin{aligned} 4\pi\left(ip + \frac{4\pi k_1}{K_1}\right)\left(\frac{x}{r}f + \frac{y}{r}g + \frac{z}{r}h\right) &= -\Sigma n(n+1)\frac{f_n(\lambda r)}{r}\,{}_1\omega_n' \\ \left(\frac{x}{r}\alpha + \frac{y}{r}\beta + \frac{z}{r}\gamma\right) &= -\Sigma n(n+1)(2n+1)\frac{f_n(\lambda r)}{r}\,{}_1\omega_n' \end{aligned}\right\}$$

$$\ldots\ldots\ldots(8).$$

The boundary conditions which we shall assume are continuity of tangential electric and magnetic force at the surface of the sphere.

Let R_0, Θ_0, and Φ_0 be the components of electrical force outside, and N_0 the normal component of magnetic force; and let the suffix 1 refer to the inside.

We have the equations

$$-\mu_0\frac{\partial N_0}{\partial t} = \frac{1}{r^2\sin\vartheta}\left\{\frac{\partial r\Phi_0}{\partial\vartheta} - \frac{\partial}{\partial\phi} r\sin\vartheta\,\Theta_0\right\}$$

$$-\mu_1\frac{\partial N_1}{\partial t} = \frac{1}{r^2\sin\vartheta}\left\{\frac{\partial r\Phi_1}{\partial\vartheta} - \frac{\partial}{\partial\phi} r\sin\vartheta\,\Theta_1\right\},$$

$$\frac{\partial}{\partial r}r^2R_0 + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}r\sin\vartheta\,\Theta_0 + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\phi}r\Phi_0 = 0,$$

$$\frac{\partial}{\partial r}r^2R_1 + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}r\sin\vartheta\,\Theta_1 + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\phi}r\Phi_1 = 0.$$

The differentiations on Θ_0, Φ_0, Θ_1, and Φ_1 are performed along the surface.

Hence it follows that, if $\Theta_0 = \Theta_1$ and $\Phi_0 = \Phi_1$ when $r = a$, we must have

$$\left.\begin{aligned} \frac{\partial}{\partial r}\, r^2 R_0 &= \frac{\partial}{\partial r}\, r^2 R_1 \\ \mu_0 \frac{\partial N_0}{\partial t} &= \mu_1 \frac{\partial N_1}{\partial t} \end{aligned}\right\} \text{..................} (9),$$

when $r = a$.

In an exactly similar manner it appears that, if the tangential components of magnetic force are continuous, we must have

$$\left.\begin{aligned} \frac{\partial}{\partial r}\, r^2 N_0 &= \frac{\partial}{\partial r}\, r^2 N_1 \\ ipK_0 R_0 &= \left(ip + \frac{4\pi k_1}{K_1}\right) K_1 R_1 \end{aligned}\right\} \text{............} (10),$$

when $r = a$.

These are the surface conditions, and we may observe that they involve the continuity of normal magnetic induction.

Since $K_0 R_0 - K_1 R_1 = 4\pi\sigma$, we have a surface density of electricity given by

$$\sigma = \frac{k_1 R_1}{ip}.$$

If p be not zero, σ will vanish when k_1 is zero, and in that case the continuity of normal electric induction is fulfilled. This is just what our preliminary considerations would lead us to expect.

On referring to the expressions (6), (7), and (8), it will be observed that the proper forms to assume in order that (9) and (10) may be satisfied are as follows:

$$\left.\begin{aligned} \varpi_n &= (2n+1) A_n \sin\vartheta \sin\phi \frac{\partial P_n}{\partial \mu} \frac{r^n}{a^n} \\ \varpi_n' &= A_n' \sin\vartheta \cos\phi \frac{\partial P_n}{\partial \mu} \frac{r^n}{a^n} \\ {}_1\varpi_n &= (2n+1) B_n \sin\vartheta \sin\phi \frac{\partial P_n}{\partial \mu} \frac{r^n}{a^n} \\ {}_1\varpi_n' &= B_n' \sin\vartheta \cos\phi \frac{\partial P_n}{\partial \mu} \frac{r^n}{a^n} \end{aligned}\right\} \text{......} (11),$$

Substituting these expressions in (6), (7), and (8), and applying conditions (9) and (10), we obtain four series. We equate the coefficients of $\frac{\partial P_n}{\partial \mu}$ to zero, and the following relations between the coefficients are obtained:

$$-\frac{1}{4\pi i p K_0}\frac{A_n'}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)+\frac{1}{4\pi\left(ip+\frac{4\pi k_1}{K_1}\right)K_1}\times\frac{B_n'}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)=-\frac{if_0}{\kappa K_0}\frac{(2n+1)}{n(n+1)}\frac{\partial}{\partial a}\frac{aS_n(\kappa a)}{i^n},$$

$$-f_n(\kappa a)A_n'+f_n(\lambda a)B_n'=\frac{4\pi p f_0}{\kappa}\frac{(2n+1)}{n(n+1)}\frac{S_n(\kappa a)}{i^n},$$

$$-\frac{A_n}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)+\frac{B_n}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)=\frac{4\pi i V_0 f_0}{\kappa}\frac{(2n+1)}{n(n+1)}\frac{\partial}{\partial a}\frac{aS_n(\kappa a)}{i^n},$$

$$-\mu_0 f_n(\kappa a)A_n+\mu_1 f_n(\lambda a)B_n=\mu_0 4\pi i V_0 f_0\frac{2n+1}{n(n+1)}\frac{S_n(\kappa a)}{i^n}.$$

From these equations we obtain the values

$$A_n'=\frac{4\pi f_0(2n+1)}{i^n n(n+1)}\times\frac{\left\{-\frac{i}{\kappa K_0}f_n(\lambda a)\frac{\partial}{\partial a}aS_n(\kappa a)-\frac{V_0}{\left(ip+\frac{4\pi k_1}{K_1}\right)K_1}\frac{S_n(\kappa a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)\right\}}{\left\{\frac{1}{\left(ip+\frac{4\pi k_1}{K_1}\right)K_1}\frac{f_n(\kappa a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)-\frac{1}{ipK_0}\frac{f_n(\lambda a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)\right\}}$$

$$A_n=\frac{4\pi i V_0 f_0}{\kappa i^n}\frac{(2n+1)}{n(n+1)}\times\frac{\left\{\mu_0\frac{S_n(\kappa a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)-\mu_1 f_n(\lambda a)\frac{\partial}{\partial a}aS_n(\kappa a)\right\}}{\left\{\mu_1\frac{f_n\lambda a}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)-\mu_0\frac{f_n(\kappa a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)\right\}},$$

$$B_n' = \frac{4\pi i f_0}{\kappa K_0 i^n} \cdot \frac{2n+1}{n(n+1)}$$

$$\times \frac{\left\{-f_n(\kappa a)\frac{\partial}{\partial a} a\, S_n(\kappa a) - \frac{S_n(\kappa a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\kappa a)\right\}}{\left\{\frac{1}{\left(ip+\frac{4\pi k_1}{K_1}\right)K_1}\frac{f_n(\kappa a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\lambda a) - \frac{1}{ipK_0}\frac{f_n(\lambda a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\kappa a)\right\}},$$

$$B_n = \frac{4\pi i V_0 f_0 (2n+1)}{\kappa i^n n(n+1)}$$

$$\times \frac{\left\{\mu_0 \frac{S_n(\kappa a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\kappa a) - \mu_1 f_n(\kappa a)\frac{\partial}{\partial a} a S_n(\kappa a)\right\}}{\left\{\mu_1 \frac{f_n(\lambda a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\kappa a) - \mu_0 \frac{f_n(\kappa a)}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\lambda a)\right\}}.$$

It will now be convenient to show that, if k_1, the conductivity, can be considered exceedingly great compared with any other large quantity involved $\left(\text{such as } p \text{ or } \frac{1}{a}\right)$, the above expressions converge to the values found by Professor Thomson for a perfect conductor.

We have

$$f_n(\lambda r) = \left(\frac{1}{\lambda r}\frac{\partial}{\partial \lambda r}\right)^n \frac{\sin \lambda r}{\lambda r}.$$

If λr is exceedingly great the most important term of $f_n(\lambda r)$ is

$$f_n(\lambda r) = \frac{\sin\{\frac{1}{2}(n\pi) + \lambda r\}}{(\lambda r)^{n+1}}.$$

Again, $\frac{1}{r^n}\frac{\partial}{\partial r} r^{n+1} f_n(\lambda r) = (n+1) f_n(\lambda r) + \lambda^2 r^2 f_{n+1}(\lambda r)$

$$= (n+1) f_n(\lambda r) + \frac{\sin\{\frac{1}{2}(n+1)\pi + \lambda r\}}{(\lambda r)^n}.$$

Thus $\frac{1}{r^n}\frac{\partial}{\partial r} r^{n+1} f_n(\lambda r)$ is very great compared with $f_n(\lambda r)$, although both are small quantities, while k_1 is proportional to λ^2.

Hence

$$\frac{1}{\left(ip+\frac{4\pi k_1}{K_1}\right)K_1}\frac{1}{r^n}\frac{\partial}{\partial r}r^{n+1}f_n(\lambda r)$$

is very small compared with $f_n(\lambda r)$.

Thus for infinite values of k_1, A_n' converges to the value

$$-4\pi V_0 f_0 \frac{2n+1}{i^n n(n+1)} \frac{\frac{\partial}{\partial a} aS_n(\kappa a)}{\frac{1}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)},$$

while A_n converges to the value

$$-\frac{4\pi i V_0 f_0}{\kappa i^n}\frac{(2n+1)}{n(n+1)}\frac{S_n(\kappa a)}{f_n(\kappa a)}.$$

These are the values found by Professor Thomson.* It remains to show that the expressions for the electric force inside the sphere converge to zero.

Returning to the expressions (5), we see that the term involving B_n in the expression for f (which is proportional to the force) is proportional to

$$\frac{\lambda^2\left(\frac{r}{a}\right)^n f_n(\lambda r)B_n}{\left(ip+\frac{4\pi k_1}{K_1}\right)},$$

that is, proportional to

$$\frac{\left(\frac{r}{a}\right)^n f_n(\lambda r)}{\left\{\mu_1\frac{f_n\lambda a}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\kappa a)-\frac{\mu_0 f_n(\kappa a)}{a^n}\frac{\partial}{\partial a}a^{n+1}f_n(\lambda a)\right\}},$$

(retaining only factors which may become very great or very small as k_1 is made infinite).

Remembering the important terms in $f_n(\lambda r)$, &c., this reduced to the limit when k_1 is infinite of

$$\frac{1}{\lambda r}\frac{\sin\left(\frac{n\pi}{2}+\lambda r\right)}{\cos\left(\frac{n\pi}{2}+\lambda a\right)}.$$

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* See *Recent Researches*, by J. J. Thomson.

Now $$\lambda^2 = p^2 K_1 \mu_1 - 4\pi i p \mu_1 k_1,$$

so that, as k_1 becomes infinite, we have

$$\lambda = \frac{1-i}{\sqrt{2}} \sqrt{(4\pi\mu_1 k_1)}.$$

Since λ is complex,

$$\cos\{\tfrac{1}{2}(n\pi) + \lambda a\}$$

cannot vanish.

Hence

$$L_\iota \frac{1}{\lambda r} \frac{\sin\{\frac{1}{2}(n\pi) + \lambda r\}}{\cos\{\frac{1}{2}(n\pi) + \lambda a\}} = L_\iota \frac{1}{i\lambda r} \frac{e^{i\lambda r} - e^{-in\pi - i\lambda r}}{e^{i\lambda a} + e^{-in\pi - i\lambda a}},$$

$$= L_\iota \frac{1}{i\lambda r} e^{i\lambda(r-a)},$$

$$= L_\iota \frac{1}{i\lambda r} e^{\sqrt{(2\pi\mu_1 k_1)}\,(r-a)\,(1+i)}.$$

Now $r < a$ so that even at the origin the limit of the expression is zero.

Again the typical term involving B_n' is proportional to

$$\frac{1}{\lambda^2}\left\{\{(n+1) f_{n-1}(\lambda r) - n\lambda^2 r^2 f_{n+1}(\lambda r)\} \frac{\partial_1 \omega_n'}{\partial x} \right.$$
$$\left. + n(2n+1)\lambda^2 x_1 \omega_n' f_{n+1}(\lambda r)\right\}.$$

The most important term in the denominator of B_n' is that involving $f_n(\lambda a)$. Thus the value depends on the limits of

$$\frac{1}{\lambda r} \frac{\sin\{\frac{1}{2}(n-1)\pi + \lambda r\}}{\cos\{\frac{1}{2}(n-1)\pi + \lambda a\}} \text{ and } \frac{1}{\lambda r} \frac{\sin\{\frac{1}{2}(n+1)\pi + \lambda r\}}{\cos\{\frac{1}{2}(n+1)\pi + \lambda a\}}.$$

These expressions converge to the limit zero for all values $a > r > 0$.

Hence each term of the expression for the force inside the sphere vanishes provided k_1 is infinite.

In dealing with actual bodies, k_1, although very great for metals, is not of sufficient magnitude to be treated as infinite. On the contrary in dealing with optical waves we may regard it as small,

$$\lambda^2 = p^2 K_1 \mu_1 - 4\pi i p \mu_1 k_1.$$

Take the case of silver, which is the best conductor we know, $\mu_1 = 1$, k_1 is of order 10^{-3}.

For sodium light p is of order 10^{15}.

Thus pk_1 is of order 10^{12}.

The other term will not be very different from κ^2, *i.e.* $\left(\frac{2\pi}{\bar{\lambda}}\right)^2$, where $\bar{\lambda}$ is now the wave length in ether. This term is of order 10^{10}, so that λ^2 is equal to the sum of a term of order 10^{10} and one of order 10^{12}. Taking a as the radius of a molecule,

$$a \text{ is of order } 10^{-9}.*$$

Thus $\lambda^2 a^2$ is at most of order 10^{-6}.

We might even take a much greater, so as to include a particle consisting of a million molecules and still have $(\lambda a)^2$ very small.

It is thus obvious that we are concerned with small values of λa and not with large values. This is the case which is of practical importance in the scattering of light from metallic particles, and so we proceed to consider the approximate value of the series under these conditions.

The numerators of A_n' and A_n vanish when $k_1 = 0$, $K_0 = K$, $\mu_0 = \mu$, while their denominators remain finite, so that for small values of differences A_n' and A_n are proportional to the first power of the differences

$$S_n(\lambda a) = (\lambda a)^n \left(\frac{1}{\lambda a}\frac{\partial}{\partial \lambda a}\right)^n \frac{\sin \lambda a}{\lambda a},$$

When (λa) is small we can expand $\sin(\lambda a)$ in powers of λa and differentiate, because the differentiated series is convergent.

Thus

$$S_n(\lambda a) = \frac{(-)^n (\lambda a)^n}{1.3\ldots 2n+1}\left\{1 - \frac{\lambda a}{2n+3}\ldots\right\},$$

$$f_n(\lambda a) = (-)^n \frac{1}{1.3\ldots 2n+1}\left\{1 - \frac{\lambda a}{2n+3}\ldots\right\},$$

$$f_n(\kappa a) = (-)^n \frac{1.3\ldots(2n-1)}{(\kappa a)^{2n+1}} + \frac{(-i)^{2n+1}}{1.3\ldots 2n+1},$$

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* I took a much larger estimate of a in the former paper, in order to be sure of the convergency of the expressions.

$$S_n(\kappa a) = (-)^n \frac{(\kappa a)^n}{1.3\ldots 2n+1} \left\{1 - \frac{\kappa a}{2n+3} \ldots\right\},$$

$$\frac{\partial}{\partial a} a S_n(\kappa a) = (-)^n \frac{(\kappa a)^n}{1.3\ldots 2n+1} \left\{n + 1 - \frac{(n+2)\,\kappa a}{2n+3} \ldots\right\}$$

$$\frac{1}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\lambda a) = (-)^n \frac{1}{1.3\ldots(2n+1)} \left\{n + 1 - \frac{(n+2)\,\lambda a}{2n+3} \ldots\right\},$$

$$\frac{1}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\kappa a) = (-)^{n+1} \frac{n.1.3\ldots 2n-1}{(\kappa a)^{2n+1}} + \frac{(-i)^{2n+1}(n+1)}{1.3\ldots 2n+1}.$$

These expressions are true when κa and λa are very small. It has been necessary to retain two terms as the first terms cancel in the numerator of A_n when $\mu_0 = \mu_1$.

$$A'_n = \frac{4\pi f_0(2n+1)}{i^n n(n+1)}$$

$$\times \frac{\left\{-\frac{i}{\kappa K_0} f_n(\lambda a) \frac{\partial}{\partial a} a S_n(\kappa a) - \frac{V_0}{(ipK_1 + 4\pi k_1)} \frac{S_n(\kappa a)}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\lambda a)\right\}}{\left\{\frac{1}{(ipK_1 + 4\pi k_1)} \frac{f_n(\kappa a)}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\lambda a) - \frac{1}{ipK_0} \frac{f_n(\lambda a)}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\kappa a)\right\}},$$

$$A_n = \frac{4\pi i V_0 f_0(2n+1)}{\kappa i^n n(n+1)} \frac{\left\{\frac{S_n \kappa a}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\lambda a) - f_n(\lambda a) \frac{\partial}{\partial a} a S_n(\kappa a)\right\}}{\left\{\frac{f_n(\lambda a)}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\kappa a) - \frac{f_n(\kappa a)}{a^n} \frac{\partial}{\partial a} a^{n+1} f_n(\lambda a)\right\}},$$

since $\mu_0 = \mu_1$ in most cases.

Substituting the approximate values of $S_n(\kappa a)$, &c., we obtain, after a little reduction,

$$A'_n = \frac{4\pi f_0 V_0 (\kappa a)^{3n+1}}{i^n n\{1.3\ldots(2n-1)\}^2} \left\{\frac{-\frac{i}{\kappa K_0} - \frac{V_0}{(ipK_1 + 4\pi k_1)}}{\frac{n}{i\kappa K_0} + \frac{(n+1)V_0}{(ipK_1 + 4\pi k_1)}}\right\},$$

$$A_n = \frac{4\pi i V_0 f_0 (\kappa a)^{3n+1} (\lambda - \kappa) a}{\kappa i^n}$$

$$\times \frac{1}{n(n-1)(1.3\ldots 2n-1)(1.3\ldots 2n+3)}.$$

Thus we observe that A_n involves κa to a higher power than does A_n'. When the values of A_n and A_n' are calculated for a perfect conductor (κa) is involved to the *same* power in each.

Now referring to the expressions (3) for f, g, and h, we see that at great distances from the origin the most important term is ω_1', ω_1 being negligible in comparison since κa is involved to a power one higher,

$$4\pi i\kappa V_0 f = \tfrac{1}{3}\left\{[2f_0(\kappa r) - \kappa^2 r^2 f_2(\kappa r)]\frac{\partial\omega_1'}{\partial x} + 3\kappa^2 f_2(\kappa r)\, x\omega_1'\right\},$$

$$4\pi i\kappa V_0 g = \tfrac{1}{3}\left\{[2f_0(\kappa r) - \kappa^2 r^2 f_2(\kappa r)]\frac{\partial\omega_1'}{\partial y} + 3\kappa^2 f_2(\kappa r)\, y\omega_1'\right\},$$

$$4\pi i\kappa V_0 h = \tfrac{1}{3}\left\{[2f_0(\kappa r) - \kappa^2 r^2 f_2(\kappa r)]\frac{\partial\omega_1'}{\partial z} + 3\kappa^2 f_2(\kappa r)\, z\omega_1'\right\}.$$

Now $$\omega_1' = A_1'\frac{r}{a}\sin\vartheta\cos\phi\,\frac{\partial P_1}{\partial\mu} = \frac{A_1'}{a}x.$$

For great values of κr we get

$$f_0(\kappa r) = \frac{e^{-i\kappa r}}{\kappa r},\quad f_1(\kappa r) = \frac{-ie^{-i\kappa r}}{(\kappa r)^2},\quad f_2(\kappa r) = \frac{-e^{-i\kappa r}}{(\kappa r)^3}.$$

Thus

$$4\pi i\kappa V_0 f = \frac{e^{-i\kappa r}}{\kappa r}\left\{1 - \frac{x^2}{r^2}\right\}\frac{A_1'}{a},$$

$$4\pi i\kappa V_0 g = \frac{e^{-i\kappa r}}{\kappa r}\left\{-\frac{xy}{r^2}\right\}\frac{A_1'}{a},$$

$$4\pi i\kappa V_0 h = \frac{e^{-i\kappa r}}{\kappa r}\left\{-\frac{xz}{r^2}\right\}\frac{A_1'}{a}.$$

Substituting for A_1' and restoring the time factor we get

$$f = f_0 e^{i(pt-\kappa r)}\frac{(y^2+z^2)}{r^3}\kappa^2 a^3\frac{\left\{\dfrac{i}{\kappa K_0} + \dfrac{V_0}{(ipK_1 + 4\pi k_1)}\right\}}{\left\{\dfrac{1}{i\kappa K_0} + \dfrac{2V_0}{(ipK_1 + 4\pi k_1)}\right\}},$$

$$g = -f_0 e^{i(pt-\kappa r)}\frac{xy}{r^3}\kappa^2 a^3\frac{\left\{\dfrac{i}{\kappa K_0} + \dfrac{V_0}{(ipK_1 + 4\pi k_1)}\right\}}{\left\{\dfrac{1}{i\kappa K_0} + \dfrac{2V_0}{(ipK_1 + 4\pi k_1)}\right\}},$$

$$h = -f_0 e^{i(pt-\kappa r)}\frac{xz}{r^3}\kappa^2 a^3\frac{\left\{\dfrac{i}{\kappa K_0} + \dfrac{V_0}{(ipK_1 + 4\pi k_1)}\right\}}{\left\{\dfrac{1}{i\kappa K_0} + \dfrac{2V_0}{(ipK_1 + 4\pi k_1)}\right\}}.$$

This is Lord Rayleigh's law of scattering, and agrees with what we found by the approximate method.

We see that the difference in the extreme cases arises from the fact that for a perfect conductor ω'_n and ω_n are of the same order in (κa), while for the best conductor we know ω_n is of order one higher than ω'_n. The reason is that the function $f(x)$ assumes very different forms for large and for small values of the argument.

It is unnecessary to pursue the matter further, as the divergence has been accounted for. The result of the investigation has been to show that for non-magnetic metals the law of scattering will, very approximately, be that given by Lord Rayleigh for insulators as regards the polarization; while the intensity will fall off as inversely as the *square* of the wave length so far as the conductivity enters.

ON THE TRANSITIVE SUBSTITUTION GROUPS OF DEGREE SEVENTEEN.

By G. A. Miller, Ph.D.

THE metacyclic group of degree 17 is of order 272 and contains one, and only one, transitive subgroup of each of the orders 136, 68, 34, and 17. These five groups are well known and follow directly from the general theorem that the metacyclic group of a prime degree (p) is of order $p(p-1)$ and contains one, and only one, subgroup for each divisor of $p-1$, and that there is no other transitive group of degree p, except those whose order exceeds $p(p-1)$ and which contain more than one subgroup of order p. Each of the other transitive groups of degree 17 must therefore contain $1+17k$ conjugate subgroups of order 17, and its order must be $17\alpha(1+17k)$, where $\alpha=2^\beta$, $0<\beta<5$, $k>0$. The alternating and the symmetric group are included among these groups, but since they are so well known they will not be considered in what follows.

Since a group of degree $2p+\lambda$ (p being any prime number and $\lambda>2$) cannot be more than λ times transitive without including the alternating group,* the required groups of degree 17 cannot be more than three times transitive. Their orders can therefore not be divisible by either 11 or 13. We

* *Bulletin of the American Mathematical Society,* Vol. IV., 1898, p. 143.

proceed to prove that the order of such a group cannot be divisible by 7. Since the operators of order p that are contained in a transitive group of degree p generate a simple group whose order is the quotient obtained by dividing the order of the entire group by some factor of $p-1$, we may confine our attention to simple groups. As these groups cannot involve any negative substitution, the given β must have one of the three values 1, 2, 3.

If the order of one of the required groups (G) were divisible by 7 each of its subgroups of order 7 would be transformed into itself by $3m$ substitutions, where m is the number of the substitutions contained in a maximal subgroup of degree 16 (G_1) that transform this subgroup of order 7 into itself. Each of the subgroups of order 7 that could occur in G would be of degree 14. Hence $3m$ could not be divisible by 9 and m would therefore be prime to 3.

G could not be triply transitive; for the intransitive subgroup of degree 14 would be formed by a simple isomorphism between two groups of degree 7 and order 7 or 14, since the subgroups of order 7 that are found in the other transitive groups of degree 7 are transformed into themselves by a multiple of three substitutions. In these two possible cases the order of G would be either 7.15.16.17 or 14.15.16.17. As neither of these two numbers is of the form $17.2^{\beta}(1+17k)$, $0<\beta<4$, there is no just triply transitive group of degree 17 whose order is divisible by 7.

If G were doubly transitive its subgroup which includes all the substitutions which do not involve two given letters would be of degree 14 or of degree 15. In the former case the order of G would be either 7.16.17 or 14.16.17; in the latter case this group of degree 15 would be formed by establishing an x, 1 isomorphism between a primitive group of degree 8 and one of the two given groups of degree 7 and order 7 or 14; hence the order of G would be 56.16.17. Since the order of G would have to be divisible by 3 if it would contain a subgroup of order 7, we have proved that there is no just doubly transitive group of degree 17 whose order is divisible by 7.

If G would be simply transitive its maximal subgroup (G_1) of degree 16 would contain substitutions of order 3. Its transitive constituents could not be of degrees 8, 8, since G^{8}_{56} is the only primitive group of degree 8 in which a subgroup of order 7 is not transformed into itself by a substitution of order 3; and if the two transitive constituents would be this group, the order of G_1 would not be divisible by 3.

Its transitive constituents could not be of degrees 7, 9, since the triply transitive groups of degree 9 do not involve any quotient group of order 7 or 14.* This completes the proof that a primitive group of degree 17 does not involve any substitution of order 7 unless it includes the alternating group.

We proceed to prove that none of the required groups contains a subgroup of order 25. If G would contain a subgroup of order 25, this subgroup would have three systems of intransitivity, each being of degree 5. Hence it would contain substitutions of degree 10 and order 5. Such a substitution would be found in 7 maximal subgroups of degree 16, and therefore in at least two groups of 25 that do not have all their elements in common. These two groups would generate an intransitive group having transitive constituents of degrees 5, 5, $5+\alpha$, where $\alpha=1$ or 2. This is impossible, since none of the primitive groups of degree 6 or 7 has a quotient group of order 5 or 25. The order of G must therefore be

$$g = 2^\alpha.3^\beta.5^\gamma.17, \quad \gamma \text{ being 0 or 1.}$$

We shall first consider the case when $\gamma=1$, that is, when G contains a system of conjugate subgroups of order 5. The substitutions of order 5 cannot be of degree 10, since the order of G is not divisible by 7; they must therefore be of degree 15. Since each subgroup of order 17 is transformed into itself by 17.2^β substitutions, G must contain $2^a 3^b 5$ subgroups of order 17, where $a < 14$, $b < 6$.† From

$$3^5 \equiv 5, \text{ mod. } 17 \text{ and } 3^{14} \equiv 2, \text{ mod. } 17,$$

we observe that $\quad 3^{14a+b+5} \equiv 1, \text{ mod. } 17.$

As 3 is a primitive root of 17, this leads to the following congruence

$$14a + b + 5 \equiv 0, \text{ mod. } 16.$$

Hence b must be odd, and the possible number of subgroups of order 17 that are included in G must be among the following six numbers: $2^3.3.5$; $2^{11}.3.5$; $2^4.3^3.5$; $2^{12}.3^3.5$; $2^5.3^5.5$; $2^{13}.3^5.5$. We shall prove that only the first of these is possible.

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* Since G_1 must contain substitutions of degree 16 and order 2, it cannot have any transitive constituent of an odd degree.

† Jordan, *Traité des Substitutions* (1870), p. 665.

If G were simply transitive its maximal subgroup of degree 16 would contain one transitive constituent of degree 10 and one of degree 6. Each of these would contain negative substitutions, since G contains substitutions of degree 16 and order 2. Hence the constituent of degree 6 would be $(abcdef)_{120}$ or $(abcdef)_{\text{all}}$. To identity of this constituent there would correspond an intransitive subgroup of the constituent of degree 10, since all the substitutions of order 5 are of degree 15. Hence the constituent of degree 10 would contain five systems of imprimitivity, and the given intransitive subgroup would contain five systems of intransitivity. From this fact, and from the fact that the highest power of 3 which divides the order of G must be odd, it follows that $(abcdef)_{\text{all}}$ could not be a constituent of G_1. Since the order of G_1 must not exceed 120, and it cannot contain any substitution whose degree is less than 6, nor any negative substitutions, $(abcdef)_{120}$ cannot be a constituent of G_1. This completes the proof that G cannot be simply transitive if its order is divisible by 5.

It is easy to see that G_1 could not be imprimitive; for it could not have two or four systems of imprimitivity, since it must contain a substitution of order 5 and degree 15. It could not have eight systems without also containing a smaller number, because every primitive group of degree 8 contains subgroups of order 7. We have now proved that G_1 must be primitive if the order of G is divisible by 5.

Since the order of none of the primitive groups of degree 16, which do not include the alternating group, is divisible by 27,* the required G must contain either $2^3.3.5$ or $2^{11}.3.5$ subgroups of order 17. The latter is impossible, because none of these primitive groups is of order $2^{11}.3.5.h$, where h is 2, 4 or 8. Hence we have proved that every primitive group of degree 17, whose order is divisible by 5, contains 120 subgroups of order 17.

The lowest possible order that such a primitive group can have is 240.17. Since there is only one primitive group (G_3) of order 240 and degree 16†, G has to contain this group for its G_1. Hence it must be triply transitive. Since G_3 contains sixteen conjugate subgroups of order 15, G must contain 136 such subgroups, and it must contain 30 substitutions that transform each of these cyclical regular subgroups into itself. As 15 of these substitutions must be of degree 16 they must

* Miller, *American Journal of Mathematics*, Vol. xx., p. 229.

† loc. cit.

transform each of the substitutions of the given regular group into its 14th power. These substitutions are therefore completely determined by the given regular group, and consequently there is not more than one primitive group of degree 17 and order 240.17.

The G_3 of the given article may be generated by H and *bfdjinohpcmlegk*. We may verify by the well-known methods that G_3 and *aq.fk.dg.ej.il.mn.co.hp* generate a group of order 240.17. This is simple, since it is generated by its subgroups of order 17. From *aq.fk.dg.ej.il.mn.co.hp* × *ac.bd.eg.fh.ik.jl.mo.np* = *aqcmpfijgbdelkhno*, and from the fact that the semi-metacyclic group which contains the last substitution transforms G itself while the metacyclic group does not have this property, it follows that this G is self-conjugate in one group of order 480.17 and in one of order 960.17, but in no other primitive group of degree 17.* It is evident that a primitive group of degree p cannot be self-conjugate in more than one group of degree p and of a given order.

Having found all the primitive groups of degree 17 that contain the simple group of order 240.17 as a self-conjugate subgroup, it remains to see whether there is any simple group whose order is either 480.17 or 960.17. If such a simple group does not exist, we have considered all the possible primitive groups of degree 17 whose order is divisible by 5.

Since there is only one primitive group of degree 16 and order 480, the required simple group of order 480.17 would have to contain this group (G_5). As it would be triply transitive it would have to contain a subgroup of order 60 which is formed by dimidiating (aq) and a group of order 60 and degree 15. The subgroup containing the 30 substitutions of G_5 that transform the substitutions of a regular subgroup of order 15 into its 1st and 4th powers would correspond to identity of (aq). Since the given subgroup of order 60 would be positive and could not contain any substitution of the form *aq.cjoe.dkpl.fhig*, the given group of order 60 would transform the substitutions of the group of order 15 into their 1st, 4th, 11th, and 14th powers. That is, a primitive group of order 480.17 must include the given simple group of order 240.17 as a self-conjugate subgroup, for G_5 includes G_3 and the required group would involve *aq.fk.dg.ej.il.mn.co.hp*.

We proceed to prove that there can be no simple group of order 960.17. It is very easy to prove that such a group

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* Mathieu, *Journal de Mathématiques*, Vol. XVIII. (1873), p. 46.

could not contain the solvable primitive group of order 960 and degree 16, for this solvable group contains 60 substitutions that transform one of its regular subgroups of order 15 into itself. The corresponding group of order 960.17 therefore contains 120 such substitutions. As this is the total number of the substitutions of degree 15 that transform the group of order 15 into itself, there is only one primitive group of degree 17 and order 960.17 that contains the solvable primitive group of order 960 and degree 16.

One of the other primitive groups of the required order (G_7) contains a conjugate system of subgroups of order 3 and degree 12. As such a subgroup could not be transformed into itself by any substitution of order 5, it could not occur in the required primitive group of degree 17. The other primitive group of order 960 (G_8) contains

$$bpo.ceg.dli.fjm.hnk.boifc.dpgnh.emjlk = bg.cm.dk.en.fl.ip.$$

Hence it must contain just 60 substitutions of degree 12. Since all of them are conjugate, each must be transformed into itself by sixteen substitutions of G_8. In the required primitive group of degree 17 each of these substitutions would have to be transformed into itself by eighty substitutions. Its systems of intransitivity would be permuted according to a group of degree 6 whose order would be divisible by 5. As no such group could be isomorphic to the given group of order 80 this is impossible. Hence there are just three primitive groups of degree 17 that do not include the alternating group, and whose order is divisible by 5.

If there is any other primitive group of degree 17 there must be a simple group of order $2^\alpha.3^\beta.17$, $\alpha < 14$, $\beta < 6$. From $3^{14} \equiv 2$, mod. 17, we have $3^{14\alpha+\beta} \equiv 1$, mod. 17, or

$$14\alpha + \beta \equiv 0, \text{ mod. } 16.$$

Hence β must be even and the possible number of subgroups of order 17 that are included in G must be among the following five numbers: 2^8, 2.3^2, $2^9.3^2$, $2^2.3^4$, $2^{10}.3^4$. The first of these is impossible because there is no simple group of order $p^\alpha q$, where p and q are any prime numbers.* We proceed to prove that the third and last are also impossible. To do this it is convenient to inquire into the largest subgroup whose order is a power of two that can occur in the required simple group.

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* Frobenius, *Berliner Sitzungsberichte* (1895), p. 190; Burnside, *Proceedings of the London Mathematical Society*, Vol. XXVI. (1895), p. 209.

Such a group cannot contain any substitution whose degree is less than 8. It is evident that its order would be less than 2^6 if all of its transitive constituents were of degree 2, and that it would be less than 2^{10} if there were no transitive constituent whose degree exceeds 8. Since the order of this group could not be less than 2^{10}, we may assume that G_1 would be transitive and of order $2^{10}.3^2$. As there is no primitive group of this order and degree, G_1 would have to contain two systems of imprimitivity. It is not difficult to see that no such imprimitive group could occur in a primitive group of degree 17. Hence it remains to consider the cases when the required simple group would contain either 2.3^2 or $2^2.3^4$ conjugate subgroups of order 17.

In the former case the subgroups of order 17 could not be transformed into themselves by just 34 substitutions, since there is no simple group of order 612. If they would be transformed into themselves by just 68 substitutions, the maximal subgroup of degree 16 would be of order 72. It would contain four conjugate subgroups of order 9, since it could not contain a self-conjugate subgroup of this order.* As these subgroups would have to be of degree 15, the primitive group of degree 17 would contain 34 conjugate subgroups of order 9. Since this primitive group would be simple, it would have to transform these 34 subgroups according to a simply isomorphic transitive group. Since the substitutions of degree 16 and order 2 could not transform any one of these 34 subgroups of order 9 into itself, they would have to correspond to negative substitutions, *i.e.*, the group of degree 34 could not be simple. Sylow proved† that the given 18 subgroups could not be transformed into themselves by 136 substitutions. Hence we have proved that there is no primitive group of degree 17 that contains 18 conjugate subgroups of order 17.

If one of the required groups would contain $2^2.3^4$ subgroups of order 17, and if the maximal group of degree 16 would contain just four subgroups of order 81, it would follow as in the preceding case that the simply isomorphic group of degree 34 would have to contain negative substitutions; and hence it could not be simple. From this it follows that the largest subgroup which transforms one of the subgroups of order 17 into itself could not be of order 34. The two possi-

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* *Proceedings of the London Mathematical Society*, Vol. XXVIII., p. 533.

† Sylow, *Videnskabs-Selskabets Skriften*, 1897, No. 9.

bilities which remain to be examined are that the subgroup of order 17 is transformed into itself by 68 or 136 substitutions when the maximal subgroup of degree 16 contains 16 subgroups of order 81. In each of these cases the required group would be simply isomorphic to a transitive group of degree 8, 17. If each of the subgroups of order 17 were transformed into itself by 136 substitutions, this group of degree 136 would contain substitutions involving 17 cycles of order 8. As such substitutions are negative, the group could not be simple.

We have left the single possibility when the order of the group would be 1296.17, and it would contain three hundred and twenty-four subgroups of order 17, while its maximal subgroup of degree 16 would contain 16 subgroups of order 81 and degree 15. As there is no primitive group of degree 16 and order 1296, this subgroup would have to be imprimitive or intransitive. It could not contain eight systems of imprimitivity without containing a smaller number of such systems, since each of the primitive groups of degree 8 contains substitutions of order 7. It could not have two systems; for the group of order 81 cannot contain any substitution of degree 3. Hence it would have to contain four systems of imprimitivity and it would have to permute them according to the alternating or the symmetric group of degree 4. It is clearly impossible to construct such a group of the required order.

If the given maximal group of degree 16 were intransitive, it would have to contain two systems of intransitivity of degree 12 and 4 respectively. The group of degree 4 would be the symmetric group, since there must be substitutions composed of four cycles of order 4. To identity of the symmetric group there would correspond a group of order 54. This would contain a self-conjugate subgroup of order 27, containing four systems of intransitivity, and therefore Abelian. Since the entire group of degree 16 wonld have to contain other similar subgroups of order 27*, the subgroups of order 81 wonld contain five systems of intransitivity, and hence the constituent of degree 12 could not be transitive. As this is contrary to the hypothesis, the required group cannot be constructed. This completes the proof that there are just eight primitive groups of degree 17 that do not include the alternating group of this degree.† Five of these are

* *Proceedings of the London Mathematical Society*, Vol. XXVIII., p. 533.

† The list in *Comptes Rendus*, Vol. LXXV., p. 1757, states that there are only six such groups.

included in the metacyclic group, and their orders are 17, 34, 68, 136, and 272 respectively. Each one of the other three contains 120 subgroups of order 17, and the orders of these three groups are 4080, 8160, and 16320 respectively. The group of order 4080 is simple, and it is a self-conjugate subgroup of each of the other two.

Cornell University,
February, 1899.

NOTE ON THE INVARIANTS OF A BINARY SEXTIC.

By H. W. RICHMOND, M.A., King's College, Cambridge.

BY invariants of the binary quantic,

$$(a_0, a_1, a_2, \ldots, a_n \between x, y)^n \equiv a_0 \Pi (x - \gamma_r y); \; (r = 1, 2, 3, \ldots, n) \quad \ldots\ldots\ldots(1)$$

are usually understood rational integral functions of the coefficients $a_0, a_1, \ldots, a_n$, which are therefore symmetrical functions of the roots $\gamma_1, \gamma_2, \ldots, \gamma_n$; but it is well known that certain unsymmetrical functions of the differences of the roots possess the property of invariance. Thus, very early in the investigation of the binary sextic, use was made of the unsymmetrical invariants such as

$$\pm a_0 (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)(\gamma_5 - \gamma_6).$$

Now it is worthy of note that, if the ambiguities of sign be rightly chosen, these fifteen quantities are expressible as the sums, two by two, of six other unsymmetrical invariants, here denoted by $k_1, k_2, k_3, k_4, k_5, k_6$: in fact, if it be agreed that the abbreviation 12.34.56 shall stand for

$$a_0 (\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)(\gamma_5 - \gamma_6), \text{ \&c.,}$$

the following system of equations will be found to be consistent:

$12.34.56 = k_1 + k_2$	$12.35.64 = k_3 + k_4$	$12.36.45 = k_5 + k_6$
$13.24.65 = k_4 + k_5$	$13.25.46 = k_1 + k_6$	$13.26.54 = k_2 + k_3$
$14.23.56 = k_3 + k_6$	$14.25.63 = k_2 + k_4$	$14.26.35 = k_1 + k_5$
$15.23.64 = k_2 + k_5$	$15.24.36 = k_1 + k_3$	$15.26.43 = k_4 + k_6$
$16.23.45 = k_1 + k_4$	$16.24.53 = k_2 + k_6$	$16.25.34 = k_3 + k_5$.

Each of the six quantities (k) is the product of $\frac{1}{2}a_0$ and a function of the roots (γ) composed of twenty terms, each term the product of three roots, ten terms having the sign $+$, and ten the sign $-$; an interchange of any two roots has the effect of interchanging the quantities (k) in pairs and multiplying each by -1. Again, it is clear that $\Sigma(k)=0$, and, since

$$(k_1+k_2)(k_1+k_3)(k_2+k_3)+(k_4+k_5)(k_4+k_6)(k_5+k_6)=0,$$

that the sum of the cubes of the quantities (k), and the sum of their products three at a time vanish: the quantities (k) are therefore the roots of an equation of the form

$$(1,\ 0,\ \lambda,\ 0,\ \mu,\ \pm\sqrt{\rho},\ \nu \between \theta,\ 1)^6=0 \quad \text{.........(2)},$$

in which λ, μ, ν, ρ are invariants of the given sextic of degrees 2, 4, 6, 10 respectively. To evaluate ρ, take the identity

$$(1,\ 0,\ \lambda,\ 0,\ \mu,\ \pm\sqrt{\rho},\ \nu \between \theta,\ 1)^6=\Pi(\theta-k_r),\ (r=1,\ 2,\ 3,\ 4,\ 5,\ 6),$$

substitute k_1 and $-k_1$ for θ and subtract; then

$$\pm 6\sqrt{\rho}=(k_1+k_2)(k_1+k_3)(k_1+k_4)(k_1+k_5)(k_1+k_6),$$

or $\quad 36\rho=a_0^{10}\Pi(\gamma_r-\gamma_s)^2=-6^6\Delta,$

if Δ denote the discriminant of the sextic: so that

$$\rho=-6^4\Delta.$$

It is usual to denote by I_2 the invariants

$$a_0a_6-6a_1a_5+15a_2a_4-10a_3^2,$$

by I_4 the *catalectricant*,

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix},$$

and by I_6 an invariant given in full by Salmon (*Higher Algebra*, 3rd edition, p. 233), or by Cayley in the *Third Memoir on Quantics*, and again in the *Collected Works*, Vol. VI., p. 375; and to treat these as fundamental invariants of degrees 2, 4, and 6, the reasons for the selection of I_6 being of little weight. The invariants λ, μ, ν may be expressed in terms of I_2, I_4, I_6 as follows:

$$\lambda = 2I_2; \quad \mu = 900I_4 + 11I_2^2;$$
$$\nu = 67500I_6 + 54000I_2I_4 - 80I_2^3.$$

Relations among the invariants I_2, I_4, I_6, I_{10} of the sextic may thus be replaced by relations among λ, μ, ν, ρ; and it seems that, if simple invariantive properties of the roots are to be so expressed, the latter form is usually simpler. For example, the vanishing of the skew invariant of degree 15 expresses the condition that the roots of the sextic should fall into three pairs in involution; hence the square of the skew invariant is a numerical multiple of the discriminant of

$$(1, 0, \lambda, 0, \mu, \pm\sqrt{\rho}, \nu \between \theta, 1)^6.$$

In cases of equality of two or more roots the values of the quantities (k) may be found and relations among the invariants deduced; and we may verify that the conditions are also sufficient.

Thus, if two roots are equal, $\rho = 0$.

If three roots are equal, $\rho = 0$ and $(1 0 \lambda 0 \mu 0 \nu \between \theta, 1)^6$ is a perfect cube.

If two pairs of roots are equal, $\rho = 0$ and this expression gives two equal values of θ^2.

The reasons which led me to consider the equation (2) arose from its connection with the equation of the fifth degree. The theory of the dependence of the solution of a quintic equation upon that of a special form of equation of degree 6 originates with Lagrange; but the simplest form of the *Resolvent* is that obtained by Messrs. Harley and Cockle (see Vol. III. of this *Journal*, pp. 343–359). In this form, however, the coefficients are not invariants of the quintic equation, for which reason a less simple form of resolvent is sometimes substituted. Now it will be seen that if we consider the quintic as a sextic equation of which one root is infinite, viz.

$$(0 a_1 a_2 a_3 a_4 a_5 a_6 \between x, y)^6 = 0,$$

the resolvent of Messrs. Harley and Cockle is the equation (2) considered above, of which the coefficients are invariants of the sextic, modified by the vanishing of a_0; and the same notion may be applied to the sextic equation one of whose roots is any known magnitude, the others being determined by a given quintic equation.

A CLASS OF LINEAR GROUPS INCLUDING THE ABELIAN GROUP.

By Dr. L. E. Dickson.

1. DENOTE by S a general linear homogeneous substitution on mq variables,

$$S:\quad x'_{ij} = \sum_{k=1}^{m} \left(\alpha^{ij}_{k1} x_{k1} + \alpha^{ij}_{k2} x_{k2} + \ldots + \alpha^{ij}_{kq} x_{kq}\right),$$

$$(i = 1, \ldots, m;\ j = 1, \ldots, q).$$

We study the group $G_{m,q}$ of substitutions S which, when operating cogrediently on q independent sets of mq variables, the j^{th} set of which may be exhibited thus

$$x^{(j)}_{i1},\ x^{(j)}_{i2},\ \ldots,\ x^{(j)}_{iq} \quad (i = 1, \ldots, m),$$

will leave absolutely invariant the function

$$\phi \equiv \sum_{i=1}^{m} \begin{vmatrix} x^{(1)}_{i1}, & x^{(1)}_{i2}, & \ldots, & x^{(1)}_{iq} \\ x^{(2)}_{i1}, & x^{(2)}_{i2}, & \ldots, & x^{(2)}_{iq} \\ \ldots & \ldots & \ldots & \ldots \\ x^{(q)}_{i1}, & x^{(q)}_{i2}, & \ldots, & x^{(q)}_{iq} \end{vmatrix}.$$

The conditions thus imposed upon S are seen to be

$$(1)\qquad \sum_{i=1}^{m} \begin{vmatrix} \alpha^{i1}_{j1}, & \alpha^{i1}_{j2}, & \ldots, & \alpha^{i1}_{jq} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha^{iq}_{j1}, & \alpha^{iq}_{j2}, & \ldots, & \alpha^{iq}_{jq} \end{vmatrix} = 1 \quad (j = 1, \ldots, m)$$

$$(2)\qquad \sum_{i=1}^{m} \begin{vmatrix} \alpha^{i1}_{j_1k_1}, & \alpha^{i1}_{j_2k_2}, & \ldots, & \alpha^{i1}_{j_qk_q} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha^{iq}_{j_1k_1}, & \alpha^{iq}_{j_2k_2}, & \ldots, & \alpha^{iq}_{j_qk_q} \end{vmatrix} = 0$$

holding for every combination of q pairs of integers $(j_1, k_1), \ldots, (j_q, k_q)$ given when

$$j_1, j_2, \ldots, j_q = 1, \ldots, m;\ k_1, k_2, \ldots, k_q = 1, \ldots, q,$$

independently, excluding only the combinations

$$(j, 1), (j, 2), \ldots, (j, q).$$

2. Theorem: *The inverse of the above substitution is*

$$S^{-1}:\quad x'_{rs} = \sum_{i=1}^{m} (A_{rs}^{i1} x_{i1} + A_{rs}^{i2} x_{i2} + \ldots + A_{rs}^{iq} x_{iq})$$

$$(r = 1, \ldots, m;\ s = 1, \ldots, q),$$

where A_{rs}^{ij} *denotes the adjoint of* α_{rs}^{ij} *in*

$$\begin{vmatrix} \alpha_{r1}^{i1}, & \alpha_{r2}^{i1}, & \ldots, & \alpha_{rq}^{i1} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{r1}^{iq}, & \alpha_{r2}^{iq}, & \ldots, & \alpha_{rq}^{iq} \end{vmatrix}.$$

Indeed, the product SS^{-1} replaces x_{rs} by

$$\sum_{\substack{i=1\ldots m \\ j=1\ldots q}} A_{rs}^{ij} \Big(\sum_{\substack{k=1\ldots m \\ l=1\ldots q}} \alpha_{kl}^{ij} x_{kl} \Big)$$

$$= \sum_{\substack{k=1\ldots m \\ l=1\ldots q}} \Big\{ \sum_{i=1\ldots m} \Big(\sum_{j=1\ldots q} A_{rs}^{ij} \alpha_{kl}^{ij} \Big) \Big\} x_{kl}.$$

The quantity in brackets is the expansion of

$$\sum_{i=1}^{m} \begin{vmatrix} \alpha_{r1}^{i1}, & \ldots, & \alpha_{rs-1}^{i1}, & \alpha_{kl}^{i1}, & \alpha_{rs+1}^{i1}, & \ldots, & \alpha_{rq}^{i1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_{r1}^{iq}, & \ldots, & \alpha_{rs-1}^{iq}, & \alpha_{kl}^{iq}, & \alpha_{rs+1}^{iq}, & \ldots, & \alpha_{rq}^{iq} \end{vmatrix},$$

and hence, by (1) or (2), equals 1 if $(k, l) = (r, s)$; but equals zero if $(k, l) \neq (r, s)$. Hence SS^{-1} replaces x_{rs} by x_{rs}.

The reciprocal of S is obtained by replacing in it α_{kl}^{ij} by A_{ij}^{kl} for $i, k = 1, \ldots, m$; $l, j = 1, \ldots, q$.

Writing the relation (1) for the substitution S^{-1}, we have

$$(3)\quad \sum_{i=1}^{m} \begin{vmatrix} A_{i1}^{j1}, & \ldots, & A_{i1}^{jq} \\ \ldots & \ldots & \ldots \\ A_{iq}^{j1}, & \ldots, & A_{iq}^{jq} \end{vmatrix} \equiv \sum_{i=1}^{m} \begin{vmatrix} \alpha_{i1}^{j1}, & \ldots, & \alpha_{iq}^{j1} \\ \ldots & \ldots & \ldots \\ \alpha_{i1}^{jq}, & \ldots, & \alpha_{iq}^{jq} \end{vmatrix}^{q-1} = 1.$$

3. For the case $q=2$ the group $G_{m,q}$ has been completely studied both as a continuous group and as a group in an arbitrary Galois field. The continuous group $G_{m,2}$ is the homogeneous form of the largest projective group in space of $2m-1$ dimensions, which leaves a linear complex invariant: a group studied by Lie. A direct study of the continuous group $G_{m,2}$, and an elementary proof of its simplicity, was given by the writer in a paper, "Systems of continuous and discontinuous simple groups," *Bulletin of the American Mathematical Society*, May 1897. In the same paper all the infinitesimal transformations of the continuous group $G_{m,q}$ were set up. The result, however, readily follows from § 6 below.

For $q=2$ the group $G_{m,2}$ of linear homogenous substitutions in a Galois field of order p^n is known as the generalized Abelian group. For $n=1$ its structure has been determined by Jordan*; for $n \gtreqless 1$ it has been determined by the writer.†

The character of the group $G_{m,q}$ is essentially different in the two cases $q=2$ and $q>2$. Henceforth we assume that $q>2$; in fact, the investigation following requires that q be greater than two.

4. Let $j_2, \ldots, j_q$ have fixed values chosen arbitrarily from $1, 2, \ldots, m$, and $k_2, \ldots, k_q$ fixed values chosen arbitrarily from $1, 2, \ldots, q$, but chosen in such a manner as to exclude the case $j_2=j_3=\ldots=j_q$; $k_2, k_3, \ldots, k_q$ all distinct. Then for $j_1=1, \ldots, m$; $k_1=1, \ldots, q$, we obtain mq equations of the form (2). Indeed, since $q>2$, the above values j_i, k_i can not lead to an equation of the form (1). These equations may be written in terms of the adjoint minors A as follows:

$$\sum_{\substack{i=1\ldots m\\ l=1\ldots q}} A^{il}_{j_1k_1} \quad \alpha^{il}_{j_1k_1} \qquad \begin{pmatrix} j_1=1\ldots m \\ k_1=1\ldots q \end{pmatrix}.$$

The determinant $\left| \alpha^{il}_{j_1k_1} \right|$ is not zero, being the determinant of the substitution S. Hence we have

$$A^{il}_{j_1k_1} \equiv \begin{vmatrix} \alpha^{ai_2}_{j_2k_2}, & \ldots, & \alpha^{ia_2}_{j_qk_q} \\ \ldots & \ldots & \ldots \\ \alpha^{ia_q}_{j_2k_2}, & \ldots, & \alpha^{ia_q}_{j_qk_q} \end{vmatrix} = 0 \quad \begin{pmatrix} i=1\ldots m \\ l=1\ldots q \end{pmatrix},$$

* *Traité des Substitutions*, pp. 171–179.

† *Quarterly Journal of Mathematics*, Vol. XXIX., pp. 169–178 (1897), and Vol. XXX., pp. 383–384 (1899).

where $a_2, a_3, \ldots, a_q$ denote the integers $1, \ldots l-1, l+1, \ldots, q$, and therefore an arbitrary combination of $q-1$ distinct integers $\overline{\gtrless} q$. If $q=3$, we have the result (4′) below. If $q>3$, we denote by $B^{ia_s}_{j_2k_2}$ the adjoint of $\alpha^{ia_s}_{j_2k_2}$ in the latter determinant, and consider the following equations:

$$\sum_{s=2}^{q} B^{ia_s}_{j_2k_2} \quad \alpha^{ia_s}_{j_2k_2} = 0.$$

Of these equations consider the mq equations in which $i, j_3, \ldots, j_q$ have fixed values chosen arbitrarily from $1, 2, \ldots, m$, and $k_3, \ldots, k_q$ fixed values chosen arbitrarily from $1, 2, \ldots, q$, but such that the case $j_3 = j_4 = \ldots = j_q$ with $k_3, k_4, \ldots, k_q$ all distinct is excluded, while lastly j_2 takes the values $1, 2, \ldots, m$, and k_2 the values $1, 2, \ldots, q$.

Since the matrix

$$\left(\alpha^{ia_s}_{j_2k_2}\right) \qquad \begin{pmatrix} j_2 = 1, \ldots, m; & k_2 = 1, \ldots, q \\ s = 2, \ldots, q & \end{pmatrix}$$

comprises $q-1$ rows of the determinant of S, not all of its determinants of order $q-1$ are zero. Hence the $q-1$ quantities $B^{ia_s}_{j_2k_2}$, which are the same in each of the mq equations, must be zero, viz.

$$\begin{vmatrix} \alpha^{ib_3}_{j_3k_3}, & \ldots, & \alpha^{ib_3}_{j_qk_q} \\ \cdots & \cdots & \cdots \\ \alpha^{ib_q}_{j_3k_3}, & \ldots, & \alpha^{ib_q}_{j_qk_q} \end{vmatrix} = 0,$$

where $b_3, \ldots, b_q$ denote any $q-2$ distinct integers $\overline{\gtrless} q$. If $q=4$, we have reached the result (4′). If $q>4$, we proceed as before. After $q-2$ such steps, we reach the set of relations

$$(4') \qquad \begin{vmatrix} \alpha^{ir}_{j_{q-1}k_{q-1}}, & \alpha^{ir}_{j_qk_q} \\ \alpha^{is}_{j_{q-1}k_{q-1}}, & \alpha^{is}_{j_qk_q} \end{vmatrix} = 0, \quad \begin{pmatrix} i = 1, \ldots, m \\ r, s = 1, \ldots, q; \; r \neq s \end{pmatrix},$$

holding also for $j_{q-1}, j_q = 1, \ldots, m$; $k_{q-1}, k_q = 1, \ldots, q$, excluding the case in which $j_{q-1} = j_q$, $k_{q-1} \neq k_q$. But (4′) evidently holds true if $j_{q-1} = j_q$, $k_{q-1} = k_q$, since it is then an identity.

We may thus state our result in the form

$$(4) \qquad \begin{vmatrix} \overset{ir}{\alpha}_{jk}, & \overset{ir}{\alpha}_{j_1k_1} \\ \overset{is}{\alpha}_{jk}, & \overset{is}{\alpha}_{j_1k_1} \end{vmatrix} = 0, \quad \begin{pmatrix} i, j, j_1 = 1, \ldots, m; \; j \neq j_1 \\ r, s, k, k_1 = 1, \ldots, q \end{pmatrix};$$

In virtue of the relations (4), the conditions (2) all reduce to identities. Indeed, in every relation (2) not an identity at least two of the j's are distinct, say $j_1 \neq j_2$, and therefore all minors formed from the first and second columns vanish in virtue of (4).

5. Theorem. *Every substitution S leaving ϕ invariant can be derived from the totality of linear substitutions of determinant unity on q indices,*

$$x'_{1j} = \sum_{k=1}^{q} \beta^{1j}_{1k} x_{1k}, \quad (j = 1 \ldots q),$$

together with the totality of literal substitutions of the type

$$P_{ij} \equiv (x_{i1}x_{j1})(x_{i2}x_{j2}) \ldots (x_{iq}x_{jq}), \quad (i, j = 1, \ldots, m).$$

We can evidently derive from these generators a literal substitution T, replacing an arbitrary index x_{kl} by any index as x_{11}. We may therefore suppose that in the product $S' \equiv TS$ the coefficient $\overset{11}{\alpha}_{11} \neq 0$. If then we set

$$\overset{11}{\alpha}_{jk} = c_{jk} \overset{11}{\alpha}_{11}, \quad (j = 2, \ldots, m; \; k = 1, \ldots, q),$$

it follows from (4) that

$$(5) \quad \overset{1s}{\alpha}_{jk} = c_{jk} \overset{1s}{\alpha}_{11}, \quad (j = 2, \ldots, m; \; k, s = 1, \ldots, q).$$

Substituting these values in the relation (3) for $j = 1$, we get

$$\begin{vmatrix} \overset{11}{\alpha}_{11}, & \ldots, & \overset{11}{\alpha}_{1q} \\ \ldots & \ldots & \ldots \\ \overset{1q}{\alpha}_{11}, & \ldots, & \overset{1q}{\alpha}_{1q} \end{vmatrix}^{q-1} + \sum_{i=2}^{m} \begin{vmatrix} c_{i1}\overset{11}{\alpha}_{11}, & \ldots, & c_{iq}\overset{11}{\alpha}_{11} \\ \ldots & \ldots & \ldots \\ c_{i1}\overset{1q}{\alpha}_{11}, & \ldots, & c_{iq}\overset{1q}{\alpha}_{11} \end{vmatrix}^{q-1} = 1.$$

It follows that

$$\begin{vmatrix} \overset{11}{\alpha}_{11}, & \ldots, & \overset{11}{\alpha}_{1q} \\ \ldots & \ldots & \ldots \\ \overset{1q}{\alpha}_{11}, & \ldots, & \overset{1q}{\alpha}_{1q} \end{vmatrix} \neq 0.$$

If, therefore, we denote by R the linear substitution on q indices,

$$R: \begin{array}{l} x'_{11} = \alpha^{11}_{11} x_{11} + \ldots + \alpha^{11}_{1q} x_{1q}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x'_{1q} = \alpha^{1q}_{11} x_{11} + \ldots + \alpha^{1q}_{1q} x_{1q}, \end{array}$$

the product $S_1 \equiv R^{-1} S'$ affects the indices $x_{11}, \ldots, x_{1q}$ as follows:

$$x'_{1j} = x_{1j} + \sum_{k=2}^{m} (\alpha^{1j}_{k1} x_{k1} + \ldots + \alpha^{1j}_{kq} x_{kq}).$$

It is evident that R leaves the equation $\phi = 0$ invariant. Hence the product S_1 leaves $\phi = 0$ invariant, and therefore satisfies the relations (2), and consequently the relations (4) derived from them.

For the substitution S_1, we have $\alpha^{1s}_{11} = 0$ $(s = 2, \ldots, q)$. Hence, by (5),

$$\alpha^{1s}_{jk} = 0 \quad (j = 2, \ldots, m;\ k = 1, \ldots, q;\ s = 2, \ldots, q).$$

Also $\alpha^{11}_{1s} = 0$ $(s = 2, \ldots, q)$, $\alpha^{1s}_{1s} \neq 0$. Hence, by the relations (4), viz.

$$\begin{vmatrix} \alpha^{11}_{jk}, & \alpha^{11}_{1s} \\ \alpha^{1s}_{jk}, & \alpha^{1s}_{1s} \end{vmatrix} = 0, \quad \begin{pmatrix} j = 2, \ldots, m;\ k = 1, \ldots, q \\ s = 2, \ldots, q \end{pmatrix},$$

we have $\alpha^{11}_{jk} = 0$. Hence every $\alpha^{1s}_{jk} = 0$, if $j > 1$.

It follows that S_1 leaves $x_{11}, x_{12}, \ldots, x_{1q}$ fixed. It must therefore leave ϕ absolutely invariant. Hence the determinant of R is unity. We proceed with the indices $x_{21}, x_{22}, \ldots, x_{2q}$ in a similar manner. It follows that

$$S = T^{-1} R S_1 = \ldots = T^{-1} R T_1^{-1} R_1 \ldots T^{-1}_{m-1} R_{m-1},$$

the substitutions R_i, T_i being derived from those given in the theorem.

6. The group $G_{m,q}$, $q > 2$, evidently has an invariant subgroup $\Gamma_{m,q}$ composed of the substitutions

$$x'_{ij} = \sum_{k=1}^{q} \beta^{ij}_{ik} x_{ik} \quad (i = 1, \ldots, m;\ j = 1, \ldots, q),$$

in which $\left| \beta_{ik}^{ij} \right| = 1 \quad (j, k = 1, \ldots, q)$

for every $i = 1, \ldots, m$. The quotient group is generated by the substitutions P_{ij} and is therefore simply isomorphic to the symmetric group on m letters. The group $\Gamma_{m,q}$ is the direct product of m (commutative) groups, each the general linear homogeneous group of determinant unity on q indices. As continuous groups the latter are simple. As groups in the Galois field of order p^n, their structure has been determined by the writer in the *Annals of Mathematics*, pp. 161–183, 1897. The structure of $G_{m,q}$ is therefore completely determined.

University of California,
January 11*th*, 1899.

ON THE FLEXURE OF HEAVY BEAMS SUBJECTED TO CONTINUOUS SYSTEMS OF LOAD.

(PART II.)

By KARL PEARSON, F.R.S., and L. N. G. FILON, M.A. (Lond.).

(Continued from Vol. XXIV., pp. 63–110.)

SECTION III.

On the flexure of an elliptic cylinder subjected to its own weight only.

31. We shall first obtain the general Cartesian equations suitable for the flexure of a heavy beam of length $2l$ and of weight $2W$, which carries a total vertical surface load $2S$ uniformly distributed. The reaction at a terminal, if built-in, will consist of a couple M_0 and the vertical force $W' = W + S$. By assuming $S = 0$, we clearly obtain the case of a heavy unloaded beam of weight $2W$ and length $2l$. Let the axis of z be horizontal and taken through the centroid of the mid-section, the axis of x vertical, and that of y horizontal. Applying the semi-inverse method, let us assume:

$$\left.\begin{aligned} \widehat{zz} &= -Ax(l^2 - z^2) + F(y, x) \\ u &= C(x^2 - y^2)(l^2 - z^2) + \chi(z) + f(y, x) \\ v &= Dxy(l^2 - z^2) + \psi(y, x) \end{aligned}\right\} \ldots\text{(lxxxiii)}.$$

where A, C, D are constants, and F, f, χ, and ψ are arbitrary functions of the variables indicated.

Now

$$(\lambda+2\mu)\frac{dw}{dz}=\widehat{zz}-\lambda\left(\frac{du}{dx}+\frac{dv}{dy}\right)$$

$$=F-\lambda\left(\frac{df}{dx}+\frac{d\psi}{dy}\right)-x\,(l^2-z^2)\,(A+\lambda D+2\lambda C),$$

whence, if

$$Q=-\frac{F}{\lambda+2\mu}+\frac{\lambda}{\lambda+2\mu}\left(\frac{df}{dx}+\frac{d\psi}{dy}\right)\ldots\text{(lxxxiv)},$$

and
$$H=\frac{A+\lambda D+2\lambda C}{\lambda+2\mu}\ldots\ldots\ldots\ldots\text{(lxxxv)},$$

we have
$$\frac{dw}{dz}=-Hx\,(l^2-z^2)-Q,$$

or
$$w=-Hx\,(l^2z-\tfrac{1}{3}z^3)-Qz+\zeta(x,y)\ldots\text{(lxxxvi)}.$$

But $w=0$, if $z=0$, for all values of x and y. Hence it follows that $\zeta(x,y)=0$, or

$$w=-Hx\,(l^2z-\tfrac{1}{3}z^3)-Qz\ldots\ldots\ldots\text{(lxxxvii)}.$$

We are now able to find θ from

$$\theta=\frac{\widehat{zz}}{\lambda}-\frac{2\mu}{\lambda}\frac{dw}{dz}$$

$$=\frac{F+2\mu Q}{\lambda}+\frac{2\mu H-A}{\lambda}x\,(l^2-z^2)\ldots\text{(lxxxviii)}.$$

We can write down the expressions for the tractions $\widehat{xx}$ and $\widehat{yy}$. These are given by

$$\left.\begin{aligned}\widehat{xx}&=(2\mu H+4\mu C-A)\,x\,(l^2-z^2)+F+2\mu Q+2\mu\frac{df}{dx}\\ \widehat{yy}&=(2\mu H+2\mu D-A)\,x\,(l^2-z^2)+F+2\mu Q+2\mu\frac{d\psi}{dy}\end{aligned}\right\}$$

$$\ldots\ldots\text{(lxxxix)}.$$

Further, we have for the shears

$$\left.\begin{aligned}\widehat{xy} &= \mu(D-2C)\,y\,(l^2-z^2)+\mu\left(\frac{df}{dy}+\frac{d\psi}{dx}\right)\\ \widehat{yz} &= -\mu z\left(\frac{dQ}{dy}+2Dxy\right)\\ \widehat{zx} &= -\mu z\left\{\frac{dQ}{dx}+2C(x^2-y^2)\right\}+\mu\frac{d\chi}{dz}-\mu H(l^2z-\tfrac{1}{3}z^3)\end{aligned}\right\}\text{...(xc).}$$

Let l, m, 0 be the direction-cosines of the normal at any point (x, y) of the contour of a cross-section. Then if there were a given surface load parallel to the plane of the cross-section, but the same for all values of z, we should have

$$\left.\begin{aligned}l\widehat{xx}+m\widehat{xy} &= \kappa_1(x, y)\\ l\widehat{xy}+m\widehat{yy} &= \kappa_2(x, y)\\ l\widehat{xz}+m\widehat{yz} &= 0\end{aligned}\right\}\text{.............(xci),}$$

for all values of z.

The coefficients of z^2 must therefore on the substitution of the stresses in the first two of these equations vanish, or we must have

$$lx(2\mu H+4\mu C-A)+my(\mu D-2\mu C)=0,$$

$$ly(\mu D-2\mu C)+mx(2\mu H+2\mu D-A)=0$$

all round the contour of the cross-section. This can only hold if the coefficients of x and y vanish, or

$$D=2C,\quad 2\mu H+4\mu C=A\text{ ....(xcii).}$$

The third equation of (xci) further shows us that we must have $d\chi/dz-H(l^2z-\frac{1}{3}z^3)=Kz$, where K is a constant, or

$$\chi=J+\tfrac{1}{2}Kz^2+H\{\tfrac{1}{2}l^2z^2-\tfrac{1}{12}z^4\}\text{ ......(xciii),}$$

where J is another arbitrary constant.

Equations (lxxxv) and (xcii) give us

$$H=A/E,\quad C=\eta A/(2E),\quad D=\eta A/E\text{ ...(xciv).}$$

We can now write down the shifts and stresses in their simplest forms, namely,

$$\left.\begin{aligned} u &= \frac{\eta A}{2E}(x^2-y^2)(l^2-z^2)+J+\tfrac{1}{2}Kz^2 \\ &\qquad\qquad + \frac{A}{E}\left(\tfrac{1}{2}l^2z^2-\tfrac{1}{12}z^4\right)+f(y,x) \\ v &= \frac{\eta A}{E}xy(l^2-z^2)+\psi(y,z) \\ w &= -\frac{A}{E}x\left(l^2z-\tfrac{1}{3}z^3\right)-Qz \end{aligned}\right\}\text{...(xcv)},$$

and

$$\left.\begin{aligned} \widehat{xx} &= F+2\mu Q+2\mu\frac{df}{dx} \\ \widehat{yy} &= F+2\mu Q+2\mu\frac{d\psi}{dy} \\ \widehat{zz} &= -Ax(l^2-z^2)+F(y,x) \\ \widehat{yz} &= -\mu z\left(\frac{dQ}{dy}+\frac{2\eta A}{E}xy\right) \\ \widehat{zx} &= -\mu z\left\{\frac{dQ}{dx}+\frac{\eta A}{E}(x^2-y^2)-K\right\} \\ \widehat{xy} &= \mu\left(\frac{df}{dy}+\frac{dx}{d\psi}\right) \end{aligned}\right\}\text{...(xcvi)}.$$

32. The arbitrary functions must now be chosen so as to satisfy the body stress-equations. Let us take first

$$\frac{d\widehat{xz}}{dx}+\frac{d\widehat{yz}}{dy}+\frac{d\widehat{zz}}{dz}=0,$$

we find
$$\frac{d^2Q}{dx^2}+\frac{d^2Q}{dy^2}=\frac{4Ax}{E}\text{ ....(xcvii)}.$$

Combining this with the third equation of (xci), or

$$l\left\{\frac{dQ}{dx}+\frac{\eta A}{E}(x^2-y^2)-K\right\}+m\left\{\frac{dQ}{dy}+\frac{2\eta A}{E}xy\right\}=0\text{...(xcviii)},$$

we can fully determine Q with the exception of certain arbitrary constants.

33. We may pause here to note how the value of Q is determined in two special cases, first remarking that the equations to determine Q are exactly of the form of those obtained by Saint-Venant to determine F, an arbitrary constant which occurs in his theory of flexure. See the *History of Elasticity*, Vol. II., p. 60.

Case I. Assume

$$Q = Q_1 + Kx + \frac{A}{E}\left\{(\eta + 2)\frac{x^3}{3} - \eta y^2 x\right\},$$

then we find

$$\frac{d^2 Q_1}{dx^2} + \frac{d^2 Q_1}{dy^2} = 0 \quad \text{..............(xcix)},$$

$$l\left[\frac{dQ_1}{dx} + \frac{2A}{E}\{(1+\eta)x^2 - \eta y^2\}\right] + m\frac{dQ_1}{dy} = 0.$$

Hence, if the cross-section be a rectangle of height $2a$ and breadth $2b$, we have to solve (xcix) subject to the conditions

$$\frac{dQ_1}{dx} = \frac{2A}{E}\{\eta y^2 - (1+\eta)a^2\}, \text{ when } x = \pm a,$$

$$\frac{dQ_1}{dy} = 0, \text{ when } y = \pm b,$$

We find as Saint-Venant

$$Q_1 = e_0 + \frac{2A}{E}\left\{\frac{\eta b^3}{3} - (1+\eta)a^2\right\}x$$

$$- \frac{2A\eta}{E} b^3 \frac{4}{\pi^3}\sum_1^\infty \frac{(-1)^{n-1}}{n^3} \frac{\sinh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}} \cos\frac{n\pi y}{b},$$

or, for Q,

$$Q = e_0 + \left[\frac{2A}{E}\left\{\frac{\eta b^2}{3} - (1+\eta)a^2\right\} + K\right]x$$

$$+ \frac{A}{E}\left\{(\eta+2)\frac{x^3}{3} - \eta y^2 x\right\}$$

$$- \frac{2A\eta}{E} b^3 \frac{4}{\pi^3}\sum_1^\infty \left\{\frac{(-1)^{n-1}}{n^3} \frac{\sinh\frac{n\pi x}{b}}{\cosh\frac{n\pi a}{b}} \cos\frac{n\pi y}{b}\right\} \text{.........(c)}.$$

Case II. Assume the contour of the cross-section to be given by the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

It will be found that a value of Q satisfying (xcvii) and (xcviii) is

$$Q = e_0 - \left\{\frac{A}{E}\,\frac{2\beta^2 + 4(1+\eta)\alpha^2}{3\alpha^2 + \beta^2}\,\alpha^2 - K\right\} x + \frac{2A}{E}\,\frac{x^3}{3}$$
$$+ \frac{A}{E}\,\frac{2\alpha^2 + \eta(\beta^2 - \alpha^2)}{3\alpha^2 + \beta^2}\left(y^2 x - \frac{x^3}{3}\right) \text{.........(ci).}$$

Compare the *History of Elasticity*, Vol. II., Art. 86.

Obviously the determination of Q corresponds exactly to the solution of Saint-Venant's flexure problem. Two undetermined constants e_0 and K occur, however, in the values of Q in our present problem.

34. We must investigate the conditions involved by the other two body stress-equations. These are

$$\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} + \rho g = 0,$$

$$\frac{d\widehat{yx}}{dx} + \frac{d\widehat{yy}}{dy} + \frac{d\widehat{yz}}{dz} = 0.$$

They lead to

$$\left.\begin{aligned} -(\lambda+\mu)\frac{dQ}{dx} + (\lambda+2\mu)\frac{dR}{dx} - \mu\frac{dP}{dy} &= \frac{\mu\eta}{E}A(x^2 - y^2) - \mu K + \rho g \\ -(\lambda+\mu)\frac{dQ}{dy} + (\lambda+2\mu)\frac{dR}{dy} + \mu\frac{dP}{dx} &= \frac{2\mu\eta}{E}Axy \end{aligned}\right\} \text{.........(cii),}$$

where

$$\left.\begin{aligned} R &= \frac{df}{dx} + \frac{d\psi}{dy} \\ P &= \frac{d\psi}{dx} - \frac{df}{dy} \end{aligned}\right\} \text{..................(ciii).}$$

Take

$$\left.\begin{aligned} P &= P_0 + \frac{\eta}{E} A \left(x^2 y + \frac{y^3}{3} \right) \\ R &= \frac{\mu}{\lambda + 2\mu} R_0 + \frac{\lambda + \mu}{\lambda + 2\mu} Q + \frac{\rho g - \mu K}{\lambda + 2\mu} x \\ &\qquad + \tfrac{2}{3} \frac{\mu}{\lambda + 2\mu} \frac{\eta}{E} A x^3 \end{aligned}\right\} \text{...(civ)},$$

and we find

$$\left.\begin{aligned} \frac{dR_0}{dx} - \frac{dP_0}{dy} = 0 \\ \frac{dR_0}{dy} + \frac{dP_0}{dx} = 0 \end{aligned}\right\} \text{.................(cvi).}$$

Thus R_0 and P_0 are what are usually termed *conjugate functions.* Hence, if ν_1, ν_2 be functions of x and y such that

$$\nu_1 + \sqrt{(-1)}\, \nu_2 = \text{function of } x + y\sqrt{(-1)},$$

we may write

$$\left.\begin{aligned} R_0 &= \phi_1 \{\nu_1 + \sqrt{(-1)}\, \nu_2\} + \phi_1 \{\nu_1 - \sqrt{(-1)}\, \nu_2\} \\ P_0 &= \frac{1}{\sqrt{(-1)}} [\phi_1 \{\nu_1 + \sqrt{(-1)}\, \nu_2\}] - \phi_1 \{\nu_1 - \sqrt{(-1)}\, \nu_2\} \end{aligned}\right\} \text{...(cvi)},$$

as a solution.

This is the starting point of the general solution in terms of conjugate functions.

Preserving Cartesian notation, we have

$$\left.\begin{aligned} R_0 &= \phi_1' \{x + \sqrt{(-1)}\, y\} + \phi_1' \{x - \sqrt{(-1)}\, y\} \\ P_0 &= \frac{1}{\sqrt{(-1)}} [\phi_1' \{x + \sqrt{(-1)}\, y\} - \phi_1' \{x - \sqrt{(-1)}\, y\}] \end{aligned}\right\} \text{...(cvii)},$$

where ϕ_1' denotes a differential of some arbitrary function ϕ_1.

Let us break up f and ψ each into two parts f_0, ψ_0 and f_1, ψ_1, where f_1, ψ_1 are the particular integrals which arise from substituting the terms in P and R which involve Q and the remaining algebraic portions. Thus:

$$\left.\begin{aligned} \frac{\mu}{\lambda + 2\mu} R_0 &= \frac{df_0}{dx} + \frac{d\psi_0}{dy} \\ P_0 &= \frac{d\psi_0}{dx} - \frac{df_0}{dy} \end{aligned}\right\} \text{............(cviii).}$$

The most general solution of these equations is given by

$$\left.\begin{aligned}
f_0 &= \phi_2\{x+\sqrt{(-1)}y\} + \phi_2\{x-\sqrt{(-1)}y\} \\
&\quad + \frac{\mu}{\lambda+2\mu}\,\frac{x}{2}\,[\phi_1'\{x+y\sqrt{(-1)}\} + \phi_1'\{x-y\sqrt{(-1)}\}] \\
&\quad - \tfrac{1}{2}x[\phi_1'\{x+y\sqrt{(-1)}\} + \phi_1'\{x-y\sqrt{(-1)}\}] \\
&\quad + \tfrac{1}{2}[\phi_1\{x+y\sqrt{(-1)}\} + \phi_1\{x-y\sqrt{(-1)}\}], \\
\psi_0 &= \frac{1}{\sqrt{(-1)}}[\phi_2\{x-y\sqrt{(-1)}\} - \phi_2\{x+y\sqrt{(-1)}\}] \\
&\quad + \frac{\mu}{\lambda+2\mu}\,\frac{x\sqrt{(-1)}}{2}[\phi_1'\{x+y\sqrt{(-1)}\} - \phi_1'\{x-y\sqrt{(-1)}\}] \\
&\quad - \frac{\mu}{\lambda+2\mu}\,\frac{\sqrt{(-1)}}{2}[\phi_1\{x+y\sqrt{(-1)}\} - \phi_1\{x-y\sqrt{(-1)}\}] \\
&\quad + \frac{x}{2\sqrt{(-1)}}[\phi_1'\{x+y\sqrt{(-1)}\} - \phi_1'\{x-y\sqrt{(-1)}\}]
\end{aligned}\right\} \quad \text{..............(cix)},$$

where ϕ_2 is another arbitrary function.

35. It remains now to determine the particular integrals corresponding to f_1 and ψ_1.

The only part of this which presents any complexity is the term in Q. Let f_2, ψ_2 be the particular integrals due to it, then by (xcix)

$$Q = Q_1 + Kx + \frac{A}{E}\left\{(\eta+2)\frac{x^3}{3} - \eta y^2 x\right\},$$

where $$\frac{d^2Q'}{dx^2} + \frac{d^2Q'}{dy^2} = 0,$$

or we must have

$$Q_1 = \chi_1'\{x+\sqrt{(-1)}y\} + \chi_2'\{x-\sqrt{(-1)}y\} \text{.....(cx)},$$

where χ_1, χ_2 are arbitrary functions, and χ_1', χ_2' their first derivatives. Hence we have to find f_2 and ψ_2:

$$\left.\begin{aligned}
\frac{\lambda+\mu}{\lambda+2\mu}\Big[\chi_1'\{x+\sqrt{(-1)}y\} + \chi_2'\{x-\sqrt{(-1)}y\} + Kx \\
+ \frac{A}{E}\left\{(\eta+2)\frac{x^3}{3} - \eta y^2 x\right\}\Big] = \frac{df_2}{dx} + \frac{d\psi_2}{dy}, \\
0 = \frac{d\psi_2}{dx} - \frac{df_2}{dy}
\end{aligned}\right\} \quad \text{..................(cxi)}.$$

After some analysis we find

$$\left.\begin{aligned}
f_2 &= \frac{\lambda+\mu}{(\lambda+2\mu)}\,\frac{x}{2}\,[\chi_1'\{x+y\sqrt{(-1)}\}+\chi_2'\{x-y\sqrt{(-1)}\}] \\
&\quad+\frac{\lambda+\mu}{\lambda+2\mu}\left[K\frac{x^2}{2}+\frac{A}{12E}\{(\eta+2)x-\eta y^4\}\right], \\
\psi_2 &= \frac{\lambda+\mu}{\lambda+2\mu}\,\frac{x\sqrt{(-1)}}{2}\,[\chi_1'\{x+y\sqrt{(-1)}\}-\chi_2'\{x-y\sqrt{(-1)}\}] \\
&\quad-\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{\sqrt{(-1)}}{2}\,[\chi_1\{x+y\sqrt{(-1)}\}-\chi_2\{x-y\sqrt{(-1)}\}] \\
&\quad-\frac{\lambda+\mu}{\lambda+2\mu}\,\frac{A\eta}{E}\,\frac{y^3x}{3}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(\text{cxii}).$$

36. For the values of the particular integrals f_1-f_2 and $\psi_1-\psi_2$ we easily deduce

$$\left.\begin{aligned}
f_1-f_2 &= \frac{\rho g-\mu K}{\lambda+2\mu}\,\frac{x^2}{2}+\tfrac{1}{6}\,\frac{\mu}{\lambda+2\mu}\,\frac{\eta}{E}Ax^4-\frac{\eta A}{12E}(x^4+y^4), \\
\psi_1-\psi_2 &= \frac{\eta A}{3E}x^3y
\end{aligned}\right\} \text{(cxiii)}.$$

Thus the values of f and ψ are fully determined. They contain two arbitrary functions ϕ_1 and ϕ_2 and two arbitrary constants introduced through the integration of χ_1' and χ_2' in Q. But these constants add together, and therefore give a single constant which may be introduced into the value of ψ, and represented by γ. Looking back at (lxxxviii) we see that for $x=y=z=0$, $v=\gamma$, or γ is the horizontal shift of the centroid of the mid-section. But this may be taken as zero, or we shall put $\gamma=0$. The arbitrary functions ϕ_1 and ϕ_2 will have to be determined from the first two equations of (xci), and with their determination we have the complete solution of the *general* problem, we have proposed, in terms of Cartesian coordinates.

The constants A, K, e_0 will be determined by the conditions at the free end. We shall proceed first to the discussion of the latter equations.

37. They are the following:

(i) The total traction when $z = l$ must vanish, or

$$\int (\widehat{zz})_{z=l} d\omega = 0.$$

Substituting from (xcvi), we have

$$\int F(y, x)\, d\omega = 0.$$

But, by (lxxxiv) and (ciii), we find

$$F = \lambda R - (\lambda + 2\mu)\, Q.$$

Hence $$\lambda \int R d\omega = (\lambda + 2\mu) \int Q d\omega.$$

Substituting from (civ), we have

$$\lambda \int R_0 d\omega = (3\lambda + 4\mu) \int Q d\omega$$
$$= (3\lambda + 4\mu) \int Q_1 d\omega \text{ ....(cxiv)},$$

where Q_1 is defined in § 33, and we have supposed the cross-section symmetrical about the axis of y.

For the cases of the rectangular and elliptic cross-sections, we have by (c) and (ci)

$$\int Q d\omega = e_0 \omega.$$

Hence we have to determine e_0:

$$e_0 = \frac{1}{\omega} \frac{\lambda}{3\lambda + 4\mu} \int R_0 d\omega \text{ ....(cxv).}$$

(ii) The horizontal total shear over the terminal cross-section must be zero, or

$$\int (\widehat{zy})_{z=l} d\omega = 0,$$

whence by (xcvi)

$$\int \left(\frac{dQ}{dy} + \frac{2\eta A}{E} xy \right) d\omega = 0.$$

But since w can only contain even powers of y, if the cross-section be symmetrical about the vertical axis, so by (lxxxvii) must Q. Hence $\frac{dQ}{dy}$ contains only odd powers and $\int \frac{dQ}{dy} d\omega = 0$. Further $\int xy d\omega = 0$, since the section is supposed to have the axes of x and y for principal axes. Hence this equation is identically satisfied.

(iii) The total vertical shear at the wall must equal $-W'$ or

$$\int(\widehat{zx})_{z=l}\,d\omega = W'.$$

Whence by (xcvi)

$$-\mu l\int\left(\frac{dQ}{dx} + \frac{\eta A}{E}(x^2 - y^2) - K\right)d\omega = W' \text{...(cxvi)}.$$

Let us work this out for the case of the elliptic cross-section, using the value of Q given in (ci) and putting for brevity

$$q = \frac{2\beta^2 + 4(1+\eta)\alpha^2}{3\alpha^2 + \beta^2} \text{ and } q' = \frac{2\alpha^2 + \eta(\beta^2 - \alpha^2)}{3\alpha^2 + \beta^2}.$$

Then we have

$$\frac{dQ}{dx} - K = \frac{A}{E}\{-q + 2x^2 + q'(y^2 - x^2\},$$

and therefore

$$\frac{W'}{\mu l} = \frac{A}{E}\int\{q - 2x^2 - (q' - \eta)(y^2 - x^2)\}\,d\omega$$

$$= \frac{A\omega}{E}\left\{q - \frac{\alpha^2}{2} - (q' - \eta)\frac{(\beta^2 - \alpha^2)}{4}\right\}$$

$$= \frac{A\omega}{E}\alpha^2(1 + \eta), \text{ after some deductions,}$$

$$= \frac{A\omega}{\mu}\frac{\alpha^2}{2} = \frac{2A\omega\kappa^2}{\mu}, \text{ if } \kappa^2 = \frac{\alpha^2}{4}.$$

Hence $$A\kappa^2 = \frac{W'}{2l\omega} \text{.....................(cxvii)}.$$

This result would also follow by using the value of Q given in (c) for a beam of rectangular section. But it may also be ascertained more readily in the following manner by taking the difference of the bending moments at the mid-section and the terminal. We find

$$\int(\widehat{zz})_{z=0}\,x\,d\omega - \int(\widehat{zz})_{z=l}\,d\omega = (W + S)\tfrac{1}{2}l - W'l,$$

or by (xcvi)

$$\int(Ax^2l^2)\,d\omega = (W + S)\tfrac{1}{2}l - W'l$$

$$-A\kappa^2\omega = \frac{W + S}{2l} - \frac{W'}{l'},$$

whence, since $W' = W + S,$

$$A\kappa^2 = \frac{W'}{2l\omega} = \frac{g\rho}{2} + \frac{S}{2l\omega} \quad \text{..........(cxviii)},$$

since $W = l\omega \times g\rho$. This agrees with (xli).

(iv) The moment of the tractions $\widehat{zz}$ at $z = l$, or M_0 must be found from

$$\begin{aligned} M_0 &= \int (\widehat{zz})_{z=l}\, x d\omega \\ &= \int F x d\omega \\ &= \lambda \int R x d\omega - (\lambda + 2\mu) \int Q x d\omega, \text{ by (lxxxiv)}, \\ &= \frac{\lambda\mu}{\lambda + 2\mu} \int R_0 x d\omega - \frac{\mu(3\lambda + 4\mu)}{\lambda + 2\mu} \int Q x d\omega \\ &= \frac{\lambda}{\lambda + 2\mu} (\rho g - \mu K)\, \kappa^2 \omega + \tfrac{2}{3} \frac{\lambda + 2\mu}{\lambda\mu} \frac{A\eta}{E} \int x^4\, d\omega \text{ ...(cxix)}. \end{aligned}$$

If the beam be simply supported, $M_0 = 0$ and (cxix) is the equation to find K. If the beam be built-in, (cxix) gives the value of the bending-moment at the built-in end in terms of the undetermined constant K. But K will then be found from the condition $du/dz = 0$, for $x = y = 0$, $z = l$. This gives us, from (xcv),

$$\left(\frac{du}{dz}\right)_{z=l} = Kl + \frac{A}{E}\left(l^3 - \frac{l^3}{3}\right) = 0.$$

or

$$\left.\begin{aligned} K &= -\tfrac{2}{3} \frac{E}{Al^2} \\ &= -\tfrac{1}{3} \frac{W'}{El\omega} \frac{l^2}{\kappa^2} \end{aligned}\right\} \text{...............(cxx)}.$$

Compare equation (liii).

38. We shall now proceed to the evaluation of M_0 for the special case of the elliptic cross-section, noting that in this case

$$\int x^4 d\omega = \tfrac{1}{8}\alpha^4\omega,\ \int x^2 y^2 d\omega = \tfrac{1}{24}\alpha^2\beta^2\omega,\ \kappa^2 = \tfrac{1}{4}\alpha^2.$$

We find, after some reductions,

$$\begin{aligned} \frac{\lambda + 2\mu}{\lambda\mu} \frac{M_0}{\omega\kappa^2} &= \frac{\int R_0 x d\omega}{\omega\kappa^2} + \frac{\rho g}{\mu} - \frac{2K}{\eta} \\ &\quad + \frac{A}{E} \frac{(40 + 26\eta - 17\eta^2)\alpha^4 + (16 - 4\eta)\alpha^2\beta^2 + (\eta^2 - 2\eta)\beta^4}{6\eta(3\alpha^2 + \beta^2)} \end{aligned}$$

.........(cxxi).

The coefficient of A/E will be found to agree with that in (lii) if we put $\alpha = \beta$.

For a doubly built-in beam K is known from (cxx), and thus M_0 is fully determined from (cxxi) so soon as we know R_0.

Let us consider the case of a doubly-supported beam. Here $M_0 = 0$, and by (cxviii)

$$\frac{\rho g}{2} = -\frac{S}{2l\omega} + A\kappa^2$$

Putting, for brevity, $\beta^2/\alpha^2 = \epsilon^2$, we find

$$K = \tfrac{1}{2}\eta \frac{\int R_0 x d\omega}{\omega\kappa^2} - \frac{S}{2\mu l\omega}$$
$$+ \frac{A\kappa^2}{E} \frac{(40 + 44\eta + \eta^2) + (16 + 2\eta + 6\eta^2)\epsilon^2 + (\eta^2 - 2\eta)\epsilon^4}{3(3 + \epsilon^2)}$$

.........(cxxii).

This determines K as soon as R_0 is known.

(v) If we suppose $u = 0$ at the centroid of the mid cross-section, we have to determine J.

$$J + f(0, 0) = 0 \quad \text{................(cxxiii).}$$

Thus if δ be the total vertical deflection, we have

$$\delta = (u)_{z=l} = \frac{l^2}{2}\left(K + \tfrac{5}{6}\frac{Al^2}{E}\right) \quad \text{........(cxxiv),}$$

which is therefore determined as soon as K is known.

39. Before proceeding to the expressions for the stresses, we may make one or two remarks with regard to the functions F, f, and ψ, &c. Since $\widehat{zz}$ does not change when we change the sign of y, F can contain only even powers of y; since w does not change with the sign of y, Q, by (xcv), can contain only even powers of y. Hence, by (lxxxiv),

$$df/dx + d\psi/dy,$$

or R can contain only even powers of y. But, by (lxxxiii), it follows in like manner that f contains only even powers of y. Thus ψ can contain only odd powers of y. From (ciii) we see that P contains only odd powers of y, and thus P_0 will contain only odd and R_0 only even powers of y. Thus $\widehat{xy}$, by (xcvi) contains only odd powers of y.

40. We will now find the value of f and ψ for the elliptic cross-section. From equations (xcix) and (ci), we find

$$\left.\begin{aligned} Q_1 &= Q - Kx - \frac{2Ax^3}{3E} - \frac{A\eta}{E}\left(\frac{x^3}{3} - y^2x\right) \\ &= \frac{e_0}{2} - \frac{Aq}{2E}\{x+y\sqrt{(-1)}\} - \frac{q''A}{6E}\{x+y\sqrt{(-1)}\}^3 \\ &+ \frac{e_0}{2} - \frac{Aq}{2E}\{x-y\sqrt{(-1)}\} - \frac{q''A}{6E}\{x-y\sqrt{(-1)}\}^3 \end{aligned}\right\}\text{...(cxxv)},$$

where

$$q = \frac{2\beta^2 + 4(1+\eta)\alpha^2}{3\alpha^2+\beta^2}\alpha^2, \quad q'' = q' + \eta = \frac{2\{\alpha^2 + \eta(\alpha^2+\beta^2)\}}{3\alpha^2+\beta^2},$$

or q and q' are the same as in § 37 (iii).

Clearly the first line in Q_1 is the $\chi_1'\{x+y\sqrt{(-1)}\}$ and the second line the $\chi_2'\{x-y\sqrt{(-1)}\}$ of equation (cx).

With the aid of (cxii) and (cxiii) we can now write down the values of the particular integrals f_1 and ψ_1 for the case of the elliptic section; we find after some reductions the following values

$$\begin{aligned} f_1 = \frac{\rho g - \mu K}{\lambda+2\mu}\frac{x^2}{2} + \tfrac{1}{6}\frac{\mu}{\lambda+2\mu}\frac{\eta A}{E}x^4 - \frac{\eta A}{12E}(x^4+y^4) \\ + \frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{e_0 x}{2} + \left(K - \frac{Aq}{E}\right)\frac{x^2}{2} + \frac{Aq''}{2E}\left(y^2x^2 - \frac{x^4}{3}\right)\right. \\ \left. + \frac{A}{12E}\{(\eta+2)x^4 - \eta y^4\}\right]\text{...(cxxvi)}, \end{aligned}$$

$$\psi_1 = \frac{\eta A}{3E}x^3y + \frac{\lambda+\mu}{\lambda+2\mu}\left\{\frac{e_0}{2}y + \frac{Aq''}{3E}x^3y - \frac{A\eta}{3E}y^3x\right\}\text{...(cxxvii)}.$$

We may now choose suitable values of ψ_0 and f_0 as given by (cix). Let us find the most general solution for ψ_0 and f_0 up to terms in x and y of the fourth order. To do this assume:

$$\left.\begin{aligned} \phi_1(\zeta) &= \tfrac{1}{2}\{b_0 + b_1\zeta + \tfrac{1}{2}b_2\zeta^2 + \tfrac{1}{3}b_3\zeta^3 + \tfrac{1}{4}b_4\zeta^4\} \\ \phi_2(\zeta) &= \tfrac{1}{2}\{a_0 + a_1\zeta + \tfrac{1}{2}a_2\zeta^2 + \tfrac{1}{3}a_3\zeta^3 + \tfrac{1}{4}a_4\zeta^4\} \end{aligned}\right\}\text{...(cxxviii)}.$$

Then we find from (cvii)

$$\left.\begin{aligned} R_0 &= b_1 + b_2 x + b_3 (x^2 - y^2) + b_4 (x^3 - 3xy^2) \\ P_0 &= b_2 y + 2b_3 xy + b_4 (3x^2 y - y^3) \end{aligned}\right\} \text{....(cxxix).}$$

Further, from (cix), we have

$$\begin{aligned} f_0 = {} & a_0 + a_1 x + \tfrac{1}{2}a_2 (x^2 - y^2) + \tfrac{1}{3}a_3 (x^3 - 3xy^2) + \tfrac{1}{4}a_4 (x^4 - 6x^2y^2 + y^4) \\ & - \frac{\lambda + \mu}{\lambda + 2\mu} \tfrac{1}{2}x \{b_1 + b_2 x + b_3 (x^2 - y^2) + b_4 (x^3 - 3xy^2)\} \\ & + \tfrac{1}{2} \{b_0 + b_1 x + \tfrac{1}{2}b_2 (x^2 - y^2) + \tfrac{1}{3}b_3 (x^3 - 3xy^2) + \tfrac{1}{4}b^4 (x^4 - 6x^2y^2 + y^4)\} \end{aligned}$$

$$\text{.........(cxxx),}$$

$$\begin{aligned} \psi_0 = {} & - \{a_1 y + a_2 xy + a_3 (yx^2 - \tfrac{1}{3}y^3) + a_4 (x^3 y - y^3 x)\} \\ & + \frac{\lambda + \mu}{\lambda + 2\mu} \tfrac{1}{2}x \{b_2 y + 2b_3 xy + 3b_4 (yx^2 - \tfrac{1}{3}y^3) \\ & + \frac{\mu}{\lambda + 2\mu} \tfrac{1}{2} \{b_1 y + b_2 xy + b_3 (yx^2 - \tfrac{1}{3}y^3) + b_4 (x^3 y - y^3 x)\} \end{aligned}$$

$$\text{.........(cxxxi).}$$

These equations contain nine constants, for a_0 and $\frac{1}{2}b_0$ add together. These constants will have to be determined by the first two equations of (xci), which still remain to be satisfied.

Since $f(0, 0) = f_0(0, 0) + f_1(0, 0) = a_0 + \frac{1}{2}b_0$, by (cxxvi) and (cxxxi), we have, from (cxxiii),

$$J = -(a_0 + \tfrac{1}{2}b_0) \text{ ....(cxxxii),}$$

which determines J when $a_0 + \frac{1}{2}b_0$ is known.

Further, by (cxv) and the first of (cxxix), we have

$$e_0 = \frac{\lambda}{3\lambda + 4\mu} \{b_1 + \tfrac{1}{4}b_3 (\alpha^2 - \beta^2)\} \text{ .....(cxxxiii).}$$

Thus the whole system of constants is determined, provided we can find

$$a_0 + \tfrac{1}{2}b_0,\ a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3, \text{ and } b_4.$$

We may record here the value of K as given by (cxxii). It will be found that

$$\int \frac{R_0 x d\omega}{\omega k_2^2} = b_2 + \tfrac{1}{2}b_4 (\alpha^2 - \beta^2),$$

whence

$$K = \tfrac{1}{2}\eta\{b_2 + \tfrac{1}{2}b_4(\alpha^2 - \beta^2)\} - \frac{S}{2\mu l\omega}$$
$$+ \frac{AK^2}{E}\frac{(40 + 44\eta + \eta^2) + (16 + 2\eta + 6\eta^2)\varepsilon^2 + (\eta^2 - 2\eta)\varepsilon^4}{3(3 + \varepsilon^2)}$$
...............(cxxxiv).

41. It will now be necessary to calculate the values of the stresses $\widehat{xx}$, $\widehat{yy}$, and $\widehat{xy}$.

From the equations:

$$\widehat{xx} = (\lambda + 2\mu)\frac{df}{dx} + \lambda\left(\frac{d\psi}{dy} - Q\right),$$

$$\widehat{yy} = (\lambda + 2\mu)\frac{d\psi}{dy} + \lambda\left(\frac{df}{dx} - Q\right),$$

$$\widehat{xy} = \mu\left(\frac{df}{dy} + \frac{d\psi}{dx}\right),$$

which follow from (xciv) and (lxxxiv) we find, by aid of (cxxvi), (cxxvii), (cxxix), and (cxxx), if $\gamma = \dfrac{\lambda + \mu}{\lambda + 2\mu}$, the values

$$\frac{\widehat{xx}}{\mu} = (1 - \gamma)e_0 + 2a_1 + \gamma b_1 + x\left\{\frac{\rho g}{\mu} - \frac{Aq}{E} + 2a_2\right\} + x^2(2a_3 - \gamma b_3)$$
$$- y^2(2a_3 + \gamma b_3) + x^3\left[2a_4 - 2\gamma b_4 + \{(3 - 2\gamma) - (1 + 2\gamma)q'\}\frac{A}{3E}\right]$$
$$- y^2x\left[6a_4 - \{(2\gamma - 1) + q'\}\frac{A}{E}\right] \text{.........(cxxxv)},$$

$$\frac{\widehat{yy}}{\mu} = (1 - \gamma)e_0 - 2a_1 + \gamma b_1 + x\left\{(2\gamma - 1)\left(\frac{\rho g}{\mu} + \frac{Aq}{E} - 2K\right)\right.$$
$$\left. + 2\gamma b_2 - 2a_2\right\} + x^2(3\gamma b_3 - 2a_3) - y^2(\gamma b_3 - 2a_3) + x^3\left[-2a_4 + 4\gamma b_4\right.$$
$$\left. + \left\{(2\gamma - 1) + (4\gamma - 1)q'\right\}\frac{A}{3E}\right] - y^2x\left[-6a_4 + 6\gamma b_4\right.$$
$$\left. + \{(2\gamma - 1) + (2\gamma - 1)q'\}\frac{A}{E}\right] \text{............(cxxxvi)},$$

$$\frac{\widehat{xy}}{\mu} = y\Big[-2a^2 + x(2\gamma b_3 - 4a_3) + x^2\Big\{6\gamma b_4 - 6a_4 + [\eta\{(1+2\gamma)\}$$

$$+ 2\gamma q'] \frac{A}{E}\Big\} - y^2\Big\{-2a_4 + \eta(1+2\gamma)\frac{A}{3E}\Big\}\Big] \ldots\ldots(\text{cxxxvii}).$$

These equations will be written for brevity

$$\left.\begin{aligned} \frac{\widehat{xx}}{\mu} &= c_1 + c_2x + c_3x^2 - g_3y^2 + c_4x^3 - g_4y^2x, \\ \frac{\widehat{yy}}{\mu} &= d_1 + d_2x + d_3x^2 - h_3y^2 + d_4x^3 - h_4y^2x, \\ \frac{\widehat{xy}}{\mu} &= y(f_2 + f_3x + f_4x^2 - k_4y^2) \end{aligned}\right\} \ldots(\text{cxxxviii}).$$

We shall now substitute these values in (xci), *supposing there to be no surface load*; we find

$$\left.\begin{aligned} \frac{x}{\alpha^2}\widehat{xx} + \frac{y}{\beta^2}\widehat{xy} = 0, \\ \frac{x}{\alpha^2}\widehat{xy} + \frac{y}{\beta^2}\widehat{yy} = 0 \end{aligned}\right\}$$

for all values of x and y on $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, or for $y^2 = \beta^2 - \epsilon^2x^2$.

The second of these equations gives

$$\epsilon^2x\{f_2 - k_4\beta^2 + f_3x + (f_4 + k_4\epsilon^2)x^2\}$$
$$+ d_1 - h_3\beta^2 + (d_2 - h_4\beta^2)x + (d_3 + h_3\epsilon^2)x^2 + (d_4 + h_4\epsilon^2)x^3 = 0.$$

Whence we have

$$d_1 - h_3\beta^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a),$$
$$(f_2 - k_4\beta^2)\epsilon^2 + (d_2 - h_4\beta^2) = 0 \ldots\ldots\ldots\ldots(b),$$
$$f_3\epsilon^2 + (d_3 + h_3\epsilon^2) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(c),$$
$$(f_4 + k_4\epsilon^2)\epsilon^2 + (d_4 + h_4\epsilon^2) = 0 \ldots\ldots\ldots(d).$$

The first equation gives

$$\epsilon^2x\{c_1 - \beta^2g_3 + (c_2 - g_4\beta^2)x + (c_3 + g_3\epsilon^2)x^2 + (c_4 + g_4\epsilon^2)x^3\}$$
$$+ (\beta^2 - \epsilon^2x^2)\{f_2 - k_4\beta^2 + f_3x + (f_4 + k_4\epsilon^2)x^2\} = 0.$$

Whence we must have

$$f_2 - k_4\beta^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(e),$$

$$\epsilon^2 (c_1 - \beta^2 g_3) + f_3\beta^2 = 0 \ldots\ldots\ldots\ldots\ldots(f),$$

$$\epsilon^2 (c_2 - g_4\beta^2) - \epsilon^2 (f_2 - k_4\beta^2) + \beta^2 (f_4 + k_4\epsilon^2) = 0 \ldots(g),$$

$$\epsilon^2 (c_3 + g_3\epsilon^2) - \epsilon^2 f_3 = 0 \ldots\ldots\ldots\ldots\ldots(h),$$

$$\epsilon^2 (c_4 + g_4\epsilon^2) - \epsilon^2 (f_4 + k_4\epsilon^2) = 0 \ldots\ldots\ldots\ldots(i),$$

Let us consider the equations with odd subscripts first. They are

$$d_1 - h_3\beta^2 = 0,$$

$$f_3\epsilon^2 + d_3 + h_3\epsilon^2 = 0,$$

$$f_3\alpha^2 + c_1 - \beta^2 g_3 = 0,$$

$$f_3 - c_3 - \epsilon^2 g_3 = 0.$$

But by inspection we see $c_3 = -h_3$, whence we deduce

$$d_1 = h_3\beta^2, \text{ or } (1-\gamma)\, e_0 - 2a_1 + \gamma b_1 = (\gamma b_3 - 2a_3)\,\beta^2,$$

$$c_1 = h_3\alpha^2, \text{ or } (1-\gamma)\, e_0 + 2a_1 + \gamma b_1 = (\gamma b_3 - 2a_3)\,\alpha^2,$$

$$d_3 = -g_3\epsilon^4, \text{ or } 3\gamma b_3 - 2a_3 = -\epsilon^4 (2a_3 + \gamma b_3),$$

$$f_3 = -h_3 + g_3\epsilon^2, \text{ or } 3\gamma b_3 - 6a_3 = \epsilon^2 (2a_3 + \gamma b_3),$$

The latter two equations can only hold if $a_3 = b_3 = 0$, whence the first two give $a_1 = 0$, and, by (cxxxiii), b_1 and e_0 both zero. Hence we conclude that if there be no surface load

$$e_0 = a_1 = b_1 = a_3 = b_3 = 0 \ldots\ldots\ldots\ldots\ldots(\text{cxl}),$$

a result which immensely simplifies the expressions for the stresses.

The equations with even subscripts give us

$$f_2 = k_4\beta^2,$$

$$d_2 = h_4\beta^2,$$

$$c_2 - g_4\beta^2 + f_4\alpha^2 + k_4\beta^2 = 0,$$

$$c_4 + g_4\epsilon^2 - f_4 - k_4\epsilon^2 = 0,$$

$$d_4 + h_4\epsilon^2 + f_4\epsilon^2 + k_4\epsilon^4 = 0.$$

We have five equations with only four unknowns a_2, b_2, a_4, b_4, and therefore unless we can show that two are identical we shall be unable to obtain a solution in the manner proposed. The third and fourth of these equations give us

$$c_2 = -\alpha^2 c_4,$$

so that we may take as our five equations

$$\left.\begin{array}{l} f_2 = k_4\beta^2, \quad d_2 = h_4\beta^2, \quad c_2 = -\alpha^2 c_4, \\ c_2 = g_4\beta^2 - f_4\alpha^2 - k_4\beta^2, \quad d_4 = -(h_4 + f_4 + k_4\epsilon^2)\,\epsilon^2 \end{array}\right\} \text{...(cxli).}$$

Let us write

$$\left.\begin{array}{ll} J_1 = \dfrac{\rho g}{\mu} - \dfrac{Aq}{E}, & J_2 = \{(3-2\gamma) - (1+2\gamma)\,q'\}\,\dfrac{A}{3E}, \\ J_3 = \{(2\gamma-1) + q'\}\,\dfrac{A}{E}, & J_4 = (2\gamma-1)\left(\dfrac{\rho g}{\mu} + \dfrac{Aq}{E}\right), \\ J_5 = \{(2\gamma-1) + (4\gamma-1)\,q'\}\,\dfrac{A}{3E}, & J_6 = \{(2\gamma-1) + (2\gamma-1)\,q'\}\,\dfrac{A}{E}, \\ J_7 = \{\eta\,(1+2\gamma) + 2\gamma q'\}\,\dfrac{A}{E}, & J_8 = \eta\,(1+2\gamma)\,\dfrac{A}{3E} \end{array}\right\} \text{.........(cxlii).}$$

The equations (clxi) then become

$$\left.\begin{array}{c} -2a_2 = \beta^2(-2a_4 + J_8), \\ 2\gamma b_2 - 2a_2 + J_4 - (4\gamma - 2)\,K = \beta^2(-6a_4 + 6\gamma b_4 + J_6), \\ J_1 + 2a_2 = -\alpha^2(2a_4 - 2\gamma b_4 + J_2), \\ J_1 + 2a_2 = \beta^2(6a_4 - J_3) - \alpha^2(6\gamma b_4 - 6a_4 + J_7) - \beta^2(-2a_4 + J_8), \\ -2a_4 + 4\gamma b_4 + J_5 = -\epsilon^2\{-6a_4 + 6\gamma b_4 + J_6 - 6\gamma b_4 - 6a_4 + J_7 + \epsilon^2(-2a_4 + J_8)\} \end{array}\right\} \text{............(cxliii).}$$

The first and second of these equations suffice to give a_2 and b_2, if a_4 and b_4 be known, by the aid of (cxxxiv). Let us eliminate by aid of the first a_2 from the third and fourth, and show that the resulting equations are *identical*. We find from the third

$$(1+\epsilon^2)\,a_4 - \gamma b_4 = \tfrac{1}{2}\left(\epsilon^2 J_8 - J_2 - \frac{J_1}{\alpha^2}\right) \text{......(cxliv),}$$

and from the fourth

$$(1+\epsilon^2)\,a_4 - \gamma b_4 = \tfrac{1}{6}\left(\epsilon^2 J_3 + J_7 + \frac{J_1}{\alpha^2}\right) \text{......(cxlv).}$$

Hence we must have

$$\epsilon^2 J_3 + J_7 + \frac{J_1}{\alpha^2} = 3\left(\epsilon^2 J_8 - J_2 - \frac{J_1}{\alpha^2}\right),$$

or

$$\epsilon^2 (J_3 - 3J_8) + J_7 + 3J_2 = -\frac{4J_1}{\alpha^2}.$$

This equation, after substituting from (cxlii) and using the values of q and q' given in (37) (iii), reduces to

$$A\kappa^2 = \frac{\rho g}{2},$$

which is the value of $A\kappa^2$ by (cxviii), since we have put the surface load S zero.

The fifth equation of (cxliii) gives us

$$a_4 (1 + 6\epsilon^2 + \epsilon^4) - 2\gamma b_4 (1 + 3\epsilon^2) = \tfrac{1}{2}\{J_5 + \epsilon^2 (J_6 + J_7) + \epsilon^4 J_8\} \quad \text{............(cxlvi)},$$

(cxliv) and (cxlvi) determine a_4 and b_4.

Whence we have

$$\left.\begin{aligned} a_4 &= \frac{(1+3\epsilon^2)\left\{\epsilon^2 J_8 - J_2 - \frac{J_1}{\alpha^2}\right\} - \frac{1}{2}\{J_5 + \epsilon^2 (J_6 + J_7) + \epsilon^4 J_8\}}{1 + 2\epsilon^2 + 5\epsilon^4}, \\ \gamma b_4 &= \frac{\frac{1+6\epsilon^2+\epsilon^4}{2}\left\{\epsilon^2 J_8 - J_2 - \frac{J_1}{\alpha^2}\right\} - \frac{1+\epsilon^2}{2}\{J_5 + \epsilon^2 (J_6 + J_7) + \epsilon^4 J_8\}}{1 + 2\epsilon^2 + 5\epsilon^4} \end{aligned}\right\} \quad \text{............(cxlvii)}.$$

Substituting the values of the J's we find, remembering (cxviii),

$$\epsilon^2 J_8 - J_2 - \frac{J_1}{\alpha^2}$$

$$= \frac{A}{3E} \frac{8\gamma + 2\eta - 3 + (10\gamma + \eta - 4)\epsilon^2 + (2\gamma + \eta - 1)\epsilon^4}{3 + \epsilon^2} \quad \text{...(cxlviii)},$$

$$J_5 + \epsilon^2 (J_6 + J_7) + \epsilon^4 J_8$$

$$= \frac{A}{3E} \frac{10\gamma + \eta - 3 + (54\gamma + 11\eta - 21)\epsilon^2 + (30\gamma + 3\eta - 15)\epsilon^4 + (2\gamma + \eta - 1)\epsilon^6}{3 + \epsilon^2} \quad \text{............(cxlix)}.$$

Whence we have from (cxlvii),

$$\left.\begin{aligned} a_4 &= \frac{A}{6E}\frac{6\gamma+3\eta-3+(14\gamma+3\eta-5)\epsilon^2+(34\gamma+5\eta-11)\epsilon^4+(10\gamma+5\eta-5)\epsilon^6}{(1+2\epsilon^2+5\epsilon^4)(3+\epsilon^2)}, \\ \gamma b_4 &= -\frac{A}{6E}\frac{2\gamma-\eta+(6\gamma-\eta-2)\epsilon^2+(14\gamma+5\eta-8)\epsilon^4+(10\gamma-3\eta-6)\epsilon^6}{(1+2\epsilon^2+5\epsilon^4)(3+\epsilon^2)} \end{aligned}\right\} \quad \text{............(cl).}$$

We now pass to the determination of a_2, b_2, and K. By (cxliii)

$$a_2 = \beta^2 (a_4 - \tfrac{1}{2}J_8),$$

whence we find

$$a_2 = \frac{A\beta^2\epsilon^2(1-2\eta)(1+3\epsilon^2)}{3E(1+2\epsilon^2+5\epsilon^4)(3+\epsilon^2)} \quad \text{............(cli)}:$$

b_2 is now known from the two equations

$$2\gamma b_2 - 2a_2 + J_4 - (4\gamma-2)K = \beta^2(-6a_4 + 6\gamma b_4 + J_6),$$

$$2K = \eta\{b_2 + \tfrac{1}{2}b_4(\alpha^2-\beta^2)\} + \frac{A\alpha^2}{6E}\frac{40+44\eta+\eta^2+(16+2\eta+6\eta^2)\epsilon^2+(\eta^2-2\eta)\epsilon^4}{3+\epsilon^2},$$

whence, after some rather long, but otherwise straightforward, reductions, we obtain

$$b_2 = \frac{1}{1+\eta}\frac{A\alpha^2}{6E} \times \frac{\left\{\begin{array}{l} -2\eta-\eta^2+\eta^3+\epsilon^2(-6-12\eta-6\eta^2)+\epsilon^4(-14-28\eta-12\eta^2-6\eta^3) \\ \quad +\epsilon^6(-18-76\eta-34\eta^2+8\eta^3)+\epsilon^8(6-42\eta-11\eta^2-3\eta^3) \end{array}\right\}}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

$$\text{.........(clii).}$$

$$\gamma K = \frac{A\alpha^2}{24E}\frac{1}{(1+\eta)(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} \times \left\{\begin{array}{l} 332\gamma-126-43\eta+\epsilon^2(720\gamma-264-84\eta)+\epsilon^4(1752\gamma-644-202\eta) \\ \qquad +\epsilon^6(240\gamma-40+28\eta)+\epsilon^8(-100\gamma+50+45\eta) \end{array}\right\}$$

$$\text{....... .(cliii).}$$

42. This completes the determination of the constants. We have now to find the stresses and strains.

Turning to the expressions (cxxxv), (cxxxvi), (cxxxvii), and using equations (cxliii) we find

$$\left.\begin{aligned}\frac{\widehat{xx}}{\mu} &= \frac{x(\alpha^2 - x^2)}{\alpha^2}(J_1 + 2a_2) - y^2 x(6a_4 - J_3),\\ \frac{\widehat{yy}}{\mu} &= x(\beta^2 - y^2)(-6a_4 + 6\gamma b_4 + J_6) + x^3(-2a_4 + 4\gamma b_4 + J_6),\\ \frac{\widehat{xy}}{\mu} &= y\{-2a_2 + x^3(6\gamma b_4 - 6a_4 + J_7) - y^2(-2a_4 + J_8)\}\end{aligned}\right\} \quad \text{.........(cliv).}$$

To find the other stresses, substitute in the values of $\widehat{yz}$ and $\widehat{zx}$ as given by (xc) the values of χ and Q from (xciii) and (ci), then, remembering that $e_0 = 0$, we find

$$\frac{\widehat{yz}}{\mu} = -\frac{4A}{E}xyz\left\{\frac{1 + \eta(1 + \varepsilon^2)}{3 + \varepsilon^2}\right\} \quad \text{..........(clv),}$$

$$\frac{\widehat{zx}}{\mu} = -\frac{zA}{E}\left\{-\alpha^2\frac{2\varepsilon^2 + 4(1+\eta)}{3 + \varepsilon^2} + 2x^2 + (x^2 - y^2)\frac{4\eta - 2}{3 + \varepsilon^2}\right\} \quad \text{...(clvi).}$$

Also $\qquad \widehat{zz} = -Ax(l^2 - z^2) + F(y, x),$

but, from (lxxxiv),

$$\begin{aligned}F &= \lambda\left(\frac{df}{dx} + \frac{d\psi}{dy}\right) - (\lambda + 2\mu)Q\\ &= \lambda R - (\lambda + 2\mu)Q\\ &= \mu\left\{\frac{\lambda}{\lambda + 2\mu}R_0 - (2\gamma + 1)Q + \frac{\lambda}{\lambda + 2\mu}\left(\frac{\rho g}{\mu} - K\right)x\right.\\ &\qquad\qquad\qquad\left. + \tfrac{2}{3}\frac{\lambda}{\lambda + 2\mu}\frac{\eta A}{E}x^3\right\}.\end{aligned}$$

This last result follows from (civ), and if now we substitute the value of R_0 given by (cxxix), putting in also the value of Q, we get

$$\begin{aligned}\frac{\widehat{zz}}{\mu} &= -\frac{A}{\mu}x(l^2 - z^2)\\ &\quad + x\left\{(2\gamma - 1)\left(b_2 + \frac{\rho g}{\mu}\right) + (2\gamma + 1)\frac{A}{E}\frac{2\varepsilon^2 + 4(1+\eta)}{3 + \epsilon^2}\alpha^2 - 4\gamma K\right\}\\ &\quad + (x^3 - 3xy^2)\left\{(2\gamma - 1)b_4 + \frac{A}{3E}(2\gamma + 1)\frac{2 + \eta(\epsilon^2 - 1)}{3 + \varepsilon^2}\right\} - \frac{2A}{3E}x^3(2 + \eta)\end{aligned}$$

.........(clvii).

If now we substitute for the a's and b's and for K in the expressions (cliv)—(clvii), we obtain the following values for the stresses

$$\frac{\widehat{xx}}{\mu} = -x(\alpha^2 - x^2)\frac{A}{3E}\frac{3+3\eta+\epsilon^2(9+3\eta)+\epsilon^4(19+13\eta)+\epsilon^6(9-3\eta)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} - y^2x\frac{A}{E}\frac{(4\eta-2)(1+\epsilon^2+2\epsilon^4)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)},$$

$$\frac{\widehat{yy}}{\mu} = -x(\beta^2 - y^2)\frac{A}{E}\frac{1+\eta+\epsilon^2(3+\eta)+\epsilon^4(5+7\eta)+\epsilon^6(7\eta-1)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} - x^3\frac{A}{3E}(4\eta-2)\frac{\epsilon^4}{1+2\epsilon^2+5\epsilon^4},$$

$$\frac{\widehat{zz}}{\mu} = -\frac{A}{\mu}x(l^2-z^2) + \frac{xA\alpha^2}{6E} \times \frac{\left\{8+7\eta-\eta^2+\epsilon^2(24+12\eta)+\epsilon^4(56+30\eta+6\eta^2) + \epsilon^6(40+4\eta-8\eta^2)+\epsilon^8(11\eta+3\eta^2)\right\}}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$
$$+ (x^3-3xy^2)\frac{A}{3E} \times \frac{4-\eta+\eta^2+(8+\eta+\eta^2)\epsilon^2+(20+5\eta-5\eta^2)\epsilon^4+(11\eta+3\eta^2)\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} - \frac{2A}{3E}x^3(2+\eta)$$

............(clviii).

$$\frac{\widehat{yz}}{\mu} = -\frac{4A}{E}xyz\frac{1+\eta(1+\epsilon^2)}{3+\epsilon^2} = \sigma_{yz},$$

$$\frac{\widehat{zx}}{\mu} = -\frac{zA}{E}\left\{(x^2-\alpha^2)\frac{2\epsilon^2+4(1+\eta)}{3+\epsilon^2} - y^2\frac{(4\eta-2)}{3+\epsilon^2}\right\} = \sigma_{zx},$$

$$\frac{\widehat{xy}}{\mu} = y\left\{(B^2-y^2)\frac{A}{3E}\frac{(4\eta-2)\epsilon^2(1+3\epsilon^2)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + \frac{x^2A}{E}\frac{\eta+1+(5\eta+1)\epsilon^2+(7\eta+5)\epsilon^4+(3\eta+1)\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}\right\} = \sigma_{xy},$$

...............(clix),

(clix) gives us both shears and slides. It remains to calculate the stretches $s_x = \frac{du}{dx}$, $s_y = \frac{dv}{dy}$, $s_z = \frac{dw}{dz}$.

We find

$$s_x = \frac{\eta A}{E} x (l^2 - z^2) + \frac{df}{dx},$$

$$s_y = \frac{\eta A}{E} x (l^2 - z^2) + \frac{d\psi}{dy},$$

$$s_z = -\frac{A}{E} x (l^2 - z^2) - \left\{ -\left(\frac{A}{E} \frac{2\epsilon^2 + 4(1+\eta)}{3 + \epsilon^2} \alpha^2 - K \right) x + \frac{2A}{E} \frac{x^3}{3} + \frac{A}{E} \frac{2 + \eta(\epsilon^2 - 1)}{3 + \epsilon^2} \left(y^2 x - \frac{x^3}{3} \right) \right\},$$

whence, after reduction,

$$\begin{aligned} s_x = {} & \frac{\eta A}{E} x (l^2 - z^2) \\ & - \frac{x\alpha^2 A}{12E} \frac{\left\{ \begin{array}{l} 6 + 14\eta + 7\eta^2 - \eta^3 + \epsilon^2 (18 + 24\eta + 6\eta^2) + \\ \epsilon^4 (38 + 64\eta + 24\eta^2 + 6\eta^3) + \epsilon^6 (18 + 4\eta - 38\eta^2 - 8\eta^3) \\ + \epsilon^8 (6\eta - 31\eta^2 + 3\eta^3) \end{array} \right\}}{(1+\eta)(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \\ & + \frac{Ax^3}{6E} \frac{\left\{ \begin{array}{l} 3 + 8\eta - \eta^2 + \epsilon^2 (9 + 14\eta - \eta^2) \\ + \epsilon^4 (19 + 36\eta + 5\eta^2) + \epsilon^6 (9 + 6\eta - 3\eta^2) \end{array} \right\}}{(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \\ & - \frac{xy^2 A}{2E} \frac{-2 + 3\eta - \eta^2 + \epsilon^2(-2 + \eta - \eta^2) + \epsilon^4(-4 - 3\eta + 5\eta^2) + \epsilon^6(-\eta - 3\eta^2)}{(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \end{aligned} \quad \text{.................(clx).}$$

$$\begin{aligned} s_y = {} & \frac{\eta A}{E} x (l^2 - z^2) \\ & - \frac{x\alpha^2 A}{12E} \frac{\left\{ \begin{array}{l} 2\eta^2 + \eta^2 - \eta^3 + \epsilon^2 (6 + 12\eta + 6\eta^2) \\ + \epsilon^4 (18 + 24\eta + 4\eta^2 + 6\eta^3) + \epsilon^6 (30 + 64\eta + 10\eta^2 - 8\eta^3) \\ + \epsilon^8 (-6 + 42\eta + 11\eta^2 + 3\eta^2) \end{array} \right\}}{(1+\eta)(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \\ & + \frac{Ax^3}{6E} \frac{5\eta - \eta^2 + \epsilon^2 (11\eta - \eta^2) + \epsilon^4(6 + 11\eta + 5\eta^2) + \epsilon^6(2 + 5\eta - 3\eta^2)}{(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \\ & + \frac{Axy^2}{2E} \frac{1 + 2\eta + \eta^2 + \epsilon^2(3 + 4\eta + \eta^2) + \epsilon^4(5 + 18\eta - 5\eta^2) + \epsilon^6(-1 + 8\eta + 3\eta^2)}{(3+\epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \end{aligned} \quad \text{...............(cxli).}$$

$$s_z = -\frac{A}{E}x(l^2 - z^2)$$

$$+ \frac{x\alpha^2 A}{12E}\,\frac{\left\{\begin{array}{l}8 + 13\eta + 5\eta^2 + \varepsilon^2(24 + 36\eta + 12\eta^2)\\ +\varepsilon^4(56+86\eta+38\eta^2) + \varepsilon^6(40+52\eta+28\eta^2) + \varepsilon^8(5\eta+45\eta^2)\end{array}\right\}}{(1+\eta)(3+\varepsilon^2)(1+2\varepsilon^2+5\varepsilon^4)}$$

$$-\frac{Ax^3}{6E}\,\frac{8 + 2\eta + \varepsilon^2(20 + 2\eta) + \varepsilon^4(48 + 6\eta) + \varepsilon^6(20 - 10\eta)}{(3+\varepsilon^2)(1+2\varepsilon^2+5\varepsilon^4)}$$

$$-\frac{Axy^2}{2E}\,\frac{4 - 2\eta + \varepsilon^2(8 - 2\eta) + \varepsilon^4(20 - 6\eta) + \varepsilon^6 10\eta}{(3+\varepsilon^2)(1+2\varepsilon^2+5\varepsilon^4)} \quad \text{......(clxii)}.$$

Finally, to find the maximum deflection δ, substitute for K in (cxxiv), and we find

$$\delta = \frac{Wl}{E\omega}\left\{\frac{5}{6}\frac{l^2}{\alpha^2} + \frac{1}{12}\,\frac{\left[\begin{array}{l}40+83\eta+43\eta^2+\epsilon^2(96+180\eta+84\eta^2)+\varepsilon^4(232+442\eta+202\eta^2)\\ \qquad + \epsilon^6(80+68\eta-28\eta^2) + \epsilon^8(-5\eta-45\eta^2)\end{array}\right]}{(1+\eta)(3+\epsilon^2)(1+2\epsilon^2+5\varepsilon^4)}\right\}$$

$$\text{...............(clxiii)}.$$

Also the curvature

$$\frac{1}{\rho} = \left(\frac{d^2u}{dz^2}\right)_{\substack{x=0\\y=0}} = K + \frac{W}{2l(E\omega k^2)}(l^2 - z^2)$$

$$= \frac{M}{E\omega k^2} + K, \text{ where } M \text{ is the bending moment}$$

$$= \frac{M}{E\omega k^2}\left\{1 + \frac{\alpha^2}{l^2 - z^2}\right.$$

$$\left.\times\frac{\left[\begin{array}{l}40+83\eta+43\eta^2+\epsilon^2(96+180\eta+84\eta^2)+\epsilon^4(232+442\eta+202\eta^2)\\ \qquad + \epsilon^6(80+68\eta-28\eta^2) + \epsilon^8(-5\eta-45\eta^2)\end{array}\right]}{12(1+\eta)(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}\right\}$$

$$\text{............(clxiv)}.$$

We see from this that there is a definite curvature K at the points of support. This correction K to the curvature is constant. Hence, if we draw a circle of radius $1/K$ passing through the two terminals of the beam, and add the ordinates of this circle to the ordinates of the "elastic line" obtained from the ordinary theory, we shall obtain the true elastic line, the determination of which is thus made to depend only on the calculation of K.

43. Before proceeding to discuss the formulæ we have obtained, we may record here the values we get for the somewhat simpler case when the material possesses uni-constant isotropy.

In this case $\lambda=\mu$, $E=\frac{5}{2}\mu$, $\eta=\frac{1}{4}$, $\gamma=\frac{2}{3}$.

We have then

$$K=\frac{A\alpha^2}{240E}\frac{1015+2340\epsilon^2+5682\epsilon^4+1524\epsilon^6-65\epsilon^8}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} \text{ ....(clxv).}$$

$$\frac{\widehat{xx}}{\mu}=-x(\alpha^2-x^2)\frac{A}{12E}\frac{15+39\epsilon^2+89\epsilon^4+33\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + \frac{y^2xA}{E}\frac{1+\epsilon^2+2\epsilon^4}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)},$$

$$\frac{\widehat{yy}}{\mu}=-x(\beta^2-y^2)\frac{A}{4E}\frac{5+13\epsilon^2+27\epsilon^4+3\epsilon^3}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}+\frac{Ax^3}{3E}\frac{\epsilon^4}{(1+2\epsilon^2+5\epsilon^4)},$$

$$\frac{\widehat{zz}}{\mu}=-\frac{A}{E}\frac{5}{2}x(l^2-z^2)+\frac{xA\alpha^2}{96E}\frac{155+432\epsilon^2+1022\epsilon^4+648\epsilon^6+47\epsilon^8}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + (x^3-3xy^2)\frac{A}{48E}\frac{61+133\epsilon^2+335\epsilon^4+47\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}-\frac{3}{2}\frac{A}{E}x^3,$$

$$\frac{\widehat{yz}}{\mu}=-\frac{A}{E}xyz\frac{5+\epsilon^2}{3+\epsilon^2},$$

$$\frac{\widehat{zx}}{\mu}=-\frac{zA}{E}\left\{(x^2-\alpha^2)\frac{5+2\epsilon^2}{3+\epsilon^2}+\frac{y^2}{3+\epsilon^2}\right\},$$

$$\frac{\widehat{xy}}{\mu}=y\left\{-(\beta^2-y^2)\frac{A}{3E}\frac{\epsilon^2(1+3\epsilon^2)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + \frac{x^2A}{4E}\frac{5+9\epsilon^2+27\epsilon^4+7\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}\right\}$$

................(clxvi).

$$s_x=\frac{A}{4E}x(l^2-z^2)-\frac{x\alpha^2A}{960E}\frac{635+1560\epsilon^2+3558\epsilon^4+1056\epsilon^6-25\epsilon^8}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + \frac{Ax^3}{96E}\frac{79+199\epsilon^2+453\epsilon^4+165\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}+\frac{Axy^2}{32E}\frac{21+29\epsilon^2+71\epsilon^4+7\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

...............(clxvii).

$$s_y = \frac{A}{4E} x (l^2 - z^2) - \frac{x\alpha^2 A}{960E} \frac{35 + 600\epsilon^2 + 1558\epsilon^4 + 2976\epsilon^6 + 335\epsilon^8}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)}$$

$$+ \frac{Ax^3}{96E} \frac{19 + 43\epsilon^2 + 145\epsilon^4 + 49\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} + \frac{Axy^2}{32E} \frac{25 + 65\epsilon^2 + 147\epsilon^4 + 19\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)}$$

............(clxviii).

$$s_z = -\frac{A}{E} x (l^2 - z^2) + \frac{x\alpha^2 A}{240E} \frac{185 + 540\epsilon^2 + 1278\epsilon^4 + 876\epsilon^6 + 65\epsilon^8}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)}$$

$$- \frac{Ax^3}{12E} \frac{17 + 41\epsilon^2 + 99\epsilon^4 + 35\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} - \frac{Axy^2}{4E} \frac{7 + 15\epsilon^2 + 37\epsilon^4 + 5\epsilon^6}{(3 + \epsilon^5)(1 + 2\epsilon^2 + 5\epsilon^4)}$$

.........(clxix).

44. Looking now at the results (clviii)—(clxiv), we see that the assumptions of the Bernoulli-Eulerian theory, viz.:

(1) That the cross-sections remain plane,

(2) That the curvature is proportional to the bending moment,

(3) That the stretch varies as the distance from the neutral axis,

(4) That the shears are all zero,

are in no case *strictly* true.

Let us consider how far our expressions for the stresses and strains differ from those given by the Bernoulli-Eulerian theory.

The Bernoulli-Eulerian theory gives

$$\widehat{zz} = -Ax(l^2 - z^2),$$

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = \widehat{yz} = \widehat{zx} = 0,$$

$$\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0,$$

$$s_x = \frac{\eta A}{E} x (l^2 - z^2), \quad s_y = \frac{\eta A}{E} x (l^2 - z^2), \quad s_z = -\frac{A}{E} x (l^2 - z^2).$$

We find in nearly all cases that the difference with the old theory is of the form

$$x(-p_0\alpha^2 + p_1 x^2 + p_2 y^2) = m, \text{ say,}$$

where p_0, p_1, and p_2 are positive quantities.

We have therefore to find the maximum or minimum of this consistent with the condition that $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} < 1$.

Let us use polar coordinates $r\cos\theta = x$, $r\sin\theta = y$, and we have

$$m = \{-p_0\alpha^2 r + (p_1\cos^2\theta + p_2\sin^2\theta)\,r^3\}\cos\theta.$$

As we go outwards from the centre

$$\frac{dm}{dr} = \{-p_0\alpha^2 + 3\,(p_1\cos^2\theta + p_2\sin^2\theta)\,r^2\}\cos\theta,$$

whence m decreases algebraically from 0 to

$$-\tfrac{2}{3}p_0\alpha^3\sqrt{\left\{\frac{p_0}{3\,(p_1\cos^2\theta + p_2\sin^2\theta)}\right\}}\cos\theta,$$

if $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, and increases algebraically to that value if $\frac{1}{2}\pi < \theta < \frac{3}{2}\pi$.

This occurs when

$$r = \alpha\sqrt{\left\{\frac{p_0}{3\,(p_1\cos^2\theta + p_2\sin^2\theta)}\right\}}\ \ldots\ldots\ldots\ldots(a),$$

but

$$r < \frac{\alpha\beta}{\sqrt{(\beta^2\cos^2\theta + \alpha^2\sin^2\theta)}}\ \ldots\ldots\ldots\ldots\ldots\ldots(b),$$

the maximum therefore occurs at (a) or at the edge of the ellipse, according as

$$\sqrt{\left\{\frac{p_0}{3\,(p_1\cos^2\theta + p_2\sin^2\theta)}\right\}} \lesseqgtr \frac{\beta}{\sqrt{(\beta^2\cos^2\theta + \alpha^2\sin^2\theta)}}.$$

If
$$\frac{p_0}{3\,(p_1\cos^2\theta + p_2\sin^2\theta)} > \frac{\beta^2}{(\beta^2\cos^2\theta + \alpha^2\sin^2\theta)},$$

then most certainly the only numerical maximum will be at the edge.

But if, on the other hand,

$$\frac{p_0}{3\,(p_1\cos^2\theta + p_2\sin^2\theta)} < \frac{\beta^2}{\beta^2\cos^2\theta + \alpha^2\sin^2\theta},$$

we may have a negative maximum, say, inside the ellipse, and then m may again become zero and increase positively, so that we get a second maximum at the edge. The question will then be, which of these two maxima is numerically greater?

In the only case where there can be a maximum inside the ellipse, its numerical value

$$= \tfrac{2}{3} p_0 \alpha^3 \sqrt{\left\{\frac{p_0}{3 (p_1 \cos^2\theta + p_2 \sin^2\theta)}\right\}} \cos\theta,$$

$$< \tfrac{2}{3} p_0 \alpha^3 \frac{\beta \cos\theta}{\sqrt{(\beta^2 \cos^2\theta + \alpha^2 \sin^2\theta)}} < \tfrac{2}{3} p_0 \alpha^3.$$

In the other case, to find the value at the edge

$$m = \alpha^2 r \cos\theta \left\{- p_0 + \frac{\beta^2 (p_1 \cos^2\alpha + p^2 \sin^2\theta)}{\beta^2 \cos^2\theta + \alpha^2 \sin^2\theta}\right\}$$

$$= \alpha^2 r \cos\theta \left\{- p_0 + p_2\epsilon^2 + (p_1 - p_2\epsilon^2) \frac{\beta^2 \cos^2\theta}{\beta^2 \cos^2\theta + \alpha^2 \sin^2\theta}\right\},$$

write $r \cos\theta = \alpha \cos\phi$, $r \sin\theta = \beta \sin\phi$, so that ϕ is the eccentric angle, and we have

$$m = \alpha^3 \cos\phi \,\{- p_0 + p_2\epsilon^2\} + \alpha^3 \cos^3\phi \,\{p_1 - p_2\epsilon^2\}.$$

Let $t = \cos\phi$, then

$$\frac{dm}{dt} = \alpha^3 \{(- p_0 + p_2\epsilon^2) + 3t^2 (p_1 - p_2\epsilon^2)\},$$

when $t = 0$, $m = 0$, and m increases therefore numerically until $t = \sqrt{\left\{\frac{p_0 - p_2\epsilon^2}{3 (p_1 - p_2\epsilon^2)}\right\}}$.

If now $\frac{p_0 - p_2\epsilon^3}{3 (p_1 - p_2\epsilon^2)}$ be negative, or numerically > 1, then the maximum value of m is when $t = 1$, and it is therefore $\alpha^3 (-p_0 + p_1)$.

If on the other hand $\frac{p_0 - p_2\epsilon^2}{3 (p_1 - p_2\epsilon^2)}$ is positive and < 1, then the maximum value

$$= \alpha^3 \sqrt{\left(\frac{p_0 - p_2\epsilon^2}{3 (p_1 - p_2\epsilon^2)}\right)} \, (- \tfrac{2}{3}) (p_0 - p_2\epsilon^2),$$

$< \tfrac{2}{3}\alpha^3 (p_0 - p_2\epsilon^2)$ numerically. Our steps will therefore be to find the value of $\frac{p_0 - p_2\epsilon^2}{3 (p_1 - p_2\epsilon^2)}$ in all cases. This will enable us to choose between $\alpha^3 (p_1 - p_0)$ and $\tfrac{2}{3}\alpha^3 (p_0 - p_2\epsilon^2)$ as limiting maxima. The one chosen must then be compared with $\tfrac{2}{3} p_0 \alpha^3$ to see which is the greater, unless indeed $p_0 > 3p_1$, and also $> 3p_2$, in which case there never can be a maximum inside the ellipse. To simplify we shall assume uni-constant isotropy, and only work out the results for the cases ϵ very small (*i.e.*, a practically flat lamina, *e.g.*, web or "joist" in its own plane); ϵ very large (*i.e.*, a practically flat lamina or thin plank bent perpendicular to its plane); and ϵ unity, which gives a circular shaft.

45. The most important stress is $\widehat{zz}$, this being the only finite stress in the Bernoulli-Eulerian theory. We find here

$$p_0 = \tfrac{1}{96}\frac{A}{E}\,\frac{155+432\epsilon^2+1022\epsilon^4+648\epsilon^6+47\epsilon^8}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)},$$

$$p_1 = \tfrac{1}{96}\frac{A}{E}\,\frac{310+742\epsilon^2+1778\epsilon^4+616\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)},$$

$$p_2 = \tfrac{1}{96}\frac{A}{E}\,\frac{366+798\epsilon^2+2010\epsilon^4+282\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)},$$

whence

$$\frac{p_0-p_2\epsilon^2}{3(p_1-p_2\epsilon^2)} = \frac{155+66\epsilon^2+224\epsilon^4-1362\epsilon^6-235\epsilon^8}{3(310+376\epsilon^2+980\epsilon^4-1394\epsilon^6-282\epsilon^8)},$$

$\epsilon = 0$ this ratio $= 1/6$, $\epsilon = 1$ it is $1152/30$, $\epsilon = \infty$ it is $235/846$. Hence, if $\epsilon = 0$, one limiting maximum is

$$(p_0 - p_2\epsilon^2)\tfrac{2}{3}\alpha^3 = \frac{\alpha^3 A}{E}\tfrac{155}{432}.$$

Similarly if $\epsilon = \infty$, remembering that in this case $\beta = \alpha\epsilon$ must remain infinite, α being then very small, the limiting maximum $= \dfrac{A}{E}\alpha\beta^2\tfrac{47}{144}$ numerically.

If on the other hand $\epsilon = 1$, the ratio being > 1, the limiting maximum is given by $\alpha^3(p_1 - p_0) = \tfrac{3}{8}\dfrac{A}{E}\alpha^3$, p_0 being in the present case clearly less than $3p_1$ or $3p_2$, we have a maximum inside the ellipse, which we have also to consider.

This we have seen is always less than $\tfrac{2}{3}p_0\alpha_1^3$, *i.e*, less than $\dfrac{\alpha^3 A}{E}\tfrac{155}{432}$ when $\epsilon = 0$, $\tfrac{1}{2}\dfrac{\alpha^3 A}{E}$ when $\epsilon = 1$, $\dfrac{\alpha\beta^2 A}{E}\tfrac{47}{720}$ if $\epsilon = \infty$.

Hence the greatest possible differences in the stress $\widehat{zz}$ from the ordinary theory are seen to be (comparing our two sets of maxima)

$$< \frac{\alpha^3 A}{E}\tfrac{155}{432}\,(\epsilon = 0), \quad < \frac{\alpha^3 A}{E}\tfrac{1}{2}\,(\epsilon = 1), \quad < \frac{\alpha\beta^2 A}{E}\tfrac{47}{144}\,(\epsilon = \infty).$$

Now the greatest stress possible on the Bernoulli-Eulerian theory is the value of $\widehat{zz}$ when $x=\alpha$, $z=0$, *i.e.*, it is given by

$$\frac{\widehat{zz}}{\mu} = \tfrac{5}{2}\frac{A}{E}l^2\alpha,$$

therefore $\dfrac{\text{greatest difference in stress } \widehat{zz}}{\text{greatest stress}}$

$$< \tfrac{2}{5}\tfrac{155}{432}\left(\frac{\alpha}{l}\right)^2 < \tfrac{31}{216}\left(\frac{\alpha}{l}\right)^2 \text{ when } \epsilon = 0,$$

$$< \tfrac{1}{5}\left(\frac{\alpha}{l}\right)^2 \text{ when } \epsilon = 1,$$

$$< \tfrac{47}{360}\left(\frac{\beta}{l}\right)^2 \text{ when } \epsilon = \infty.$$

Suppose now that the greatest dimension of the cross-section to be not greater than 1/10th of the length of the beam, then the greatest difference in the stress $\widehat{zz}$ is less than $\frac{31}{216}$, $\frac{1}{5}$, $\frac{47}{360}$ per cent. of the greatest stress in the three cases taken.

These give differences of .14, .2, .13 per cent. of the greatest stress on the old theory, and those are comparatively negligible.

Proceeding in an exactly similar manner, we find that for $\dfrac{\widehat{xx}}{\mu}$ the differences are always less than

$$\frac{\alpha^3 A}{E}\tfrac{5}{18} \text{ when } \epsilon = 0,$$

$$\frac{\alpha^3 A}{E}\tfrac{11}{36} \text{ when } \epsilon = 1.$$

In the case $\epsilon = \infty$ the ratio of the greatest difference to the greatest stress is found to be multiplied by the quantity $\left(\dfrac{1}{\epsilon}\right)^2$, and is therefore vanishingly small.

Taking, as before $\alpha/l < 1/10$, we find that the divergences in $\widehat{xx}$ due to the corrected theory are less than .11 and .12 per cent. of the greatest stress for the "joist" and for the shaft respectively.

In the case of the stress $\widehat{yy}$, the difference is vanishingly small when $\epsilon = 0$. It is always less than .15 per cent. of the greatest stress for the shaft, and less than .06 per cent. in the case of the thin plank.

46. The stretches s_x, s_y, s_z may be investigated by means of the same formulæ.

The maximum values of these three stretches on the Bernoulli-Eulerian theory are

$$\frac{A}{4E}\alpha l^2, \; \frac{A}{4E}\alpha l^2, \; -\frac{A\alpha l^2}{E}$$

respectively.

In the case of s_x the greatest deviation is always less than .59, .59, and .13 per cent. of the greatest stretch in the three cases $\epsilon = 0$, 1, and ∞ respectively.

In the case of s_y the greatest deviation is always less than .03, .48, and .19 per cent. of the greatest stretch in the same three cases.

In the case of S_z the greatest deviation is less than ·17, ·26 and ·13 per cent of the greatest stretch in those three cases.

Of course, in all the above, the greatest dimension of the cross-section is to be taken less than one-tenth of the length of the beam.

47. There remain the slides and shears. Here the divergences are not of the same form as those we have treated of above.

Keeping to the case of uni-constant isotropy, we have

$$\sigma_{xy} = y\left\{-(\beta^2 - y^2)\frac{A}{3E}\frac{\epsilon^2(1+3\epsilon^2)}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)} + \frac{x^2}{4}\frac{A}{E}\frac{5+9\epsilon^2+27\epsilon^4+7\epsilon^6}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}\right\}.$$

Now σ_{xy} is clearly increased if all the terms in the curled bracket are made positive, and their coefficients are given their greatest value.

Hence

$$\sigma_{xy} < \beta \left\{ \frac{\beta^2 A}{3E} \frac{\epsilon^2 (1 + 3\epsilon^2)}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} + \frac{\alpha^2 A}{4E} \frac{5 + 9\epsilon^2 + 27\epsilon^4 + 7\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)} \right\}$$

$$< \beta \frac{\alpha^2 A}{12E} \frac{15 + 27\epsilon^2 + 85\epsilon^4 + 33\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)}.$$

The greatest stress on the Bernoulli-Eulerian theory being given by

$$\frac{\widehat{zz}}{\mu} = \tfrac{5}{2} \frac{A}{E} l^2 \alpha,$$

we have

$$\frac{\text{greatest value of } \widehat{xy}}{\text{greatest stress}} = \frac{\alpha\beta}{l^2} \frac{1}{30} \frac{15 + 27\epsilon^2 + 85\epsilon^4 + 33\epsilon^6}{(3 + \epsilon^2)(1 + 2\epsilon^2 + 5\epsilon^4)};$$

when ϵ is either very small or very large, this is small. When $\alpha = \beta < \frac{1}{10} l$ the greatest value of $\widehat{xy}$ is less than ·17 per cent. of the greatest stress.

The longitudinal shears $\widehat{yz}$ and $\widehat{zx}$ are much more important,

$$\frac{\widehat{zx}}{\mu} = -\frac{zA}{E} \left\{ (x^2 - \alpha^2) \frac{5 + 2\epsilon^2}{3 + \epsilon^2} + \frac{y^2}{3 + \epsilon^2} \right\}.$$

We see, however, that the most important values of $\widehat{zx}$ occur near the terminal. But the state of affairs at the terminal we are not able to deal accurately with, because we have made use of Saint-Venant's principle of equipollent loads.

Consider the term in curled brackets

$$(x^2 - \alpha^2) \frac{5 + 2\epsilon^2}{3 + \epsilon^2} + \frac{y^2}{3 + \epsilon^2}.$$

There is a negative maximum at the centre, the value of which is $-\frac{5 + 2\epsilon^2}{3 + \epsilon^2} . \alpha^2$. The expression then varies continuously in one sense as we go outwards from the centre.

Along the circumference

$$x^2 - \alpha^2 = -y^2/\epsilon^2.$$

Therefore our expression becomes at the boundary

$$\frac{y^2}{3+\epsilon^2}\left(1-\frac{5}{\epsilon^2}-2\right),$$

and we have therefore another negative maximum when $y=\beta$.

Its value is $-\alpha^2\frac{5+\epsilon^2}{3+\epsilon^2}$. Hence numerically $\frac{\widehat{zx}}{\mu}$ is greatest at the centre.

The ratio to the greatest stress

$$=\tfrac{2}{5}\frac{z}{l}\frac{\alpha}{l}\frac{5+2\epsilon^2}{3+\epsilon^2}$$

$$=\tfrac{2}{3}\frac{z}{l}\frac{\alpha}{l}\text{ or }\tfrac{7}{10}\frac{z}{l}\frac{\alpha}{l}\text{ or }\tfrac{4}{5}\frac{z}{l}\frac{\alpha}{l},$$

according as $\epsilon=0$, 1, or ∞.

In the latter case α/l is exceedingly small, because $\beta/l<\frac{1}{10}$ and α/β is supposed very small.

In the first and second cases the ratio is $<\frac{20}{3}\frac{z}{l}$ and $<7\frac{z}{l}$ per cent. respectively, *i.e.* for $z=\frac{1}{2}l$, *i.e.* the quarter and three-quarter spans, the shear may be as much as 3.3 and 3.5 per cent. of the greatest stress in the whole beam.

Again $$\frac{\widehat{yz}}{\mu}=-\frac{A}{E}xyz\frac{5+\epsilon^2}{3+\epsilon^2},$$

This is zero at the centre and can be always increased numerically, so long as we are not on the boundary.

On the boundary we have

$$xy=m=\tfrac{1}{2}\alpha\beta\sin 2\phi,$$

where ϕ is the eccentric angle.

This is the greatest when $\phi=\pi/4$, and the greatest values of $\frac{\widehat{yz}}{\mu}$ occur therefore at the points $x=\pm\alpha/\sqrt{2}$, $y=\pm\beta/\sqrt{2}$ and are numerically equal to

$$\frac{A}{E}\frac{\alpha\beta z}{2}\frac{5+\epsilon^2}{3+\epsilon^2}.$$

The ratio of this to greatest stress

$$=\tfrac{1}{5}\frac{z}{l}\frac{\beta}{l}\frac{5+\epsilon^2}{3+\epsilon^2}.$$

This is negligible in the case of the joist, β being then exceedingly small.

In the case of the shaft and the plank, the greatest shear $\widehat{yz}$ is less than $\frac{3z}{l}$ and $\frac{2z}{l}$ per cent. of the greatest stress, respectively, *i.e.* less than 1·5 and 1 per cent. at the quarter and three-quarter spans, which are comparatively small values.

48. Looking back upon these results, we see that, although the assumptions of the Bernoulli-Eulerian theory are none of them strictly true, yet in the case of the stretches and tractions, and of the transverse shear and slide, they give at all events a close approximation to the truth.

In the case of the longitudinal shears and slides (which produce the distortion) the discrepancies are more serious. We see, however, that even then the amount of shear called into play is not large.

49. There exists no "Neutral Axis," properly so-called, the stretches and tractions not varying as the distance from a fixed straight line. The neutral axis of the old theory, however, still possesses the property that in crossing it the tractions and stretches change sign: thus the upper half of the beam is in compression and the lower half in tension. Further the result that the stretch in any "fibre" varies as its distance from the neutral axis is only modified by terms of a small magnitude, when the greatest dimension of the cross-section is less than one-tenth of the length of the beam.

50. The curvature is not proportional to the bending-moment, but differs from it by a constant quantity K, which gives the curvature at the points of inflexion.

We may note the values of K for uni-constant isotropy in in the three cases $\epsilon = 0$, $\epsilon = 1$, and $\epsilon = \infty$.

They are

$$(K)_{\epsilon=0} = \tfrac{203}{144} \frac{A\alpha^2}{E},$$

$$(K)_{\epsilon=1} = \tfrac{41}{30} \frac{A\alpha^2}{E},$$

$$(K)_{\epsilon=\infty} = -\tfrac{13}{240} \frac{A\beta^2}{E}.$$

Hence from (clxiv) the percentage error in the curvature at the mid point is in these three cases, taking

$$\alpha = l/10 \text{ or } \beta = l/10,$$

$$\tfrac{203}{144},\ \tfrac{41}{30},\ -\tfrac{13}{240} \text{ per cent.};$$

thus we may get a sensible deviation in the curvature in the case of the joist and circle.

This deviation becomes even more sensible when we come to deal with the maximum deflection, the percentage error being seen from (clxiii) to be 6/5 of the percentage error in the curvature. Thus the deviations are

$$\tfrac{203}{120},\ \tfrac{41}{25},\ -\tfrac{13}{200},$$

i.e. 1·69, 1·64, and – ·065 per cent.

The first two will be quite sensible, especially in the case of the joist. We see therefore that we may get an error of nearly 1·7 per cent. in the maximum deflection. This error, although small, is yet appreciable, and might be detected by careful experiment.

51. The third result brings out a point of some interest, namely, that for a very flat lamina or plank the curvature at the terminals is *negative*, or there are two points of inflexion rather close to the terminals even when the plank is simply-supported. This, however, might not be borne out by experiment, inasmuch as we have made use of the principle of equipollent loads, and it may be that this last result depends less upon the *total* terminal shear than upon the actual *distribution* of shear over terminal cross-section, with which we are not able to deal.

52. Finally we have to investigate the limiting strain. To do this we notice that in the Bernoulli-Eulerian theory, the greatest strain is the value of s_z when $x = \alpha$, $y = 0$, $z = 0$. The differences with the Bernoulli-Eulerian theory being *small*, the greatest strain will still occur very near that point and will still be s_z. Moreover, from (clxii), we see $\dfrac{ds_z}{dx} = -\dfrac{A}{E}(l^2 - z^2)$ + very small terms of order α^2, x^2, y^2. Therefore $\dfrac{ds_z}{dx}$ is always negative within the limits of the beam. Hence s_z numerically increases right up to the boundary. Therefore we see that the maximum of s_z is not merely *near* the point $(a, 0, 0)$, but

exactly at it, and the greatest strain s_0 is the value of s_z at this point, *i.e.*

$$s_0 = -\frac{A}{E}\alpha l^2 - \frac{A}{E}\frac{\alpha^3}{12}$$

$$\frac{\left\{\begin{array}{l}8+7\eta-\eta^2+\epsilon^2(16+8\eta-8\eta^2)+\epsilon^4(40+22\eta-26\eta^2)\\ \qquad\qquad +\epsilon^6(-32\eta-48\eta^2)+\epsilon^8(-5\eta-45\eta^2)\end{array}\right\}}{(1+\eta)(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

..............(clxx),

$$= -\frac{A}{E}\alpha l^2 - \frac{A}{E}\frac{\alpha^3}{240}\frac{155+280\epsilon^2+702\epsilon^4-176\epsilon^6-65\epsilon^8}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

...............(clxxi)

in the case of uni-constant isotropy.

For the case of the heavy joist

$$s_0 = -\frac{A\alpha}{E}l^2\left(1+\frac{\alpha^2}{l^2}\tfrac{31}{144}\right).$$

For the case of the shaft

$$s_0 = -\frac{A}{E}\alpha l^2\left(1+\frac{\alpha^2}{l^2}\tfrac{7}{30}\right).$$

For the case of the plank

$$s_0 = -\frac{A}{E}\alpha l^2\left(1-\frac{\beta^2}{l^2}\tfrac{13}{240}\right).$$

We see therefore that if $\overline{s_0}$ be the limiting stretch for safe elastic loading, then

$$\left.\begin{array}{l}\dfrac{A\alpha l^2}{E}\left(1+\dfrac{\alpha^2}{l^2}\tfrac{31}{144}\right)<\overline{s_0}\\[2ex] \dfrac{A\alpha l^2}{E}\left(1+\dfrac{\alpha^2}{l^2}\tfrac{7}{30}\right)<\overline{s_0}\\[2ex] \dfrac{A\alpha l^2}{E}\left(1-\dfrac{\beta^2}{l^2}\tfrac{13}{240}\right)<\overline{s_0}\end{array}\right\}\text{in these three cases.}$$

But, according to the Bernoulli-Eulerian theory, the necessary condition that the material may not be strained beyond the elastic limit is

$$\frac{A\alpha l^2}{E} < \overline{s_0}.$$

Hence, in the case of the joist, the weight must for safety be *less* than the ordinary theory allows by a fraction about $\frac{\alpha^2}{l^2}\frac{31}{144}$ of the limiting load. This is less than .215 per cent. if $\alpha < l/10$.

Again, for the shaft, the load must be *reduced* .233 per cent.

For the plank, on the other hand, the ordinary theory errs on the side of safety, as the limiting load may be increased .054 per cent. But we notice that the ordinary theory errs *more* on the side of *danger* in the first two cases than it errs on the side of safety in the last.

52. In the two extreme cases of the joist and the plank, the expressions for the stretches and stresses simplify to a great extent. In the first case we may neglect all terms in ϵ^2 and β^2, and therefore also in y^2. We then find

$$\left.\begin{aligned} \frac{\widehat{xx}}{\mu} &= -x(\alpha^2 - x^2)\frac{A}{3E}(1+\eta) \\ \frac{\widehat{yy}}{\mu} &= 0 \\ \frac{\widehat{zz}}{\mu} &= -\frac{A}{\mu}x(l^2 - z^2) + \frac{xA}{18E}(8 + 7\eta - \eta^2)(\alpha^2 - x^2) \end{aligned}\right\} \text{......(clxxii)}.$$

$$\left.\begin{aligned} \frac{\widehat{yz}}{\mu} &= -\frac{4A}{3E}xyz(1+\eta) \\ \frac{\widehat{zx}}{\mu} &= \frac{4Az}{3E}(\alpha^2 - x^2)(1+\eta) \\ \frac{\widehat{xy}}{\mu} &= yx^2\frac{A}{3E}(1+\eta) \end{aligned}\right\} \begin{array}{l}\text{(retaining first power of } y\text{)} \\ \text{..................(clxxiii).}\end{array}$$

$$\left.\begin{aligned}
s_x &= \frac{\eta A}{E}x(l^2 - z^2) - \frac{x\alpha^2 A(6 + 8\eta - \eta^2)}{36E} + \frac{x^3 A}{18E}(3 + 8\eta - \eta^2) \\
s_y &= \frac{\eta A}{E}x(l^2 - z^2) - \frac{x\alpha^2 A(2\eta - \eta^2)}{36E} + \frac{x^3 A}{18E}(5\eta - \eta^2) \\
s_z &= -\frac{A}{E}x(l^2 - z^2) + \frac{x\alpha^2 A(8 + 5\eta)}{36E} - \frac{x^3 A}{18E}(8 + 2\eta)
\end{aligned}\right\} \quad \text{.........(clxxiv).}$$

In the second case we may neglect all terms in $x\epsilon^{-2}$, x^3, $\alpha^2 x$, &c., retaining, however, second powers of x and ϵ^{-1}, we have

$$\left.\begin{aligned}
\frac{\widehat{xx}}{\mu} &= 0 \\
\frac{\widehat{yy}}{\mu} &= -x(\beta^2 - y^2)\frac{A}{5E}(7\eta - 1) \\
\frac{\widehat{zz}}{\mu} &= -\frac{A}{\mu}x(l^2 - z^2) - \frac{xA}{5E}(11\eta + 3\eta^2)(y^2 - \beta^2/6)
\end{aligned}\right\} \quad \text{.....(clxxv).}$$

$$\left.\begin{aligned}
\frac{\widehat{yz}}{\mu} &= -\frac{4A}{E}xyz\eta \\
\frac{\widehat{zx}}{\mu} &= -z\frac{A}{E}\{2(x^2 - \alpha^2) - (4\eta - 2)\}y^2/\epsilon^2 \\
\frac{\widehat{xy}}{\mu} &= y\frac{A}{5E}\{(4\eta - 2)(\beta^2 - y^2)/\epsilon^2 + x^2(3\eta + 1)\}
\end{aligned}\right\} \quad \text{......(clxxvi).}$$

$$\left.\begin{aligned}
s_x &= \frac{\eta A}{E}x(l^2 - z^2) + \frac{y^2 xA}{10E}(\eta + 3\eta^2) - \frac{x\beta^2 A}{60E}\left(\frac{6\eta - 31\eta^2 + 3\eta^3}{1 + \eta}\right) \\
s_y &= \frac{\eta A}{E}x(l^2 - z^2) + \frac{y^2 xA}{10E}(-1 + 8\eta + 3\eta^2) \\
&\qquad - \frac{x\beta^2 A}{60E}\,\frac{(-6 + 42\eta + 11\eta^2 + 3\eta^3)}{(1 + \eta)} \\
s_z &= -\frac{A}{E}x(l^2 - z^2) - \frac{y^2 xA\eta}{E} + \frac{x\beta^2 A}{12E}\,\frac{\eta + 9\eta^2}{1 + \eta}
\end{aligned}\right\} \quad \text{............(clxxvii).}$$

Looking at the above results we see that in both cases any stress which acts parallel to the small dimension of the cross-section is of the first order of small quantities with regard to the other stresses, and of the second order of small quantities if it act parallel to this small dimension across a plane perpendicular to this dimension (*e.g.*, $\widehat{yy}$ in the first case and $\widehat{xx}$ in the second). The dimensions of the other stresses are not affected, except that in the second case all stresses (and also stretches) are small.

54. An interesting point is the distortion of the cross-section in its own plane.

If we write out the values of u and v given in (xcv) and substitute for f and ψ, using (cxxiii) and (cxxvi)—(cxxxi), and remembering that e_0 and the odd l's and a's vanish, we obtain

$$u = \frac{\eta A}{2E}(x^2 - y^2)(l^2 - z^2) + \tfrac{1}{2}Kz^2 + \frac{A}{E}\left(\frac{l^2z^2}{2} - \frac{z^4}{12}\right)$$

$$+ \left\{\frac{\rho g - \mu K}{\lambda + 2\mu} + \gamma\left(K - \frac{Aq}{E}\right) + a_2 - \gamma b_2 + \frac{b_2}{2}\right\}\frac{x^2}{2} - (a_2 + b_2)\tfrac{1}{2}y^2$$

$$+ \left\{-\frac{\lambda\eta}{\lambda + 2\mu}\cdot\frac{A}{12E} - \frac{A}{6E}\gamma q'' + \frac{A}{12E}(\eta + 2) + \frac{a_4}{4} - \tfrac{1}{2}\gamma b_4 + \tfrac{1}{8}b_4\right\}x^4$$

$$+ \left\{\frac{\gamma A q''}{2E} - \tfrac{3}{2}a_4 + \tfrac{3}{2}\gamma b_4 - \tfrac{3}{2}b_4\right\}x^2y^2 + \left\{-\frac{\eta A}{6E} + \frac{a_4}{4} + \frac{b_4}{8}\right\}y^4$$

..........(clxxviii).

$$v = \frac{\eta A}{E}xy(l^2 - z^2) + \psi.$$

Consider first only the central section $z = 0$, ψ and f being of order a^2/b^2 compared with the first terms in u and v, to a first approximation; we may take

$$u = \frac{\eta A}{2E}(x^2 - y^2)l^2, \quad v = \frac{\eta A}{E}xyl^2.$$

Thus $u = 0$ along the straight lines $x = \pm y$.

Along $x = +y$, $\quad v = \dfrac{\eta A}{E}y^2l^2$, $u = 0$, along $x=-y$, $\qquad v=-\frac{\eta A}{E}y^2l^2,\ u=0,$

$v=0$ along the coordinate axes,

along $x=0$, $\qquad v=0,\ u=-\frac{\eta A}{2E}l^2y^2,$

along $y=0$, $\qquad v=0,\ u=\frac{\eta Ax^2}{2E}l^2.$

These results show us that the horizontal line through the centroid is bent into the form of a parabola, which is curved downwards. All horizontal stright lines in the cross-section are bent into equal parabolas. Any such straight lines in the upper half of the beam are pushed farther apart, while those in the lower half close up.

Similarly the results for v along $x=\pm y$ show that the upper half of the beam expands horizontally, while the lower half contracts.

The distortion is represented in fig. 1. The full curve represents the original cross-section and the dotted lines the distorted one.

55. From this result we see that, if α be small enough, the extremities of the y-axis will be brought to a lower level than the lower extremity of the x-axis, or, in the case of a flat elliptic section, not only will the central line bend downwards, but the actual contour of the cross-section will have downwards curvature, and the 'curling' at the sides of the lamina will be outwardly noticeable.

This will occur when

$$(u)_{y=0,\ x=-\alpha}-(u)_{x=0,\ y=\pm\beta}>\alpha,$$

i.e.

$$\frac{\eta A\alpha^2}{2E}(l^2-z^2)+\left\{\frac{\rho g-\mu K}{\lambda+2\mu}+\gamma\left(K-\frac{Aq}{E}\right)+a_2-\gamma b_2+\frac{b_2}{2}\right\}\frac{\alpha^2}{2}$$

$$+(a_2+b_2)\frac{\beta^2}{2}+\frac{\eta A}{2E}\beta^2(l^2-z^2)$$

$$+\left\{-\frac{\lambda\eta}{\lambda+2\mu}\frac{A}{12E}-\frac{A}{6E}\gamma q''+\frac{A}{12E}(\eta+2)+\frac{a_4}{4}-\tfrac{1}{2}\gamma b_4+\tfrac{1}{8}b_4\right\}\alpha^4$$

$$-\left\{-\frac{\eta A}{6E}+\frac{a_4}{4}+\frac{b_4}{8}\right\}\beta^4>\alpha;$$

whence, after reduction, we find

$$\frac{\eta A}{2E}(\alpha^2+\beta^2)(l^2-z^2) > \alpha$$

$$+\frac{A\alpha^4}{24E(1+\eta)}\frac{\begin{Bmatrix}3+3\eta+\epsilon^2(9+\eta-7\eta^2+\eta^3)+\epsilon^4(19+9\eta-15\eta^2+\eta^3)\\ \epsilon^6(9-11\eta-41\eta^2-5\eta^3)+\epsilon^8(6\eta-31\eta^2+3\eta^3)\end{Bmatrix}}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

$$+\frac{A\alpha^2\beta^2}{12E}\frac{\begin{Bmatrix}2\eta+\eta^2-\eta^3+(6+12\eta+6\eta^2)\epsilon^2+(12+30\eta+16\eta^2+6\eta^3)\epsilon^4\\ +(12+82\eta+46\eta^2-8\eta^3)\epsilon^6+(-6+42\eta+11\eta^2+3\eta^3)\epsilon^8\end{Bmatrix}}{(1+\eta)(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

$$-\frac{A\beta^4}{24E}\frac{\begin{Bmatrix}-6\gamma+4+8\eta+\eta^2+(-14\gamma+6+26\eta+\eta^2)\epsilon^2\\ +(-34\gamma+10+76\eta-5\eta^2)\epsilon^4+(-10\gamma+4+18\eta+3\eta^2)\epsilon^6\end{Bmatrix}}{(3+\epsilon^2)(1+2\epsilon^2+5\epsilon^4)}$$

............(clxxix).

The long terms are comparatively small.

Also the curling will be most noticeable at the central section, where $z=0$.

We can easily see that there will be no curling at the terminal, for there the left-hand side of (clxxix) is zero. Now if α be fairly large the term α settles the sign of the right-hand side, which is therefore positive and the inequality cannot hold. If on the other hand α is very small, we need only consider the coefficients of the highest powers of ϵ in the last two terms of the right-hand side, and these are seen to give, in the case of uniconstancy, the quantity

$$\alpha+\frac{A\beta^4}{E}\text{ (a positive coefficient)},$$

on the right-hand side.

Hence there can never be curling at the terminal.

To the first approximation, we have, at the central section

$$\frac{\eta A}{2E}(\alpha^2+\beta^2)\,l^2 > \alpha,$$

but $$\frac{A}{2E}=\frac{W}{E\omega}\,\frac{1}{l\alpha^2};$$

therefore $$\frac{W\eta}{E\omega}(1+\epsilon^2)\,l > \alpha,$$

i.e. $$\frac{\alpha}{l} < (1+\epsilon^2)\frac{\eta W}{E\omega};$$

therefore $$\frac{\beta}{l} < \eta\frac{W}{E\omega}(\epsilon+\epsilon^3);$$

therefore $$\epsilon^3+\epsilon > \frac{E\omega}{\eta W}\frac{\beta}{l},$$

and if $\beta/l < \frac{1}{10}$,

$$\epsilon^3+\epsilon > \frac{E\omega}{\eta W}\frac{1}{10}$$

ensures that the condition is satisfied.

This generally gives a high value of ϵ.

We may therefore neglect ϵ as compared with ϵ^3, and we have, to the first approximation,

$$\epsilon > \left(\frac{E\omega}{\eta W}\frac{1}{10}\right)^{\frac{1}{3}} > \left(\frac{E}{\rho g l \eta}\frac{1}{10}\right)^{\frac{1}{3}} \quad \text{......(clxxx).}$$

To the second approximation, let us retain only terms of order β^4, terms of order $\alpha^2\beta^2$ being neglected. We find

$$\frac{\eta A}{lE}(\alpha^2+\beta^2)(l^2-z^2) > \alpha + \frac{A\beta^4}{120E}\frac{20\gamma-21+62\eta+\eta^2+3\eta^3}{1+\eta}.$$

For uni-constant isotropy this is

$$\frac{A}{8E}(\alpha^2+\beta^2)(l^2-z^2) > \alpha + \frac{A\beta^4}{150E}(7\cdot94),$$

and for the central section

$$\alpha < \frac{A}{8E}\beta^2 l^2(1+\epsilon^{-2}) - \frac{A\beta^4}{150E}(7\cdot94)$$

$$< \frac{W}{4E\omega}l\epsilon^2(1+\epsilon^{-2}) - \frac{W}{75E\omega}\frac{\beta^2\epsilon^2}{l}(7\cdot94).$$

Therefore $$\left(\frac{1}{\epsilon}\right)^3 < \frac{E\omega}{W}\left\{\frac{1}{4}\frac{l}{\beta}(1+\epsilon^{-2}) - \frac{7\cdot94}{75}\cdot\frac{\beta}{l}\right\};$$

substitute the first approximation on the right-hand side, and we have

$$\left(\frac{1}{\epsilon}\right)^3 < \frac{W}{E\omega}\left\{\frac{1}{4}\frac{l}{\beta}\left(1+\left[\frac{Wl}{4E\omega\beta}\right]^{\frac{2}{3}}\right) - \frac{7\cdot 94}{75}\frac{\beta}{l}\right\} \text{...(clxxxi).}$$

The alteration due to the second term is only about $\frac{32}{75}\frac{\beta^2}{l^2}$ of the whole, and when we take the cube root makes a change of

$$\tfrac{32}{225}\frac{\beta^2}{l^2} \text{ of the whole,}$$

i.e. $< \cdot 14$ per cent., if $\beta < l/10$, which change is quite negligible, so that we see the equation

$$\epsilon^3 + \epsilon = \frac{E\omega}{\eta W}\frac{\beta}{l} \text{..................(clxxxii)}$$

is practically quite sufficient to determine the limiting value of ϵ in order that the curling should be apparent externally.

56. Reviewing generally the results obtained in this paper we arrive at practically the same conclusions for beams of elliptic cross-section, which had previously been reached for beams of a circular cross-section (see *Quarterly Journal of Mathematics*, No. 93, 1889), namely, that the Bernoulli-Eulerian theory, considered as an exact theory, is false in the case of beams continuously loaded. On the other hand its *results* approximate closely to those obtained from the true theory.

Moreover, it seems not impossible, by careful experiments with flat horizontal laminæ or thin planks, to test the practical value of the present theory. The differences from the ordinary theory in the maximum deflection ought to be noticeable, the smallness of the percentage deviation in the case of the horizontal lamina being counter-balanced by the large increase in the deflection itself.

The curling of a flat lamina in a plane at right angles to the plane of bending might possibly be also made the subject of experiment.

RESIDUES OF BINOMIAL-THEOREM COEFFICIENTS WITH RESPECT TO p^3.

By J. W. L. Glaisher.

§ 1. On pp. 150–153* of Vol. XXX. of the *Quarterly Journal*, it was shown that, if $n = kp + q$, $r = gp + s$, where p is a prime, q and $s < p$, $s \overline{\overline{<}} q$, $g \overline{\overline{<}} k$, then $(n)_r$, the number of combinations of n things taken r together, $= (k)_g \times$ a quantity $\equiv (q)_s$, mod. p; and that, if

$$n = q_l p^l + q_{l-1} p^{l-1} + \ldots + q_1 p + q,$$

$$r = s_l p^l + s_{l-1} p^{l-1} + \ldots + s_1 p + s,$$

where all the q's and s's are $< p$, and any of them may be zero, *i.e.* if n and r expressed in the scale of radix p, are

$$n = (q_l q_{l-1} \ldots q_1 q), \quad r = (s_l s_{l-1} \ldots s_1 s);$$

and if each s, say s_i, $\overline{\overline{<}}$ the corresponding q, viz. q_i, then the number $(n)_r$ is not divisible by p, and

$$(n)_r \equiv (q_l)_{s_l} \times (q_{l-1})_{s_{l-1}} \times \ldots \times (q_1)_{s_1} \times (q)_s, \text{ mod. } p.$$

It was also shown that, if t of the s's are $>$ the corresponding q's, then $(n)_r$ is divisible by p^t; and in a subsequent paper in the same volume,† the residue of $\frac{(n)_r}{p^t}$ in this case was obtained.

The object of this paper is to extend the results given in these two papers from mod. p to mod. p^3.

§ 2. This extension is rendered comparatively simple by the result given on p. 21 of the present volume, viz. that m being any positive integer, and p any prime,

$$(mp+1)(mp+2)\ldots(mp+p-1) \equiv (p-1)!, \text{ mod. } p^3.$$

In forming therefore, as on pp. 150–152 of Vol. XXX., the value of the quotient $(n)_r = \frac{n!}{r!\,(n-r)!}$, where $n = kp + q$,

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* 'On the residue of a binomial-theorem coefficient with respect to a prime modulus.'

† 'On the residue with respect to p^{n+1} of a binomial-theorem coefficient divisible by p^n,' pp. 349–360.

$r = gp + s$, q and $s < p$,* $s \gtreqless q$, $g \gtreqless k$, we may, to modulus p^3, ignore all the products of the form

$$(mp+1)(mp+2)\dots(mp+p-1),$$

for these products are congruent to one another, mod. p^3, and the number of such products is the same in the numerator as in the denominator.

Thus $\qquad (n)_r \equiv (k)_g \times A$, mod. p^3,

where

$$A \equiv \frac{(kp+1)(kp+2)\dots(kp+q)}{(gp+1)(gp+2)\dots(gp+s)\times\{(k-g)p+1\}\dots\{(k-g)p+q-s\}}, \text{ mod. } p^3.$$

The numerator on the right-hand side

$$\equiv q!\,\{1 + kpH_q + k^2p^2K_q\}, \text{ mod. } p^3,$$

where
$$H_q = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{q},$$

and
$$K_q = \Sigma \frac{1}{ij},$$

i and j having all values from 1 to q, but not both the same value in the same term.

Thus, if

$$J_q = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{q^2},$$

we have
$$K_q = \tfrac{1}{2}\{H_q^2 - J_q\}.$$

Expressing the products in the denominator in the same manner, we have

$$\begin{aligned} A \equiv (q)_s \{1 + kpH_q + k^2p^2K_q\} & \\ \times \{1 - gpH_s + g^2p^2H_s^2 - g^2p^2K_s\} & \\ \times \{1 - (k-g)pH_{q-s} + (k-g)^2p^2H_{q-s}^2 - (k-g)^2p^2K_{q-s}\}&, \text{ mod. } p^3. \end{aligned}$$

Performing the multiplications and replacing the K's by their values in terms of H's and J's, we see that, to mod. p^3, the expression multiplying $(q)_s$

$$\equiv 1 + \{kH_q - gH_s - (k-g)H_{q-s}\}p$$
$$+ \tfrac{1}{2}[\{kH_q - gH_s - (k-g)H_{q-s}\}^2 - \{k^2J_q - g^2J_s - (k-g)^2J_{q-s}\}]p^2.$$

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* Throughout the paper p is always prime, and all the q's and s's are supposed to be $< p$.

If then we put

$$H\binom{k,\ q}{g,\ s} = kH_q - gH_s - (k-g)\,H_{q-s},$$

$$J\binom{k,\ q}{g,\ s} = k^2J_q - g^2H_s - (k-g)^2J_{q-s},$$

it being supposed in these definitions that $s \overline{\gtrless} q$, we may express this result in the form

$$(n)_r = (k)_g \times (q)_s \times \text{a quantity} \equiv$$

$$1 + H\binom{k,\ q}{g,\ s}p + \tfrac{1}{2}\left\{H^2\binom{k,\ q}{g,\ s} - J\binom{k,\ q}{g,\ s}\right\}p^2, \text{ mod. } p^3.$$

§ 3. Now let $k = k_1p + q_1$, $g = g_1p + s_1$, and $s_1 \overline{\gtrless} q$, then the preceding expression evidently

$$\equiv 1 + H\binom{q_1,\ q}{s_1,\ s}p + \tfrac{1}{2}\left\{H^2\binom{q_1,\ q}{s_1,\ s} - J\binom{q_1,\ q}{s_1,\ s} + 2H\binom{k_1,\ q}{g_1,\ s}\right\}p^2,$$

$$\text{mod. } p^3.$$

Also we have, as in the preceding section,

$$(k)_g = (k_1)_{g_1} \times (q_1)_{s_1} \times \text{a quantity} \equiv$$

$$1 + H\binom{k_1,\ q_1}{g_1,\ s_1}p + \tfrac{1}{2}\left\{H^2\binom{k_1,\ q_1}{g_1,\ s_1} - J\binom{k_1,\ q_1}{g_1,\ s_1}\right\}p^2, \text{ mod. } p^3;$$

whence, multiplying, we find that, if

$$n = k_1p^2 + q_1p + q,$$

$$r = g_1p^2 + s_1p + s,$$

where the s's $\overline{\gtrless}$ their q's, then

$$(n)_r = (k_1)_{g_1} \times (q_1)_{s_1} \times (q)_s \times \text{a quantity} \equiv$$

$$1 + \left\{H\binom{k_1,\ q_1}{g_1,\ s_1} + H\binom{q_1,\ q}{s_1,\ s}\right\}p$$

$$+ \tfrac{1}{2}\left[\left\{H\binom{k_1,\ q_1}{g_1,\ s_1} + H\binom{q_1,\ q}{s_1,\ s}\right\}^2\right.$$

$$\left. - J\binom{k_1,\ q_1}{g_1,\ s_1} - J\binom{q_1,\ q}{s_1,\ s} + 2H\binom{k_1,\ q}{g_1,\ s}\right], \text{ mod. } p^3.$$

If we put

$$H\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} = H\begin{pmatrix} q_1, & q \\ s_1, & s \end{pmatrix} + H\begin{pmatrix} k_1, & q_1 \\ g_1, & s_1 \end{pmatrix},$$

$$J\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} = J\begin{pmatrix} q_1, & q \\ s_1, & s \end{pmatrix} + J\begin{pmatrix} k_1, & q_1 \\ g_1, & s_1 \end{pmatrix},$$

this congruence may be written

$$(n)_r \equiv (k_1)_{g_1} \times (q_1)_{s_1} \times (q)_s \times \left[1 + H\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} p\right.$$

$$\left. + \tfrac{1}{2}\left\{H^2\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} - J\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} + 2H\begin{pmatrix} k_1, & q \\ g_1, & s \end{pmatrix}\right\}\right], \text{ mod. } p^3.$$

§ 4. We thus see that, in general, if n and r when expressed in the scale of radix p are

$$n = (q_l q_{l-1} \ldots q_2 q_1 q), \quad r = (s_l s_{l-1} \ldots s_2 s_1 s),$$

in which all the s's $\overline{\overline{<}}$ their q's, then

$$(n)_r \equiv (q_l)_{s_l} (q_{l-1})_{s_{l-1}} \ldots (q_1)_{s_1} (q)_s \left[1 + H\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix} p\right.$$

$$+ \tfrac{1}{2}\left\{H^2\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix} - J\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix}\right.$$

$$\left.\left. + 2H'\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix}\right\} p^2\right], \text{ mod. } p^3,$$

where, all the s's being $\overline{\overline{<}}$ their q's,

$$H\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix} = q_1 H_q - s_1 H_s - (q_1 - s_1) H_{q-s}$$

$$+ q_2 H_{q_1} - s_2 H_{s_1} - (q_2 - s_2) H_{q_1 - s_1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$+ q_l H_{q_{l-1}} - s_l H_{s_{l-1}} - (q_l - s_l) H_{q_{l-1} - s_{l-1}},$$

$$J\begin{pmatrix} q_l, & q_{l-1}, & \ldots, & q_1, & q \\ s_l, & s_{l-1}, & \ldots, & s_1, & s \end{pmatrix} = q_1^2 J_q - s_1^2 J_s - (q_1 - s_1)^2 J_{q-s}$$

$$+ q_2^2 J_{q_1} - s_2^2 J_{s_1} - (q_2 - s_2)^2 J_{q_1 - s_1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$+ q_l^2 J_{q_{l-1}} - s_l^2 J_{s_{l-1}} - (q_l - s_l)^2 J_{q_{l-1} - s_{l-1}},$$

$$H'\binom{q_l,\ q_{l-1},\ \dots,\ q_1,\ q}{s_l,\ s_{l-1},\ \dots,\ s_1,\ s} = q_2H_q - s_2H_s - (q_2 - s_2)H_{q-s}$$
$$+ q_3H_{q_1} - s_3H_{s_1} - (q_3 - s_3)H_{q_1-s_1}$$
$$\dots\dots\dots\dots\dots\dots,$$
$$+ q_lH_{q_{l-2}} - s_lH_{s_{l-2}} - (q_l - s_l)H_{q_{l-2}-s_{l-2}},$$

the quantities denoted by H_m, J_m being (§ 2)

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m},$$

$$J_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{m^2}.$$

§ 5. As an example of this general formula, let $n = 39$, $r = 8$, $p = 5$; then, expressing n and r in the scale of radix p, we have

$$n = (124),\quad r = (13),\quad p = 5,$$

and $\quad (n)_r = (39)_8 = \binom{124}{013}^* \equiv 1_0 \times 2_1 \times 4_3 \times A$, mod. p^3,

where $\quad A \equiv 1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta + 2\gamma)p^2$, mod. p^3,

and $\quad \alpha = 2H_4 - 1H_3 - 1H_1 + 1H_2 - 0H_1 - 1H_1,$

$$\beta = 4J_4 - 1J_3 - 1J_1 + 1J_2 - 0J_1 - 1J_1,$$

$$\gamma = 1H_4 - 0H_3 - 1H_1.$$

Thus

$$\alpha = 2(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4}) - (1 + \tfrac{1}{2} + \tfrac{1}{3}) - 1 + (1 + \tfrac{1}{2}) - 1$$
$$= \tfrac{11}{6} \equiv 1 + p, \text{ mod. } p^2,$$

$$\beta = 4(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16}) - (1 + \tfrac{1}{4} + \tfrac{1}{9}) - 1 + (1 + \tfrac{1}{4}) - 1$$
$$\equiv 4, \text{ mod. } p,$$

$$\gamma = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - 1 \equiv 4, \text{ mod. } p,$$

---

* As in previous papers (*e.g.*, Vol. XXX., p. 363) the suffix is written beneath the number and inside the bracket when the numbers are expressed in the scale whose radix is the modulus p.

whence $A \equiv 1 + (1+p)p + \frac{1}{2}(1-4+3)p^2$, mod. p^3

$$\equiv 1 + p + p^2, \text{ mod. } p^3,$$

and therefore

$$(n)_r \equiv 8(1+p+p^2) \equiv (3+p)(1+p+p^2), \text{ mod. } p^3$$

$$\equiv 3 + 4p + 4p^2, \text{ mod. } p^3.$$

The value of $(39)_8$ is 61523748, which, divided three times by 5, gives the successive remainders 3, 4, 4.

§ 6. As a second example let $r = 11$, n and p remaining as before. In this case

$$n = (124), \quad r = (21), \quad p = 5,$$

and

$$(n)_r = (39)_{11} = \binom{124}{021} \equiv 1_0 2_2 4_1 \{1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta + 2\gamma)p^2\}, \text{ mod. } p^3,$$

where $\alpha = 2H_4 - 2H_1 - 0H_3 + 1H_2 - 0H_2 - 1H_0$

$$= \tfrac{11}{3} \equiv 2 + 2p, \text{ mod. } p^2,$$

$$\beta = 4J_4 - 4J_1 - 0J_3 + 1J_2 - 0J_2 - 1J_0 \equiv 1, \text{ mod. } p$$

$$\gamma = 1H_4 - 0H_1 - 1H_3 \equiv 4, \text{ mod. } p.$$

Thus

$$(n)_r \equiv 4\{1 + (2+2p)p + \tfrac{1}{2}(4-1+3)p^2\}, \text{ mod. } p^3$$

$$\equiv 4(1+2p) \equiv 4 + 3p + p^2, \text{ mod. } p^3.$$

The value of $(39)_{11}$ is 1676056044, which, divided three times by 5, gives the successive remainders 4, 3, 1.

It will be noticed that it is necessary to calculate α, the coefficient of p, to mod. p^2; but it is sufficient to find the residues of β and γ to mod. p.

§ 7. In the particular case in which n, expressed in the scale of radix p, consists of only two digits the term in H' does not occur and the formula is simply

$$(n)_r = \binom{q_1 q}{s_1 s} \equiv (q_1)_{s_1} (q)_s \{1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta)p^2\}, \text{ mod. } p^3,$$

where $\alpha = q_1 H_q - s_1 H_s - (q_1 - s_1) H_{q-s},$

$$\beta = q_1^2 J_q - s_1^2 J_s - (q_1 - s_1)^2 J_{q-s}.$$

As an example, let $n = 26$, $r = 9$, $p = 7$; then

$$(26)_9 = \binom{35}{12} \equiv 3_1 \times 5_2 \times \{1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta) p^2\}, \text{ mod. } p^3,$$

where $\alpha = 3H_5 - 1H_2 - 2H_3 = \frac{101}{60} \equiv 6 + 3p$, mod. p^2,

$$\beta = 9J_5 - 1J_2 - 4J_3 \equiv 2, \text{ mod. } p,$$

whence

$$(n)_r \equiv 30\{1 + (6 + 3p)p + \tfrac{1}{2}(36 - 2)p^2\} \text{ mod. } p^3$$

$$\equiv (2 + 4p)(1 + 6p + 6p^2) \equiv 2 + 2p + 3p^2, \text{ mod. } p^3.$$

The value of $(26)_9$ is 3124550 and the remainders are 2, 2, 3.

§ 8. If only the residue to mod. p^2 is required the general theorem (§ 4) gives, omitting the term in p^2,

$$(n)_r \equiv (q_l)_{s_l} \cdots (q_1)_{s_1} (q)_s \left\{1 + H\begin{pmatrix} q_l, \ldots, q_1, q \\ s_l, \ldots, s_1, s \end{pmatrix} p\right\}, \text{ mod. } p^2,$$

which in the particular case of

$$n = (q_1 q), \quad r = (s_1 s),$$

becomes

$$(n)_r \equiv (q_1)_{s_1} (q)_s [1 + \{q_1 H_q - s_1 H_s - (q_1 - s_1) H_{q-s}\} p], \text{ mod. } p^2.$$

This theorem was stated in the note on p. 382, of Vol. XXX., where also several numerical examples are given.

§ 9. I now consider the case in which

$$n = kp + q, \quad r = gp + s,$$

but $s > q$.

Proceeding as on pp 150–152 of Vol. XXX. and observing as in § 2 that every product of the form

$(mp + 1)(mp + 2) \ldots (mp + p - 1) \equiv (p - 1)!$, mod. p^3, we find

$$(n)_r = p \times (k - g)(k)_g \times A,$$

where

$$A \equiv \frac{(p-1)! \times (kp+1)(kp+2)\dots(kp+q)}{(gp+1)(gp+2)\dots(gp+s) \times \{(k-g-1)p+1\}\dots\{(k-g-1)p+p+q-s\}},$$

mod. p^3.

Now $(p-1)!$

$$\equiv \{(k-g-1)p+1\}\{(k-g-1)p+2\}\dots\{(k-g-1)p+p-1\}, \text{ mod. } p^3,$$

and therefore

$$\frac{(p-1)!}{\{(k-g-1)p+1\}\dots\{(k-g-1)p+p+q-s\}}$$

$$\equiv \{(k-g)p-1\}\{(k-g)p-2\}\dots\{(k-g)p-(s-q-1)\}, \text{ mod. } p^3.$$

Thus A

$$\equiv \frac{(kp+1)\dots(kp+q) \times \{(k-g)p-1\}\dots\{(k-g)p-(s-q-1)\}}{(gp+1)(gp+2)\dots(gp+s)},$$

mod. p^3,

$$\equiv q!\{1+kpH_q+k^2p^2K_q\}$$

$$\times (-1)^{s-q-1}(s-q-1)!\{1-(k-g)pH_{s-q-1}+(k-g)^2p^2K_{s-q-1}\}$$

$$\times \frac{1}{s!}\{1-gpH_s+g^2p^2H_s^2-g^2p^2K_s\}, \text{ mod. } p^3,$$

where H and K have the same meanings as in § 2.

We thus find that $A = \dfrac{(-1)^{s-q-1}}{(s-q)(s)_q} \times$ a quantity $\equiv$

$$1+\{kH_q-gH_s-(k-g)H_{s-q-1}\}p$$

$$+\tfrac{1}{2}[\{kH_q-gH_s-(k-g)H_{s-q-1}\}^2$$

$$-\{k^2J_q-g^2J_s+(k-g)^2J_{s-q-1}\}]p^2,$$

where, as in § 2,

$$H_m = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{m},$$

$$J_m = 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{m^2}.$$

If then, s being $> q$, we put*

$$H\binom{k,\ q}{g,\ s} = kH_q - gH_s - (k-g)\,H_{s-q-1},$$

$$J\binom{k,\ q}{g,\ s} = k^2J_q - g^2J_s + (k-g)^2J_{s-q-1},$$

we may write the final result

$$(n)_r \equiv p \times (k-g)\,(k)_g \times \frac{(-1)^{s-q-1}}{(s-q)\,(s)_q}$$

$$\times \left[1 + H\binom{k,\ q}{g,\ s}p + \tfrac{1}{2}\left\{H^2\binom{k,\ q}{g,\ s} - J\binom{k,\ q}{g,\ s}\right\}p^2\right], \text{ mod. } p^4.$$

§ 10. Now, let $k = k_1p + q_1$, $g = g_1p + s_1$,

then, if $s_1 \mathrel{\overline{\underline{<}}} q_1$,

$$(k)_g \equiv (q_1)_{s_1} \times (k_1)_{g_1} \times A, \text{ mod. } p^3,$$

and, if $s_1 > q_1$,

$$(k)_g \equiv p \times \frac{(-1)^{s_1-q_1-1}}{(s_1-q_1)\,(s_1)_{q_1}} \times (k_1-g_1)\,(k_1)_{g_1} \times A, \text{ mod. } p^4,$$

where, in both cases,

$$A = 1 + H\binom{k_1,\ q_1}{g_1,\ s_1}p + \tfrac{1}{2}\left\{H^2\binom{k_1,\ q_1}{g_1,\ s_1} - J\binom{k_1,\ q_1}{g_1,\ s_1}\right\}p^2.$$

§ 11. Thus, if $\quad n = k_1p^2 + q_1p + q,$

$$r = g_1p^2 + s_1p + s,$$

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* The value of $H\binom{k,\ q}{g,\ s}$, $s > q$, differs from the value when $s \mathrel{\overline{\underline{<}}} q$ only in the suffix in the third term which is $q - s$ in the latter case and $s - q - 1$ in the former. The corresponding values of $J\binom{k,\ q}{g,\ s}$ differ in the same way, as well as by a change of sign in the last term.

we have, if $s > q$, $s_1 \leqq q_1$,

$$(n)_r \equiv p \times \frac{(-1)^{s-q-1}}{(s-q)(s)_q} \times (q_1)_{s_1} \times (k_1)_{g_1}$$
$$\times \{q_1 - s_1 + (k_1 - g_1)p\} \times A, \text{ mod. } p^4,$$

and, if $s > q$, $s_1 > q_1$,

$$(n)_r \equiv p^2 \times \frac{(-1)^{s-q-1}}{(s-q)(s)_q} \times \frac{(-1)^{s_1-q_1-1}}{(s_1-q_1)(s_1)_{q_1}} \times (k_1)_{g_1}$$
$$\times \{(q_1 - s_1) + (k_1 - g_1)p\} \times (k_1 - g_1) \times A, \text{ mod. } p^4,$$

where, in both cases,

$$A = 1 + H\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} p + \tfrac{1}{2}\left\{H^2\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix}\right.$$
$$\left. - J\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} + 2H\begin{pmatrix} k_1, & q \\ g_1, & s \end{pmatrix}\right\} p^2,$$

and
$$H\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix} = H\begin{pmatrix} q_1, & q \\ s_1, & s \end{pmatrix} + H\begin{pmatrix} k_1, & q_1 \\ g_1, & s_1 \end{pmatrix}$$

with a similar definition for $J\begin{pmatrix} k_1, & q_1, & q \\ g_1, & s_1, & s \end{pmatrix}$.

§ 11. We are now in a position to enunciate the general theorem, which includes that in § 4, and may be stated as follows: If n and r when expressed in the scale of radix p are

$$n = (q_l q_{l-1} \ldots q_2 q_1 q), \quad r = (s_l s_{l-1} \ldots s_2 s_1 s),$$

then
$$(n)_r \equiv P \times A, \text{ mod. } p^{t+3},$$

where t is the number of s's which are greater than their q's, + the number of s's which are equal to their q's when the preceding s which is not equal to its q is greater than its q; and P consists of a product of factors, viz., for each $s_i \leqq q_i$ the factor $(q_i)_{s_i}$, which $= 1$ when $s_i = q_i$, and for each $s_i > q_i$ the factors

$$\frac{(-1)^{s_i-q_i-1}}{(s_i - q_i)(s_i)_{q_i}} \times p \times \{(q_{i+j} - s_{i+j})p^{j-1} + (q_{i+j+1} - s_{i+j+1})p^j$$
$$+ (q_{i+j+2} - s_{i+j+2})p^{j+1}\},$$

s_{i+j} being the next s to s_i which, is not equal to its q; and $A =$

$$1 + H\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s} p + \tfrac{1}{2}\left\{H^2\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s} - J\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s} + 2H'\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s}\right\} p^2.$$

The quantities

$$H\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s},\quad J\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s},\quad H'\binom{q_l, \ldots, q_1, q}{s_l, \ldots, s_1, s}$$

denote respectively

$$H\binom{q_1, q}{s_1, s} + H\binom{q_2, q_1}{s_2, s_1} + \ldots + H\binom{q_l, q_{l-1}}{s_l, s_{l-1}},$$

$$J\binom{q_1, q}{s_1, s} + J\binom{q_2, q_1}{s_2, s_1} + \ldots + J\binom{q_l, q_{l-1}}{s_l, s_{l-1}},$$

$$H\binom{q_2, q}{s_2, s} + H\binom{q_3, q_1}{s_3, s_1} + \ldots + H\binom{q_l, q_{l-2}}{s_l, s_{l-2}},$$

and we have

$$H\binom{a,\ u}{b,\ v} = aH_u - bH_v - (a-b)\,H_{u-v}, \quad \text{if } v \leqq u, \ (\S\ 2)$$

$$= aH_u - bH_v - (a-b)\,H_{v-u-1}, \text{ if } v > u, \ (\S\ 9),$$

and

$$J\binom{a,\ u}{b,\ v} = a^2J_u - b^2J_v - (a-b)^2J_{u-v}, \quad \text{if } v \leqq u, \ (\S\ 2)$$

$$= a^2J_u - b^2J_v + (a-b)^2J_{v-u-1}, \text{ if } v > u,^* \ (\S\ 9),$$

where $$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m},$$

$$J_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{m^2}.$$

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* It will be noticed that the *form* of the values of $H\binom{a,\ u}{b,\ v}$ and $J\binom{a,\ u}{b,\ v}$ depends only upon the relative magnitude of the numbers in the second column, and not at all upon the first column.

§ 12. Taking the same example as that employed on p. 353 of Vol. XXX. to illustrate the theorem giving the residue to mod. p^{t+1}, let n and r, when expressed in the scale of radix p, be

$$n = (q_9 q_8 q_7 q_6 q_5 q_4 q_3 q_2 q_1 q),$$

$$r = (\rho_9 \rho_8 q_7 q_6 \sigma_5 \sigma_4 \rho_3 q_2 \rho_1 \sigma),$$

where the ρ's are less than their q's, and the σ's greater than their q's. Then the theorem gives

$$(n)_r \equiv P \times A, \text{ mod. } p^8,$$

where
$$P \equiv \frac{(-1)^{\sigma-q-1}}{(\sigma-q)(\sigma)_q} \times p \times \{q_1 - \rho_1 + (q_3 - \rho_3)p^2\}$$

$$\times (q_1)_{\rho_1} \times 1 \times \frac{(-1)^{\sigma_4 - q_4 - 1}}{(\sigma_4 - q_4)(\sigma_4)_{q_4}} \times p \times \{q_5 - \sigma_5\}$$

$$\times \frac{(-1)^{\sigma_5 - q_5 - 1}}{(\sigma_5 - q_5)(\sigma_5)_{q_5}} \times p \times \{(q_8 - \rho_8)p^2 + (q_9 - \rho_9)p^3\}$$

$$\times 1 \times 1 \times (q_8)_{\rho_8} \times (q_9)_{\rho_9}, \text{ mod. } p^8,$$

and
$$A \equiv 1 + \alpha p + \tfrac{1}{2}\{\alpha^2 - \beta + 2\gamma\}p^2, \text{ mod. } p^3,$$

where
$$\alpha = H\begin{pmatrix} q_9, & q_8, & q_7, & q_6, & q_5, & q_4, & q_3, & q_2, & q_1, & q \\ \rho_9, & \rho_8, & q_7, & q_6, & \sigma_5, & \sigma_4, & \rho_3, & q_2, & \rho_1, & \sigma \end{pmatrix},$$

and β, γ are the J and H' with the same arguments.

It is evident that, for mod. p^6, P reduces to the expression obtained on p. 353 of Vol. XXX.

§ 13. As numerical examples:

(i) Let $n = 32$, $r = 9$, $p = 5$,

then
$$(n)_r = (32)_9 = \begin{pmatrix} 112 \\ 014 \end{pmatrix} \equiv P \times A, \text{ mod. } p^5,$$

where
$$P = \frac{(-1)^{4-2-1}}{(4-2)4_2} \times p \times (0+p) \times 1_1 \times 1_0$$

$$= p^2 \times -\tfrac{1}{12} \equiv p^2 \times (2 + 0p + 2p^2), \text{ mod. } p^5,$$

and
$$A = 1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta + 2\gamma)p^2,$$

where

$$\alpha = H\binom{1,\ 1,\ 2}{0,\ 1,\ 4} = 1H_2 - 1H_4 - 0H_1 + 1H_1 - 0H_1 - 1H_0$$

$$= H_2 - H_4 + H_1 = -\tfrac{1}{3} - \tfrac{1}{4} + 1 = \tfrac{5}{12} \equiv 0 + 3p, \text{ mod. } p^2,$$

$$\beta = J\binom{1,\ 1,\ 2}{0,\ 1,\ 4} = 1J_2 - 1J_4 + 0J_1 + 1J_1 - 0J_1 - 1J_0$$

$$= J_2 - J_4 + J_1 = -\tfrac{1}{9} - \tfrac{1}{16} + 1 \equiv 1, \text{ mod. } p,$$

$$\gamma = H\binom{1,\ 2}{0,\ 4} = 1H_2 - 0H_4 - 1H_1 = \tfrac{1}{2} \equiv 3, \text{ mod. } p.$$

Thus

$$A \equiv 1 + (0 + 3p)\,p + \tfrac{1}{2}(0 - 1 + 6)\,p^2 \equiv 1 + 0p + 3p^2, \text{ mod } p^3,$$

and $\quad (n)_r \equiv p^2(2 + 0p + 2p^2)(1 + 0p + 3p^2)$, mod. p^5.

$$\equiv p^2(2 + 0p + 3p^2), \text{ mod. } p^5.$$

The value of $(32)_9$ is $28048800 = 5^2 \times 1121952$; and when 1121952 is divided by 5 three times, the successive remainders are 2, 0, 3.

(ii) Let $n = 32,\ r = 14,\ p = 5$.

Then $\quad (n)_r = (32)_{14} = \binom{112}{024} \equiv P \times A$, mod. p^5,

where

$$P = \frac{(-1)^{4-2-1}}{2.4_2} \times p \times \{(1-2) + (1-0)\,p\} \times \frac{(-1)^{2-1-1}}{1.2_1} \times p \times \{1\}$$

$$= p^2 \times -\tfrac{1}{24}(p-1) \equiv p^2 \times (4 + 0p + 4p^2), \text{ mod. } p^5,$$

and $\qquad A = 1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta + 2\gamma)\,p^2$, mod. p^3,

where

$$\alpha = 1H_2 - 2H_4 - (1-2)\,H_1 + 1H_1 - 0H_2 - 1H_0$$

$$= -\tfrac{2}{3} \equiv 1 + 3p, \text{ mod. } p^2,$$

$$\beta = 1J_2 - 4J_4 + 1J_1 + 1J_1 \equiv 2, \text{ mod. } p,$$

$$\gamma = 1H_2 - 0H_4 - 1H_1 = \tfrac{1}{2} \equiv 3, \text{ mod. } p.$$

Thus

$$A \equiv 1 + (1 + 3p)\,p + \tfrac{1}{2}(1 - 2 + 6)\,p^2 \equiv 1 + p + 3p^2, \text{ mod. } p^3.$$

Therefore

$$(n)_r \equiv p^2(4+0p+4p^2)(1+p+3p^2), \text{ mod. } p^5$$
$$\equiv p^2(4+4p+p^2), \text{ mod. } p^5.$$

The value of $(32)_{14}$ is $471435600 = 5^2 \times 18857424$; and when 18857424 is divided by 5 three times the successive remainders are 4, 4, 1.

(iii) Let $n = 26$, $r = 19$, $p = 5$.

Then $(n)_r = (26)_{19} = \binom{101}{034} \equiv P \times A$, mod. p^5,

and

$$P = \frac{(-1)^{4-1-1}}{3.4_1} \times p \times \{(0-3)+(1-0)p\} \times \frac{(-1)^{3-0-1}}{3.3_0} \times p \times \{1\}$$
$$= p^2 \times \tfrac{1}{36}(p-3) \equiv p^2 \times (2+p+0p^2), \text{ mod. } p^5,$$

and

$$\alpha = 0H_1 - 3H_4 - (0-3)H_2 + 1H_0 - 0H_3 - (1-0)H_2$$
$$= -\tfrac{13}{4} \equiv 3 + 0p, \text{ mod. } p^2,$$
$$\beta = 0J_1 - 9J_4 + 9J_2 + 1J_0 - 0J_3 + 1J_2 \equiv 0, \text{ mod. } p,$$
$$\gamma = 1H_1 - 0H_4 - 1H_2 = -\tfrac{1}{2} \equiv 2, \text{ mod. } p,$$

so that

$$A \equiv 1 + 3p + \tfrac{1}{2}(9 - 0 + 4)p^2 \equiv 1 + 3p + 4p^2, \text{ mod. } p^3,$$

and $$(n)_r \equiv p^2(2+p+0p^2)(1+3p+4p^2), \text{ mod. } p^5$$
$$\equiv p^2(2+2p+2p^2), \text{ mod. } p^5.$$

The value of $(26)_{19}$ is $657800 = 5^2 \times 26312$; and when 26312 is divided by 5 three times, the successive remainders are 2, 2, 2.

§ 14. In the particular case in which n expressed in the scale of radix p consist of only two digits, so that $n = (q_1 q)$, $r = (s_1 s)$, and when $s > q$, the formula, corresponding to that in § 7, is

$$(n)_r = \binom{q_1 q}{s_1 s} \equiv \frac{(-1)^{s-q-1}}{(s-q)(s)_q} \times p \times (q_1 - s_1)(q_1)_{s_1}$$
$$\times \{1 + \alpha p + \tfrac{1}{2}(\alpha^2 - \beta)p^2\}, \text{ mod. } p^4,$$

where $$\alpha = q_1 H_q - s_1 H_s - (q_1 - s_1) H_{s-q-1},$$
$$\beta = q_1^2 J_q - s_1^2 J_s + (q_1 - s_1)^2 J_{s-q-1}.$$

As an example, let $n = 29$, $r = 20$, $p = 7$; then

$$(29)_{20} = \binom{41}{26} \equiv \frac{(-1)^4}{5.6_1} \times p \times 2 \times 4_2 \times A = \tfrac{2}{5} \times p \times A, \text{ mod. } p^4$$
$$\equiv p \times (6 + 2p + p^2) \times A, \text{ mod. } p^4,$$

and $$\alpha = 4H_1 - 2H_6 - 2H_4 = -\tfrac{76}{15} \equiv 1 + p, \text{ mod. } p^2,$$
$$\beta = 16J_1 - 4J_6 + 4J_4 \equiv 4, \text{ mod. } p,$$

so that

$$(n)_r \equiv p \times (6 + 2p + p^2) \times (1 + p + 3p^2), \text{ mod. } p^3$$
$$\equiv p \times (6 + p + p^2), \text{ mod. } p^3.$$

The value of $(29)_{20}$ is $10015005 = 7 \times 1430715$; and when 1430715 is divided by 7 three times successively, the remainders are 6, 1, 1.

In this case, in which n contains only two digits, the theorem is not of so much value, as there is less disparity between the amounts of calculation required for the actual determination of $(n)_r$ and for the numerical evaluation of the formula.

§ 15. The general theorem is remarkable on account of the comparatively slight additional complication produced by the extension from mod. p^{t+1} to mod. p^{t+3}, p^t being the highest power of p by which $(n)_r$ is divisible; for the factor P (§ 11) differs from the residue to mod. p^{t+1}* only by the substitution of

$$(q_{i+j} - s_{i+j})\, p^{j-1} + (q_{i+j+1} - s_{i+j+1})\, p^j + (q_{i+j+2} - s_{i+j+2})\, p^{j+1}$$

for its first term $(q_{i+j} - s_{i+j})\, p^{j-1}$; and the new factor A is much more simple in form than might have been anticipated, having regard to the nature of the quantities to be taken into account. This simplicity is mainly due to the fact that all products of the form $(mp+1)(mp+2)\ldots(mp+p-1)$ are congruent to one another, mod. p^3. Thus the coefficients H, J, H' involve only q's and s's, and their actual values are independent of the modulus p.

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* *Quarterly Journal*, Vol. xxx., p. 352.

ON THE FIGURE OF SIX POINTS IN SPACE OF FOUR DIMENSIONS.

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THE history of the development of the idea that light is thrown on the theorems of projective geometry by the conception of a geometry of a higher number of dimensions affords a very striking illustration of Cayley's mathematical insight. The valuable and ever increasing series of investigations concerning the synthetic geometry of n dimensions, which we owe almost entirely to Italian mathematicians, has its source in Veronese's memoir, *Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens*, (*Math. Ann.* XIX., 1882); but, as *Gino Loria* points out in his admirable historical monograph, *Il passato e il presente delle principali teori geometriche*, (Memorie d. R. Accad. d. Scienze di Torino, Series II. Vol. 38), the value of the method of Veronese had been clearly grasped by Cayley more than thirty years earlier and used by him in a paper published in 1846, in Vol. XXXI. of *Crelle's Journal*. (*Collected Works*, Vol. I., p. 317).

In comparison with much that has already been effected, the aim of the present paper is a modest one; viz. to consider a very simple figure in four-dimensional space in more detail than has apparently yet been done, and than is possible in the case of the analogous figure in n dimensions; and to derive by projection and section certain figures in space or in a plane and their properties. It may be said that if $n+1$ or more points be taken in a space of n dimensions, and a line or plane or other locus of the first order determined by each combination of the points, two, three,......n at a time, the nature of the configurations derived from this by projection or section is fully treated in Veronese's memoir. Three points in a plane, or four in space, or $n+1$ in space of n dimensions, give us nothing beyond such a system of loci of the first order; the smallest number of points in space of n dimensions which give anything more, or which form a figure having any properties of its own (other than axiomatic ones) is $n+2$. Now from four points in a plane we derive the one-dimensional theory of involution and harmonic section, and from five points in space the two-dimensional principle of homology, each as an intuitive consequence of the fundamental axioms of geometry. What, if anything, can we infer by similar means from the conception of a figure formed by six points in space of four dimensions? The result of the inquiry, though not of the fundamental and indispensable nature of the foregoing cases, is yet sufficiently remarkable; we find that a necessary consequence of the conception is that families of fifteen lines must exist in a plane which possess all the long list of properties that have slowly accumulated round the figure formed by joining six points of a conic, commonly known as Pascal's Hexagram. The fifteen lines do not in general join six points, and it is necessary to investigate their nature by analytical methods: they prove to be in most cases of a class already familiar to mathematicians in connexion with curved loci of the fourth order; and the discovery of a means of obtaining them by linear methods is therefore a simplification. Incidentally too we are led to regard some theorems on quartic curves, cubic surfaces, and quartic surfaces having fifteen or sixteen nodes, from a new point of view, viz. in their connection with the four-dimensional figure.

There is at present no recognised system of nomenclature for loci in hypergeometry, the names *plane* and *surface* for example being employed in different senses by different writers. In what follows I shall have to deal almost exclusively with descriptive properties of space of four dimensions, and shall adopt the usage of Italian mathematicians, Veronese, Segre and others;—denoting by S_n a space of n dimensions, and, when four-dimensional loci are under consideration, applying the term a *line* to an S_1, the term a *plane* to an S_2, and the term a *space* (without specification of the number of dimensions) to an S_3. The names *curve* and *surface* refer respectively to curved loci of one and two dimensions, and the word *variety* is confined to curved loci of three dimensions. Thus hereafter, when analytical methods are introduced, loci in S_4 the coordinates of whose points satisfy one or two or three relations, are respectively varieties, surfaces, and curves; but should the relations be all of the first degree they are respectively spaces, planes and lines. Elementary properties of S_4, such as that two planes have as a rule only one common point, and that one and only one line meets three given lines, will be assumed: they are investigated in Veronese's *Fondamenti di Geometria*, (Padua, 1891), Part II., Book I., pp. 457-500: (German translation by Schepp, entitled *Grundzüge der Geometrie* etc., Leipzig 1894).

Section I.

On the nature of a Hexastigm.

Let six points denoted by symbols 1, 2, 3, 4, 5, 6, be chosen at random in a space of four dimensions; excluding exceptional cases, we assume that no five of them lie in an S_3, and *à fortiori* that no four are coplanar, no three collinear and no two coincident. Each pair of these points determines a line, each set of three a plane, and each set of four a space, and to the figure thus constituted I give the name *Hexastigm*, (following Townsend, *Modern Geometry*, Dublin, 1863). The foundation of a Hexastigm is the set of six fundamental points or *vertices*, 1, 2, 3, 4, 5, 6 ; it comprises in addition fifteen lines or *edges*, 12, 13,......; twenty planes or faces, 123, 124,......; and fifteen spaces 1234, 1235....... Two faces such as 123, 456, are said to be opposite to each other, and the edge 12 is said to be opposite to the space 3456 ; and, by a slight extension of the meaning of the word, three edges such as 12, 34, 56, are described as three opposite edges of the Hexastigm.

The section of the Hexastigm by a space, that is to say the figure formed by those parts of the Hexastigm which lie in an arbitrarily chosen S_3, consists of fifteen points, twenty lines, and fifteen planes, derived respectively from the edges, faces and spaces of the Hexastigm. Through any one of the fifteen points, (for example that derived from the edge 12), pass four of the twenty lines, (viz. those derived from the faces 123, 124, 125, 126), on each of which lie two more of the fifteen points, (derived from the edges 13, 14, 15, 16, and 23, 24, 25, 26). Consider now the two tetrahedra of which these eight points are vertices; corresponding vertices lie on four concurrent lines, as we have just seen; corresponding edges meet in six coplanar points, and corresponding faces in four coplanar lines, and the complete system of fifteen points, twenty lines and fifteen planes is thus accounted for. We have in fact obtained, by a method clearly capable of further development, (which we owe to Prof. Veronese), a very natural and simple analogue in space of the plane figure of two perspective triangles and their axis of homology. The plane sections of the three-dimensional figure formed of the lines and planes determined by five arbitrarily chosen points consist of ten lines and ten points, and can be resolved in ten distinct ways into a pair of perspective triangles and their axis of homology: the *space-sections* (if I may be allowed to coin a convenient word) of the four-dimensional figure formed of the lines, planes, and spaces determined by six arbitrarily chosen points may be resolved in fifteen distinct ways into two perspective tetrahedra and their plane of homology. It may further be pointed out that a plane figure may be derived from that of the two perspective tetrahedra either by projecting it on a plane or by cutting it by a plane: that derived by the latter method consists of twenty points which lie by fours on fifteen lines, and may be resolved in twenty different ways into a set of three perspective triangles and their three concurrent axes of homology. But before discussing such matters further it is best to study the nature and properties of a Hexastigm in S_4 more fully. For analogues in S_n see Veronese, *Fondamenti di Geometria*, Part II., Book II., Chap. II., pp. 550-561, but specially §2. p. 558: see also the memoir in *Math. Ann.* XIX., and, for a different method of investigation, Whitehead, *Universal Algebra*, pp. 139-142.

Returning to the consideration of the Hexastigm, we notice that each pair of opposite faces has in common a single point, and each edge intersects the opposite space in a single point. The latter family of fifteen points will be called *diagonal points* of the Hexastigm, the diagonal point which lies on the edge 12 being denoted by P_{12}, and so for the other edges. A second point Q_{12} is taken on each edge, viz. that which with the diagonal point divides the edge harmonically; Q_{12} will be called the *harmonic point* of the edge 12. The point common to two opposite faces 123 and 456 is written indiscriminately P_{123} or P_{456}, but is of minor importance. In any selected face of the Hexastigm, for example the face 135, we have now, besides the triangle 135, three diagonal points, P_{35}, P_{51}, P_{13}, and three harmonic points Q_{35}, Q_{51}, Q_{13}, upon its sides 35, 51, 13 respectively; and also the point P_{135} where our selected face is intersected by the opposite face 246. The last point is the intersection of three lines which join the vertices of the triangle 135 to the diagonal points of the opposite sides; and so by a well known property of the triangle the following four sets of three points are collinear; Q_{35}, P_{51}, P_{13}; P_{35}, Q_{51}, P_{13}; P_{35}, P_{51}, Q_{13}; Q_{35}, Q_{51}, Q_{13}.

Theorem. The fifteen diagonal points lie by threes on fifteen straight lines. For clearly the diagonal points of three opposite edges of the Hexastigm (such as 12, 34, 56) lie each of them in the three opposite spaces of the Hexastigm (3456, 5612, 1234) and are therefore collinear. This line is the only line which intersects each of the three opposite edges of the

Hexastigm and will be called a *transversal* of the Hexastigm: the three transversals which meet any one edge all meet it in its diagonal point.

For the sake of realizing clearly the nature of the Hexastigm, its diagonal points, harmonic points, and transversals, it is convenient to consider separately such parts of the complete four-dimensional figure as fall within some chosen three-dimensional space. Thus in one of the spaces of the Hexastigm determined by four vertices 1, 2, 3, 4, are contained a tetrahedron 1234 whose vertices, edges and faces are vertices, edges and faces of the Hexastigm; the diagonal points, harmonic points, etc. which belong to those edges and faces, and, in addition, P_{56} the diagonal point of the opposite edge 56. In order to describe the configuration in the simplest way I allow myself to make use of the language of metrical geometry, and describe a particular case (from which the general case may be derived projectively) thus:—under special circumstances the point P_{56} is the centre of mean position of the points 1, 2, 3, 4; the diagonal points P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34} are the middle points of the edges of the tetrahedron 1234; the harmonic points are at infinity; the centres of the faces of the tetrahedron are the points where they are met by the opposite faces of the Hexastigm, and the lines which join the middle points of opposite edges of the tetrahedron are transversals of the Hexastigm. Hence, by projection, the following sets of points are in all cases collinear:—

(1) The vertices 1, 2, and P_{12}, Q_{12}.

(2) P_{12}, P_{34}, P_{56}.

(3) P_{12}, P_{13}, Q_{23}.

(4) Q_{12}, Q_{13}, Q_{23}.

And the following are coplanar:—

(5) The vertices 1, 2, 3, and P_{12}, P_{23}, P_{31}, Q_{12}, Q_{23}, Q_{31}.

(6) The vertices 1, 2, and P_{12}, Q_{12}, P_{34}, P_{56}.

(7) P_{56}, P_{12}, P_{34}, P_{14}, P_{23}, Q_{13}, Q_{24}.

(8) P_{12}, P_{13}, P_{14}, Q_{23}, Q_{34}, Q_{42}.

(9) Q_{12}, Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{34}.

As a consequence of (9) we learn that the harmonic points of the ten edges which join any five vertices of the Hexastigm lie in an S_3 and that the fifteen harmonic points are the points of intersection four by four of six spaces. Thus, from the family of six random points in an S_4 we have worked round to a family of six spaces; but it would be equally simple to develop the figure from six random spaces of an S_4 and end with a family of six points. The Hexastigm is therefore reproduced by the principle of duality, and in fact will be shewn later to be its own polar reciprocal with respect to a certain imaginary quadric variety. To state this matter more explicitly, the principle of duality establishes a correspondence in the Hexastigm between each vertex and the space containing the harmonic points of the ten edges which join the remaining five vertices; between the harmonic point of any edge and the opposite space; between the diagonal point P_{12} and the space containing Q_{12}, Q_{34}, Q_{35}, Q_{36}, Q_{45}, Q_{46}, Q_{56}; between the transversal which meets the edges 12, 34, 56, and the plane containing Q_{12}, Q_{34}, Q_{56}; and so forth.

Again the space which contains the coplanar system of points (7) and the point P_{16} contains of necessity the following nine diagonal points and six harmonic points,—

$$P_{12},\ P_{14},\ P_{16};\ P_{32},\ P_{34},\ P_{36};\ P_{52},\ P_{54},\ P_{56};$$

$$Q_{35},\ Q_{51},\ Q_{13};\ Q_{46},\ Q_{62},\ Q_{24}.$$

To construct this three-dimensional figure, (of which the Hexastigm contains ten examples), take any three lines in space and any three others intersecting the former three: the diagonal points lie at the nine intersections of these lines, which are thus transversals of the Hexastigm and must also be generators of a quadric surface: the harmonic points are now easily determined, since Q_{13} is common to the lines which join P_{12} to P_{32}, P_{14} to P_{34}, P_{16} to P_{36}. It will be observed that these ten spaces meet every edge of the Hexastigm either in its diagonal point or its harmonic point: they will be called *cardinal spaces* of the Hexastigm, and constitute an extremely interesting configuration in S_4, whether regarded as part of the Hexastigm or as distinct from it. The principle of duality explained in the last paragraph connects these ten spaces reciprocally with the family of ten points each of which is common to two opposite faces of the Hexastigm, and each of which we agreed to denote by either of two symbols P_{123} or P_{456}. That cardinal space which contains the harmonic points of six edges situated in the opposite faces 123, 456 of the Hexastigm and the diagonal points of the remaining nine edges will be denoted by the symbol $C(123, 456)$. Each cardinal space contains as we have seen nine diagonal points, six harmonic points, and six transversals which are generators of a quadric surface; conversely each diagonal point lies in six cardinal spaces, each harmonic point in four and each transversal also in four. Any two cardinal spaces intersect in a plane,—one of a system of forty-five,—in which lie two intersecting transversals: thus the two spaces

$$C\,(123,\ 456) \text{ and } C\,(124,\ 356)$$

intersect in a plane containing the two transversals on which lie P_{34}, P_{15}, P_{26}, and P_{34}, P_{16}, P_{25}. In this plane the points P_{15}, P_{16}, P_{26}, P_{25}, form a quadrangle whose opposite sides intersect in the points P_{34}, Q_{12}, Q_{56}. The analytical methods of III. will be found of great help in such investigations as this.

Up to this point the only way of describing a particular transversal of the Hexastigm and distinguishing it from its fellows has been to mention the diagonal points through which it passes or the edges which it meets. The following considerations suggest a simpler notation. It will be found that from the fifteen transversals it is possible in six distinct ways to select a set of five which meet all fifteen edges of the Hexastigm: call them set *a*, set *b*, set *c*, set *d*, set *e*, set *f*; each transversal enters into two of these sets, and therefore the symbol *ab* is suitable as a means of representing that transversal which belongs both to set *a* and set *b*. The constitution of the six sets will readily be inferred from the appended tables; in the former of which the new symbol for each transversal is followed by a list of the edges which it meets; and in the latter, after each edge is written a list of the transversals which meet it:—

Table I.

ab	12, 34, 56	*bc*	16, 24, 35	*ce*	14, 23, 56
ac	13, 25, 46	*bd*	15, 23, 46	*cf*	15, 26, 34
ad	14, 26, 35	*be*	13, 26, 45	*de*	16, 25, 34
ae	15, 24, 36	*bf*	14, 25, 36	*df*	13, 24, 56
af	16, 23, 45	*cd*	12, 36, 45	*ef*	12, 35, 46

Table II.

12	*ab, cd, ef*	23	*af, bd, ce*	35	*ad, bc, ef*
13	*ac, be, df*	24	*ae, bc, df*	36	*ae, bf, cd*
14	*ad, bf, ce*	25	*ac, bf, de*	45	*af, be, cd*
15	*ae, bd, cf*	26	*ad, be, cf*	46	*ac, bd, ef*
16	*af, bc, de*	34	*ab, cf, de*	56	*ab, ce, df*

The first column of table I contains the transversals belonging to set *a*, and the members of other sets may be selected without difficulty when necessary. Three transversals which pass through a diagonal point are represented by symbols such as *ab*, *cd*, *ef*, in which all six letters occur; and generally two transversals do or do not intersect according as their representative symbols do not or do possess a letter in common: in other words two transversals which belong to the same set do not intersect, and two which do not belong to the same set must do so. Six transversals which lie in one of the ten cardinal spaces of the Hexastigm are therefore denoted by symbols such as *bc*, *ca*, *ab*, *ef*, *fd*, *de*; a fact which suggests a second notation for the cardinal spaces, to be used concurrently with that already explained, viz. the symbol $C\,(abc.\,def)$ to denote the above space, which would be written C (145. 236) in the previous system. The connexion between the two notations is shewn in the following table:

Table III.

$C\,(abc.def) = C\,(145.236)$	$C\,(ace.bdf) = C\,(126.345)$
$C\,(abd.cef) = C\,(136.245)$	$C\,(acf.bde) = C\,(124.356)$
$C\,(abe.cdf) = C\,(146.235)$	$C\,(ade.bcf) = C\,(123.456)$
$C\,(abf.cde) = C\,(135.246)$	$C\,(adf.bce) = C\,(125.346)$
$C\,(acd.bef) = C\,(156.234)$	$C\,(aef.bcd) = C\,(134.256)$

[A certain reciprocity (which however is really illusory, and becomes misleading if pursued too far), will be noticed in these tables between the symbols a and 1, b and 2, c and 3, d and 4, e and 5, f and 6: thus the transversal ce meets the edges 14, 23, 56, and reciprocally the edge 35 is met by transversals ad, bc, ef, and so throughout: it does not appear that this reciprocity can be followed out to any result of value, but the construction of the foregoing tables is considerably facilitated by it].

Numerous instances of harmonic section and involution will be found in the Hexastigm, so many indeed that the wisest plan appears to be to pass them over for the present, since after the introduction of the methods of analysis such properties are far more easily discerned: for the same reason further investigation of other details is postponed; but it will be well, as a conclusion to this part of the subject, briefly to consider how the Hexastigm may be built up from a different foundation, viz. the family of ten cardinal spaces. In section II. when we turn to the contemplation of the nature of space-sections of the Hexastigm, we shall no longer be able to regard the family of six random points as the basis of the figure, for the points of the four-dimensional figure will be lost: its lines, planes and spaces will however persist in the shape of points, lines and planes in the S_3 by which the section is made, and on this account the suggested change of standpoint from which we view the structure we have raised becomes desirable. It is necessary to retain both notations for the cardinal spaces, the 1, 2, 3,......notation to shew their relations with the edges and faces of the Hexastigm, the a, b, c,......notation to shew their relations with the transversals; constant references must therefore be made to the three tables given above.

Each of the ten cardinal spaces corresponds to one of the ten ways in which the six letters a, b, c, d, e, f, or the six figures 1, 2, 3, 4, 5, 6, can be subdivided into two triads (or sets of three), as the representative symbols shew. From either symbol for any one space the symbol for any of the other nine spaces may be derived by an interchange of two letters or figures, one from each triad: and deleting the interchanged members we obtain a convenient symbol for the forty-five *pairs* of cardinal spaces: thus the pair

$$C\,(123.456) = C\,(ade.bcf) \text{ and } C\,(135.246) = C\,(abf.cde)$$

is represented by either $C\,(13.46)$ or $C\,(bf.de)$. The latter symbol at once informs us that the transversals bf, de are common to the two spaces; the former that the points Q_{13}, Q_{46}, P_{14}, P_{16}, P_{34}, P_{36}, P_{25} lie in each. The six cardinal spaces which contain the diagonal point P_{24} (the intersection of transversals *ae*, *bc*, *df*) fall into three pairs C (13.56), C (15.36), C (16.35) or C (*ae*.*bc*), C (*bc*.*df*), C (*ae*.*df*); the remaining four spaces contain the point Q_{24} and have the figures 2, 4, in the same triad. On the other hand the transversal *bd*, which meets the edges 15, 23, 46, is cut by two spaces in each of its diagonal points, viz. by the pair C (23.46) = C (*ac*.*cf* in P_{15}, by the pair (C 15.46) = C *af*.*ce*) in P_{23}, and by the pair C (15.23) = C (*ac*.*ef*) in eP_{26}; and it lies in the other four cardinal spaces, of which it is characteristic that the letters *b*, *d* are members of the same triad.

Three cardinal spaces intersect either in a transversal of the Hexastigm, through which a fourth space also passes, or in one of a family of sixty lines which join the diagonal points of two intersecting edges of the Hexastigm: thus the three spaces

C (123.456) = C (*ade*.*bcf*);

C (124.356) = C (*acf*.*bde*); C (125.346) = C (*adf*.*bce*);

intersect in the line joining P_{16} to P_{26}, which passes also through Q_{12}. For a reason to be justified later I call these the sixty *Pascal lines* of the Hexastigm; the line just quoted is also common to three planes which contain the following pairs of transversals, *ad* and *bc*, *de* and *cf*, *eb* and *fa*. We are thus led to study the figure formed of lines which join the diagonal points of different edges: if the edges do not intersect, the line is a transversal and contains a third diagonal point; if the edges do intersect, the line is a Pascal line of the Hexastigm and contains a harmonic point. The sixty Pascal lines fall into six sets of ten lines, which may be called set 1, set 2,......set 6, the members of set 1 being the lines which join each two of the five points P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, the diagonal points of the five edges of the Hexastigm which meet in the vertex 1. But in space of four dimensions a figure built up from five points is equally well determined by the five spaces which contain four of the points, just as in space of three dimensions a tetrahedron is equally well defined by its faces or its vertices. The Pascal lines thus lie by threes in the faces of these figures and by sixes in their spaces: they lie also by threes in the faces of the Hexastigm and by twelves in its spaces.

Finally it may be pointed out that not only is the notation for various loci absolutely symmetrical, but that it is legitimate to interchange in any way either two or more of the symbols 1, 2, 3, 4, 5, 6, or again two or more of the symbols a, b, c, d, e, f; in consequence of this symmetry it is only necessary to give a single example of any type of locus that we discover, since the complete system of similar loci may be obtained by such interchanges. Any correspondence between various loci will be shewn by either scheme of notation just as effectively as though it were explicitly defined in geometrical language. With regard to the curious reciprocity in the groupings of the two sets of six symbols, all that is needed in its application to our present purpose is shewn in tables I, II, III.

Section II.

On space-sections of the Hexastigm, and their projections on a plane.

In considering the three-dimensional figure composed of those parts of the Hexastigm which lie in an arbitrary space, (which for brevity I describe as a space-section of the Hexastigm), I shall, as has been already stated, regard as the foundation of the figure the ten planes which are sections of the ten cardinal spaces; I shall speak of them as the ten cardinal planes of the three-dimensional figure. Since the cardinal spaces of the Hexastigm pass by fours through the the transversals, the cardinal planes of a space-section intersect by fours in fifteen points, (which I shall call the *principal* points), each derived from one transversal. With each cardinal plane and each principal point will be associated the same symbol as with the cardinal space or transversal of which it is the section; thus, in the $a, b, c,$......notation, to each principal point is assigned a symbol $ab, ac,$......ef, and to each cardinal plane a symbol such as $C\,(abc.def)$ in such a way that the points which lie in this plane are ab, bc, ca, de, ef, fd; moreover these six points, being derived from six lines which are generators of a quadric surface, must lie on a conic section. But before attempting to establish properties of the space-section, we must realize more exactly the configuration of the ten cardinal planes; as before I shall allow myself to lapse into the language of metrical geometry when it appears to me that clearness of description is gained by so doing.

To this end we observe that, in their relations to any space and opposite edge of the Hexastigm, as for example the edge 12 and the space 3456, the ten cardinal spaces divide themselves into a set of four spaces and another of six. The former meet the edge 12 in its harmonic point Q_{12}, and have no other common point; the latter meet the edge 12 in its diagonal point P_{12} which is also the intersection of the three transversals *ab*, *cd*, *ef*, situated in the space 3456; and two of them contain *cd* and *ef*, two contain *ef* and *ab*, and two contain *ab* and *cd*: moreover each of the other twelve transversals lies in two out of the set of six cardinal spaces and two out of the set of four. It follows that, in a space-section of the Hexastigm, if we select the triangle of principal points *ab*, *cd*, *ef*, two cardinal planes will be found to pass through each of its sides. Thus these six planes form a figure to which, in the special case when *ab*, *cd*, *ef* are at infinity, we should apply the title parallelepiped, and which we may describe in other cases by the phrase a *projected parallelepiped.* The other twelve principal points lie one on each of the twelve edges of the parallelepiped; they lie also two by two on the edges of the tetrahedron formed by the remaining four cardinal planes, the principal points which lie on two opposite edges of the parallelepiped being on the same edge of the tetrahedron. We deduce that the vertices of the tetrahedron are upon the four diagonals of the parallelepiped. This arrangement of the ten cardinal planes is a very convenient one to bear in mind: not only is it easy to picture mentally but it lends itself also to the construction of a model of the cardinal planes and principal points, either simply by marks on the edges of a rectangular box, or better by means of a wire framework in the form of the edges of a parallelepiped, with silk threads passed through holes bored one in each edge at its principal point. It will appear that none of the twelve points need lie on the edges produced, a matter of no small importance in making such a model. It must however always be remembered that the cardinal planes form an absolutely symmetrical system; that no plane or pair of planes can possess any descriptive property which is not possessed equally by every plane or pair of planes of the system. Fig. 2 is copied from such a model, and may be described in the following manner:—

Let SP', SQ', SR', be three concurrent edges of a parallelepiped; $S'P$, $S'Q$, $S'R$, the opposite edges: O the point of concurrence of the diagonals PP', QQ', RR', SS'; p, q, r, s, arbitrary points on these diagonals,—(for compactness it is best to take them upon OP', OQ', OR', OS' respectively). Then the faces of the parallelepiped and of the tetrahedron p, q, r, s shew the configuration of ten cardinal planes of a space-section of a Hexastigm in a (projectively) quite general form; and the principal points are on the edges of the parallelepiped either at infinity or at the points where the edges of the tetrahedron meet them, the following being one of the many possible schemes:—

ce on $Q'R$ and qr; *df* on $R'Q$ and qr;

cf on PS' and ps; *de* on SP' and ps;

ea on $R'P$ and rp; *fb* on $P'R$ and rp;

eb on QS' and qs; *fa* on SQ' and qs;

ac on $P'Q$ and pq; *bd* on $Q'P$ and pq;

ad on RS' and rs; *bc* on SR' and rs;

besides which there lie at infinity,

ab on SP', PS', QR', RQ'; *cd* on SQ', QS', RP', PR';

ef on SR', RS', PQ', QP',

[Some new facts present themselves to our notice in this figure. We observe that the points which lie on the sides of each of the four skew hexagons that we can form of the edges of the parallelepiped, *e.g.* $PQ'RP'QR'P$, lie in a cardinal plane: again we see that the three principal points of three edges which meet in a point form a triangle whose sides are parallel to those of the triangle formed by the principal points of the three opposite edges: these properties however, stated in their projected form, will be found included in the following list. One warning also is needed; fig. 2 is drawn according to the conventional system of perspective used in mathematical diagrams; *i.e.* the eye of the reader is supposed to be indefinitely remote, and yet to see the figure as of finite size, so that the diagram is an orthogonal projection of the configuration in space. But the eye being at infinity is situated in the plane which contains the principal points *ab*, *cd*, *ef*. Thus a mental picture, called up by fig 2, of a three-dimensional configuration of lines, points, and planes, is (projectively) quite general: regarded as a diagram of the projection of this three-dimensional configuration, fig. 2 represents a special case, since in it the points *ab*, *cd*, *ef* have been so projected as to become collinear].

Properties of a space-section of a Hexastigm.

We proceed to make a list of a few of the more important properties of the space-sections of a Hexastigm, illustrating them by references to the foregoing figure. It will be found occasionally that the means of arriving at some line or plane or space in the four-dimensional figure ceases to be available, and a new construction must be devised. It is established, as a consequence of our conception of space of four-dimensions, or of our axioms concerning it, that *There exist in space of three dimensions families of fifteen points which possess the following properties:—*

(α) With each one of the fifteen points may be associated one of the fifteen pairs that can be formed with six symbols, in such a way that any symmetry or correspondence between certain sets of the points is effectively shewn by symmetry or correspondence among the associated symbols; an important consequence of this theorem being that, if we agree to denote each point by the associated pair of six selected symbols, we may, by interchange of the six symbols, alter the representative symbols of different points of the family without affecting the truth of any statement of a property possessed by the points.

Let it be agreed that the fifteen points be called principal points, and that the letters a, b, c, d, e, f, be chosen as the six symbols: each of the fifteen points is denoted one of the symbols, ab, ac,......ef.

(β) Six points ab, bc, ca, de, ef, fd, are coplanar and lie on a conic section: there are in all ten such planes, called cardinal planes, four of which pass through each principal point.

(γ), The intersection of three cardinal planes is either a principal point, through which a fourth cardinal plane will also pass, or one of a set of sixty points, called *Pascal* points: on the line of intersection of any two cardinal planes lie two principal points and four Pascal points; for example the line SP' in figure 2 passes through two principal points ab and de and four Pascal points viz. S', P', and its intersection with pqr, qrs. Each Pascal point, for example S, is the intersection of three lines of intersection of two cardinal planes, SP', SQ', SR', which join ab to de, bc to ef, and cd to fa. Each Pascal point may be described as the intersection of lines joining opposite vertices of a skew hexagon whose vertices are ab, bc, cd, de, ef, fa; and therefore each corresponds to one of the sixty reversible cyclical arrangements of the symbols a, b, c, d, e, f.

(δ) The Pascal points fall into six sets of ten points, the members of each set constituting the vertices of a figure formed by five planes. The ten points of a set therefore lie by threes in ten lines and by sixes in the five planes. An illustration of such a set is furnished by the four points P, Q, R, S, of figure 2, and the six points where the lines PS, QS, RS, QR, RP, PQ, meet ps, qs, rs, qr, rp, pq, respectively: the tetrahedra $PQRS$ and $pqrs$ are perspective; the lines Pp, Qq, Rr, Ss, joining corresponding vertices meet in O; the six points of intersection of corresponding edges just mentioned lie in a plane which with the faces of the tetrahedron $PQRS$ forms the set of five planes referred to.

(ϵ) The Pascal points lie also by threes on twenty lines and by twelves in fifteen planes, (derived respectively from the faces and spaces of the Hexastigm), which form the figure described at the beginning of I., resoluble in fifteen ways into a pair of perspective tetrahedra. In figure 2, the set of planes derived from the spaces of the Hexastigm is made up of the plane at infinity, the six planes which pass through two opposite edges of the parallelepiped, and eight others each containing the principal points of three concurrent edges: of these eight, the four which pass through the principal points of edges meeting in P, Q, R, S, form one tetrahedron whose vertices lie on PP', QQ', RR', SS' respectively, and the remaining four form a second tetrahedron whose vertices lie also on these lines and whose faces are parallel to those of the former: the two tetrahedra have O as centre of perspective and the plane at infinity as plane of homology. The twelve Pascal points which lie at infinity are intersections of the three lines at infinity of the faces of the parallelepiped with the faces of $pqrs$.

(ζ) Each of the six sets of ten Pascal lines in (δ) was determined by section of the edges of a figure defined by five diagonal points of the Hexastigm such as P_{12}, P_{13}, P_{14}, P_{15}, P_{16}: the faces of these six figures intersect by threes in the lines such as Q_{56}, Q_{64}, Q_{45}, and the spaces intersect by twos in the planes similar to that containing Q_{34}, Q_{35}, Q_{36}, Q_{45}, Q_{46}, Q_{56}. But we have seen that the harmonic points are determined fully by the intersection of six spaces; hence, returning to the space-section of the Hexastigm, we infer that there is a certain figure of six planes through each of whose edges pass two faces of the figures (δ), and through each of whose vertices pass three edges of these figures.

The plane figure.

Instead of considering the plane figure derived from the space-section of a Hexastigm by projection, we find it rather more convenient to treat of its reciprocal, that is, a plane section of the figure derived by the principle of duality from that which we have just discussed. The six vertices and ten cardinal spaces of the Hexastigm being both unavailable, we look on the fifteen lines derived from the transversals as the foundation of the plane figure. The reason for describing the lines and points of the figure by the names of various mathematicians will appear shortly, if it is not already recognized.

A necessary consequence of our conception of space of four dimensions is that there exist in a plane families of fifteen lines possessing the following properties:—

(α) Each line may be associated with, and denoted by, one of the fifteen symbols *ab*, *ac*,.........*ef*, (formed by selecting two out of six symbols *a*, *b*, *c*, *d*, *e*, *f*, in all possible ways), in such a manner that the validity of any statement concerning the lines is unaffected by an interchange of these six symbols, and any symmetry or correspondence of various sets of the lines is shewn effectively by their representative symbols.

(β) Six lines such as *ab*, *bc*, *ca*, *de*, *ef*, *fd*, touch a conic; there being in all ten such conics, any two of which have two of the fifteen lines as common tangents.

(γ) Corresponding to the different permutations of the six letters *a*, *b*, *c*, *d*, *e*, *f*, (*e.g. acebdf*) we may form sixty hexagons such as *ac*, *ce*, *eb*, *bd*, *df*, *fa*; the three intersections of opposite sides of any such hexagon are collinear; I call these lines the sixty *Pascal* lines.

(δ) The sixty Pascal lines fall into six sets of ten lines, each set forming the well known configuration of ten lines and ten points, which may be resolved in ten different ways into a pair of perspective triangles. The ten Pascal lines of a set thus meet by threes in ten points which I call Kirkman points. That there is a correspondence between each of the sixty Pascal lines and one of the sixty Kirkman points which is a member of the same set, is obvious from the nature of the configuration; for any separate one of the ten points of a set may be taken as a centre of perspective of two triangles, and their axis of homology is the corresponding Pascal line.

(ϵ) The Pascal lines also meet by threes in twenty *Steiner* points which lie by fours on fifteen *Plücker* lines. These form another well known configuration, the projection of the intersections of six planes in space, which may be resolved in twenty distinct ways into three triangles whose vertices lie on three concurrent lines, and the three concurrent axes of homology of each pair.

(ζ) When three Pascal lines meet in a Steiner point their three corresponding Kirkman points lie on one of twenty *Cayley-Salmon* lines, which meet by fours in fifteen *Salmon* points, and form a configuration reciprocal to that in (ϵ).

Now these six theorems, which may be obtained without difficulty from previous results concerning the Hexastigm and its space-sections, have a very familiar form; a reference to the note at the end of Salmon's Conic Sections shews that, if a, b, c, d, e, f, denote six points of a curve of the second degree, the fifteen lines ab, ac,......ef, which join each two of them possess numerous properties included in the above theorems. It would be extremely rash to assume that the fifteen lines ab, ac......ef, which we have derived from the Hexastigm, necessarily join six points of a conic; but it is clearly advisable to see to what extent their properties (intuitive consequences of the nature of a simple four-dimensional figure) are in agreement with those better known results which in the middle half of the present century were accumulated round the celebrated theorem discovered by Pascal more than two hundred and fifty years ago, and still known by his name.

The development of the figure now commonly known as the *Pascal Hexagram* dates from 1828, when Steiner drew attention to the important fact that, from the same six points of a conic section, sixty distinct hexagons can be formed, each with its own Pascal line. During the next fifty years the figure formed of these sixty lines aroused wide interest, Steiner, Plücker, Hesse, v. Staudt, Schöter, Cayley, Salmon, Kirkman, and many others applying themselves to the study of its properties. To dwell in detail on the advances made by each of these mathematicians is superfluous, since the results of their labours have been summed up and extended by Veronese, in a masterly memoir, *Nuovi teoremi sull' Hexagrammum Mysticum*, (*Atti d. R. Accad. dei Lincei*, 1877, Vol. I, Series III, pp. 642-703), which is prefaced by an excellent historical sketch, with full references to the works of earlier writers, and contains proofs not only of all previously known theorems but of a large number of new and original ones. The names used in (γ), (δ), (ϵ), (ζ) are adopted from Veronese's memoir, and the subdivision of the Pascal lines and Kirkman points in (δ) into six sets is the most important of his original contributions to the theory.

The results of a thorough investigation of Veronese's memoir may be stated as follows;—*If we join in all possible ways, by lines, planes, and spaces, the diagonal points and harmonic points of a Hexastigm; take a space-section of the figure so formed; reciprocate it; and take a plane section of the reciprocal; we obtain a plane figure built up from fifteen lines, coextensive with that built up by Veronese for the special case when the fifteen lines join in pairs six points of a conic, together with proofs (intuitive consequences of the nature of the four-dimensional figure) of all his theorems:* (there must of course be exceptions to so sweeping an assertion as this, but they are so few and so trivial that it seems justifiable to ignore them).

By far the most important addition to the subject since the publication of Veronese's memoir is due to Cremona and will be referred to later: so far as I am aware nothing has appeared which renders inadmissible the statement, that, *the existence in a plane of other families of fifteen lines, which possess practically all known properties of the fifteen lines that join six points of a conic, is a necessary consequence of our conception of space of four dimensions;* or more precisely of the axiomatic law that in it, lines, planes, and spaces are determined by two, three, and four points respectively, and are cut by a space of three dimensions in points, lines, and planes. It must be admitted that the later properties of Veronese's memoir become tedious when considered in detail and are of less importance than the earlier results: the properties quoted (α), (β), (γ), (δ), (ϵ), (ζ), carry us as far into the subject of these families of fifteen lines as it seems advisable to penetrate. Far more important than the extension of the long list of elementary geometrical results concerning them is the enquiry as to the nature of these lines in the general case; for the discussion of this I call in the aid of Analytical methods.

The transition from the set of six points in S_4 to the fifteen lines in a plane has been accomplished by three operations, (1) a section, (2) a reciprocation, (3) a section: but the operations of section, reciprocation and projection are commutative, if we allow for the fact that a reciprocation interchanges the other two. Instead of the above process we may make the passage from four to two dimensions by first reciprocating, then taking a section and finally projecting, or in many other ways; the initial and final figures being always the same, but the intermediate ones of quite different types.

Section III.

Analytical Methods.

Whatever be the number of dimensions of the space we are considering, the *coordinates* of its points will be denoted by letters x, X, and the *equations* of its points by letters u, U, with suffixes added; the capitals being used for fixed, the small letters for current coordinates.

In a space of four dimensions S_4, the equations of any six points must be connected by one identical linear relation; in the case of the vertices of the Hexastigm it was stipulated that no five were to lie in a space of three dimensions, and we are therefore at liberty to represent the vertices 1, 2,......6, by equations $u_1=0$, $u_2=0$, ... $u_6=0$, which satisfy the identity

$$u_1+u_2+u_3+u_4+u_5+u_6\equiv 0;$$

or

$$\Sigma(u_r)\equiv 0;\ (r=1, 2, 3, 4, 5, 6).$$

The equations which determine any points, lines, planes or spaces in the Hexastigm now become apparent; the edge 12 is $u_1=0$, $u_2=0$; the face 123 is $u_1=0$, $u_2=0$, $u_3=0$; and the space 1234 is $u_1=0$, $u_2=0$, $u_3=0$, $u_4=0$: the diagonal point P_{12}, common to the edge 12 and the space 3456 is $u_1+u_2=0$, and the harmonic point Q_{12} is therefore $u_1-u_2=0$. The transversal P_{12}, P_{34}, P_{56} is $u_1+u_2=0$, $u_3+u_4=0$, $u_5+u_6=0$; and the cardinal space C (123 . 456) has coordinates

$$u_1=u_2=u_3=-u_4=-u_5=-u_6:\ \text{etc. etc.}$$

As regards coordinates of points (x), we choose them in the first instance to satisfy the indentity

$$\Sigma(u_r x_r)\equiv 0,\ (r=1, 2, 3, 4, 5, 6),$$

and make no further condition: on account of the former identical relation $\Sigma(u_r)\equiv 0$, each coordinate x is liable to be increased by the same quantity, and we can therefore only expect to obtain relations among the *differences* of the coordinates x: for example the space 3456 is $x_1=x_2$, the face 456 is $x_1=x_2=x_3$, and the edge 56 is $x_1=x_2=x_3=x_4$: the transversal and the cardinal space quoted above are represented

by $x_1=x_2$; $x_3=x_4$; $x_5=x_6$; and by $x_1+x_2+x_3=x_4+x_5+x_6$;

respectively. We have perfect right to impose another condition on the coordinates x but at present there is no advantage gained by doing so.

The case is altered when we come to the harmonic points, which we proved were determined by the intersections of six spaces, viz. $u_2 = u_3 = u_4 = u_5 = u_6$, or $6x_1 = \Sigma(x_r)$ etc. By imposing the condition $\Sigma(x_r) \equiv 0$, the equations of these spaces become $x_1 = 0$, $x_2 = 0$,......$x_6 = 0$; not only is the figure self-dualistic as stated in I., but it is actually its own polar reciprocal with respect to the imaginary quadric variety

$$\Sigma(x_r^2) = 0; \text{ or } \Sigma(u_r^2) = 0; \ (r = 1, 2, 3, 4, 5, 6);$$

as may be easily verified: the harmonic point of each edge of the Hexastigm is the pole of the opposite space with respect to this quadric, and the diagonal point is the pole of the space which contains the harmonic point of that edge and of each of the opposite space. Thus, when our coordinates satisfy the identities $\Sigma(x_r) \equiv 0$; $\Sigma(u_r) \equiv 0$; $\Sigma(u_r x_r) \equiv 0$; from the six points $u_r = 0$ we work round to the six spaces $x_r = 0$; and the six spaces would serve equally well as the foundation of the figure. A very important consequence is that, if we discuss fully the space-sections of the complete Hexastigm, we may pass over its projections; for the projections of the figure derived from six points in S_4 are merely reciprocals of the space-sections of the figure derived from six spaces in S_4.

The following are the equations, in both systems of coordinates x and u, of loci connected with the Hexastigm:

$$\Sigma(x_r) \equiv 0; \ \Sigma(u_r) \equiv 0; \ \Sigma(u_r x_r) \equiv 0; \ (r = 1, 2, 3, 4, 5, 6).$$

Vertex 1; $u_1 = 0$; $x_2 = x_3 = x_4 = x_5 = x_6$:

Edge 12; $u_1 = u_2 = 0$; $x_3 = x_4 = x_5 = x_6$:

Face 123; $u_1 = u_2 = u_3 = 0$; $x_4 = x_5 = x_6$:

Space 1234; $u_1 = u_2 = u_3 = u_4 = 0$; $x_5 = x_6$:

Diagonal point P_{12}; $u_1 + u_2 = 0$; $x_1 = x_2$; $x_3 = x_4 = x_5 = x_6$:

Harmonic point Q_{12}; $u_1 = u_2$; $x_3 = x_4 = x_5 = x_6 = 0$.

Transversal line P_{12}, P_{34}, P_{56}; $u_1 + u_2 = u_3 + u_4 = u_5 + u_6 = 0$;

or $x_1 = x_2$; $x_3 = x_4$; $x_5 = x_6$:

reciprocal to this is the plane containing Q_{12}, Q_{34}, Q_{56}.

Cardinal space $C(123.456)$; $u_1 = u_2 = u_3 = -u_4 = -u_5 = -u_6$;

or $x_1 + x_2 + x_3 = x_4 + x_5 + x_6 = 0$:

reciprocal to this is the point common to the faces 123, 456.

The symmetry of the two systems of coordinates u and x is so perfect that, while the name Hexastigm is retained to denote the complete series of loci, ranging between the six points $u_r = 0$ on the one hand and the six spaces $x_r = 0$ on the other, it is clearly desirable to recognize as fully as possible the equal claim of the two systems to be regarded as the basis of the figure. We may describe the two as the six-point system and the six-space system respectively, and, just as we have derived from the six-point system diagonal points, harmonic points, transversal lines and cardinal spaces, we derive reciprocal loci from the six-space system, and call them diagonal spaces, harmonic spaces, transversal planes and cardinal points. A diagonal space for example contains the plane common to two spaces of the six-space system and the point common to the remaining four. Certain loci it will be seen appear under different names in the two systems; the harmonic spaces, for instance, of the six-space system are identical with the spaces containing four of the six original points of the six-point system; but the equations we have given will prevent us from overlooking such facts as this. The symmetry is lost when we take a space-section of the Hexastigm but reappears in a less perfect form in the two-dimensional figure derived by projection. The difficulty in the four-dimensional figure is how to connect in a simple geometrical manner the two reciprocal systems—(reciprocation with respect to an imaginary quadric cannot well be used)—and it would be of great use to us for this purpose if a closer connexion between the two systems existed.

We have yet to find the equations of the Pascal lines of the Hexastigm; but in so doing it is best to keep in mind the lines and points they lead to in the plane figure. A Pascal line in the six-point system was determined by two diagonal points such as P_{12}, P_{13}, and its equations are therefore

$$-u_1 = u_2 = u_3\,;\ \text{or}\ x_4 = x_5 = x_6 = x_2 + x_3 - x_1 = -\tfrac{1}{2}x_1$$

The Pascal lines which join P_{12}, P_{13}, P_{23} lie in one of the planes 123 of the six-point system, and lead to three Pascal lines in the plane figure which meet in a Steiner point; the Steiner points and Plücker lines in the plane figure being derived from the edges and faces of the six-point system. The Pascal lines and Kirkman points of one of the six sets which Veronese discovered are derived from the edges and faces of the figures formed by joining P_{12}, P_{13}, P_{14}, P_{15}, P_{16}, (or a similar set of diagonal points), in all possible ways; the Pascal line derived from joining P_{12} and P_{13} corresponding to the Kirkman point derived from the plane P_{14}, P_{15}, P_{16}.

Thus a Veronese set of Pascal lines and Kirkman points is derived from either five points or five spaces, *e.g.*

$$u_1 + u_2 = 0\,;\; u_1 + u_3 = 0\,;\ldots\ldots \qquad u_1 + u_6 = 0\,;$$

or $$x_1 + 2\,x_2 = 0\,;\; x_1 + 2\,x_3 = 0\,;\ldots\ldots \qquad x_1 + 2\,x_6 = 0:$$

and it is clear that, when three Pascal lines lie in one of the six-point system, the corresponding planes pass through the reciprocal edge of the six-space system; but the Pascal lines are unfortunately not reciprocals of the planes which lead to the corresponding Kirkman points.

Corresponding investigations in the six-space system are taken for granted. It seems worth while to digress here for a moment in order to point out that it is possible by projective methods to bring six random points in S_4 to a form in which the distance of each two is the same. With the vertices of the Hexastigm arranged thus, the diagonal points bisect the edges, and the figure acquires many beautiful metrical properties. Reduction to this form is not possible in Euclidean space, for the equation $\Sigma\,(u_r^2) = 0$ or $\Sigma\,(x_r^2) = 0$ must represent the Absolute.

Space-sections of a four-dimensional figure.

The coordinates x may be applied at once to space-sections, the sole difference being that they have to satisfy a second identical linear relation, viz. the equation of the space by which the section is made. If U_1, U_2,......U_6 denote the coordinates of this space, the equations of all loci in the section may be deduced at once from the foregoing formulae in terms of six coordinates x_1, x_2,......x_6, connected by two identical linear relations

$$\Sigma\,(x_r) \equiv 0\,;\; \Sigma\,(U_r x_r) \equiv 0\,;\; (r = 1, 2, 3, 4, 5, 6).$$

But if we wish to use coordinates u, difficulties beset us: on account of the identity $\Sigma\,(U_r x_r) = 0$, each of the quantities u_r is liable to be increased by the same multiple of U_r, and thus we shall only be able to interpret equations in u's which are not altered when $u_r + \lambda\,U_r$ is substituted for u_r: we are in just the same case as when at the beginning of this III. we saw that only equations in the differences of the coordinates x were to be expected to arise: as then, we have every right to simplify our equations, if possible, by assuming that the quantities u satisfy a new linear identity. When several equations in u coordinates determine a locus, the space-section by the space $(U_1, U_2,\ldots U_6)$ is found by writing $u_r + \lambda\,U_r$ for

u_r in the equations and eliminating λ. [The matter is more easily explained in its reciprocal form, viz. when we are projecting loci in S_4 upon an S_3 from a centre of projection whose coordinates are $X_1, X_2, \ldots X_6$. Equations in u coordinates here present no difficulty; but, given a locus defined by two or more equations in x coordinates, we first write $x_r + \lambda X_r$ for x_r and eliminate λ: this represents a locus generated by lines which join the point X to each point of the given locus: we may now if we wish, assume a new relation among the coordinates x, i.e. specify a particular S_3 as the space on which the projection is made; but it is seldom advisable to do this]. Obviously, in discussing the space sections of the Hexastigm, we must keep as far as possible to x coordinates. As has been stated above, on account of the perfect reciprocity of the Hexastigm, it will not be necessary to discuss its projections into space of three dimensions; for all that concerns them may be obtained by the principle of duality from a space-section, provided the latter be considered in its relation both to the six-point and to the six-space systems: but first we shall turn our attention to a certain variety in S_4 which throws light on the nature of the planes, lines, and points of a space-section, and shews that they have already become to some extent familiar to mathematicians.

On Segre's cubic variety.

Intimately connected with any six-space system in S_4 is a certain variety of the third order, some of whose properties, studied without the aid of analysis, form the subject of a note by its discoverer, Corrado Segre, in vol. XXII of the *Atti della R. Accad. delle Scienze di Torino* 1887. p. 547. This variety, which appears to me to possess far more beautiful properties than any cubic surface in three-dimensional space, is of the fourth class, is rational, has no independent invariant, has the maximum finite number of double points possible in a cubic variety, namely ten. When the equations of the six planes are so prepared that their sum is identically zero, the equation of the variety expresses that the sum of their cubes vanishes. With our six-space system, $x_r = 0$, is associated the Segre's cubic variety, (to be denoted in what succeeds by V_3),

$$\Sigma (x_r)^3 = 0\ ;\ \Sigma (x_r) \equiv 0\ ;\ (r = 1, 2, 3, 4, 5, 6,):$$

the equation may however be also written in ten forms similar to

$$(x_2 + x_3)(x_3 + x_1)(x_1 + x_2) + (x_5 + x_6)(x_6 + x_4)(x_4 + x_5) = 0.$$

The cardinal points ($x_1 = x_2 = x_3 = -x_4 = -x_5 = -x_6$, etc). of the six-space system lie on V_3, and an attempt to determine their tangent spaces shews that each is a double point. The transversal planes, which form a system of fifteen planes situated by threes in the fifteen diagonal spaces lie on V_3, and are part of it: each diagonal space thus cuts V_3 in three planes. Now in S_4 we can draw through any ordinary point of a variety six lines having four-point contact at the point, and in the case of Segre's cubic V_3, these lines must lie wholly on the variety, and must therefore meet each one of the fifteen diagonal spaces in one of the three transversal planes contained in it. Reasoning from this we are able to attach a more definite geometrical significance to the symbols a, b, c, d, e, f of Tables 1, 2, 3, than has hitherto been possible; viz. that the six lines which pass through any point of V_3 and lie wholly on V_3 are of six distinct types, a, b, c, d, e, f, those of type a meet one set of five transversal planes, those of type b another set, and so on: the symbol ab associated with the transversal plane 12, 34, 56 in Table 1, shews that this plane is met by all lines of the types a and b.

Section IV.

On the Pascal Hexagram.

In order to pass from the transversal lines of a six-point system to the plane families of fifteen lines which possess the properties of Pascal's Hexagram, it has been said that three operations are required. One of these is necessarily a reciprocation, but it may be either the first or second or third of the series. In virtue of III., the simplest way of making the transition is to take the reciprocation first, for this merely changes the transversal lines of the six-point system into transversal planes of an equally simple six-space system; projecting a space-section of this family of planes we arrive at the plane figure desired. The derivation of Pascal's Hexagram from the six-space system in this way is the subject of the present section; we have first to consider space-sections of the fifteen planes which formed part of Segre's cubic variety V_3 in general.

Now the section of V_3 by an arbitrary space is a cubic surface of quite general type: for Cremona has shewn (*Math. Annalen* XIII. p. 301) that the equation of a non-singular cubic surface may be reduced to the form

$$\Sigma (x_r^3) = 0; \ \Sigma (x_r) \equiv 0; \ \Sigma (U_r x_r) \equiv 0; \ (r = 1, 2, 3, 4, 5, 6).$$

in thirty six different ways: see *Salmon-Fiedler*, p. 403, section 310. For special positions of the space of section, *i.e.* for special values of $U_1, U_2, \ldots U_6$, the cubic surface may possess singularities; but of such cases I shall consider only one, viz. when the space of section touches V_3 at an ordinary point and the cubic surface therefore has a double point at the point of contact; *Salmon-Fiedler*. p. 412, and foot-note, section 341. The space-sections of the fifteen transversal planes of the six-space system are a family of fifteen lines which lie by threes in fifteen planes, and also lie on the cubic surface; they are a set of fifteen of the twenty-seven lines of the surface such as is left when we omit a double-six; *Salmon-Fiedler*. p. 401, section 308. Schläfli, *Quarterly Journal*, Vol. 2 p. 116. In the special case of section by a space which touches V_3, they are the fifteen lines of the surface which do not go through the nodal point. We thus arrive at a theorem due in part to Cremona, viz.

The plane systems of lines which possess the properties proved for the lines which join six points of a conic are projections of fifteen lines of a cubic surface such as are left when we exclude a double-six. It is necessary to include a statement of the special case, for this arises when the members of the rejected double-six coalesce two by two in six lines through the double point.

Mention was made at the end of II. of a memoir by Cremona: it is to be found in the same volume of the *Atti d. R. Accad. dei Lincei* as that of Veronese; pp. 854-874. On reading Veronese's manuscript Cremona was led to seek another basis for the existence of this vast series of theorems, and found it in the three-dimensional system of lines that lie on a cubic surface having a double point. On such a surface lie six lines which pass through the nodal point, and are generators of the tangent cone, and fifteen others, one in the plane of each pair of the foregoing: by projecting these on a plane, from the nodal point as centre of projection, Cremona obtained fifteen lines joining six points of a conic, and, having established the fact that these lines lie by threes in fifteen planes, shewed that all Veronese's theorems were intuitive consequences. In conclusion, he observes that the projections of any family of fifteen lines which lie by threes in fifteen planes would possess these properties, and that the lines of a cubic surface supply examples of such families, but goes no further, overlooking the fact that Geiser had discussed (*Math. Ann.* I. p. 129) the projections of the lines of a cubic surface, with valuable and well known consequences. Cremona, then, stopped after taking a very important step in the direction of the simplification of the vast figure which Veronese had constructed, in that he shewed how it could be derived, and all its properties established, from a comparative simple figure in space: a discussion of the Hexagram from Cremona's point of view will be found in the *Transactions of the Cambridge Philosophical Society*, vol. XV. p. 207, in which it is pointed out that the complete figure as developed by Veronese was the result, save in a few unimportant details, of projecting the intersections of the two systems of planes analogous to $x_1 \pm x_2 = 0$.

The method followed in the present paper derives the plane figure, and establishes its properties, from one of the simplest possible (descriptive) figures in space of four dimensions, by purely linear methods; it leads us to notice that other systems of coplanar lines and points possess all these properties, of which systems the Pascal Hexagram is an extremely special case: and it will be seen that the transition from four to two dimensions may be made by a different route with no less interesting results. That the true cause for the existence of families of coplanar lines and points, possessed of all Veronese's long category of properties, is to be found in the figures in S_4 cannot be doubted; although we shall find that, as a matter of history, most of these families have been already discovered, and some of their properties obtained, by other means, chiefly in connection with the study of curves and surfaces of the fourth degree. We will now consider how the above fifteen lines in space lead us, under special conditions, to Pascal's Hexagram, and then treat the most general case of a projection of a space-section of the fifteen transversal planes of a six-space system in S_4.

In order to derive from the transversal planes, such as

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 0,$$

of a system of six spaces $x_r = 0$, $(r = 1, 2, 3, 4, 5, 6,)$ where $\Sigma(x_r) \equiv 0$, the configuration of lines which join six points of a conic section, let the S_4 in which the spaces lie be cut by a space which touches Segre's variety V_3, $\Sigma(x_r^3) = 0$, in some point K. As explained at the end of III., six lines lying wholly on V_3 pass through K, and these are members of six different families distinguishable by six symbols a, b, c, d, e, f; in such a way that the symbols ab, ac etc., associated with the transversals in Table I, shew us which two of the six lines through K each transversal plane intersects: the first line in that table associates ab with 12, 34, 56; therefore the plane quoted above intersects the lines through K which are of type a and type b. In the space-section by the tangent space at K, V_3 is represented by a cubic surface having a double point at K, and the transversal planes by fifteen lines on the surface; but the lines of V_3 which pass through K persist in the space-section as lines, still distinguishable by letters a, b, c, d, e, f, which lie on the cubic surface, and pass through its double point; and each of the former fifteen lines intersects two of the latter according to a scheme shewn immediately by reference to Table 1.

The analytical formulae are discussed at length in my paper in the *Cambridge Phil. Trans.* to which I have referred. If the coordinates of the point of contact of the space of section with V_3 be denoted by X, we have the following system of equations for a three-dimensional cubic surface endowed with one nodal point K:

Equation of the surface, $\Sigma (x_r^3) = 0$; $(r = 1, 2, 3, 4, 5, 6)$:

Coordinates of K, the double point (X_r):

And the relations which connect the coordinates and constants are

$$\Sigma (x_r) \equiv 0; \ \Sigma (X_r^2 x_r) \equiv 0; \ \Sigma (X_r) = 0; \ \Sigma (X_r^3) = 0.$$

The six lines a, b, c, d, e, f, which lie on the surface and pass through K, are generators of the quadric cone $\Sigma (X_r x_r^2) = 0$; but it does not appear that the separate equations can be exhibited in a simple form. The plane through any two, for example c and e, can be found at once, for it meets the surface in a third line ce, whose equation is shewn by Table 1 to be

$$x_1 + x_4 = x_2 + x_3 = x_5 + x_6 = 0;$$

the equation of the plane through this line ce, c, e and K is therefore

$$\frac{x_1 + x_4}{X_1 + X_4} = \frac{x_2 + x_3}{X_2 + X_3} = \frac{x_5 + x_6}{X_5 + X_6}.$$

Through each of the fifteen lines ab, ac,......ef, pass three of the planes $x_1 + x_2 = 0$, sections of the fifteen diagonal spaces of the six-space system, according to the scheme shewn in Table 2; and the intersection of two such planes, *e.g.* $x_1 + x_2 = 0$ and $x_1 + x_6 = 0$, where the former contains ab, cd, ef, and the latter af, bc, de, passes through the points of meeting of ab with de, of bc with ef, and of cd with fa.

Project this system on a plane from K as centre of projection: the lines a, b, c, d, e, f, cut the plane in six points of a conic, also denoted by a, b, c, d, e, f, and the lines ab, ac,... ef project into lines which join each two of the six, and are therefore naturally still denoted by symbols ab, ac,...ef. The projection of the line of intersection of the planes $x_1 + x_2 = 0$, $x_1 + x_6 = 0$, still contains the intersections of ab with de, of bc with ef, of cd with fa, and is a Pascal line of the Hexagram.

As regards the equations of loci in the plane figure, we may, from the equations of any line (or curve) in the three-dimensional figure, derive the equation of the plane (or cone) formed by joining each point to K, and thus obtain equations of a system of geometrical loci in space wholly generated by lines through K; practically we do this by writing $x_r + \lambda X_r$ in place of x_r in the equations of the lines (or curve) in space, and eliminating λ. The section of this system by any plane is the projection of the three-dimensional figure on the plane from K as vertex of projection. As a rule it is not desirable to specify a particular plane as the plane of section; yet, as it is convenient to be able to use the nomenclature of plane geometry, we alway suppose such a section made. The outcome of these considerations is that in the plane figure lines (or curves) are given by one, points by two, homogeneous equations in six coordinates x_r subject to the relations $\Sigma(x_r) \equiv 0$, $\Sigma(X_r^2 x_r) \equiv 0$, and, further, the equations are of such a form that the substitution of $x_r + \lambda X_r$ for x_r does not alter them. For instance we speak of the equation of the plane through K and the line ce of the figure in space, found a short way back, as being the equation of the line ce of the projected plane figure of Pascal's Hexagram.

The verification of Veronese's theorems concerning the Pascal Hexagram by means of these equations is usually instantaneous and in no case presents any difficulty, but there is no reason to consider the theorems in detail. Cremona realised that the whole series of propositions were in truth only the relics of the simpler properties of a three-dimensional figure, and we have gone further in connecting them with four-dimensional space. To quote the theorems one by one is wearisome; but to be able to describe the properties of a set of coplanar lines by the phrase Veronese's properties of Pascal's Hexagram is so convenient for my purpose that some consideration of the meaning and origin of the phrase was called for. That all the properties (α) (β) (γ) (δ) (ϵ) (ζ) of II. do hold when a, b, c, d, e, f denote six points on a conic will in future be taken as proved by the foregoing investigation. It may be said that the equations used by Cremona in his paper on Pascal's Hexagram are very inconvenient, while those which we have used, discovered also by Cremona at a later time but not applied to this subject, are perfectly symmetrical. [A system of equations in u coordinates may also be employed for this case, and there is no reason why each kind of coordinate should not be used both here and in the general case of projections of space-sections of a Hexastigm. For if loci in the S_4 be first cut by a space whose coordinates are U_r and then projected from a point of this space whose coordinates are X_r, the resulting plane loci will be defined by equations in coordinates x_r or u_r subject to conditions

$$\Sigma(x_r) \equiv 0;\ \Sigma(u_r x_r) \equiv 0;\ \Sigma(u_r) \equiv 0;$$
$$\Sigma(U_r x_r) \equiv 0;\ \Sigma(X_r u_r) \equiv 0;$$
$$\Sigma(X_r) = 0;\ \Sigma(U_r X_r) = 0;\ \Sigma(U_r) = 0:$$

the equations being always of such a nature that a substitution of $x_r + \lambda X_r$ for x_r or of $u_r + \mu U_r$ for u_r does not affect them].

Section V.

Generalization of these results.

First Method.

In the course of the last section it was observed that the space-sections of the transversal planes of a six-space system in S_4 were a set of lines already familiar to mathematicians in connexion with surfaces of the third order; they form in the most general case such a set of lines as remains when from the twenty-seven lines of the surface we reject a double-six. The properties of these lines in space which are consequences of the nature of the Hexastigm, prove to be well known deductions from this new mode of defining them and need not detain us; the equations which we obtained for them from the Hexastigm were obtained by Cremona from three-dimensional considerations. The plane systems of fifteen lines which possess the properties (α), (β), (γ), (δ), (ϵ), (ζ) of II., and all the rest of Veronese's properties of Pascal's Hexagram, are then projections of certain sets of lines of a cubic surface and have been discussed by Geiser, (*Math. Ann.* I. p. 129).

When a cubic surface is given, and lines drawn through a point K to touch the surface, their points of contact lie on a quadric surface, the polar of K; each line that lies on the cubic surface meets this quadric in two points and therefore touches in two distinct points the cone with vertex K that envelopes the cubic surface. The projections from centre K of the lines which lie on a cubic surface are double tangents of the section of the cone with vertex K which envelopes the surface. When the cubic surface is a space-section of Segre's cubic variety V_3, the cone is a space-section of the cone in S_4 formed of lines which pass through K and touch V_3; if (X_r) be the coordinates of K, the equation of the cone is the discriminant of

$$\Sigma\,(x_r + \lambda\, X_r)^3 = 0,$$

and is in general of the sixth order. If the equation of the cone be required, a considerable simplification is effected by imposing the condition $\Sigma\,(X_r^2 x_r) \equiv 0$, upon the coordinates x, *i.e.* by projecting upon the polar plane of K. But we may state at once that, if K is not on the cubic surface, the section of the enveloping cone is a sextic curve of a particular type, distinguished by its having six cusps which lie on a conic. Since the cubic surface has thirty-six double-sixes of lines upon it, we infer that, from the twenty-seven double tangents which Plücker's equations shew this curve to possess, we may select thirty-six sets of fifteen lines which possess all Veronese's properties of Pascal's Hexagram. I do not know of any discussion of the properties of this sextic curve; its interest appears to be due wholly to its relation to the surface of the third order.

A far more important series of results springs from the particular case when K lies on the cubic surface. We cannot now choose the polar plane of K as the plane on which we project, for it now passes through K; it is best not to specify any plane for the purpose. The tangent cone from K is now of the fourth order, and if, as before (X_r) be the coordinates of K, and (U_r), those of the space by which the four-dimensional figure is cut, we have the following system of equations for the enveloping cone:—

$$3\,\{\Sigma\,(X_r\, x_r^2)\}^2 = 4\,\{\Sigma\,(x_r^3)\}\,\{\Sigma\,(X_r^2\, x_r)\}\,;$$

$$\Sigma\,(x_r) \equiv 0\,;\;\; \Sigma\,(U_r\, x_r) \equiv 0\,;\;\; \Sigma\,(X_r) = 0\,;\;\; \Sigma\,(U_r\, X_r) = 0\,;$$

$$\Sigma\,(X_r^3) = 0.$$

In the special case when K lies on the cubic surface, the section of the cone with vertex K which envelopes the surface is a curve of the fourth order without singularities, twenty-seven of whose double tangents are projections from vertex K of the lines of the cubic surface, the remaining double tangent being the intersection of the tangent plane at K with the plane of the curve. Now the system of double tangents of a quartic curve has been widely studied, and receives very thorough treatment from a point of view suited to the present geometrical investigation in Salmon's *Higher Plane Curves.* It is there shewn that the twenty eight double tangents may be denoted by the pairs of eight symbols a, b, c, d, e, f, g, h; (Salmon uses 1, 2, 3, 4, 5, 6, 7, 8); but the complete symmetry of the system is not fully shewn by this notation. A rule has been given by Cayley, founded on Hesse's investigations, called the rule of the *bifid substitution*, which removes this defect. The simplest of many possible ways of connecting the notations for the lines of the cubic surface which we have used and that just explained for double tangents of the quadric is to denote the double tangent derived from the tangent plane at K by gh; those derived by projection of members of a double six by ag, bg, eg, dg, eg, fg; ah, bh, ch, dh, eh, fh respectively. The remaining fifteen double tangents, (which form a set derivable by projection from the space-section of the transversal planes of a six-space system) are represented by the same symbols associated with each in Table I. Thus we arrive at a theorem concerning double tangents of a plane quartic which may be stated in the following curious form:—

The fifteen double tangents of a plane curve of the fourth order, denoted in Hesse's Algorithm by symbols formed of pairs of six symbols a, b, c, d, e, f, possess all the properties of the Pascal Hexagram formed by lines, (*naturally represented by the same symbols*), *which join each two of six points a, b, c, d, e, f, of a conic section.* That some of these properties should have been discovered independently is not to be wondered at: Salmon quotes (p. 234) two sets of six double tangents studied by Steiner and Hesse, the former set of which the six ab, bc, ca, de, ef, fd, are typical touch a conic, as we saw in (β); the latter set, ab, bc, cd, de, ef, fa, have their intersections on a line, which we call a Pascal line. We find ourselves in possession of an immense extension of Steiner's and Hesse's results, and have also a much clearer view of the inner principle on which these results rest, than can be obtained by slowly developing elementary geometrical properties of the lines.

A remarkable fact, not however without parallel, comes to light when we seek, by aid of the rule of the bifid substitution, for a distinctive geometrical property of such a set of fifteen double tangents. Selecting any pair of double tangents (ag

and *ah* for example), we find five other pairs (*bg*, *bh*; *cg*, *ch*; *dg*, *dh*; *eg*, *eh*; *fg*, *fh*), such that the eight contacts of any two of the six pairs lie on a conic: such a system of six pairs may be chosen in sixty-three ways. Of the remain double tangents, *any* fifteen possess all the properties of the Hexagram. Thus whereas, in the case of Pascal's Hexagram or of the double tangents of the sextic curve above described, we have to deal with sets of fifteen lines which possess a long series of properties on account of a quite definite cause, we here find the fifteen lines joined by a sixteenth, which forms with them an absolutely symmetrical family, any fifteen of whose members possess all the properties of the former sets. Our sense of symmetry alone shews the necessity for considering sets of sixteen double tangents of the quartic rather than fifteen; but the discussion may be postponed. It will be seen that the statement that the fifteen double tangents of a quartic curve *ab*, *ac*,...*ef*, possess all Veronese's properties of Pascal's Hexagram, does not include all their properties: the statement in fact deals which only forty-five of their intersections and ignores the remaining sixty, which are of equal importance in the case of the quartic curve, but coalesce by tens in the Hexagram. For instance the rule of the bifid substitution shows that in the case of a quartic curve the points of intersection of *ab* with *ac*, of *ad* with *ae*, and of *bc* with *de*, are collinear: the same theorem is nugatory in the Hexagram and does not hold in the case of the sextic curve.

The relations between the different families of fifteen double tangents of the sextic or of sixteen double tangents of the quartic; how far the Pascal lines, Kirkman points, Steiner points, etc. of different families are common, and so forth, will be passed over entirely. Between the very special case of Pascal's Hexagram and the general quartic (or the above mentioned sextic) are numerous other special cases; for example, if the quartic have a node, the properties of the Hexagram belong to any fifteen of the sixteen double tangents, and in a modified form to certain sets of lines composed partly of double tangents and partly of tangents from the node.

Second Method.

The two operations (section and projection), by means of which the plane families of lines just considered were derived from the transversal planes, may be taken in the reverse order without altering the final result; the intermediate stage, the figure in three-dimensions through which we pass, will be quite different from the former. But, in place of applying the process of projection followed by that of section to the six-space system, it is more convenient to turn to the reciprocal problem, viz., that of deriving a set of fifteen points in a plane from the transversal lines of a six-point system by first taking a space-section of the figure and then projecting on a plane. The resulting family of points will necessarily possess properties reciprocal to Veronese's properties of Pascal's Hexagram, and will also be necessarily reciprocal to a family of lines such as we have just been discussing; the interest of this second method lies in the intermediate stage, which is of quite a new character, not in the final stage, which is bound to be simply reciprocal to that of the earlier method. One advantage of arranging the two methods in this manner is that we may follow the two simultaneously, by taking a space-section of the complete Hexastigm, (which comprises both a six-space and a six-point system), and then projecting the complete section on a plane: the space-section will thus include fifteen lines derived from the transversal planes of the six-space system, and fifteen points derived from the transversal lines of the six-point system. What is the nature of these fifteen points? Do they form a configuration already known? We have seen that they lie by sixes in ten cardinal planes, and we have to some extent discussed their properties under the title principal points in II. A better clue for the purpose of connecting them with known results is furnished by the variety reciprocal to V_3 which is associated with the six-point system in the same manner that V_3 is associated with the six-space system. The reciprocal of a cubic variety with ten double points is of order $3.2^3-2.10=4$, and we therefore denote it by V_4.

In the coordinates u the equation of V_4 is $\Sigma(u_r^3)=0$, and therefore in coordinates x the equation is found by eliminating λ and the quantities u_r from

$$x_r+\lambda=u_r^2;\ \Sigma(u_r)\equiv 0,\ \Sigma(u_r^3)=0;\ (r=1, 2, 3, 4, 5, 6),$$

and, if advisable, using $\Sigma(x_r)\equiv 0$ to simplify the result: the quantities u_r being thus roots of a sextic equation in some variable θ lacking terms in θ^5 and θ^3; the quantities x_r are roots of an equation which differs from a perfect square only in its two last terms: from this we deduce that

$$\{\Sigma(x_r^2)\}^2=4\Sigma(x_r^4);\quad \Sigma(x_r)\equiv 0;$$

but many other forms may be given to the result. The variety V_4 is of the fourth order, has each transversal of the six-point system as a double line and each cardinal space as a singular tangent space, *i.e.* is cut by each cardinal space in a quadric surface taken twice. A space-section of V_4 is therefore a quartic surface which has fifteen double points and ten singular tangent planes, the principal points and cardinal planes of the space-section of the six-point system.

Conversely, if a quartic surface have fifteen double points, it may be shown that it must be a space-section of a quartic variety such as V_4; it does not seem necessary to give the proof; incidentally we notice some other forms to which the equation of V_4 may be reduced, such as

$$\{(x_1-x_4)^2-(x_2+x_5-x_3-x_6)^2\}^{\frac{1}{2}}+\{(x_2-x_5)^2-(x_3+x_6-x_1-x_4)^2\}^{\frac{1}{2}} + \{(x_3-x_6)^2-(x_1+x_4-x_2-x_5)^2\}^{\frac{1}{2}};$$

or again the equation of V_4 is the discriminant of

$$(\lambda+x_2+x_3-x_1)(\lambda+x_3+x_1-x_2)(\lambda+x_1+x_2-x_3) - (\lambda+x_5+x_6-x_4)(\lambda+x_6+x_4-x_5)(\lambda+x_4+x_5-x_6).$$

It is not, however, my purpose to develop properties of V_4, except in so far as they throw light on the families of lines and points we have discovered. What concerns us at present is that we have obtained a second quite new way of arriving at the plane families of lines, or rather the families of points derived by the principle of duality; in fact we may assert:—

The projections on a plane of the fifteen double points of a quartic surface form a family of points possessing properties reciprocal to Veronese's series of properties of Pascal's Hexagram.

Of the two methods the latter is to be preferred; the fifteen principal points of a space-section of the six-point system are determined by ten cardinal planes which form a figure in space quite readily pictured mentally if we conceive the planes to be disposed as in Figure 2. The method of the fifteen lines which lie by threes in fifteen planes, as Cremona expressed it, or of fifteen lines of a cubic surface excluding a double six, which formed the intermediate stage in the first method are by no means so easy to realize, even after a model has been studied. As to the two-dimensional figure there is nothing to choose: in fact it becomes more and more apparent that the plane figure must be considered simply as a projection of a space-figure, and its properties thus derived; any attempt to think of the plane figure by itself, purely as a two-dimensional system, entangles us in a maze of elementary theorems absolutely bewildering in their numbers. The space-figures are not so complex as to confuse us and can be realized with a slight effort; if the four-dimensional figure could be pictured mentally, to discuss even the space-figures would be superfluous. It might be thought that the fact that the first method depended on a cubic surface and the second on a quartic told in favour of the former; but even this appears to me to be untrue, for these quartic surfaces are of particular interest.

An important special case presents itself when the space of section touches V_4 at some point K; for the resulting surface must then have another node at K or sixteen in all. The surface is in fact the much-studied Kummer's surface; not only are the fifteen nodes ab, ac, ..., ef joined by a new node K, but the ten singular tangent planes are joined by six others which pass through K, reciprocals of the six lines which were proved to pass through each point of Segre's variety V_3 and lie on the surface. These six lines were denoted by a, b, c, d, e, f in such a way that a met the five planes ab, ac, ad, ae, af in the six-space system; if the six planes which pass through K, reciprocal to these lines, be here denoted by a, b, c, d, e, f, we find ourselves making use of the ordinary notation for a Kummer's configuration (see Reye, *Geometrie der Lage*, latest edition; or Sturm, *Liniengeometrie*, Vol. 2) viz.

(1) K, one of the nodal points;

(2) a, b, c, d, e, f the six planes which pass through K;

(3) ab, ac, ..., ef, the other fifteen points, so named that ab lies on the planes a and b.

(4) $C\,(abc.def)$, or simply $abc.def$, the remaining ten planes, each containing six of the points (3).

The fifteen points (3) and the ten planes (4) retain all their previous properties, but also acquire some new ones, *e.g.* that ab, ac, ad, ae, af are now coplanar; but *any* fifteen of the sixteen nodes possess all the properties proved for these fifteen; any fifteen, for example, have sixty Pascal points, &c., not, however, all of them distinct, for it appears that there are in all only two hundred and forty Pascal points; each in fact belongs to four different sets of fifteen points. We observe also that, by taking the vertex of projection at K, the projections of the fifteen points ab, ac, ..., ef are intersections of six tangents to a conic; and it is clear that, as Kummer's surface is self-dualistic, we may derive, by cutting its singular planes by a plane, a set of sixteen lines, which has all the properties of the lines reciprocal to the projections of its nodes.

Each method thus defines the most general family of lines which possess Veronese's properties of the Pascal Hexagram, or the reciprocal family of points, by means of double tangents of a sextic curve, or by projection of the nodes of a surface of the sixth class, (for the class of a quartic with fifteen nodes is 6): each shows the existence of an important special case of double tangents of a quartic when the phenomenon of the appearance of a new member of the system occurs; and finally there is the case of Pascal's Hexagram, or its reciprocal, in which the plane figure is so approached that the new member of the family is made indeterminate (by choice of a centre of projection coinciding with it, or some such means). The case of the quartic curve obviously demands further study as regards the mutual relations of the Pascal lines, Kirkman points, Steiner points, &c., &c., of the different sets of fifteen lines: the sets of sixteen double tangents in question are represented either by gh and $ab, ac, \ldots, ef$ as before, or by one of four symbols a, b, c, d associated with one of the four e, f, g, h. It is, however, clear that the second method of passing from four to two dimensions enables us to discuss the matter in connexion with the comparatively simple three-dimensional figure instead of the complex plane figure. For the fifteen nodes and ten singular planes of a quartic surface which has fifteen nodes may be investigated by means of a perfectly symmetrical set of symbols and equations; their properties are directly connected with the plane figure on the one hand and with the simpler four-dimensional figure on the other, and may be developed with a very slight amount of labour. The effect of the sixteenth node on these and the symmetry of the system is better dealt with in space than in the plane; at the same time it does not depend upon space of four dimensions; (except in so far as the symmetrical system of equations for the quartic with fifteen nodes was suggested by considerations of an S_4); and so does not fall within the range of this paper.

That any tangent plane of a cubic surface, and the twenty-seven planes through the point of contact and the twenty-seven lines of the surface should form an absolutely symmetrical set of planes was once pointed out to me by Professor Cayley as a remarkable fact which must imply a series of quite unknown properties of the cubic surface. The same interesting fact appears in Segre's cubic variety, but I can suggest no explanation.

King's College,
Cambridge,
March 30th, 1899.

ON LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD AND FOURTH ORDERS IN WHOSE SOLUTIONS EXIST CERTAIN HOMOGENEOUS RELATIONS.

By D. F. Campbell, A.B., Ph.D.

IN *Acta Mathematica*, Tome XIV., 1890, is a Memoir by Professor Brioschi, entitled "Les invariants des équations différentielles linéaires," in which are considered in a number of special cases the question of homogeneous relations in the solutions of a linear differential equation. I propose to adopt the methods employed in this Memoir and to show that, in at least three cases, the reasoning can be extended with profit. The subject will be so treated that a previous knowledge of the Memoir is unnecessary. When my work coincides with that of Professor Brioschi's I indicate the fact by means of the symbol [B].

1. *Definitions.* In this work accents or exponents enclosed in brackets mean differentiation with respect to x. When the derivative is taken with respect to any other variable it is written in full.

An n-ary *form* of the m^{th} degree, $f(y_1, y_2, \ldots, y_n)$ is a rational integral homogeneous expression in the variables $y_1, y_2, \ldots, y_n$, with constant coefficients. A *form* equated to zero is spoken of as a *relation.*

If there exist p relations, $f_l(y_1, y_2, \ldots, y_n) = 0$, $(l = 1.2\ldots p)$, we say they are linearly dependent if $\sum_{l=1}^{l=p} C_l f_l(y_1, y_2, \ldots, y_n) \equiv 0$, where C_l are suitably chosen constants not all zero. We say they are linearly independent when C_l can not be chosen, not all zero, such that $\sum_{l=1}^{l=p} C_l f_l(y_1, y_2, \ldots, y_n) \equiv 0$.

2. Let

$$Y^{(n)} + nP_1 Y^{(n-1)} + \frac{n(n-1)}{2!} P_2 Y^{(n-2)} + \ldots + nP_{n-1} Y' + P_n Y = 0 \qquad \ldots\ldots\ldots (A)$$

be a linear differential equation of the n^{th} order.

The transformation

$$Y = e^{-\int P_1 dx} y$$

transforms it to the form

$$y^{(n)} + \frac{n(n-1)}{2!} p_2 y^{(n-2)} + \ldots + n p_{n-1} y' + p_n y = 0 \ldots (B),$$

where $p_2, \ldots, p_n$ are rational integral functions of $P_1, P_2, \ldots, P_n$ and derivatives of P_1.

The p's in the cases $n = 3$ and $n = 4$ are, in each case:

$$p_2 = P_2 - P_1^2 - P_1',$$

$$p_3 = P_3 - 3P_1P_2 + 2P_1^3 - P_1'';$$

in the case $n = 4$,

$$p_4 = P_4 - P_1P_3 - 6P_1'P_2 + 6P_1^2P_2 - 4P_1P_1'' + 12P_1'P_1^2 - 4P_1^4.$$

If equation (B) be transformed by means of the transformation

$$\left.\begin{array}{l} y = \rho . v \\ z = \phi(x) \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C),$$

where ρ is a function of x and v is a function of the new independent variable Z, and if ρ and z are connected by the equation

$$\frac{\rho'}{\rho} = -\frac{n-1}{2}\frac{z''}{z'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (D),$$

the transformed equation will have the form

$$\frac{d^n v}{dz^n} + \frac{n(n-1)}{2!} q_2 \frac{d^{n-2} v}{dz^{n-2}} + \ldots + n q_{n-1} \frac{dv}{dz} + q_n v = 0 \ldots (E).$$

[B]. There are thus $n-1$ equations connecting the q's with the p's, of which the first three are

$$q_2 z'^2 = p_2 + \frac{n+1}{2.3} R_2,$$

$$q_3 z'^3 = p_3 - 3p_2 Z + \frac{n+1}{4} R_3,$$

$$q_4 z'^4 = p_4 - 6p_3 Z + 9p_2 Z^2 + (n+5) p_2 R_2 + \frac{(n+1)(5n+7)}{3.4.5} R_2^2 + \frac{3(n+1)}{2.5} R_4,$$

where $\frac{z''}{z'} = Z, \; -Z' + \frac{1}{2}Z^2 = R_2, \; R_{r+1} = R_r' - rZR_r.$

These relations, together with (D), are the only ones needed in the subsequent work.

Differentiate the first with respect to x.

Therefore

$$\frac{dq_2}{dz} z'^3 = p_2' - 2p_2 Z + \frac{n+1}{2.3} R_3.$$

Multiply this result by $\frac{3}{2}$, and subtract from the second equation.

Therefore

$$\left(q_3 - \frac{3}{2}\frac{dq_2}{dz}\right) z'^3 = (p_3 - \tfrac{3}{2}p_2').$$

$a_3 = p_3 - \frac{3}{2}p_2'$ or $\alpha_3 = q_3 - \frac{3}{2}\frac{dq_2}{dz}$ has been named by Laguerre an *invariant.*

3. A rational integral function of the coefficients of a linear differential equation and their derivatives, such that when the same function is formed for the transformed differential equation, the two functions are equal, save as to a positive integral power of z', is called an *invariant.** An invariant is said to be of order r when its index is r.

The invariant a_3 has already been found to be

$$a_3 = p_3 - \tfrac{3}{2}p_2'.$$

The invariant a_4 can readily be calculated. It is

$$a_4 = p_4 - 2p_3' + \tfrac{6}{5}p_2'' - \tfrac{3}{5}.\frac{5n+7}{n+1} p_2^2.$$

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* A full treatment of the subject of invariants of linear differential equations is given by Forsyth in the *Philosophical Transactions of the Royal Society*, Vol. CLXXIX(A), 1888.

When expressed in terms of the coefficients of equation (A), they are

$$a_3 = P_3 - 3P_1P_2 + 2P_1^3 + \tfrac{1}{2}P_1'' - \tfrac{3}{2}P_2' + 3P_1P_1',$$

$$a_4 = P_4 - P_1P_3 - 2\left(\tfrac{9}{5}\right)^2 P_1'P_2 - 3\left(\tfrac{2}{5}\right)^2 P_1^2P_2$$

$$- \frac{2^5}{5} P_1P_1'' + 2\left(\tfrac{9}{5}\right)P_1'P_1^2 - \tfrac{19}{25}P_1^4 + \left(\tfrac{9}{5}\right)^2 P_2^2$$

$$- 2P_3' + 6P_1P_2' + \tfrac{4}{5}P_1''' + \tfrac{6}{5}P_2'' + \tfrac{21}{25}P_1'^2.$$

[B]. If $\alpha_r z'^r = a_r$ and $\alpha_s z'^s = a_s$ are two invariants of order r and s respectively, it can be easily seen that

$$A_{r,s} = \frac{12rs}{n+1} p_2 a_r a_s + \frac{(2r+1)(2s+1)}{r+s+1} a_r' a_s'$$

$$- \frac{r(2r+1)}{r+s+1} a_r a_s'' - \frac{s(2s+1)}{r+s+1} a_s a_r'',$$

and $$C_{r,s} = s a_s a_r' - r a_r a_s',$$

are also invariants and of order $r+s+2$ and $r+s+1$ respectively.

§ 4. Theorem. If there are g functions Y_i $(i = 1.2\ldots g)$ of an independent variable, among which ν and only ν linearly independent relations of the first degree exist, then

$$u = \Sigma a_i Y_i (i = 1.2\ldots g),$$

a_i arbitrary constants, is the general solution of a linear differential equation of order $g - v$.

The proof of this theorem is not difficult.

§ 5. Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of the linear differential equation (B). Let $u = f(y_1, y_2, \ldots, y_n)$ be a form of the mth degree $(m > 1)$ in these solutions. If equation (B) is of the second order, then no matter what the linearly independent solutions y_1 and y_2 may be, the coefficients of $u = f(y_1, y_2)$ cannot be so chosen, not all zero, that u is identically zero; for suppose $f(y_1, y_2) = 0$, all the coefficients not zero; $f(y_1, y_2)$ can be factored into m linear factors in y_1 and y_2. Now $f(y_1, y_2) = 0$ for all values of x, therefore one at least of these linear factors must vanish an infinite number of times in some range of values of x. Each factor is a solution of the linear differential equation, and it is a well-known theorem that if a solution of the linear differential equation vanishes an infinite number of times in a certain range of values of the independent variable, it vanishes identically. Therefore y_1 and y_2 are linearly dependent, contrary to hypothesis.

If, however, n is greater than 2, the coefficients $p_2, \ldots, p_n$ may be such that the coefficients in $u = f(y_1, y_2, \ldots, y_n)$ can be so chosen that u is identically zero. This can be seen by the following example: Let $y_1 = 1, y_2 = x, \ldots, y_n = x^{n-1}$, then $y_1, y_2, \ldots, y_n$ satisfy the linear differential equation

$$\frac{d^n y}{dx^n} = 0,$$

and in these solutions exist the quadratic relations

$$y_r^2 - y_{r-1} y_{r+1} = 0,$$

the cubic relations $y_r^3 - y_r y_{r-1} y_{r+1} = 0$, and so on.

8. THEOREM. If p and only p linearly independent relations of degree m exist in the solutions of the linear differential equation (B), then p and only p linearly independent relations of degree m exist in the solutions of equation (A).

Let $f_l(y_1, y_2, \ldots, y_n) = 0$ be a relation of degree m in the solutions of equation (B). Let $f_l(Y_1, Y_2, \ldots, Y_n)$ be the form in the solutions of (A) got by substituting $Y_i (i = 1.2 \ldots n)$ for $y_i (i = 1.2 \ldots n)$ in the form $f_l(y_1, y_2, \ldots, y_n)$. The equation connecting the two differential equations is $Y = e^{-\int P_1 dx} y$.

$$\text{Therefore } f_l(Y_1, Y_2, \ldots, Y_n) = e^{-m\int P_1 dx} f_l(y_1, y_2, \ldots, y_n),$$

and by means of this identity, since $e^{-m\int P_1 dx}$ cannot vanish, the theorem can be established immediately.

9. Let

$$f(y_1, y_2, \ldots, y_n) \equiv a_{1,1,1\cdots 1} y_1^m + m \sum_{i=2}^{i=n} a_{1,1\cdots 1,i} y_1^{m-1} y_i +$$

$$+ \frac{m(m-1)}{2!} \sum_{i=2}^{i=n} y_i \sum_{j=2}^{j=n} a_{1,1\cdots 1,i,j} y_1^{m-2} y_j$$

$$+ \frac{m(m-1)(m-2)}{3!} \sum_{i=2}^{i=n} y_i \sum_{j=2}^{j=n} y_j \sum_{k=2}^{k=n} a_{1,1\cdots 1,i,j,k} y_1^{m-3} y_k + \ldots$$

be a form of degree m, with arbitrary coefficients, in the solutions $y_1, y_2, \ldots, y_n$ of the linear differential equation

$$y^{(n)} + \frac{n(n-1)}{2!} p_2 y^{(n-2)} + \ldots + np_{n-1} y' + p_n y = 0.$$

In this form the number of subscripts attached to each a is m, and those a's are supposed equal which would be equal if the subscripts were interchanged in any desiredmanner, *e.g.*,

$$a_{1,1\cdots 1,i,j} = a_{1,1\, 1\cdots 1,j,i}\,.$$

Take the expression $(\sum_{i=1}^{i=n} \alpha_i y_i)^m$ or its equal

$$\alpha_1^m y_1^m + m \sum_{i=2}^{i=n} \alpha_1^{m-1} y_1^m y_i + \frac{m(m-1)}{2!} \alpha_1^{m-2} y_1^2 \sum_{i=2}^{i=n} \alpha_i y_i \sum_{j=2}^{j=n} \alpha_i y_j$$

$$+ \frac{m(m-1)(m-2)}{3!} \alpha_1^{m-3} y_1^3 \sum_{i=2}^{i=n} \alpha_i y_i \sum_{j=2}^{j=n} \alpha_j y_j \sum_{k=2}^{k=n} \alpha_k y_k + \ldots\ldots (1).$$

Comparing (1) with $f(y_1, y_2, \ldots, y_n)$ we see that any term in $f(y_1, y_2, \ldots, y_n)$ can be got from the corresponding term in (1) by writing, instead of the α's, an a with subscripts the same as the subscripts of the α's in that term, and that therefore any derivative of any term in $f(y_1, y_2, \ldots, y_n)$ can be got from the corresponding derivative of the corresponding term in (1) in the same way. Therefore

$$\frac{df(y_1, y_2, \ldots, y_n)}{dx} = m\,(\sum_{i=1}^{i=n} \alpha_i y_i)^{m-1} (\sum_{i=1}^{i=n} \alpha_i y_i')$$

symbolically, *i.e.*,

$\frac{df(y_1, y_2, \ldots, y_n)}{dx}$ can be got from $m\,(\sum_{i=1}^{i=n} \alpha_i y_i)\,(\sum_{i=1}^{i=n} \alpha_i y_i')$ by

multiplying the factors together and substituting the a's for the α's.

Similarly,

$$\frac{d^2 f(y_1, y_2, \ldots, y_n)}{dx^2} = m(m-1)\left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{m-2} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^2 + m\left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{m-1} \left(\sum_{i=1}^{i=n} \alpha_i y_i''\right),$$

and, in general,

$$\frac{d\left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^{r_1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(j)}\right)^{r_j} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)}\right)^{r_{n-1}}}{dx}$$

$$= \sum_{j=0}^{j=n-2} r_j \left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^{r_1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(j)}\right)^{r_j - 1}$$

$$\times \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(j+1)}\right)^{r_{j+1}+1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)}\right)^{r_{n-1}}$$

$$+ r_{n-1} \left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^{r_1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)}\right)^{r_{n-1}-1} \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n)}\right) \ldots (2).$$

Since y_i is a solution of the differential equation

$$y_i^{(n)} = -\frac{n(n-1)}{2!} p_2 y_i^{(n-2)} \ldots - n p_{n-1} y_i' - p_n y_i,$$

therefore

$$r_{n-1} \left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^{r_1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)}\right)^{r_{n-1}-1} \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n)}\right)$$

becomes

$$-r_{n-1}\left[\sum_{j=2}^{j=n} \frac{n(n-1)\ldots(n-j+1)\,p_j}{1.2\ldots j} \times \left(\sum_{i=1}^{i=n} \alpha_i y_i\right)^{r_0} \left(\sum_{i=1}^{i=n} \alpha_i y_i'\right)^{r_1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-j)}\right)^{r_{n-j}+1} \ldots \left(\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)}\right)^{r_{n-1}-1}\right].$$

It is evident that $\sum_{j=0}^{j=n-1} r_j = m,$

therefore $r_0 = m - r$, where $r = \sum_{j=1}^{j=n-1} r_j$.

In formula (2) the only term the sum of whose exponents, not including r_0, is greater than r is

$$r_0 (\sum_{i=1}^{i=n} \alpha_i y_i)^{r_0-1} (\sum_{i=1}^{i=n} \alpha_i y_i')^{r_1+1} (\sum_{i=1}^{i=n} \alpha_i y_i'')^{r_2} \ldots (\sum_{i=1}^{i=n} \alpha_i y_i^{(n-1)})^{r_{n-1}}.$$

When $r=m$, r_0 is zero and therefore this term is zero. Then each term is zero the sum of whose exponents, not including r_0, is greater than m.

I shall employ the following symbolic notation:—

I shall omit $(\sum_{i=1}^{i=n} \alpha_i y_i)^{r_0}$ because it can readily be supplied when the other factors are known. Instead of writing the other factors in full I shall indicate each by its power, being careful to keep them in the order

$$(\sum_{i=1}^{i=n} \alpha_i y_i^{(j)})^{r_j} \quad j=1.2.3\ldots n-1.$$

With this notation the results of this section can be stated in the following formulæ:

[B]. $f(y_1, y_2, \ldots, y_n) = [0, 0, 0, \ldots, 0]$,

$[r_1, r_2, \ldots, r_{n-1}] = 0$ when $r = r_1 + r_2 + \ldots + r_{n-1} > m$,

$$\frac{d[r_1, r_2, \ldots, r_{n-1}]}{dx} = \sum_{j=1}^{j=n-2} r_j [r_1 \ldots r_j - 1,\ r_{j+1} + 1 \ldots r_{n-1}]$$

$$+ (m-r)[r_1 + 1 \ldots r_{n-1}]$$

$$- r_{n-1} \sum_{j=2}^{j=n} \frac{n(n-1)\ldots(n-j+1)}{1.2\ldots j} p_j [r_1, r_2, \ldots, r_{n-j}+1 \ldots r_{n-1}-1] \ldots (3).$$

10. I shall now apply the foregoing theorems to linear differential equations of the third order.

Let $Y''' + 3P_1 Y'' + 3P_2 Y' + P_3 Y = 0 \quad (A)$

be such an equation, and

$$y''' + 3p_2 y' + p_3 y = 0 \quad (B)$$

the equation into which (A) transforms by means of the transformation $Y = e^{-\int P_1 dx} y$.

Let Y_1, Y_2, Y_3 be three linearly independent solutions of (A), and define y_1, y_2, y_3 by means of the equations

$$Y_i = e^{-\int P_1 dx} y_i \quad (i = 1.2.3).$$

Then y_1, y_2, y_3 are linearly independent solutions of equation (B).

Let $f(y_1, y_2, y_3) = u(x)$ be a form of degree m, with arbitrary coefficients, in the solutions y_1, y_2, y_3.

Formulæ (3) applied to this form are:

$$f(y_1, y_2, y_3) = [0.0],$$

$$[r_1, r_2] = 0, \quad r_1 + r_2 > m,$$

$$\frac{d[r_1, r_2]}{dx} = r_1[r_1 - 1, r_2 + 1] + (m - r_1 - r_2)[r_1 + 1, r_2]$$
$$- 3p_2 r_2 [r_1 + 1, r_2 - 1] - p_3 r_2 [r_1, r_2 - 1] \ldots\ldots\ldots (4).$$

11. Let the symbol $[2, 0]$ be denoted by λ.

I shall first prove a theorem necessary to the subsequent investigation. It is:—*The* λ *of any relation of degree* m $(m = 2$ or $3)$ *that may exist in the solutions of equation* (B) *cannot vanish identically.*

Let $f_1(y_1, y_2, y_3) = 0$ be a relation. Let h be the Hessian of $f_1(y_1, y_2, y_3)$. Let $f_1(y_1, y_2, y_3) = (\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3)^m = (\Sigma)^m$.

$$\Delta = \begin{vmatrix} y_1, & y_2, & y_3 \\ y_1', & y_2', & y_3' \\ y_1'', & y_2'', & y_3'' \end{vmatrix},$$

$$h = \begin{vmatrix} \alpha_1^2 (\Sigma)^{m-2}, & \alpha_1\alpha_2 (\Sigma)^{m-2}, & \alpha_1\alpha_3 (\Sigma)^{m-2} \\ \alpha_1\alpha_2 (\Sigma)^{m-2}, & \alpha_2^2 (\Sigma)^{m-2}, & \alpha_2\alpha_3 (\Sigma)^{m-2} \\ \alpha_1\alpha_3 (\Sigma)^{m-2}, & \alpha_2\alpha_3 (\Sigma)^{m-2}, & \alpha_3^2 (\Sigma)^{m-2} \end{vmatrix},$$

and by multiplication of determinants,

$$h\Delta^2 = \begin{vmatrix} (\Sigma)^m, & (\Sigma)^{m-1}(\Sigma)', & (\Sigma)^{m-1}(\Sigma)'' \\ (\Sigma)^{m-1}(\Sigma)', & (\Sigma)^{m-2}(\Sigma)'^2, & (\Sigma)^{m-2}(\Sigma)'(\Sigma)'' \\ (\Sigma)^{m-1}(\Sigma)'', & (\Sigma)^{m-2}(\Sigma)'(\Sigma)'', & (\Sigma)^{m-2}(\Sigma)''^2 \end{vmatrix},$$

$$= \begin{vmatrix} [0.0], & [1.0], & [0.1] \\ [1.0], & [2.0], & [1.1] \\ [0.1], & [1.1], & [0.2] \end{vmatrix}.$$

Formulæ (6) applied to this form give

$$[0.0]=0,\quad \frac{d[0.0]}{dx}=m[1.0]=0,\quad \frac{d[1.0]}{dx}=(m-1)\lambda+[0.1]=0,$$

therefore $[0.1]=-(m-1)\lambda$. Also $[1.1]=-\lambda'$, $[0.2]=3p_2\lambda$. Therefore $h\Delta^2=-(m-1)^2\lambda^3$.

Now Δ is constant and not zero, therefore the Hessian is a constant multiple of λ^3, therefore λ cannot vanish when the Hessian does not vanish.

In the case $m=2$ the Hessian of the form is the discriminant, and its vanishing would express the condition that $f_1(y_1, y_2, y_3)$ could be expressed as the product of two linear factors. Then, since $f_1(y_1, y_2, y_3)=0$, it can be shown as in § 5 that one of these factors must vanish identically, and this would violate the condition of linear independence of y_1, y_2, y_3.

Therefore in this case the Hessian cannot vanish, and therefore λ cannot vanish.

In the case $m=3$ the relation $f_1(y_1, y_2, y_3)=0$ can, by a linear transformation whose determinant is not zero, be transformed to the form

$$3u_3u_2^2+au_1^3+3bu_1^2u_3+3cu_1u_3^2+du_3^3=0.$$

The Hessian of this form is

$$h=\begin{vmatrix} au_1+bu_3, & 0, & bu_1+cu_3 \\ 0\quad, & u_3, & u_2 \\ bu_1+cu_3, & u_2, & cu_1+du_3 \end{vmatrix}.$$

Suppose $h=0$, then

$$(ae-b^2)\,u_1^2u_3+(ad-bc)\,u_1u_3^2-(au_1+bu_3)\,u_2^2+(bd-c^2)$$

Eliminate u_2 between these two equations, therefore

$$6acu_1^2u_3^2+4adu_1u_3^3+(4bd-3c^2)\,u_3^4+a^2u_1^4+4abu_1^3u_3=0.$$

Since this is a binary relation, and u_1 and u_3 are linearly independent, the coefficient of each term must be zero; therefore

$$6ac=0,$$

$$4ad=0,$$

$$4bd-3c^2=0,$$

$$a^2=0,$$

and $$4ab=0.$$

Since $a^3 = 0$, therefore $a = 0$. Then the given equation is

$$u_3 (3u_2^2 + 3bu_1^2 + 3cu_1u_3 + du_3^2) = 0.$$

Now $3u_2^2 + 3bu_1^2 + 3cu_1u_3 + du_3^2)$ is the product of two linear factors, since $(4bd - 3c^2) = 0$.

Then the left-hand member of the given equation breaks up into three linear factors, and therefore, as in § 5, one of these factors must vanish identically, and this violates the condition of linear independence of u_1, u_2, u_3 and therefore of y_1, y_2, y_3. Therefore the Hessian does not vanish identically and therefore λ does not.

12. Let $u = f(y_1, y_2, y_3)$ be a *quadratic* form in the solutions of equation (B) with arbitrary coefficients.

Apply formulæ (4); then

$$[0, 0] = u,$$

$$\frac{d[0, 0]}{dx} = 2[1.0].$$

Also $$\frac{d[0, 0]}{dx} = u'.$$

Therefore $$[1, 0] = \tfrac{1}{2}u',$$

$$\frac{d[1, 0]}{dx} = [0, 1] + \lambda.$$

Also $$\frac{d[1, 0]}{dx} = \tfrac{1}{2}u''.$$

Therefore $$[0, 1] = \tfrac{1}{2}u'' - \lambda,$$

$$\frac{d[0, 1]}{dx} = [1, 1] - 3p_2[1, 0] - p_3[0, 0].$$

Also $$\frac{d[0, 1]}{dx} = \tfrac{1}{2}u''' - \lambda'.$$

Therefore $$[1, 1] = \tfrac{1}{2}u''' + \tfrac{3}{2}p_2u' + p_3u - \lambda',$$

$$\frac{d[2, 0]}{dx} = 2[1, 1].$$

Also $$\frac{d[2, 0]}{dx} = \lambda'.$$

Therefore $[1, 1] = \frac{1}{2}\lambda'$.

Therefore $u''' + 3p_2u' + 2p_3u - 3\lambda' = 0$ ....(5),

$$\frac{d[1.1]}{dx} = [0, 2] - 3p_2\lambda - \tfrac{1}{2}p_3u'.$$

Also $\dfrac{d[1.1]}{dx} = \frac{1}{2}\lambda''$.

Therefore $[0.2] = \frac{1}{2}\lambda'' + 3p_2\lambda + \frac{1}{2}p_3u'$,

$$\frac{d[0, 2]}{dx} = -6p_2[1, 1] - 2p_3[0, 1],$$

$$= -3p_2\lambda' + 2p_3\lambda - p_3u'',$$

$$\frac{d[0.2]}{dx} = \tfrac{1}{2}\lambda''' + 3p_2\lambda' + 3p_2'\lambda + \tfrac{1}{2}p_3u'' + \tfrac{1}{2}p_3'u'.$$

Therefore $3p_3u'' + p_3'u' + \lambda''' + 12p_2\lambda' - 4a_3\lambda = 0$ ......(6),

where a_3 is the linear invariant of this differential equation.

There are thus two equations connecting λ and u from which λ can be eliminated. Differentiate (5), therefore

$$u^{iv} + 3p_2u'' + (3p_2' + 2p_3)u' + 2p_3'u - 3\lambda'' = 0 \text{...(7)}.$$

Differentiate (7), therefore

$$u^{v} + 3p_2u''' + (6p_2' + 2p_3)u'' + (3p_2'' + 4p_3')u' + 2p_3''u - 3\lambda''' = 0$$

......(8).

By means of equations (5) and (8), eliminate λ''' and λ' from (6), therefore

$$u^{v} + 15p_2u''' + (6p_2' + 11p_3)u'' + (3p_2'' + 7p_3' + 36p_2^2)u'$$
$$+ (24p_2p_3 + 2p_3'')u = 12a_3\lambda \text{...(9)}.$$

Equation (9) shows that, if a_3 vanishes, u satisfies a linear differential equation of the fifth order. Therefore $f(y_1, y_2, y_3)$, a form with six arbitrary constants, satisfies an equation of order less than 6. Therefore there is at least *one* quadratic relation $f_1(y_1, y_2, y_3) = 0$. Conversely, if there is a quadratic relation $f(y_1, y_2, y_3) = 0$, the u of this relation is zero and equation (9) shows that a_3 vanishes, since λ cannot.

Therefore a necessary and sufficient condition that at least one quadratic relation exists in the solutions of the given differential equation (B) is that the invariant a_3 vanishes identically.

To prove that there can be only one quadratic relation in the solutions of equation (B).

Let $f_1(y_1, y_2, y_3) = 0$ be a relation. Then equation (5) shows that the λ of this relation must satisfy the equation $\lambda' = 0$, therefore λ is constant $= C_1$ (suppose).

Suppose there exists another quadratic relation,

$$f_2(y_1, y_2, y_3) = 0.$$

Then the λ of this relation must satisfy the equation $\lambda' = 0$, therefore $\lambda = C_2$ (suppose).

Then the quadratic relation

$$C_2 f_1(y_1, y_2, y_3) - C_1 f_2(y_1, y_2, y_3) = 0$$

has its λ equal to zero, therefore

$$C_2 f_1(y_1, y_2, y_3) - C_1 f_2(y_1, y_2, y_3) \equiv 0,$$

and the two relations are linearly dependent.

Therefore no more than one quadratic relation can exist in the solutions of the linear differential equation (B).

Then, by aid of § 8, the following theorems can be enunciated concerning a linear differential equation in the most general form (A).

No more than one quadratic relation can exist in its solutions.

A necessary and sufficient condition for the existence of a quadratic relation in its solutions is that the invariant a_3 vanishes.

13. When a_3 vanishes there is one and only one quadratic relation in the solutions of equation (B).

Therefore in this case a quadratic form with arbitrary coefficients in the solutions of equation (B) is the general solution of a linear differential equation of order five (v. § 4). This equation is seen from (9) to be:

$$u^{v} + 15p_2 u''' + (6p_2' + 11p_3)\, u'' + (3p_2'' + 7p_3' + 36p_2^2)\, u'$$
$$+ (24p_2 p_3 + 2p_3'')\, u = 0 \ldots\ldots\ldots (10).$$

When a_3 does not vanish, no quadratic relation exists in the solutions of the given differential equation (B). Therefore in this case a quadratic form, with arbitrary coefficients in the solutions of equation (B), is the general solution of a linear differential equation of order six. It is found by combining (9) and (5) to be

$$u^{vi} - \frac{a_3'}{a_3} u^{v} + 15p_2 u^{iv} + (21p_2' + 11p_3 - 15\frac{a_3'}{a_3}p_2 - 4a_3) u'''$$

$$+ [9p_2'' + 18p_3' + 36p_2^2 - \frac{a_3'}{a_3}(6p_2' + 11p_3)] u'' + [3p_2''' + 9p_3''$$

$$+ 72p_2p_2' + 24p_2p_3 - \frac{a_3'}{a_3}(3p_2'' + 7p_3' + 36p_2^2 - 12a_3p_2)] u'$$

$$+ [24p_2p_3' + 24p_2'p_3 + 2p_3''' - 8a_3p_3 - \frac{a_3'}{a_3}(24p_2p_3 + 2p_3'')] u = 0$$

$$\ldots\ldots\ldots\ldots(11).$$

Let $U = f(Y_1, Y_2, Y_3)$, where Y_1, Y_2, Y_3 are solutions of equation (A). Then $u = e^{2\int P_1 dx} U$.

When this transformation is made in equations (10) and (11), the resulting equations is, in each case, the differential equation of which U is the general solution. These resulting equations give the following theorems:

When the invariant a_3 of a linear differential equation of the third order $\left\{\begin{matrix}\text{vanishes} \\ \text{does not vanish}\end{matrix}\right\}$, a quadratic form, with arbitrary coefficients in the solutions of this differential equation, is the general solution of a linear differential equation of order $\left\{\begin{matrix}\text{five} \\ \text{six}\end{matrix}\right\}$, the coefficients of which are rational functions of the coefficients of the given differential equation and their derivatives.

14. Let $u = f(y_1, y_2, y_3)$ be a form of the third degree, with arbitrary coefficients, in the solutions of the linear differential equation (B).

Apply formulæ (4) to this form.

Therefore $$u = [0.0],$$

$$\frac{d[0.0]}{dx} = 3\,[1.0].$$

Also $$\frac{d[0.0]}{dx} = u'.$$

Therefore $$[1.0] = \tfrac{1}{3}u',$$

$$\frac{d[1.0]}{dx} = [0.1] + 2\lambda.$$

Also $$\frac{d[1.0]}{dx} = \tfrac{1}{3}u''.$$

Therefore $$[0.1] = \tfrac{1}{3}u'' - 2\lambda,$$

$$\frac{d[0.1]}{dx} = 2[1.1] - 3p_2[1.0] - p_3[0.0]$$

$$= 2[1.1] - p_2u' - p_3u.$$

Also $$\frac{d[0.1]}{dx} = \tfrac{1}{3}u''' - 2\lambda'.$$

Therefore $$[1.1] = \tfrac{1}{6}u''' + \tfrac{1}{2}p_2u' + \tfrac{1}{2}p_3u - \lambda',$$

$$\frac{d[2.0]}{dx} = 2[1.1] + [3.0].$$

Also $$\frac{d[2.0]}{dx} = \lambda'.$$

Therefore $$[3.0] = 3\lambda' - \tfrac{1}{3}u''' - p_2u' - p_3u,$$

$$\frac{d[3.0]}{dx} = 3[2.1].$$

Also

$$\frac{d[3.0]}{dx} = 3\lambda'' - \tfrac{1}{3}u^{iv} - p_2u'' - (p_2' + p_3)u' + p_3'u.$$

Therefore

$$[2.1] = \lambda'' - \tfrac{1}{9}u^{iv} - \tfrac{1}{3}p_2u'' - \tfrac{1}{3}(p_2' + p_3)u' - \tfrac{1}{3}p_3'u,$$

$$\frac{d[1.1]}{dx} = [0.2] + [2.1] - 3p_2\lambda - p_3[1.0],$$

$$= [0.2] - \tfrac{1}{9}u^{iv} - \tfrac{1}{3}p_2u'' - \tfrac{1}{3}(p_2' + 2p_3)u' - \tfrac{1}{3}p_3'u + \lambda'' - 3p_2\lambda.$$

Also

$$\frac{d[1.1]}{dx} = \tfrac{1}{6}u^{v} + \tfrac{1}{2}p_2u'' + \tfrac{1}{2}(p_2' + p_3)u' + \tfrac{1}{2}p_3'u - \lambda''.$$

Therefore

$$[0.2] = \tfrac{5}{18}u^{iv} + \tfrac{5}{6}p_2u'' + (\tfrac{5}{6}p_2' + \tfrac{7}{6}p_3)u' + \tfrac{5}{6}p_3'u - 2\lambda'' + 3p_2\lambda.$$

$$\frac{d[0.2]}{dx} = [1.2] - 6p_2[1.1] - 2p_3[0.1]$$

$$= [1.2] - p_2u''' - \tfrac{2}{3}p_3u'' - 3p_2^2u' - 3p_2p_3u + 6p_2\lambda' + 4p_3\lambda.$$

Also

$$\frac{d[0.2]}{dx} = \tfrac{5}{18}u^{v} + \tfrac{5}{6}p_2u''' + (\tfrac{5}{3}p_2' + \tfrac{7}{6}p_3)u'' + (\tfrac{5}{6}p_2'' + 2p_3')u'$$
$$+ \tfrac{5}{6}p_3''u - 2\lambda''' + 3p_2\lambda' + 3p_2'\lambda.$$

Therefore

$$2[1.2] = \tfrac{5}{9}u^{v} + \tfrac{11}{3}p_2u''' + (\tfrac{10}{3}p_2' + \tfrac{11}{3}p_3)u'' + (\tfrac{5}{3}p_2'' + 4p_3' + 6p_2^2)u$$
$$+ (\tfrac{5}{3}p_3'' + 6p_2p_3)u - 4\lambda''' - 6p_2\lambda' + (6p_2' - 8p_3)\lambda.$$

$$\frac{d[2.1]}{dx} = 2[1.2] - 3p_2[3.0] - p_3[2.0]$$

$$= 2[1.2] + p_2u'''' + 3p_2^2u' + 3p_2p_3u - 9p_2\lambda' - p_3\lambda.$$

Also

$$\frac{d[2.1]}{dx} = -\tfrac{1}{9}u^{v} - \tfrac{1}{3}p_2u''' - (\tfrac{2}{3}p_2' + \tfrac{1}{3}p_3)u'' - (\tfrac{1}{3}p_2'' + \tfrac{2}{3}p_3')u'$$
$$- \tfrac{1}{3}p_3''u + \lambda'''.$$

Therefore

$$2[1.2] = -\tfrac{1}{9}u^{v} - \tfrac{4}{3}p_2u''' - \tfrac{1}{3}(2p_2' + p_3)u'' - \tfrac{1}{3}(p_2'' + 2p_3' + 9p_2^2)u'$$
$$- (\tfrac{1}{3}p_3'' + 3p_2p_3)u + \lambda''' + 9p_2\lambda' + p_3\lambda.$$

Therefore

$$\lambda''' + 3p_2\lambda' + \tfrac{3}{5}(3p_3 - 2p_2')\lambda = \tfrac{2}{15}u^{v} + p_2u''' + \tfrac{4}{5}(p_2' + p_3)u''$$
$$+ \tfrac{1}{5}(2p_2'' + 14p_3' + 9p_2^2)u'$$
$$+ \tfrac{1}{5}(2p_3'' + 9p_2p_3)u \text{ ....} (12).$$

By combination with (12),

$$[1.2]=\tfrac{1}{90}u^{\mathrm{v}}-\tfrac{1}{6}p_2u'''+(\tfrac{1}{15}p_2'+\tfrac{7}{30}p_3)u''+(\tfrac{1}{30}p_2''+\tfrac{2}{15}p_3'-\tfrac{3}{5}p_2^2)u'$$
$$+(\tfrac{1}{30}p_3''-\tfrac{3}{5}p_2p_3)u+3p_2\lambda'-\tfrac{2}{5}a_3\lambda.$$

$$\frac{d[1.2]}{dx}=[0.3]-6p_2[2.1]-2p_3[1.1]$$
$$=[0.3]+\tfrac{2}{3}p_2u^{\mathrm{iv}}-\tfrac{1}{3}p_3u'''+2p_2^2u''+(2p_2p_2'+p_2p_3)u'$$
$$+(2p_2p_3'-p_3^2)u-6p_2\lambda''+2p_3\lambda'.$$

Also

$$\frac{d[1.2]}{dx}=\tfrac{1}{90}u^{\mathrm{vi}}-\tfrac{1}{6}p_2u^{\mathrm{iv}}-(\tfrac{1}{10}p_2'-\tfrac{7}{30}p_3)u'''+(\tfrac{1}{10}p_2''+\tfrac{11}{30}p_3'-\tfrac{3}{5}p_2^2)u''$$
$$+(\tfrac{1}{30}p_3'''+\tfrac{1}{6}p''-\tfrac{6}{5}p_2p_2'-\tfrac{3}{5}p_2p_3)u'$$
$$+(\tfrac{1}{30}p_3'''-\tfrac{3}{5}p_2'p_3-\tfrac{3}{5}p_2p_3')u+3p_2\lambda''+(3p_2'-\tfrac{2}{5}a_3)\lambda'-\tfrac{2}{5}a_3'\lambda.$$

Therefore

$$[0.3]=\tfrac{1}{90}u^{\mathrm{vi}}-\tfrac{5}{6}p_2u^{\mathrm{iv}}-(\tfrac{1}{10}p_2'-\tfrac{17}{30}p_3)u'''+(\tfrac{1}{10}p_2''+\tfrac{11}{30}p_3'-\tfrac{13}{5}p_2^2)u'$$
$$+(\tfrac{1}{30}p_2'''+\tfrac{1}{6}p_3''-\tfrac{16}{5}p_2p_2'-\tfrac{8}{5}p_2p_3)u'$$
$$+(\tfrac{1}{30}p_3'''-\tfrac{3}{5}p_2'p_3-\tfrac{13}{5}p_2p_3'+p_3^2)u+9p_2\lambda''-\tfrac{12}{5}a_3\lambda'-\tfrac{2}{5}a_3'\lambda,$$

$$\frac{d[0.3]}{dx}=-9p_2[1.2]-3p_3[0.2]$$
$$=-\tfrac{1}{10}p_2u^{\mathrm{v}}-\tfrac{5}{6}p_3u^{\mathrm{iv}}+\tfrac{3}{2}p_2^2u'''-\tfrac{3}{5}(p_2p_2'+\tfrac{23}{5}p_2p_3)u''$$
$$-(\tfrac{3}{10}p_2p_2''+\tfrac{6}{5}p_2p_3'-\tfrac{27}{5}p_2^3+\tfrac{5}{2}p_2'p_3+\tfrac{7}{2}p_3^2)u'$$
$$-(\tfrac{5}{2}p_3p_3'+\tfrac{3}{10}p_2p_3''-\tfrac{27}{5}p_2^2p_3)u+6p_2\lambda''-27p_2^2\lambda'$$
$$-(9p_2p_3-\tfrac{18}{5}a_3p_2)\lambda.$$

Also

$$\frac{d[0.3]}{dx}\tfrac{1}{90}u^{\mathrm{vii}}-\tfrac{5}{6}p_2u^{\mathrm{v}}-(\tfrac{14}{15}p_2'-\tfrac{17}{30}p_3)u^{\mathrm{iv}}$$
$$+(\tfrac{14}{15}p_3'-\tfrac{13}{5}p_2^2)u'''+(\tfrac{2}{15}p_2'''+\tfrac{8}{15}p_3''-\tfrac{42}{5}p_2p_2'-\tfrac{8}{5}p_2p_3)u''$$
$$+(\tfrac{1}{30}p_2^{\mathrm{iv}}+\tfrac{1}{5}p_3'''-\tfrac{16}{5}p_2p_2''-\tfrac{16}{5}p_2'^2-\tfrac{21}{5}p_2p_3'-\tfrac{11}{5}p_2'p_3+p_3^2)u'$$
$$+(\tfrac{1}{30}p_3^{\mathrm{iv}}-\tfrac{3}{5}p_2''p_3-\tfrac{16}{5}p_2'p_3'-\tfrac{13}{5}p_2p_3''+2p_3p_3')u$$
$$+9p_2\lambda'''+(9p_2'-\tfrac{12}{5}a_3)\lambda''-\tfrac{14}{5}a_3'\lambda'-\tfrac{2}{5}a_3''\lambda.$$

Equate the values of $\dfrac{d\,[0.3]}{dx}$, making use of (12).

Therefore $42a_3\lambda'' + 14a_3'\lambda' + (2a_3'' + 54p_2a_3)\lambda$

$$= \tfrac{1}{18}u^{v'''} + \tfrac{7}{3}p_2u^{v} - (\tfrac{14}{3}p_2' - \tfrac{4}{3}p_3)\,u^{iv} + (\tfrac{14}{3}p_3' + \tfrac{39}{2}p_2^2)\,u'''$$
$$+ (\tfrac{2}{3}p_2''' + \tfrac{8}{3}p_2'' - 3p_2p_2' + 51p_2p_3)\,u'' + (\tfrac{1}{6}p_2^{iv} + p_3''' + \tfrac{7}{2}p_2p_2''$$
$$- 16p_2'^2 + 27p_2p_3' + 54p_2^3 + \tfrac{45}{2}p_3^2 + \tfrac{3}{2}p_2'p_3)\,u'$$
$$+ (\tfrac{1}{6}p_3^{iv} - 3p''p_3 - 16p_2'p_3' + 54p_2^2p_3 + \tfrac{13}{2}p_2p_3'' + \tfrac{45}{2}p_3p_3')\,u \quad \ldots\ldots\ldots\ldots(13).$$

For brevity write (12) and (13),

$$\lambda''' + 3p_2\lambda' + \tfrac{3}{5}(3p_3 - 2p_2')\,\lambda = A_5 \quad \ldots\ldots(12),$$
$$42a_3\lambda'' + 14a_3'\lambda' + (2a_3'' + 54p_2a_3)\,\lambda = 2B_7 \ldots\ldots(13).$$

Differentiate (13).
Therefore

$$21a_3\lambda''' + 28a_3'\lambda'' + (8a_3'' + 27p_2a_3)\,\lambda' + (a_3''' + 27p_2'a_3 + 27p_2a_3')\,\lambda = B_7' \ldots\ldots(14).$$

Multiply (12) by $21a_3$, and subtract from (14).
Therefore

$$28a_3'\lambda'' + (8a_3'' - 36p_2a_3)\,\lambda' + (a_3''' + 27p_2a_3' - \tfrac{189}{5}p_3a_3 + \tfrac{261}{5}p_2'a_3)\,\lambda = C_8 \ldots\ldots(15).$$

Let $b_8 = 27p_2a_3^2 + 7a_3'^2 - 6a_3a_3''$. Then b_8 is invariant (v. § 3). Subtract (13) multiplied by $2a_3'$ from (15) multiplied by $3a_3$.

Therefore $\quad 4b_8\lambda' + \left(\tfrac{1}{2}b_8' + \dfrac{3^4.7}{5}a_3^3\right)\lambda = D_8 \ldots\ldots\ldots\ldots(16).$

Differentiate (16).
Therefore

$$4b_8\lambda'' + \left(\tfrac{9}{2}b_8' + \frac{3^4.7}{5}a_3^3\right)\lambda' + \left(\tfrac{1}{2}b_8'' + \frac{3^5.7}{5}a_3^2a_3'\right)\lambda = D_8' \ldots\ldots(17).$$

Subtract (13) multiplied by $4b_8$ from (17) multiplied by $21a_3$.

Therefore

$$\left(\tfrac{189}{2}a_3b_8' - 28a_3'b_8 + \frac{3^5.7^2}{5}a_3^4\right)\lambda' + \left(\tfrac{21}{2}a_3b_8'' + \frac{3^6.7^2}{5}a_3^3a_3'\right.$$

$$\left. - 4a_3''b_8 - 108p_2a_3b_8\right)\lambda = E_9 \;\ldots\ldots\ldots(18).$$

Let $$C_{8,3} = 8b_8a_3' - 3a_3b_8',$$

$$p_{3,8} = 72p_2a_3b_8 + \frac{7.17}{12}a_3'b_8' - \tfrac{7}{4}a_3b_8'' - \frac{2.17}{3}b_8a_3'',$$

$$q_{8,8} = 192p_2b_8^2 + 17b_8'^2 - 16b_8b_8''.$$

$C_{8,3}$, $p_{3,8}$, $q_{8,8}$ are invariants of order 12, 13, 18 respectively (v. § 3).

Subtract (16) multiplied by the coefficient of λ' in (18) from (18) multiplied by $4b_8$.

Therefore $$\Omega_{21}\lambda = F_9 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

where

$$\Omega_{21} = \tfrac{1}{17}(32b_8p_{3,8} - 63a_3q_{8,8}) + 4.7^2.3^3a_3^3C_{8,3} - \frac{4.7^3.3^8}{5^2}a_3^7.$$

Differentiate (19).

Therefore $$\Omega_{21}\lambda' + \Omega_{21}'\lambda = F_9' \;\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Combine (20), (19), and (16).

Therefore

$$4b_8\Omega_{21}F_9' - \left(4b_8\Omega_{21}' - \tfrac{1}{2}b_8'\Omega_{21} - \frac{3^4.7}{5}a_3^3\Omega_{21}\right)F_9 - \Omega_{21}^2D_8 = 0 \;(21).$$

This is a differential equation of the tenth order satisfied by u.

It is to be noted that if any one of the functions A_5, B_7, B_7', C_8, D_8, D_8', E_9, F_9, F_9' were equated to zero, the resulting equation would be a homogeneous linear differential equation in u, with coefficients rational integral functions of the coefficients of the differential equation (B) and their derivatives.

15. The equations (12) to (21) hold when the coefficients of $u = f(y_1, y_2, y_3)$ are arbitrary. They therefore hold if the coefficients can be so chosen that u is zero. Then the λ of any relation of the third degree that can exist in the solutions of equation (B) must be a solution of the equations

$$\lambda''' + 3p_2\lambda' + \tfrac{9}{5}(a_3 + \tfrac{5}{6}p_2')\lambda = 0 \quad \text{.........(22)},$$

$$21a_3\lambda'' + 7a_3'\lambda' + (a_3'' + 27p_2a_3)\lambda = 0 \quad \text{......(23)},$$

$$4b_8\lambda' + \left(\tfrac{1}{2}b_8' + \frac{3^4.7}{5}a_3^{\,3}\right)\lambda = 0 \quad \text{...........(24)},$$

$$\Omega_{21}\lambda = 0 \quad \text{......................(25)}.$$

Also the λ's of linearly independent relations of the third degree in the solutions of equation (B) must be linearly independent; for, suppose $f_l(y_1, y_2, y_3) = 0$ $(l = 1.2\ldots p)$ are p linearly independent relations of the third degree in the solutions of equation (B). Let their λ's be λ_l $(l = 1.2\ldots p)$. If $\sum_{l=1}^{l=p} C_l\lambda_l \equiv 0$, C_l constant, not all zero, then $\sum_{l=1}^{l=p} C_l f_l(y_1, y_2, y_3) = 0$, which is a relation of the third degree, gives rise to the zero λ, $\sum_{l=1}^{l=p} C_l\lambda_l$. Then its coefficients must all be zero. Therefore $f_l(y_1, y_2, y_3) = 0$ $(l = 1.2\ldots p)$ are linearly dependent, contrary to hypothesis.

Therefore the λ's of linearly independent relations of third degree in the solutions of equation (B) are linearly independent.

Suppose the invariant a_3 vanishes. Then $u = f(y_1, y_2, y_3)$ is a solution of the linear differential equation, $B_7 = 0$ (v. equation 13). Then there must be at least three linearly independent relations of the third degree in the solutions of equation (B); for, if there were less than three linearly independent relations, it would be the general solution of a linear differential equation of order higher than seven (v. § 4), and therefore could not be a solution of a linear differential equation of order seven. Conversely, if there are three linearly independent relations of the third degree in the solutions of equation (B), then a_3 vanishes; for, the λ's of linearly independent relations of the third degree are linearly independent, and each must satisfy equation (23). Then the coefficients of equation (23) must be zero, therefore a_3 vanishes.

Therefore a necessary and sufficient condition that three linearly independent relations of the third degree exist in the solutions of equation (B) is that the invariant a_3 vanishes.

There cannot exist more than three linearly independent relations of the third degree in the solutions of equation (*B*); for the coefficients of equation (22) are not identically zero, and therefore there cannot exist more than three linearly independent λ's. Then, since the λ's of linearly independent relations are linearly independent, there cannot exist more than three linearly independent relations.

Next, suppose b_8 vanishes, but a_3 does not vanish; then equation (16) becomes

$$\frac{3^4.7}{5} a_3^3\lambda = D_8.$$

Under this supposition there can be no relation of the third degree in the solutions of equation (*B*), because any relation would have to give a λ which satisfies the equation, $\frac{3^4.7}{5} a_3^3\lambda = 0$, and the only value that can satisfy this equation is $\lambda = 0$.

Next, suppose Ω_{21} vanishes but a_3 does not. Then u is a solution of the linear differential equation of order 9, $F_9 = 0$ (v. equation 19). But u is a form with ten arbitrary coefficients. Therefore there is at least one relation in the solutions of equation (*B*). There cannot exist more than one, because the λ's of linearly independent relations are linearly independent and each would have to be a solution of equation (24), where b_5 does not vanish.

Conversely, if there exists one, and only one, relation of the third degree in the solutions of equation (*B*), then Ω_{21} vanishes but a_3 does not. For the λ of any relation of the third degree in the solutions of (*B*) must satisfy equation (27). Therefore Ω_{21} vanishes. a_3 cannot vanish, for if it did there would be three linearly independent relations.

Therefore a necessary and sufficient condition that one and only one relation of the third degree exists in the solutions of equation (B) is that Ω_{21} vanishes but a_3 does not vanish.

Then, by the aid of the theorems of § 8, the following theorems can be enunciated concerning a linear differential equation in its most general form (*A*).

No more than three linearly independent relations of the third degree can exist in its solutions.

Two, and only two linearly independent relations of the third degree cannot exist in its solutions.

A necessary and sufficient condition that three linearly independent relations exist in its solutions is that the invariant a_3 *vanishes.*

A necessary and sufficient condition that one, and only one, relation of the third degree exists in its solutions is that the invariant Ω_{21} *vanishes but that the invariant* a_3 *does not vanish.*

16. The equation of which $u = f(y_1, y_2, y_3)$ is the general solution in each of the cases (first, when a_3 vanishes; second, when Ω_{21} vanishes and a_3 does not; third, when Ω_{21} does not vanish) can be found as in § 13.

Let $$U = f(Y_1, Y_2, Y_3) = e^{-\int P_1 dx} u.$$

Make the substitution in these equations and there result the equations of which U is the general solution.

From these equations the following theorems immediately present themselves.

In a linear differential equation of the third order when the invariant $\left\{\begin{array}{l} a_3 \text{ vanishes} \\ \Omega_{21} \text{ vanishes and } b_8 \text{ does not} \\ \Omega_{21} \text{ does not vanish} \end{array}\right\}$, *a form of the third degree in its solutions, with arbitrary coefficients, is the general solution of a linear differential equation of order* $\left\{\begin{array}{c} \text{seven} \\ \text{nine} \\ \text{ten} \end{array}\right\}$, *the coefficients of which are rational functions of the coefficients of the given differential equation and their derivatives.*

17. [B]. Suppose Ω_{21} vanishes. Then there is at least one relation of the third degree in the solutions of equation (B). Let λ be the λ of a relation.

It must be a solution of equations (22) to (25), and in particular of

$$\lambda''' + 3p_2\lambda' + \tfrac{3}{5}(3p_3 - 2p_2')\lambda = 0 \quad \text{........(22)}.$$

Replace ρ in transformation (C), § 2 by this λ; then

$$Z = -\frac{\lambda'}{\lambda} = \frac{z''}{z'}.$$

Substitute in (22), therefore

$$R_3 - 3p_2 Z + \tfrac{3}{5}(3p_3 - 2p_2') = 0 \quad \text{........(26)}.$$

The first equation connecting the p's and q's in § 2 is

$$q_2 z'^2 = p_2 + \tfrac{2}{3}(-Z' + \tfrac{1}{2}Z^2) \quad \text{...........(27)}.$$

The second is

$$q_3 z'^3 - p_3 + 3p_2 Z - R_3 = 0 \quad \text{...........(28)}.$$

Combine equations (26) and (28), therefore

$$q_3 z'^3 + \tfrac{4}{5}(p_3 - \tfrac{3}{2}p_2') = 0.$$

Therefore $$q_3 z'^3 + \tfrac{4}{5}\left(q_3 - \tfrac{3}{2}\frac{dq_2}{dz}\right) z'^3 = 0.$$

Therefore $$q_3 = \tfrac{2}{3}\frac{dq_2}{dz}.$$

Then the given differential equation (B) is transformed to the form

$$\frac{d^3 v}{dz^3} + 3q_2 \frac{dv}{dz} + \tfrac{2}{3}\frac{dq_2}{dz} v = 0 \quad \text{.............(29)}.$$

When a_3 vanishes there are three relations of the third degree, or one of the second degree, in the solutions of equation (B). In this case (B) is of the form

$$y''' + 3p_2 y' + \tfrac{3}{2}p_2' y = 0,$$

and it is known* that it can be transformed to the form $\dfrac{d^3 v}{dz^3} = 0$ by means of the transformation

$$y = \xi^2 . v, \quad z = \int \frac{dx}{\xi^2},$$

where ξ is a solution of the equation

$$\xi'' + \tfrac{3}{4}p_2 \xi = 0,$$

* Schlesinger's, *Handbuch der linearen Differential gleichungen*, Vol. II., § 180.

and the general solution is a binary form of the second degree in ξ_1 and ξ_2, where ξ_1 and ξ_2 are linearly independent solutions of the above differential equation in ξ.

When Ω_{21} vanishes, and a_3 does not, the equation

$$4b_8\lambda' + \left(\tfrac{1}{2}b_8' + \frac{3^4.7}{5}a_3^3\right)\lambda = 0 \text{ ....(24)}$$

gives a value of λ, and corresponding to it is a relation of the third degree in the solutions of equation (B). Choose this λ as ρ in transformation (C), § 2. Then the equation is transformed to the form (29), where q_3 is determined from equation (27).

18. Let

$$Y^{iv} + 4P_1Y''' + 6P_2Y'' + 4P_3Y' + P_4y = 0 \quad (A)$$

be a linear differential equation of the fourth order, and let

$$y^{iv} + 6p_2y'' + 4p_3y' + p_4y = 0 \quad (B)$$

be the linear differential equation into which (A) transforms by means of the transformation

$$Y = e^{-\int P_1 dx}.y.$$

Let four linearly independent solutions of (A) be Y_i ($i = 1.2.3.4$), and define y_i ($i = 1.2.3.4$) by means of the equation $Y_i = e^{-\int P_1 dx}.y_i$. Then y_i ($i = 1.2.3.4$) are linearly independent solutions of equation (B).

Let $u = f(y_1, y_2, y_3, y_4)$ be a *quadratic* form in the solutions of equation (B) with arbitrary coefficients.

Formulæ (4) in this case are:

$$u = [0.0.0],$$

$$[r_1, r_2, r_3] = 0, \quad r = r_1 + r_2 + r_3 > 2,$$

$$\frac{d[r_1, r_2, r_3]}{dx} = r_1[r_1 - 1, r_2 + 1, r_3] + r_2[r_1, r_2 - 1, r_3 + 1]$$
$$+ (2 - r)[r_1 + 1, r_2, r_3] - 6r_3p_2[r_1, r_2 + 1, r_3 - 1]$$
$$- 4r_3p_3[r_1 + 1, r_2, r_3 - 1] - r_3p_4[r_1, r_2, r_3 - 1].$$

19. Let the symbol [2.0.0] be denoted by λ. I shall first prove that the λ of any quadratic relation that can exist in the solutions of equation (B) cannot vanish identically.

Suppose there exists a relation $f(y_1, y_2, y_3, y_4) = 0$.

Let $f_1(y_1, y_2, y_3, y_4) = (\sum_{i=1}^{i=4} \beta_i y_i)^2 = (\Sigma)^2$. Let h be the Hessian of $(\Sigma)^2$ and let

$$\Delta = \begin{vmatrix} y_1, & y_2, & y_3, & y_4 \\ y_1', & y_2', & y_3', & y_4' \\ y_1'', & y_2'', & y_3'', & y_4'' \\ y_1''', & y_2''', & y_3''', & y_4''' \end{vmatrix},$$

Then

$$h\Delta^2 = \begin{vmatrix} (\Sigma)^2, & (\Sigma)(\Sigma)', & (\Sigma)(\Sigma)'', & (\Sigma)(\Sigma)''' \\ (\Sigma)(\Sigma)', & (\Sigma)'^2, & (\Sigma)'(\Sigma)'', & (\Sigma)'(\Sigma)''' \\ (\Sigma)(\Sigma)'', & (\Sigma)'(\Sigma)'', & (\Sigma)''^2, & (\Sigma'')(\Sigma''') \\ (\Sigma)(\Sigma)''', & (\Sigma)'(\Sigma)''', & (\Sigma)''(\Sigma)''', & (\Sigma)'''^2 \end{vmatrix},$$

$$= \begin{vmatrix} [0.0.0], & [1.0.0], & [0.1.0], & [0.0.1] \\ [1.0.0], & \lambda, & [1.1.0], & [1.0.1] \\ [0.1.0], & [1.1.0], & [0.2.0], & [0.1.1] \\ [0.0.1], & [1.0.1], & [0.1.1], & [0.0.2] \end{vmatrix}.$$

Formulæ (4) applied to $(\Sigma)^2$ gives

$$[0.0.0] = 0,\ [1.0.0] = 0,\ [0.1.0] = -\lambda,\ [0.0.1] = -\tfrac{3}{2}\lambda',$$

$$[1.1.0] = \tfrac{1}{2}\lambda',\ [1.0.1] = -\tfrac{3}{2}\lambda'' - 6p_2\lambda,\ [0.2.0] = 2\lambda'' + 6p_2\lambda,$$

$$[0.1.1] = \tfrac{3}{5}p_2\lambda' + (\tfrac{8}{5}p_3 - \tfrac{3}{5}p_2')\lambda,$$

$$[0.0.2] = \tfrac{63}{5}p_2\lambda'' + \tfrac{18}{5}p_3\lambda' + (\tfrac{8}{5}p_3' + \tfrac{3}{5}p_2'' + 36p_2^2 - p_4)\lambda.$$

These investigations show that λ, or one of its derivatives, enters into each term of $h\Delta^2$. Now Δ is constant and not zero. Therefore λ does not vanish when the Hessian does not vanish. It remains to consider the case where the Hessian or (since the form is quadratic) the discriminant of $(\Sigma)^2$ vanishes.

From the theory of quadratic forms with zero discriminant it is known that $(\Sigma)^2 = \nu_1^2 - 2\nu_2\nu_3$ by virtue of a certain transformation

$$\left.\begin{aligned} y_1 &= \gamma_{11}\nu_1 + \gamma_{12}\nu_2 + \gamma_{13}\nu_3 + \gamma_{14}\nu_4 \\ y_2 &= \gamma_{21}\nu_1 + \gamma_{22}\nu_2 + \gamma_{23}\nu_3 + \gamma_{24}\nu_4 \\ y_3 &= \gamma_{31}\nu_1 + \gamma_{32}\nu_2 + \gamma_{33}\nu_3 + \gamma_{34}\nu_4 \\ y_4 &= \gamma_{41}\nu_1 + \gamma_{42}\nu_2 + \gamma_{43}\nu_3 + \gamma_{44}\nu_4 \end{aligned}\right\} \ldots\ldots\ldots(30),$$

the determinant of whose γ's is not zero.

Indicate by $(\sigma)^2$ the result of substituting (30) in $(\Sigma)^2$.

Then $\qquad \lambda$ or $(\Sigma)'^2 = (\sigma)'^2 = \nu_1'^2 - 2\nu_2'\nu_3'$.

Now, since $\qquad \nu_1^2 - 2\nu_2\nu_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a),$

its derivative, $\qquad \nu_1\nu_1' - \nu_2\nu_3' - \nu_2'\nu_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(b).$

Suppose also $\qquad \nu_1'^2 - 2\nu_2'\nu_3' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(c).$

Therefore from (*b*), $\quad \nu_1'^2 = \dfrac{(\nu_2'\nu_3 + \nu_3'\nu_2)^2}{\nu_1^2}$.

Equate with value of $\nu_1'^2$ in (*c*).

Therefore $\qquad (\nu_2'\nu_3 + \nu_2\nu_3')^2 = 4\nu_2\nu_3\nu_2'\nu_3'$.

Therefore $\qquad (\nu_2'\nu_3 - \nu_2\nu_3')^2 = 0.$

Therefore $\qquad \dfrac{\nu_2'}{\nu_2} = \dfrac{\nu_3'}{\nu_3}$.

Therefore $\qquad \nu_2 = C.\nu_3$ (C constant).

Therefore $\nu_1, \nu_2, \nu_3, \nu_4$ are linearly dependent; but y_1, y_2, y_3, y_4, and therefore $\nu_1, \nu_2, \nu_3, \nu_4$, are linearly independent.

Therefore $\nu_1'^2 - 2\nu_2'\nu_3'$ or λ cannot vanish identically.

It ought to be noted that it was allowable to divide by ν_1, ν_2, ν_3, since none of them can be zero.

20. Apply formulæ (4) to $u = f(y_1, y_2, y_3, y_4)$.

The work is so similar to that done in the cases previously considered that I shall merely state the results.

We get the following two equations:

$$\lambda''' + \tfrac{12}{5}p_2\lambda' - \tfrac{2}{5}(4p_3 - 9p_2')\lambda = \tfrac{3}{10}u^{v} + \tfrac{9}{5}p_2u'''$$
$$+ \tfrac{2}{5}(\tfrac{9}{2}p_2' + 3p_3)u'' + \tfrac{2}{5}(3p_3' + 2p_4)u' + \tfrac{3}{5}p_4'u,$$

$$\tfrac{27}{5}p_2\lambda''' - (\tfrac{63}{5}p_2' - \tfrac{42}{5}p_3)\lambda'' - (\tfrac{26}{5}p_3' - \tfrac{3}{5}p_2'' - 4p_4)\lambda'$$
$$- (\tfrac{8}{5}p_3'' - \tfrac{3}{5}p_2''' - p_4')\lambda$$
$$= \tfrac{1}{20}u^{v''} + \tfrac{33}{10}p_2u^{v} + (\tfrac{21}{5}p_3 - \tfrac{21}{10}p_2')u^{iv}$$
$$+ (\tfrac{9}{10}p_2'' + \tfrac{3}{5}p_3' + \tfrac{9}{5}p_4 + 18p_2^2)u'''$$
$$+ (\tfrac{3}{10}p_2''' + \tfrac{3}{5}p_3'' + \tfrac{6}{5}p_4' + 36p_2p_3)u''$$
$$+ (\tfrac{1}{5}p_3''' + \tfrac{1}{2}p_4'' - 12p_2'p_3 + 12p_2p_3'$$
$$+ 12p_2p_4 + 16p_3^2)u' + (\tfrac{1}{10}p_4''' + 6p_2p_4' - 6p_2'p_4 + 8p_3p_4)u.$$

For brevity write them as

$$\lambda''' + \tfrac{12}{5}p_2\lambda' - \tfrac{2}{5}(4p_3 - 9p_2')\lambda = A_5,$$

$$\tfrac{27}{5}p_2\lambda''' - (\tfrac{63}{5}p_2' - \tfrac{42}{5}p_3)\lambda'' - (\tfrac{26}{5}p_3' - \tfrac{3}{5}p_2'' - 4p_4)\lambda'$$
$$- (\tfrac{8}{5}p_3'' - \tfrac{3}{5}p_2''' - p_4')\lambda = B_7.$$

In the elimination of λ there result the following equations:

$$\tfrac{42}{5}a_3\lambda'' + (4a_4 + \tfrac{14}{5}a_3')\lambda' + \left(a_4' + \tfrac{2}{5}a_3'' + \frac{3^3.2^3}{5^2}p_2a_3\right)\lambda = C_7,$$

$$(4a_4 + \tfrac{56}{5}a_3')\lambda'' + \left(5a_4' + \tfrac{16}{5}a_3'' - \frac{2^5.3^2}{5^2}p_2a_3\right)\lambda',$$

$$+ \left(a_4'' + \tfrac{2}{5}a_3''' + \frac{3^3.2^3}{5^2}p_2a_3' - \frac{3^3.2^2}{5}p_2'a_3 + \frac{3.2^4.7}{5^2}p_3a_3\right)\lambda = D_8,$$

$$8t_8\lambda' + (t_8' + k_9)\lambda = E_8,$$

$$\left(\frac{3^3.2.7}{5}a_3t_8' + \frac{3.2.7}{5}k_9a_3 - 2^5a_4t_8 - \frac{2^4.7}{5}a_3't_8\right)\lambda'$$

$$+ \left(\frac{3.2.7}{5}a_3t_8'' + \frac{3.2.7}{5}a_3k_9' - 2^3a_4't_8 - \frac{2^4}{5}a_3''t_8 - \frac{3^3.2^6}{5^2}p_2a_3t_8\right)\lambda = F_9,$$

$$\overline{\Omega}_{21}\lambda = G_9,$$

$$8t_8\overline{\Omega}_{21}G_9' - \tfrac{1}{2}\overline{\Omega}_{21}^2E_8 - (8t_8\overline{\Omega}_{21}' - t_8'\overline{\Omega}_{21} - k_9\overline{\Omega}_{21})G_9 = 0,$$

where

$$b_{3,3}=\frac{3^3.2^3}{5}p_2a_3^2+7a_3'^2-6a_3a_3'',$$

$$c_{3,4}=4a_3'a_4-3a_3a_4',$$

$$t_8=25a_4^2+7b_{3,3}+\frac{7.5^3}{8}c_{3,4},$$

$$b_{3,4}=\frac{3^2.2^2}{5}p_2a_3a_4+\frac{7.3^2}{2^3}a_3'a_4'-\frac{3.7}{2^3}a_3a_4''-\tfrac{3}{2}a_4a_3'',$$

$$k_9=15b_{3,4}-\frac{3^2.2^4.7^2}{5}a_3^3,$$

$$q_{8,8}=\frac{2^8.3}{5}p_2t_8^2+17t_8'^2-2^4t_8t_8'',$$

$$p_{3,8}=\frac{2^5.3^2}{5}p_2t_8a_3+\frac{7.17}{12}a_3't_8'-\tfrac{7}{4}a_3t_8''-\frac{2.17}{3}t_8a_3'',$$

$$c_{8,4}=4a_4t_8'-8t_8a_4',$$

$$c_{9,3}=3a_3k_9'-9k_9a_3',$$

$$c_{8,3}=3a_3t_8'-8t_8a_3',$$

$$\bar{\Omega}_{21}=\left\{\frac{2.3}{5.17}\left[2^5p_{3,8}t_8-3^2.7q_{8,8}a_3\right]+2^3t_8\left[c_{8,4}+\frac{2.7}{5}c_{9,3}\right]\right.$$
$$\left.-2^2.7k_9c_{8,3}-\frac{3.2.7}{5}k_9^2a_3+2^5a_4k_9t_8\right\},$$

all of which are invariants (v. § 3).

With the aid of these equations, by reasoning similar to that employed in the case of relations of the third degree in the solutions of an equation of the third order, the following results can be obtained concerning a linear differential equation in the most general form (A).

There cannot exist more than three linearly independent quadratic relations in its solutions.

A necessary and sufficient condition that three linearly independent quadratic relations exist in its solutions is that the invariants a_3 and a_4 vanish.

A necessary and sufficient condition that two and only two linearly independent quadratic relations exist in its solutions is that the invariants t_8 and Ω_{21} vanish, but neither of the invariants a_3 and a_4 vanish.

A necessary and sufficient condition that one and only one quadratic relation exists in its solutions is that the invariant $\overline{\Omega}_{21}$ vanishes, but the invariant t_8 does not vanish.

A necessary and sufficient condition that no quadratic relation exists in its solutions is that $\overline{\Omega}_{21}$ does not vanish.

21. The linear differential equation of which

$$u = f(y_1, y_2, y_3, y_4)$$

is the general solution can be found in each of the above cases as in § 13.

Let
$$U = f(Y_1, Y_2, Y_3, Y_4),$$
then
$$u = e^{2\int P_1 dx}.U.$$

Make this substitution in each of the equations thus found, and there results, in each case, the linear differential equation of which a quadratic form with arbitrary coefficients in the solutions of equation (A) is the general solution.

These equations will give immediately the following theorems:

In the linear differential equation (A) if the

$$\left\{\begin{array}{c} \text{invariants } a_3 \text{ and } a_4 \text{ vanish} \\ \text{invariants } t_8 \text{ and } \overline{\Omega}_{21} \text{ vanish and } a_3 \text{ does not vanish} \\ \text{invariant } \overline{\Omega}_{21} \text{ vanishes and } t_8 \text{ does not vanish} \\ \text{invariant } \overline{\Omega}_{21} \text{ does not vanish} \end{array}\right\},$$

a quadratic form in its solutions with arbitrary coefficients is the general solution of a linear differential equation of order $\left\{\begin{array}{l}\text{seven}\\ \text{eight}\\ \text{nine}\\ \text{ten}\end{array}\right\}$, the coefficients of which are rational functions of the coefficients of the given differential equation and their derivatives.

22. Suppose $\overline{\Omega}_{21}$ vanishes, then there is at least one quadratic relation in the solutions of equation (B). Let λ be the λ of a quadratic relation. Then it must be a solution of the

left-hand member of those equations involving λ in § 20, set equal to zero, and in particular of

$$[B] \qquad \lambda''' + \tfrac{12}{5}p_2\lambda' - \tfrac{2}{5}(4p_3 - 9p_2')\lambda = 0 \text{ ....(31).}$$

Let $Z = -\dfrac{\lambda'}{\lambda}$, and substitute in (31), therefore

$$\tfrac{5}{4}R_3 - 3p_2Z - 2p_3 + \tfrac{9}{2}p_2' = 0 \text{ (v. § 2) ....(32).}$$

In the transformation (C), § 2, let $\rho = \lambda^{\frac{3}{2}}$.
Then equation (D) becomes

$$-\frac{\lambda'}{\lambda} = \frac{z''}{z'}.$$

Therefore $\lambda z' =$ a constant, not zero. Suppose $= 1$.
The other equations connecting the q's and p's in § 2 give

$$q_2 = \{p_2 + \tfrac{5}{6}(-Z' + \tfrac{1}{2}Z^2)\}\lambda^2 \text{ ....(33),}$$

$$q_3 z'^3 = \tfrac{5}{4}R_3 - 3p_2Z + p_3 \text{ ....(34).}$$

Combine equations (32) and (34).

Therefore
$$q_3 z'^3 = 3a_3,$$
$$= \left(3q_3 - \tfrac{9}{2}\frac{dq_2}{dz}\right)z'^3.$$

Therefore
$$q_3 = \tfrac{9}{4}\frac{dq_2}{dz}.$$

Also, let $Z = -\dfrac{\lambda'}{\lambda}$ in $h\Delta^2$.

Therefore

$$h\Delta^2 = \lambda^4 [\tfrac{27}{10}p_2Z^2 - \tfrac{27}{5}p_2Z' - \tfrac{6}{5}p_3Z - \tfrac{8}{5}p_3' + \tfrac{3}{5}p_2'' + p_4$$
$$+ \tfrac{9}{5}p_2'Z - \tfrac{9}{4}Z^2Z' + \tfrac{9}{16}Z^4 + \tfrac{9}{4}Z'^2]$$
$$= \lambda^4 z'^4 \left[q_4 - \tfrac{8}{5}\frac{dq_3}{dz} + \tfrac{3}{5}\frac{d^2q_2}{dz^2}\right].$$

Now $h\Delta^2$ is constant. Suppose $= C$.

Therefore
$$q_4 = 3\frac{d^2q_2}{dz^2} + C.$$

Therefore the differential equation (B), in whose solutions exist a quadratic relation, can be transformed to the form

$$\frac{d^4v}{dz^4}+6q_2\frac{d^2v}{dz^2}+9\frac{dq_2}{dz}\cdot\frac{dv}{dz}+\left(3\frac{d^2q_2}{dz^2}+C\right)v=0^* \quad \ldots(35).$$

The solutions of an equation in the form of equation (35), when C is not zero, are the products of the solutions of the two linear differential equations of the second order

$$\left.\begin{aligned}\frac{d^2\psi}{dz^2}&=\{-\tfrac{3}{2}q_2-\tfrac{1}{2}\sqrt{(C)}\,\psi\}\\ \frac{d^2\chi}{dz^2}&=\{-\tfrac{3}{2}q_2+\tfrac{1}{2}\sqrt{(C)}\,\chi\}\end{aligned}\right\}\ldots\ldots\ldots\ldots(36).$$

For, let $\nu=\psi.\chi$.

Therefore $$\frac{d\nu}{dz}=\psi\frac{d\chi}{dz}+\chi\frac{d\psi}{dz}.$$

Therefore $$\frac{d^2\nu}{dz^2}=-3q_2\nu+2\frac{d\psi}{dz}\cdot\frac{d\chi}{dz}.$$

Therefore

$$\frac{d^3\nu}{dz^3}=-6q_2\frac{d\nu}{dz}-3\frac{dq_2}{dz}.\nu+\sqrt{(C)}\left(\chi\frac{d\psi}{dz}-\psi\frac{d\chi}{dz}\right).$$

Therefore

$$\frac{d^4\nu}{dz^4}+6q_2\frac{d^2\nu}{dz^2}+9\frac{dq_2}{dz}\cdot\frac{d\nu}{dz}+\left(3\frac{d^2q_2}{dz^2}+C\right)\nu=0.$$

When C is zero, equation (35) is

$$\frac{d^4\nu}{dz^4}+6q_2\frac{d^2\nu}{dz^2}+9\frac{dq_2}{dz}\cdot\frac{d\nu}{dz}+3\frac{d^2q_2}{dz^2}\nu=0 \quad \ldots\ldots(37).$$

This equation can be written

$$\frac{d}{dz}\left(\frac{d^3\nu}{dz^3}+6q_2\frac{d\nu}{dz}+3\frac{dq_2}{dz}.\nu\right)=0,$$

* See article in *Acta Mathematica*, Tome III., by Halphen.

or $$\frac{d^3\nu}{dz^3} + 6q_2\frac{d\nu}{dz} + 3\frac{dq_2}{dz}.\nu = 6C_1,$$

C_1 being an arbitrary constant. The invariant a_3 of the equation, got by setting the left-hand member equal to zero, vanishes. Therefore let

$$\nu = \xi^2.v, \quad \zeta = \int\frac{dx}{\xi^2},$$

where ξ is a solution of the equation

$$\frac{d^2\xi}{dz^2} + \tfrac{3}{2}q_2\xi = 0,$$

and the equation becomes

$$\frac{d^3v}{d\zeta^3} = 6C_1.$$

Therefore the general solution of (37) is

$$C_1\xi^2\left(\int\frac{dx}{\xi'^2}\right)^3 + C_2\xi_1^2 + C_3\xi_2^2 + C_4\xi_1\xi_2,$$

where ξ_1 and ξ_2 are linearly independent solutions of the above equation in ξ.

Then a linear differential equation of the fourth order, in whose solutions exist a quadratic relation, can be solved by solving equations of lower order as follows:

Transform to the form (B). Find a solution of equation (31). It can easily be proved that it is the λ of some quadratic relation in the solutions of equation (B). Then by using this value of λ in the transformation performed at the first of this section, the equation is transformed to the form (35) or (37), according as C is not or is zero. If form (35) results, the solutions are got by solving the two equations of the second order (36). If form (37) results, the solutions can be got by solving the equation of the second order

$$\frac{d^2\xi}{dz^2} + \tfrac{3}{2}q_2\xi = 0.$$

Cambridge, Mass., U.S.A.,
January 30*th*, 1899.

ON THE VALUES OF THE SERIES $x^n+(x-q)^n+(x-2q)^n+\ldots+r^n$ and $x^n-(x-q)^n+(x-2q)^n-\ldots\pm r^n$.

By J. W. L. GLAISHER.

Value of the series $x^n+(x-q)^n+\ldots+r^n$ in powers of x,
§§ 1–4.

§ 1. THE Bernoullian function $V_n(x)$ possesses the property*

$$V_n(x+1)-V_n(x)=nx^{n-1},$$

whence, putting $\frac{x}{q}$ for x and $n+1$ for n,

$$x^n=\frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x+q}{q}\right)-V_{n+1}\left(\frac{x}{q}\right)\right\},$$

Similarly, putting $x-q$, $x-2q$, ..., for x,

$$(x-q)^n=\frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x}{q}\right)-V_{n+1}\left(\frac{x-q}{q}\right)\right\},$$

$$(x-2q)^n=\frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x-q}{q}\right)-V_{n+1}\left(\frac{x-2q}{q}\right)\right\},$$

&c. &c.

whence, by addition,

$$x^n+(x-q)^n+(x-2q)^n+\ldots+r^n=\frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x}{q}+1\right)-V_{n+1}\left(\frac{r}{q}\right)\right\}.$$

This equation is true for any value of r less than x, and subject to the condition $x\equiv r$, mod. q. In this paper q will be supposed to be a positive integer and r to have one of the values 0, 1, 2, 3, ..., q.

§ 2 It is convenient now to separate the cases of even and uneven exponents.

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* *Quarterly Journal*, Vol. XXIX., p. 118.

Since $V_n(x+1) = nx^{n-1} + V_n(x)$, we have*

$$V_{2n}(x+1) = x^{2n} + \tfrac{1}{2}(2n)_1 x^{2n-1} + (2n)_2 B_1 x^{2n-2} - \ldots + (-1)^{n-1}(2n)_{2n} B_n,$$

and

$$V_{2n+1}(x+1) = x^{2n+1} + \tfrac{1}{2}(2n+1)_1 x^{2n} + (2n+1)_2 B_1 x^{2n-1} - \ldots + (-1)^{n-1}(2n+1)_{2n} B_n x,$$

where n_r denotes the Binomial-theorem coefficient

$$\frac{n(n-1)\ldots(n-r+1)}{r!},$$

and B_r is the rth Bernoullian number.

§ 3. Substituting these values for $V_n\left(\dfrac{x}{q}+1\right)$ in § 1, we obtain the results

$$x^{2n-1} + (x-q)^{2n-1} + \ldots + r^{2n-1}$$
$$= \frac{1}{2nq}\left\{x^{2n} + (2n)_1 \frac{q}{2} x^{2n-1} + (2n)_2 B_1 q^2 x^{2n-2} - (2n)_4 B_2 q^4 x^{2n-4} + \ldots + (-1)^n (2n)_{2n-2} B_{n-1} q^{2n-2} x^2\right\} + G_r,$$

where

$$G_r = \frac{q^{2n-1}}{2n}\left\{V_{2n}(0) - V_{2n}\left(\frac{r}{q}\right)\right\}, \quad V_{2n}(0) = (-1)^{n-1} B_n,$$

$$x^{2n} + (x-q)^{2n} + \ldots + r^{2n}$$
$$= \frac{1}{(2n+1)q}\left\{x^{2n+1} + (2n+1)_1 \frac{q}{2} x^{2n} + (2n+1)_2 B_1 q^2 x^{2n-1} - (2n+1)_4 B_2 q^4 x^{2n-3} + \ldots + (-1)^{n-1}(2n+1)_{2n} B_n q^{2n} x\right\} + K_r,$$

where $$K_r = -\frac{q^{2n}}{2n+1} V_{2n+1}\left(\frac{r}{q}\right).$$

In both formulæ the terms on the right-hand side run regularly after the second.

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* *Quarterly Journal, loc. cit.*, p. 116.

In the first formula n must be > 1; if $n = 1$, the formula is

$$x + (x-q) + \ldots + r = \frac{1}{2q}(x^2 + qx) + G_r,$$

where
$$G_r = \frac{q}{2}\left\{V_2(0) - V_2\left(\frac{r}{q}\right)\right\}.$$

In the second formula n must be > 0.

We may regard the quantities $V_{2n}\left(\frac{r}{q}\right)$ and $V_{2n+1}\left(\frac{r}{q}\right)$ as known for the values 2, 3, 4, 6 of q*, so that for these values of q the constants G_r and K_r are known quantities. In general, the constant is expressed by the Bernoullian function, and it can therefore be calculated for any given values of q and r by means of the general formulæ connected with that function.

§ 4. It may be remarked that since

$$x^n + (x-1)^n + \ldots + 1^n = \frac{V_{n+1}(x+1) - V_{n+1}(1)}{n+1},$$

we find, by putting $\frac{x}{q}$ for x,

$$x^n + (x-q)^n + \ldots + q^n = \frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x}{q}+1\right) - V_{n+1}(1)\right\},$$

x being supposed $\equiv 0$, mod. q. We thus see that it is only in the term independent of x that the value of the series

$$x^n + (x-q)^n + \ldots + r^n$$

differs from that derived from $x^n + (x-1)^n + \ldots + 1^n$ by putting $\frac{x}{q}$ for x and multiplying by q^n. The part of the value of the former series which depends upon x may therefore be derived from that of the latter simply by the substitution of $\frac{x}{q}$ for x, and multiplication by q^n. The constant term, which necessarily depends upon r may be determined independently, by assigning some special value to x, or otherwise.

Similar remarks apply to the series $x^n - (x-q)^n + \ldots \pm r^n$ (§ 12).

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* *Quarterly Journal*, Vol. XXIX. p. 131. The values of $V_n(\frac{1}{8})$ and $V_n(\frac{1}{12})$ are also known (Vol. XXIX., pp. 61, 65, 73, 75).

The cases $q=2, 3, 4$, §§ 5–9.

§ 5. Putting $q=2, 3, 4$ in the first formula of § 3, we obtain the following general formulæ for exponent $2n-1$:

$$x^{2n-1}+(x-2)^{2n-1}+\ldots+r^{2n-1}=\frac{1}{4n}\{x^{2n}+(2n)_1 x^{2n-1}+(2n)_2 2^2 B_1 x^{2n-2}$$
$$-(2n)_4 2^4 B_2 x^{2n-4}+\ldots+(-1)^n (2n)_{2n-2} 2^{2n-2} B_{n-1} x^2\}+G_r,$$

where $$G_1=(-1)^{n-1}\frac{(2^{2n}-1)B_n}{2n},\quad G_2=0.$$

If we put (§ 11) $\alpha_r=2(2^{2r}-1)B_r$, the former equation may be written

$$G_1=(-1)^{n-1}\frac{\alpha_n}{4n};$$

$$x^{2n-1}+(x-3)^{2n-1}+\ldots+r^{2n-1}$$
$$=\frac{1}{6n}\{x^{2n}+\tfrac{3}{2}(2n)_1 x^{2n-1}+(2n)_2 3^2 B_1 x^{2n-2}-\ldots\}+G_r,$$

where $$G_1=G_2=(-1)^{n-1}\frac{(3^{2n}-1)B_n}{4n},\quad G_3=0;$$

$$x^{2n-1}+(x-4)^{2n-1}+\ldots+r^{2n-1}$$
$$=\frac{1}{8n}\{x^{2n}+2(2n)_1 x^{2n-1}+(2n)_2 4^2 B_1 x^{2n-2}-\ldots\}+G_r,$$

where $$G_1=G_3=(-1)^{n-1}\frac{(2^{4n-1}+2^{2n-1}-1)B_n}{4n},$$
$$G_2=(-1)^{n-1}\frac{2^{2n-1}(2^{2n}-1)B_n}{2n},\quad G_4=0.$$

§ 6. The corresponding formulæ for exponent $2n$, derived from § 3, are

$$x^{2n}+(x-2)^{2n}+\ldots+r^{2n}$$
$$=\frac{1}{2(2n+1)}\{x^{2n+1}+(2n+1)_1 x^{2n}+(2n+1)_2 2^2 B_1 x^{2n-1}$$
$$-(2n+1)_4 2^4 B_2 x^{2n-3}+\ldots+(-1)^{n-1}(2n+1)_{2n} 2^{2n} B_n x\}+K_r,$$

where $$K_1=0,\quad K_2=0;$$

$$x^{2n}+(x-3)^{2n}+\ldots+r^{2n}$$
$$=\frac{1}{3(2n+1)}\{x^{2n+1}+\tfrac{3}{2}(2n+1)x^{2n}+(2n+1)_2 3^2 B_1 x^{2n-1}+\ldots\}+K_r,$$

where $K_1=(-1)^n\tfrac{1}{3}I_n,\quad K_2=(-1)^{n-1}\tfrac{1}{3}I_n,\quad K_3=0;$

$$x^{2n}+(x-4)^{2n}+\ldots+r^{2n}$$

$$=\frac{1}{4(2n+1)}\{x^{2n+1}+2(2n+1)x^{2n}+(2n+1)_2 4^2B_1x^{2n-1}+\ldots\}+K_r,$$

where $K_1=(-1)^n\tfrac{1}{4}E_n$, $K_2=0$, $K_3=(-1)^{n-1}\tfrac{1}{4}E_n$, $K_4=0$.

The quantities E_n are the Eulerian numbers. The quantities I_n have been considered in the *Quarterly Journal*, Vol. XXIX., p. 35.

§ 7. The following is a list of the particular cases of the formulæ in § 5 obtained by putting $n=1$ and $n=2$, and substituting for G_r its value:

$$x+(x-2)+\ldots+1=\tfrac{1}{4}x^2+\tfrac{1}{2}x+\tfrac{1}{4},$$
$$x+(x-2)+\ldots+2=\tfrac{1}{4}x^2+\tfrac{1}{2}x,$$

$$x^3+(x-2)^3+\ldots+1^3=\tfrac{1}{8}x^4+\tfrac{1}{2}x^3+\tfrac{1}{2}x^2-\tfrac{1}{8},$$
$$x^3+(x-2)^3+\ldots+2^3=\tfrac{1}{8}x^4+\tfrac{1}{2}x^3+\tfrac{1}{2}x^2,$$

$$x+(x-3)+\ldots+1=\tfrac{1}{6}x^2+\tfrac{1}{2}x+\tfrac{1}{3},$$
$$x+(x-3)+\ldots+2=\tfrac{1}{6}x^2+\tfrac{1}{2}x+\tfrac{1}{3},$$
$$x+(x-3)+\ldots+3=\tfrac{1}{6}x^2+\tfrac{1}{2}x,$$

$$x^3+(x-3)^3+\ldots+1^3=\tfrac{1}{12}x^4+\tfrac{1}{2}x^3+\tfrac{3}{4}x^2-\tfrac{1}{3},$$
$$x^3+(x-3)^3+\ldots+2^3=\tfrac{1}{12}x^4+\tfrac{1}{2}x^3+\tfrac{3}{4}x^2-\tfrac{1}{3},$$
$$x^3+(x-3)^3+\ldots+3^3=\tfrac{1}{12}x^4+\tfrac{1}{2}x^3+\tfrac{3}{4}x^2,$$

$$x+(x-4)+\ldots+1=\tfrac{1}{8}x^2+\tfrac{1}{2}x+\tfrac{3}{8},$$
$$x+(x-4)+\ldots+2=\tfrac{1}{8}x^2+\tfrac{1}{2}x+\tfrac{1}{2},$$
$$x+(x-4)+\ldots+3=\tfrac{1}{8}x^2+\tfrac{1}{2}x+\tfrac{3}{8},$$
$$x+(x-4)+\ldots+4=\tfrac{1}{8}x^2+\tfrac{1}{2}x,$$

$$x^3+(x-4)^3+\ldots+1^3=\tfrac{1}{16}x^4+\tfrac{1}{2}x^3+x^2-\tfrac{9}{16},$$
$$x^3+(x-4)^3+\ldots+2^3=\tfrac{1}{16}x^4+\tfrac{1}{2}x^3+x^2-1,$$
$$x^3+(x-4)^3+\ldots+3^3=\tfrac{1}{16}x^4+\tfrac{1}{2}x^3+x^2-\tfrac{9}{16},$$
$$x^3+(x-4)^3+\ldots+4^3=\tfrac{1}{16}x^4+\tfrac{1}{2}x^3+x^2.$$

§ 8. The corresponding formulæ derived from § 6 for the values $n=1$ and $n=2$ are:

$$x^2 + (x-2)^2 + \ldots + 1^2 = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x,$$

$$x^2 + (x-2)^2 + \ldots + 2^2 = \tfrac{1}{6}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{3}x,$$

$$x^4 + (x-2)^4 + \ldots + 1^4 = \tfrac{1}{10}x^5 + \tfrac{1}{2}x^4 + \tfrac{2}{3}x^3 - \tfrac{4}{15}x,$$

$$x^4 + (x-2)^4 + \ldots + 2^4 = \tfrac{1}{10}x^5 + \tfrac{1}{2}x^4 + \tfrac{2}{3}x^3 - \tfrac{4}{15}x,$$

$$x^2 + (x-3)^2 + \ldots + 1^2 = \tfrac{1}{9}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{1}{9},$$

$$x^2 + (x-3)^2 + \ldots + 2^2 = \tfrac{1}{9}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{2}x + \tfrac{1}{9},$$

$$x^2 + (x-3)^2 + \ldots + 3^2 = \tfrac{1}{9}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{2}x,$$

$$x^4 + (x-3)^4 + \ldots + 1^4 = \tfrac{1}{15}x^5 + \tfrac{1}{2}x^4 + x^3 - \tfrac{9}{10}x + \tfrac{1}{3},$$

$$x^4 + (x-3)^4 + \ldots + 2^4 = \tfrac{1}{15}x^5 + \tfrac{1}{2}x^4 + x^3 - \tfrac{9}{10}x - \tfrac{1}{3},$$

$$x^4 + (x-3)^4 + \ldots + 3^4 = \tfrac{1}{15}x^5 + \tfrac{1}{2}x^4 + x^3 - \tfrac{9}{10}x,$$

$$x^2 + (x-4)^2 + \ldots + 1^2 = \tfrac{1}{12}x^3 + \tfrac{1}{2}x^2 + \tfrac{2}{3}x - \tfrac{1}{4},$$

$$x^2 + (x-4)^2 + \ldots + 2^2 = \tfrac{1}{12}x^3 + \tfrac{1}{2}x^2 + \tfrac{2}{3}x,$$

$$x^2 + (x-4)^2 + \ldots + 3^2 = \tfrac{1}{12}x^3 + \tfrac{1}{2}x^2 + \tfrac{2}{3}x + \tfrac{1}{4},$$

$$x^2 + (x-4)^2 + \ldots + 4^2 = \tfrac{1}{12}x^3 + \tfrac{1}{2}x^2 + \tfrac{2}{3}x,$$

$$x^4 + (x-4)^4 + \ldots + 1^4 = \tfrac{1}{20}x^5 + \tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 - \tfrac{32}{15}x + \tfrac{5}{4},$$

$$x^4 + (x-4)^4 + \ldots + 2^4 = \tfrac{1}{20}x^5 + \tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 - \tfrac{32}{15}x,$$

$$x^4 + (x-4)^4 + \ldots + 3^4 = \tfrac{1}{20}x^5 + \tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 - \tfrac{32}{15}x - \tfrac{5}{4},$$

$$x^4 + (x-4)^4 + \ldots + 4^4 = \tfrac{1}{20}x^5 + \tfrac{1}{2}x^4 + \tfrac{4}{3}x^3 - \tfrac{32}{15}x.$$

§ 9. It may be remarked that the general formulæ for the cases $n=1$, $n=2$, q being unrestricted, which include the particular cases of the two preceding sections are

$$x + (x-q) + \ldots + r = \frac{x^2}{2q} + \tfrac{1}{2}x + G_r,$$

where $$G_r = \frac{q}{2}\left\{V_2(0) - V_2\left(\frac{r}{q}\right)\right\}, \quad V_2(0) = \tfrac{1}{6};$$

$$x^3 + (x-q)^3 + \ldots + r^3 = \frac{x^4}{4q} + \tfrac{1}{2}x^3 + \tfrac{1}{4}qx^2 + G_r,$$

where $$G_r = \frac{q^3}{4}\left\{V_4(0) - V_4\left(\frac{r}{q}\right)\right\}, \quad V_4(0) = -\tfrac{1}{30};$$

$$x^2 + (x-q)^2 + \ldots + r^2 = \frac{x^3}{3q} + \tfrac{1}{2}x^2 + \tfrac{1}{6}qx + K_r,$$

where $$K_r = -\frac{q^2}{3} V_3\left(\frac{r}{q}\right);$$

$$x^4 + (x-q)^4 + \ldots + r^4 = \frac{x^5}{5q} + \tfrac{1}{2}x^4 + \tfrac{1}{3}qx^3 - \tfrac{1}{30}q^3 x + K_r,$$

where $$K_r = -\frac{q^4}{5} V_5\left(\frac{r}{q}\right).$$

Value of the series $x^n - (x-q)^n + \ldots \pm r^n$, §§ 10–12.

§ 10. From the equation*

$$U_n(x+1) + U_n(x) = nx^{n-1},$$

we find, by proceeding as in § 1, that

$$x^n - (x-q)^n + \ldots + (-1)^{\frac{x-r}{q}} r^n$$
$$= \frac{q^n}{n+1}\left\{U_{n+1}\left(\frac{x}{q}+1\right) + (-1)^{\frac{x-r}{q}} U_{n+1}\left(\frac{r}{q}\right)\right\}.$$

§ 11. Now†

$$U_{2n}(x+1) = \tfrac{1}{2}\{(2n)_1 x^{2n-1} + (2n)_2 \alpha_1 x^{2n-2} - (2n)_4 \alpha_2 x^{2n-4} + \ldots + (-1)^{n-1}\alpha_n\},$$

and

$$U_{2n+1}(x+1) = \tfrac{1}{2}\{(2n+1)_1 x^{2n} + (2n+1)_2 \alpha_1 x^{2n-1}$$
$$- (2n+1)_4 \alpha_2 x^{2n-3} + \ldots + (-1)^{n-1}(2n+1)_{2n}\alpha_n x\},$$

where $$\alpha_r = 2(2^{2r}-1)B_r.$$

* *Quarterly Journal, loc. cit.*, p. 118. † *Loc. cit.*, p. 117.

§ 12. We thus obtain the general formulæ

$$x^{2n-1}-(x-q)^{2n-1}+\ldots+(-1)^{\frac{x-r}{q}}r^{2n-1}=\frac{1}{4n}\{(2n)_1x^{2n-1}+(2n)_2\alpha_1qx^{2n-2}$$

$$-(2n)_4\alpha_2q^3x^{2n-4}+\ldots+(-1)^n(2n)_{2n-2}\alpha_{n-1}q^{2n-3}x^2$$

$$+(-1)^{n+1}\alpha_nq^{2n-1}\}+(-1)^{\frac{x-r}{q}}g_r,$$

where $$g_r=\frac{q^{2n-1}}{2n}U_{2n}\left(\frac{r}{q}\right);$$

$$x^{2n}-(x-q)^{2n}+\ldots+(-1)^{\frac{x-r}{q}}r^{2n}$$

$$=\frac{1}{2(2n+1)}\{(2n+1)_1x^{2n}+(2n+1)_2\alpha_1qx^{2n-1}$$

$$-(2n+1)_4\alpha_2q^3x^{2n-3}+\ldots+(-1)^{n-1}(2n+1)_{2n}\alpha_nq^{2n-1}x\}+(-1)^{\frac{x-r}{q}}k_r,$$

where $$k_r=\frac{q^{2n}}{2n+1}U_{2n+1}\left(\frac{r}{q}\right).$$

In both formulæ the terms run regularly after the first. As before, x must be $\equiv r$, mod. q.

The quantities $U_{2n}\left(\frac{r}{q}\right)$ and $U_{2n+1}\left(\frac{r}{q}\right)$ may be regarded as known for the values 2, 3, 4, 6 of q,* so that for these values of q the constants g_r and k_r are known quantities. For other values of q the constant may be calculated by means of the formula relating to the function $U_n(x)$.

The cases $q=2, 3, 4$, §§ 13–17.

§ 13. When $q=2, 3, 4$ the general formulæ for exponent $2n-1$ are:

$$x^{2n-1}-(x-2)^{2n-1}+\ldots+(-1)^{\frac{1}{2}(x-r)}r^{2n-1}$$

$$=\frac{1}{4n}\{(2n)_1x^{2n-1}+2(2n)_2\alpha_1x^{2n-2}-(2n)_4 2^3\alpha_2x^{2n-4}+\ldots$$

$$+(-1)^{n-1}2^{2n-1}\alpha_n\}+(-1)^{\frac{1}{2}(x-r)}g_r,$$

where $$g_1=0,\quad g_2=(-1)^{n-1}\frac{2^{2n-3}\alpha_n}{n};$$

* *Loc. cit.*, p. 131.

$$x^{2n-1}-(x-3)^{2n-1}+\ldots+(-1)^{\frac{1}{3}(x-r)}r^{2n-1}$$

$$=\frac{1}{4n}\{(2n)_1 x^{2n-1}+3(2n)_2\alpha_1 x^{2n-2}-\ldots\}+(-1)^{\frac{1}{3}(x-r)}g_r,$$

where $$g_1=-g_2=(-1)^n\frac{(2^{2n}-1)(3^{2n-1}-1)B_n}{4n},$$

$$g_3=(-1)^{n-1}\frac{3^{2n-1}\alpha_n}{4n};$$

$$x^{2n-1}-(x-4)^{2n-1}+\ldots+(-1)^{\frac{1}{4}(x-r)}r^{2n-1}$$

$$=\frac{1}{4n}\{(2n)_1 x^{2n-1}+4(2n)_2\alpha_1 x^{2n-2}-\ldots\}+(-1)^{\frac{1}{4}(x-r)}g_r,$$

where $$g_1=-g_3=(-1)^n\tfrac{1}{2}Q_n,$$

$$g_2=0,\ g_4=(-1)^{n-1}\frac{4^{2n-2}\alpha_n}{n}.$$

The quantities Q_n were considered in the *Quarterly Journal*, Vol. XXIX., p. 64.

§ 14. The corresponding formulæ for exponent $2n$ are:

$$x^{2n}-(x-2)^{2n}+\ldots+(-1)^{\frac{1}{2}(x-2)}r^{2n}$$

$$=\frac{1}{2(2n+1)}\{(2n+1)_1 x^{2n}+2(2n+1)_2\alpha_1 x^{2n-1}$$

$$-(2n+1)_4 2^3\alpha_2 x^{2n-3}+\ldots+(-1)^{n-1}(2n+1)_{2n}2^{2n-1}\alpha_n x\}+(-1)^{\frac{1}{2}(x-r)}k_r,$$

where $$k_1=(-1)^n\tfrac{1}{2}E_n,\quad k_2=0;$$

$$x^{2n}-(x-3)^{2n}+\ldots+(-1)^{\frac{1}{3}(x-r)}r^{2n}$$

$$=\frac{1}{2(2n+1)}\{(2n+1)_1 x^{2n}+3(2n+1)_2\alpha_1 x^{2n-1}-\ldots\}+(-1)^{\frac{1}{3}(x-r)}k_r,$$

where $$k_1=k_2=(-1)^n\tfrac{1}{3}H_n,\quad k_3=0;$$

$$x^{2n}-(x-4)^{2n}+\ldots+(-1)^{\frac{1}{4}(x-r)}r^{2n}$$

$$=\frac{1}{2(2n+1)}\{(2n+1)_1 x^{2n}+4(2n+1)_2\alpha_1 x^{2n-1}-\ldots\}+(-1)^{\frac{1}{4}(x-r)}k_r,$$

where $$k_1=k_3=(-1)^n\tfrac{1}{2}P_n,\ k_2=(-1)^n 2^{2n-1}E_n,\ k_4=0.$$

The quantities E_n are the Eulerian numbers. The quantities H_n and P_n were considered in the *Quarterly Journal*, Vol. XXIX., pp. 47, 60.

§ 15. The following is a list of the particular cases corresponding to $n=1$ and $n=2$, derived from § 13:

$$x-(x-2)+\ldots+(-1)^{\frac{1}{2}(x-1)}1=\tfrac{1}{2}x+\tfrac{1}{2},$$

$$x-(x-2)+\ldots+(-1)^{\frac{1}{2}(x-2)}2=\tfrac{1}{2}x+\tfrac{1}{2}+(-1)^{\frac{1}{2}(x-2)}\tfrac{1}{2},$$

$$x^3-(x-2)^3+\ldots+(-1)^{\frac{1}{2}(x-1)}1^3=\tfrac{1}{2}x^3+\tfrac{3}{2}x^2-1,$$

$$x^3-(x-2)^3+\ldots+(-1)^{\frac{1}{2}(x-2)}2^3=\tfrac{1}{2}x^3+\tfrac{3}{2}x^2-1-(-1)^{\frac{1}{2}(x-2)}1,$$

$$x-(x-3)+\ldots+(-1)^{\frac{1}{3}(x-1)}1=\tfrac{1}{2}x+\tfrac{3}{4}-(-1)^{\frac{1}{3}(x-1)}\tfrac{1}{4},$$

$$x-(x-3)+\ldots+(-1)^{\frac{1}{3}(x-2)}2=\tfrac{1}{2}x+\tfrac{3}{4}+(-1)^{\frac{1}{3}(x-2)}\tfrac{1}{4},$$

$$x-(x-3)+\ldots+(-1)^{\frac{1}{3}(x-3)}3=\tfrac{1}{2}x+\tfrac{3}{4}+(-1)^{\frac{1}{3}(x-3)}\tfrac{3}{4},$$

$$x^3-(x-3)^3+\ldots+(-1)^{\frac{1}{3}(x-1)}1^3=\tfrac{1}{2}x^3+\tfrac{9}{4}x^2-\tfrac{27}{8}+(-1)^{\frac{1}{3}(x-1)}\tfrac{13}{8},$$

$$x^3-(x-3)^3+\ldots+(-1)^{\frac{1}{3}(x-2)}2^3=\tfrac{1}{2}x^3+\tfrac{9}{4}x^2-\tfrac{27}{8}-(-1)^{\frac{1}{3}(x-2)}\tfrac{13}{8},$$

$$x^3-(x-3)^3+\ldots+(-1)^{\frac{1}{3}(x-3)}3^3=\tfrac{1}{2}x^3+\tfrac{9}{4}x^2-\tfrac{27}{8}-(-1)^{\frac{1}{3}(x-3)}\tfrac{27}{8},$$

$$x-(x-4)+\ldots+(-1)^{\frac{1}{4}(x-1)}1=\tfrac{1}{2}x+1-(-1)^{\frac{1}{4}(x-1)}\tfrac{1}{2},$$

$$x-(x-4)+\ldots+(-1)^{\frac{1}{4}(x-2)}2=\tfrac{1}{2}x+1,$$

$$x-(x-4)+\ldots+(-1)^{\frac{1}{4}(x-3)}3=\tfrac{1}{2}x+1+(-1)^{\frac{1}{4}(x-3)}\tfrac{1}{2},$$

$$x-(x-4)+\ldots+(-1)^{\frac{1}{4}(x-4)}4=\tfrac{1}{2}x+1+(-1)^{\frac{1}{4}(x-4)}1,$$

$$x^3-(x-4)^3+\ldots+(-1)^{\frac{1}{4}(x-1)}1^3=\tfrac{1}{2}x^3+3x^2-8+(-1)^{\frac{1}{4}(x-1)}\tfrac{11}{2},$$

$$x^3-(x-4)^3+\ldots+(-1)^{\frac{1}{4}(x-2)}2^3=\tfrac{1}{2}x^3+3x^2-8,$$

$$x^3-(x-4)^3+\ldots+(-1)^{\frac{1}{4}(x-3)}3^3=\tfrac{1}{2}x^3+3x^2-8-(-1)^{\frac{1}{4}(x-3)}\tfrac{11}{2},$$

$$x^3-(x-4)^3+\ldots+(-1)^{\frac{1}{4}(x-4)}4^3=\tfrac{1}{2}x^3+3x^2-8-(-1)^{\frac{1}{4}(x-4)}8.$$

§ 16. The following list, with exponents 2 and 4, is derived from § 14:

$$x^2-(x-2)^2+\ldots+(-1)^{\frac{1}{2}(x-1)}\,1^2=\tfrac{1}{2}x^2+x-(-1)^{\frac{1}{2}(x-1)}\,\tfrac{1}{2},$$

$$x^2-(x-2)^2+\ldots+(-1)^{\frac{1}{2}(x-2)}\,2^2=\tfrac{1}{2}x^2+x,$$

$$x^4-(x-2)^4+\ldots+(-1)^{\frac{1}{2}(x-1)}\,1^4=\tfrac{1}{2}x^4+2x^3-4x+(-1)^{\frac{1}{2}(x-1)}\,\tfrac{5}{2},$$

$$x^4-(x-2)^4+\ldots+(-1)^{\frac{1}{2}(x-2)}\,2^4=\tfrac{1}{2}x^4+2x^3-4x,$$

$$x^2-(x-3)^2+\ldots+(-1)^{\frac{1}{3}(x-1)}\,1^2=\tfrac{1}{2}x^2+\tfrac{3}{2}x-(-1)^{\frac{1}{3}(x-1)}\,1,$$

$$x^2-(x-3)^2+\ldots+(-1)^{\frac{1}{3}(x-2)}\,2^2=\tfrac{1}{2}x^2+\tfrac{3}{2}x-(-1)^{\frac{1}{3}(x-2)}\,1,$$

$$x^2-(x-3)^2+\ldots+(-1)^{\frac{1}{3}(x-3)}\,3^2=\tfrac{1}{2}x^2+\tfrac{3}{2}x,$$

$$x^4-(x-3)^4+\ldots+(-1)^{\frac{1}{3}(x-1)}\,1^4=\tfrac{1}{2}x^4+3x^3-\tfrac{27}{2}x+(-1)^{\frac{1}{3}(x-1)}\,11,$$

$$x^4-(x-3)^4+\ldots+(-1)^{\frac{1}{3}(x-2)}\,2^4=\tfrac{1}{2}x^4+3x^3-\tfrac{27}{2}x+(-1)^{\frac{1}{3}(x-2)}\,11,$$

$$x^4-(x-3)^4+\ldots+(-1)^{\frac{1}{3}(x-3)}\,3^4=\tfrac{1}{2}x^4+3x^3-\tfrac{27}{2}x,$$

$$x^2-(x-4)^2+\ldots+(-1)^{\frac{1}{4}(x-1)}\,1^2=\tfrac{1}{2}x^2+2x-(-1)^{\frac{1}{4}(x-1)}\,\tfrac{3}{2},$$

$$x^2-(x-4)^2+\ldots+(-1)^{\frac{1}{4}(x-2)}\,2^2=\tfrac{1}{2}x^2+2x-(-1)^{\frac{1}{4}(x-2)}\,2,$$

$$x^2-(x-4)^2+\ldots+(-1)^{\frac{1}{4}(x-3)}\,3^2=\tfrac{1}{2}x^2+2x-(-1)^{\frac{1}{4}(x-3)}\,\tfrac{3}{2},$$

$$x^2-(x-4)^2+\ldots+(-1)^{\frac{1}{4}(x-4)}\,4^2=\tfrac{1}{2}x^2+2x,$$

$$x^4-(x-4)^4+\ldots+(-1)^{\frac{1}{4}(x-1)}\,1^4=\tfrac{1}{2}x^4+4x^3-32x+(-1)^{\frac{1}{4}(x-1)}\,\tfrac{57}{2},$$

$$x^4-(x-4)^4+\ldots+(-1)^{\frac{1}{4}(x-2)}\,2^4=\tfrac{1}{2}x^4+4x^3-32x+(-1)^{\frac{1}{4}(x-2)}\,40,$$

$$x^4-(x-4)^4+\ldots+(-1)^{\frac{1}{4}(x-3)}\,3^4=\tfrac{1}{2}x^4+4x^3-32x+(-1)^{\frac{1}{4}(x-3)}\,\tfrac{57}{2},$$

$$x^4-(x-4)^4+\ldots+(-1)^{\frac{1}{4}(x-4)}\,4^4=\tfrac{1}{2}x^4+4x^3-32x.$$

§ 17. The general formulæ for the cases $n=1$, $n=2$, q being unrestricted, corresponding to those in § 9 are

$$x-(x-q)+\ldots+(-1)^{\frac{x-r}{q}}\,r=\tfrac{1}{2}x+\tfrac{1}{4}q+(-1)^{\frac{x-r}{q}}\,g_r,$$

where

$$g_r=\frac{q}{2}\,U_2\left(\frac{r}{q}\right);$$

$$x^3-(x-q)^3+\ldots+(-1)^{\overline{\frac{x-r}{q}}}r^3=\tfrac{1}{2}x^3+\tfrac{3}{4}qx^2-\tfrac{1}{8}q^3+(-1)^{\overline{\frac{x-r}{q}}}g_r,$$

where
$$g_r=\frac{q^3}{4}\,U_4\left(\frac{r}{q}\right);$$

$$x^2-(x-q)^2+\ldots+(-1)^{\overline{\frac{x-r}{q}}}r^2=\tfrac{1}{2}x^2+\tfrac{1}{2}qx+(-1)^{\overline{\frac{x-r}{q}}}k_r,$$

where
$$k_r=\frac{q^2}{3}\,U_3\left(\frac{r}{q}\right);$$

$$x^4-(x-q)^4+\ldots+(-1)^{\overline{\frac{x-r}{q}}}r^4=\tfrac{1}{2}x^4+qx^3-\tfrac{1}{2}q^3x+(-1)^{\overline{\frac{x-r}{q}}}k_r,$$

where
$$k_r=\frac{q^4}{5}\,U_5\left(\frac{r}{q}\right).$$

The constants G_r, K_r, g_r, k_r, §§ 18–22.

§ 18. The constant G_r

$$=\frac{q^{2n-1}}{2n}\left\{V_{2n}(0)-V_{2n}\left(\frac{r}{q}\right)\right\},\quad (§\ 3).$$

Therefore $G_q=0$; and, since

$$V_{2n}(x)=V_{2n}(1-x),$$

we have

$$\text{(i)}\quad G_r=G_{q-r}.$$

Also, since

$$V_n\left(\frac{1}{q}\right)+V_n\left(\frac{2}{q}\right)+\ldots+V_n\left(\frac{q-1}{q}\right)=-\frac{(q^{n-1}-1)}{q^{n-1}}\,V_n(0),^*$$

we have

$$G_1+G_2+\ldots+G_{q-1}=\frac{q^{2n-1}}{2n}\left\{(q-1)\,V_{2n}(0)+\frac{(q^{2n-1}-1)}{q^{2n-1}}\,V_{2n}(0)\right\}$$
$$=\frac{q^{2n}-1}{2n}\,V_{2n}(0),$$

so that

$$\text{(ii)}\quad G_1+G_2+\ldots+G_{q-1}=(-1)^{n-1}\frac{(q^{2n}-1)\,B_n}{2n}.$$

* *Quarterly Journal*, Vol. XXIX., p. 118.

§ 19. If q is a composite number there are other relations between the constants, of the same kind as (ii), corresponding to the different divisors of q. For example, if $q = 12$,

$$V_{2n}(\tfrac{4}{12}) + V_{2n}(\tfrac{8}{12}) = V_{2n}(\tfrac{1}{3}) + V_{2n}(\tfrac{2}{3}) = -\frac{3^{2n-1}-1}{3^{2n-1}} V_{2n}(0),$$

so that

$$G_4 + G_8 = \frac{12^{2n-1}}{2n}\left\{2V_{2n}(0) + \frac{3^{2n-1}-1}{3^{2n-1}} V_{2n}(0)\right\}$$

$$= (-1)^n \frac{4^{2n-1}(3^{2n}-1)B_n}{2n}.$$

Similarly

$$V_{2n}(\tfrac{3}{12}) + V_{2n}(\tfrac{6}{12}) + V_{2n}(\tfrac{9}{12}) = -\frac{4^{2n-1}-1}{4^{2n-1}} V_{2n}(0),$$

giving

$$G_3 + G_6 + G_9 = (-1)^n \frac{3^{2n-1}(4^{2n}-1)B_n}{2n};$$

and so on.

§ 20. The constant K_r is defined by the equation

$$K_r = -\frac{q^{2n}}{2n+1} V_{2n+1}\left(\frac{r}{q}\right), \quad (§ 3),$$

and therefore $K_q = 0$, and

(i) $K_r = -K_{q-r}$,

(ii) $K_1 + K_2 + \ldots + K_{q-1} = 0$.

The relation (ii) follows at once from (i), for if q be even $K_{\frac{1}{2}q} = 0$.

There are also relations similar to those in § 19: *ex. gr.* for $q = 12$,

$$K_4 + K_8 = -\frac{12^{2n}}{2n+1}\{V_{2n+1}(\tfrac{1}{3}) + V_{2n+1}(\tfrac{2}{3})\} = 0,$$

$$K_3 + K_6 + K_9 = 0, \quad \&c.$$

§ 21. The constants g_r and k_r are defined by the equations

$$g_r = \frac{q^{2n-1}}{2n} U_{2n}\left(\frac{r}{q}\right), \quad k_r = \frac{q^{2n}}{2n+1} U_{2n+1}\left(\frac{r}{q}\right), \quad (\S\ 12),$$

so that, since $\qquad U_n(x) = (-1)^{n-1} U_n(1-x)$,

we have $\qquad$ (i) $g_r = -g_{q-r}$,

$\qquad\qquad\qquad$ (ii) $k_r = k_{q-r}$.

Also, if q be uneven,

$$U_n\left(\frac{1}{q}\right) - U_n\left(\frac{2}{q}\right) + U_n\left(\frac{3}{q}\right) - \ldots - U_n\left(\frac{q-1}{q}\right) = \frac{q^{n-1}-1}{q^{n-1}} U_n(0),^*$$

whence, q being uneven, we find

$$\text{(iii)} \quad g_1 - g_2 + g_3 - \ldots - g_{q-1} = (-1)^n \frac{q^{2n-1}-1}{4n} \alpha_n,$$

$$\text{(iv)} \quad k_1 - k_2 + k_3 - \ldots - k_{q-1} = 0.$$

If q be a composite uneven number there are also other formulæ of the same kind as (iii) and (iv) corresponding to the different divisors of q. These results may be readily obtained by proceeding as in § 19.

§ 22. The forms of the constants in the values of

$$\phi(x) + \phi(x-q) + \ldots + \phi(r)$$

have been considered by Mr. Richmond in Vol. XXI. of the *Messenger of Mathematics*†. He has there given the general relations between the G's and between the K's, which were obtained in §§ 18–20, and also the special values of the G's for $q = 2, 3, \ldots$ contained in § 5. I know of no other paper in which the subject is dealt with.

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* *Quarterly Journal*, Vol. XXIX., p. 118.

† "Note on the sum of functions of quantities which are in arithmetical progression," pp. 29–34 (1891). Mr. Richmond's paper was suggested by a note of mine in the previous volume of the *Messenger* ("Note on the sums of even powers of even and uneven numbers," Vol. XX., pp. 172–176). It may be noticed that the $\phi(k)$ of that note (p. 175) in the notation of the present paper is $\frac{2^{2r}}{2r+1} V_{2r+1}(\frac{1}{2}k+1)$, so that the relation (p. 175) $\phi(2k) = 2^{2r}\{\phi(k) + \phi(k-1)\}$ is equivalent to

$$V_{2r+1}(k+1) = 2^{2r}\{V_{2r+1}(\tfrac{1}{2}k+1) + V_{2r+1}(\tfrac{1}{2}k+\tfrac{1}{2})\},$$

which is included in the formula $V_n(2x) = 2^{n-1}\{V_n(x) + V_n(x+\frac{1}{2})\}$.

Values of the series in powers of $x+p$, §§ 23–25.

§ 23. In the preceding sections the values of the series

$$x^n + (x-q)^n + \ldots + r^n \text{ and } x^n - (x-q)^n + \ldots \pm r^n$$

in powers of x have been considered. The values of the series in powers of $x+p$, p being any constant, may be easily derived from some general formulæ given in Vol. XXIX. of the *Quarterly Journal*,* viz.

$$q\,\Sigma_q \phi\,(x+q) = C + \int \phi\,(x+p)\,dx - qV_1\left(\frac{p}{q}\right)\phi\,(x+p)$$

$$+ \frac{q^2 V_2\left(\frac{p}{q}\right)}{2\,!}\phi'\,(x+p) - \frac{q^3 V_3\left(\frac{p}{q}\right)}{3\,!}\phi''\,(x+p) + \&\text{c.},$$

and

$$q\Sigma_q\,(-1)^{\frac{x}{q}+1}\,\phi\,(x+q) = C + (-1)^{\frac{x}{q}}\left\{qU_1\left(\frac{p}{q}\right)\phi\,(x+p)\right.$$

$$\left. - \frac{q^2 U_2\left(\frac{p}{q}\right)}{2\,!}\phi'\,(x+p) + \frac{q^3 U_3\left(\frac{p}{q}\right)}{3\,!}\phi''\,(x+p) - \&\text{c.}\right\}.$$

§ 24. For, putting $\phi\,(x) = x^n$ in the first formula, we have

$$x^n + (x-q)^n + \ldots + r^n = C_r + \frac{1}{(n+1)\,q}\left\{(x+p)^{n+1} - (n+1)_1 qV_1\left(\frac{p}{q}\right)(x+p)^n\right.$$

$$\left. + (n+1)_2 q^2 V_2\left(\frac{p}{q}\right)(x+p)^{n-1} - \ldots + (-1)^n (n+1)_n q^n V_n\left(\frac{p}{q}\right)(x+p)\right\}.$$

Now the left-hand side

$$= \frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{x}{q}+1\right) - V_{n+1}\left(\frac{r}{q}\right)\right\}, \quad (\S\,1).$$

whence, putting $x = -p$, we find

$$C_r = \frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{q-p}{q}\right) - V_{n+1}\left(\frac{r}{q}\right)\right\}.$$

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* pp. 325, 327.

§ 25. Similarly, from the second equation in § 23, we have

$$x^n-(x-q)^n+\ldots+(-1)^{\frac{x-r}{q}}r^n=\frac{1}{(n+1)q}\left\{(n+1)_1 qU_1\left(\frac{p}{q}\right)(x+p)^n\right.$$

$$\left.-(n+1)_2 q^2 U_2\left(\frac{p}{q}\right)(x+p)^{n-1}+\ldots+(-1)^n q^{n+1}U_{n+1}\left(\frac{p}{q}\right)\right\}+(-1)^{\frac{x-r}{q}}C_r.$$

The left-hand side

$$=\frac{q^n}{n+1}\left\{U_{n+1}\left(\frac{x}{q}+1\right)+(-1)^{\frac{x-r}{q}}U_{n+1}\left(\frac{r}{q}\right)\right\},\quad (\S\,10),$$

when putting $x=-p$, and noticing that

$$U_{n+1}\left(\frac{q-p}{q}\right)=(-1)^n U_{n+1}\left(\frac{p}{q}\right),$$

we find

$$C_r=\frac{q^n}{n+1}U_{n+1}\left(\frac{r}{q}\right).$$

§ 26. It may be remarked that, if we put

$$A_n(x)=\frac{V_n(x)}{n},\quad A_n'(x)=\frac{U_n(x)}{n},^*$$

the formulæ of the two preceding sections may be written

$$x^n+(x-q)^n+\ldots+r^n=\frac{1}{q}\left\{\frac{(x+p)^{n+1}}{n+1}-qA_1\left(\frac{p}{q}\right)(x+p)^n\right.$$

$$\left.+n_1q^2A_2\left(\frac{p}{q}\right)(x+p)^{n-1}-\ldots+(-1)^{n+1}n_nq^{n+1}A_{n+1}\left(\frac{p}{q}\right)\right\}-q^nA_{n+1}\left(\frac{r}{q}\right),$$

$$x^n-(x-q)^n+\ldots+(-1)^{\frac{x-r}{q}}r^n=\frac{1}{q}\left\{qA_1'\left(\frac{p}{q}\right)(x+p)^n-n_1q^2A_2'\left(\frac{p}{q}\right)(x+p)^{n-1}\right.$$

$$\left.+\ldots+(-1)^n n_n q^{n+1}A'_{n+1}\left(\frac{p}{q}\right)\right\}+(-1)^{\frac{x-r}{q}}q^nA'_{n+1}\left(\frac{r}{q}\right).$$

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* This notation is used in *Quarterly Journal*, Vol. XXIX., p. 18 *et seq*; p. 93 *et seq*; and in *Messenger*, Vol. XXVI., pp. 152–182, Vol. XXVII., pp. 20 97.

Values of the series in powers of $x+\frac{1}{2}q$, §§ 27–34.

§ 27. The most interesting particular case of these formulæ is when $p=\frac{1}{2}q$; for in this case, since $V_n(\frac{1}{2})$ vanishes when n is uneven and $U_n(\frac{1}{2})$ vanishes when n is even, each expression consists wholly of even or uneven powers of $x+\frac{1}{2}q$.

Thus, putting $p=\frac{1}{2}q$ and noticing that

$$2^{2n}V_{2n}(\tfrac{1}{2})=(-1)^n\beta_n, \text{ where } \beta_n=(2^{2n}-2)B_n,$$

we obtain the formulæ

$$\text{(i) } x^{2n-1}+(x-q)^{2n-1}+\ldots+r^{2n-1}=\frac{1}{2nq}\left\{\left(x+\frac{q}{2}\right)^{2n}-(2n)_2\beta_1\left(\frac{q}{2}\right)^2\left(x+\frac{q}{2}\right)^{2n-2}\right.$$
$$\left.+\ldots+(-1)^{n-1}(2n)_{2n-2}\beta_{n-1}\left(\frac{q}{2}\right)^{2n-2}\left(x+\frac{q}{2}\right)^2\right\}+F_r,$$

where $F_r=\dfrac{q^{2n-1}}{2n}\left\{V_{2n}(\tfrac{1}{2})-V_{2n}\left(\dfrac{r}{q}\right)\right\}$, $V_{2n}(\tfrac{1}{2})=(-1)^n\dfrac{\beta_n}{2^{2n}}$;

$$\text{(ii) } x^{2n}+(x-q)^{2n}+\ldots+r^{2n}$$
$$=\frac{1}{(2n+1)q}\left\{\left(x+\frac{q}{2}\right)^{2n-1}-(2n+1)_2\beta_1\left(\frac{q}{2}\right)^2\left(x+\frac{q}{2}\right)^{2n-1}+\ldots\right.$$
$$\left.+(-1)^n(2n+1)_{2n}\beta_n\left(\frac{q}{2}\right)^{2n}\left(x+\frac{q}{2}\right)\right\}+K_r,$$

where
$$K_r=-\frac{q^{2n}}{2n+1}V_{2n+1}\left(\frac{r}{q}\right).$$

The constant K_r is the same as K_r of § 3. The constant F_r differs from G_r only by a quantity independent of r, for

$$F_r-G_r=\frac{q^{2n-1}}{2n}\{V_{2n}(\tfrac{1}{2})-V_{2n}(0)\}=(-1)^n\frac{q^{2n-1}}{2n}\frac{\alpha_n}{2^{2n}}.$$

§ 28. Since

$$2^{2n+1}U_{2n+1}(\tfrac{1}{2})=(-1)^n(2n+1)E_n,$$

E_n being the n^{th} Eulerian number, we derive from the formula of § 25, or more easily from the second formula of § 26, the results:

(i) $x^{2n-1}-(x-q)^{2n-1}+\ldots+(-1)^{\frac{x-r}{q}}r^{2n-1}$

$$=\tfrac{1}{2}\left\{\left(x+\frac{q}{2}\right)^{2n-1}-(2n-1)_2E_1\left(\frac{q}{2}\right)^2\left(x+\frac{q}{2}\right)^{2n-3}+\ldots\right.$$

$$\left.+(-1)^{n-1}(2n-1)_{2n-2}E_{n-1}\left(\frac{q}{2}\right)^{2n-2}\left(x+\frac{q}{2}\right)\right\}+(-1)^{\frac{x-r}{q}}g_r,$$

where $$g_r=\frac{q^{2n-1}}{2n}U_n\left(\frac{r}{q}\right);$$

(ii) $x^{2n}-(x-q)^{2n}+\ldots+(-1)^{\frac{x-r}{q}}r^{2n}$

$$=\tfrac{1}{2}\left\{\left(x+\frac{q}{2}\right)^{2n}-(2n)_2E_1\left(\frac{q}{2}\right)^2\left(x+\frac{q}{2}\right)^{2n-2}+\ldots\right.$$

$$\left.+(-1)^n(2n)_{2n}E_n\left(\frac{q}{2}\right)^{2n}\right\}+(-1)^{\frac{x-r}{q}}k_r,$$

where $$k_r=\frac{q^{2n}}{2n+1}U_{2n+1}\left(\frac{r}{q}\right).$$

The constants g_r and k_r are the same as in § 12.

§ 29. Putting $n=1$ and 2 in the formulæ of § 27, we find, q being unrestricted,

$$x+(x-q)+\ldots+r=\frac{1}{2q}\left(x+\frac{q}{2}\right)^2+F_r,$$

where $$F_r=\frac{q}{2}\left\{V_2(\tfrac{1}{2})-V_2\left(\frac{r}{q}\right)\right\},\quad V_2(\tfrac{1}{2})=-\tfrac{1}{12};$$

$$x^3+(x-q)^3+\ldots+r^3=\frac{1}{4q}\left(x+\frac{q}{2}\right)^4-\tfrac{1}{8}q\left(x+\frac{q}{2}\right)^2+F_r,$$

where $$F_r=\frac{q^3}{4}\left\{V_4(\tfrac{1}{2})-V_4\left(\frac{r}{q}\right)\right\},\quad V_4(\tfrac{1}{2})=\tfrac{7}{240};$$

$$x^2+(x-q)^2+\ldots+r^2=\frac{1}{3q}\left(x+\frac{q}{2}\right)^3-\tfrac{1}{12}q\left(x+\frac{q}{2}\right)+K_r,$$

where $$K_r=-\frac{q^2}{3}V_3\left(\frac{r}{q}\right);$$

$$x^4+(x-q)^4+\ldots+r^4=\frac{1}{5q}\left(x+\frac{q}{2}\right)^5-\tfrac{1}{6}q\left(x+\frac{q}{2}\right)^3+\tfrac{7}{240}q^3\left(x+\frac{q}{2}\right)+K_r,$$

where $$K_r=-\frac{q^4}{5}V_5\left(\frac{r}{q}\right).$$

These formulæ correspond to those in § 9 with which they can be easily identified. The constants K_r are the same as in that section.

§ 30. The following list of particular cases in which the values are expressed in powers of $x+\frac{1}{2}q$ corresponds to the list in § 7 in which they are expressed in powers of x:

$$x+(x-2)+\ldots+r=\tfrac{1}{4}(x+1)^2+F_r,$$

where $$F_1=0,\quad F_2=-\tfrac{1}{4};$$

$$x^3+(x-2)^3+\ldots+r^3=\tfrac{1}{8}(x+1)^4-\tfrac{1}{4}(x+1)^2+F_r,$$

where $$F_1=0,\quad F_2=\tfrac{1}{8};$$

$$x+(x-3)+\ldots+r=\tfrac{1}{6}(x+\tfrac{3}{2})^2+F_r,$$

where $$F_1=F_2=-\tfrac{1}{24},\quad F_3=-\tfrac{3}{8};$$

$$x^3+(x-3)^3+\ldots+r^3=\tfrac{1}{12}(x+\tfrac{3}{2})^4-\tfrac{3}{8}(x+\tfrac{3}{2})^2+F_r,$$

where $$F_1=F_2=\tfrac{17}{192},\quad F_3=\tfrac{27}{64};$$

$$x+(x-4)+\ldots+r=\tfrac{1}{8}(x+2)^2+F_r,$$

where $$F_1=F_3=-\tfrac{1}{8},\quad F_2=0,\quad F_4=-\tfrac{1}{2};$$

$$x^3+(x-4)^3+\ldots+r^3=\tfrac{1}{16}(x+2)^4-\tfrac{1}{2}(x+2)^2+F_r,$$

where $$F_1=F_3=\tfrac{7}{16},\quad F_2=0,\quad F_4=1.$$

§ 31. The following list in which the powers in the series are even corresponds to that in § 8:

$$x^2+(x-2)^2+\ldots+r^2=\tfrac{1}{6}(x+1)^3-\tfrac{1}{6}(x+1)+K_r,$$

where $$K_1=0,\quad K_2=0;$$

$$x^4+(x-2)^4+\ldots+r^4=\tfrac{1}{10}(x+1)^5-\tfrac{1}{3}(x+1)^3+\tfrac{7}{30}(x+1)+K_r,$$

where $\qquad K_1=0,\quad K_2=0;$

$$x^2+(x-3)^2+\ldots+r^2=\tfrac{1}{9}(x+\tfrac{3}{2})^3-\tfrac{1}{4}(x+\tfrac{3}{2})+K_r,$$

where $\qquad K_1=-\tfrac{1}{9},\quad K_2=\tfrac{1}{9},\quad K_3=0;$

$$x^4+(x-3)^4+\ldots+r^4=\tfrac{1}{15}(x+\tfrac{3}{2})^5-\tfrac{1}{2}(x+\tfrac{3}{2})^3+\tfrac{63}{80}(x+\tfrac{3}{2})+K_r,$$

where $\qquad K_1=\tfrac{1}{3},\quad K_2=-\tfrac{1}{3},\quad K_3=0;$

$$x^2+(x-4)^2+\ldots+r^2=\tfrac{1}{12}(x+2)^3-\tfrac{1}{3}(x+2)+K_r,$$

where $\qquad K_1=-\tfrac{1}{4},\quad K_2=0,\quad K_3=\tfrac{1}{4},\quad K_4=0;$

$$x^4+(x-4)^4+\ldots+r^4=\tfrac{1}{20}(x+2)^5-\tfrac{2}{3}(x+2)^3+\tfrac{28}{15}(x+2)+K_r,$$

where $\qquad K_1=\tfrac{5}{4},\quad K_2=0,\quad K_3=-\tfrac{5}{4},\quad K_4=0.$

The values of the constants K_r are the same as in § 8.

§ 32. By putting $n=1$ and 2 in § 28, we obtain the following formulæ which correspond to those in § 17. The values of the constant g_r or k_r in each equation are the same as in the corresponding equation of § 17:

$$x-(x-q)+\ldots+(-1)^{\frac{x-r}{q}}r=\tfrac{1}{2}\left(x+\frac{q}{2}\right)+(-1)^{\frac{x-r}{q}}g_r,$$

$$x^3-(x-q)^3+\ldots+(-1)^{\frac{x-r}{q}}r^3=\tfrac{1}{2}\left(x+\frac{q}{2}\right)^3-\tfrac{3}{8}q^2\left(x+\frac{q}{2}\right)+(-1)^{\frac{x-r}{q}}g_r,$$

$$x^2-(x-q)^2+\ldots+(-1)^{\frac{x-r}{q}}r^2=\tfrac{1}{2}\left(x+\frac{q}{2}\right)^2-\tfrac{1}{8}q^2+(-1)^{\frac{x-r}{q}}k_r,$$

$$x^4-(x-q)^4+\ldots+(-1)^{\frac{x-r}{q}}r^4$$
$$=\tfrac{1}{2}\left(x+\frac{q}{2}\right)^4-\tfrac{3}{4}q^2\left(x+\frac{q}{2}\right)^2+\tfrac{5}{32}q^4+(-1)^{\frac{x-r}{q}}k_r.$$

These formulæ may, of course, be at once identified with those in § 17.

§ 33. The following list corresponds to §§ 15 and 16. The values of the constants g_r and k_r have been omitted as they are the same as in the corresponding equation of § 15 or § 16:

$$x - (x-2) + \ldots + (-1)^{\frac{1}{2}(x-r)} r = \tfrac{1}{2}(x+1) + (-1)^{\frac{1}{2}(x-r)} g_r,$$

$$x^3 - (x-2)^3 + \ldots + (-1)^{\frac{1}{2}(x-r)} r^3$$
$$= \tfrac{1}{2}(x+1)^3 - \tfrac{3}{2}(x+1) + (-1)^{\frac{1}{2}(x-r)} g_r,$$

$$x - (x-3) + \ldots + (-1)^{\frac{1}{3}(x-r)} r = \tfrac{1}{2}(x+\tfrac{3}{2}) + (-1)^{\frac{1}{2}(x-r)} g_r,$$

$$x^3 - (x-3)^3 + \ldots + (-1)^{\frac{1}{3}(x-r)} r^3$$
$$= \tfrac{1}{2}(x+\tfrac{3}{2})^3 - \tfrac{27}{8}(x+\tfrac{3}{2}) + (-1)^{\frac{1}{3}(x-1)} g_r,$$

$$x - (x-4) + \ldots + (-1)^{\frac{1}{4}(x-r)} r = \tfrac{1}{2}(x+2) + (-1)^{\frac{1}{4}(x-r)} g_r,$$

$$x^3 - (x-4)^3 + \ldots + (-1)^{\frac{1}{4}(x-r)} r^3$$
$$= \tfrac{1}{2}(x+2)^3 - 6(x+2) + (-1)^{\frac{1}{4}(x-r)} g_r,$$

$$x^2 - (x-2)^2 + \ldots + (-1)^{\frac{1}{2}(x-r)} r^2 = \tfrac{1}{2}(x+1)^2 - \tfrac{1}{2} + (-1)^{\frac{1}{2}(x-r)} k_r,$$

$$x^4 - (x-2)^4 + \ldots + (-1)^{\frac{1}{2}(x-r)} r^4$$
$$= \tfrac{1}{2}(x+1)^4 - 3(x+1)^2 + \tfrac{5}{2} + (-1)^{\frac{1}{2}(x-r)} k_r,$$

$$x^2 - (x-3)^2 + \ldots + (-1)^{\frac{1}{3}(x-r)} r^2 = \tfrac{1}{2}(x+\tfrac{3}{2})^2 - \tfrac{9}{8} + (-1)^{\frac{1}{3}(x-r)} k_r,$$

$$x^4 - (x-3)^4 + \ldots + (-1)^{\frac{1}{3}(x-r)} r^4$$
$$= \tfrac{1}{2}(x+\tfrac{3}{2})^4 - \tfrac{27}{4}(x+\tfrac{3}{2})^2 + \tfrac{405}{32} + (-1)^{\frac{1}{3}(x-r)} k_r,$$

$$x^2 - (x-4)^2 + \ldots + (-1)^{\frac{1}{4}(x-r)} r^2 = \tfrac{1}{2}(x+2)^2 - 2 + (-1)^{\frac{1}{4}(x-r)} k_r,$$

$$x^4 - (x-4)^4 + \ldots + (-1)^{\frac{1}{4}(x-r)} r^4$$
$$= \tfrac{1}{2}(x+2)^4 - 12(x+2)^2 + 40 + (-1)^{\frac{1}{4}(x-r)} k_r.$$

§ 34. The constants F_r satisfy relations of the same kind as the G-relations in § 18. For, since

$$F_r = \frac{q^{2n-1}}{2n}\left\{V_{2n}(\tfrac{1}{2}) - V_{2n}\left(\frac{r}{q}\right)\right\}, \quad (\S\ 27),$$

we have

$$\text{(i)}\quad F_r = F_{q-r},$$

$$\text{(ii)}\quad F_q = (-1)^n \frac{(2^{2n}-1)\,q^{2n-1}B_n}{2^{2n}n} = (-1)^n \frac{q^{2n-1}}{2n}\,\frac{\alpha_n}{2^{2n}},$$

$$\text{(iii)}\quad F_1 + F_2 + \ldots + F_q = (-1)^n \frac{\{(2^{2n-1}-1)\,q^{2n} + 2^{2n-1}\}\,B_n}{2^{2n}n},$$

with relations similar to those in § 19.

It may be noticed that, if q is even, $F_{\frac{1}{2}q} = 0$.

Values of the series in powers of $x^2 + qx$, §§ 35–37.

§ 35. The expressions for the series in § 27 are of the same form as those for the series $\Sigma_1^{\,x} x^{2n-1}$ and $\Sigma_1^{\,x} x^{2n}$ which were given in a paper* in Vol. XXX. (p. 167), and they can be expressed in terms of $x^2 + qx$ by exactly the same process as the latter were expressed in terms of $x^2 + x$.

The resulting formulæ are found to be

$$\text{(i)}\quad x^{2n-1} + (x-q)^{2n-1} + \ldots + r^{2n-1}$$

$$= \frac{1}{2nq}\left\{(x^2+qx)^r - Q_{2n,1}\left(\frac{q}{2}\right)^2 (x^2+qx)^{n-1} + Q_{2n,2}\left(\frac{q}{2}\right)^4 (x^2+qx)^{n-2} - \ldots \right.$$

$$\left. + (-1)^{n-2}\, Q_{2n,n-2}\left(\frac{q}{2}\right)^{2n-4} (x^2+qx)^2\right\} + T_r.$$

$$\text{(ii)}\quad x^{2n} + (x-q)^{2n} + \ldots + r^{2n}$$

$$= \frac{x+\frac{1}{2}q}{(2n+1)\,q}\left\{(x^2+qx)^n - P_{2n+1,1}\left(\frac{q}{2}\right)^2 (x^2+qx)^{n-1}\right.$$

$$\left. + P_{2n+1,2}\left(\frac{q}{2}\right)^4 (x^2+qx)^{n-2} - \ldots + (-1)^{n-1} P_{2n+1,n-1}\left(\frac{q}{2}\right)^{2n-2} (x^2+qx)\right\} + S_r,$$

where the coefficients $Q_{2n,r}$ and $P_{2n+1,r}$ are the same as in Vol. XXX., p. 181, viz.

$$Q_{2n,r} = (-1)^r \{n_{n-r} - (n-1)_{n-r}\,(2n)_2\,\beta_1 + (n-2)_{n-r}\,(2n)_4\,\beta_2 - \ldots$$

$$+ (-1)^r\,(n-r)_{n-r}\,(2n)_{2r}\,\beta_r\},$$

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* "On the sums of the series $1^n + 2^n + \ldots + x^n$ and $1^n - 2^n + \ldots \pm x^n$," Vol. XXX., pp. 166–204. The present paper is a sequel to this paper, of which it originally formed a second part.

$$P_{2n+1,r} = (-1)^r \{n_{n-r} - (n-1)_{n-r}(2n+1)_2 \beta_1 + (n-2)_{n-r}(2n+1)_4 \beta_2 - \ldots$$
$$+ (-1)^r (n-r)_{n-r} (2n+1)_{2r} \beta_r\},$$

and the values of the constants T_r and S_r are

$$T_r = \frac{q^{2n-1}}{2n} \left\{ V_{2n}(0) - V_{2n}\left(\frac{r}{q}\right) \right\},$$

$$S_r = -\frac{q^{2n}}{2n+1} V_{2n+1}\left(\frac{r}{q}\right).$$

These constants may be determined by putting $x = 0$ or $x = -q$, the left-hand side of the equation being replaced by its value in terms of the V-function (§ 1).

§ 36. Treating in the same manner the expressions in § 28, we find:

(i) $x^{2n+1} - (x-q)^{2n+1} + \ldots + (-1)^{\frac{x-r}{q}} r^{2n+1}$

$$= \frac{x + \frac{1}{2}q}{2} \left\{ (x^2 + qx)^n - p_{2n+1,1} \left(\frac{q}{2}\right)^2 (x^2 + qx)^{n-1} \right.$$

$$\left. + p_{2n+1,2} \left(\frac{q}{2}\right)^4 (x^2 + qx)^{n-2} - \ldots + (-1)^n p_{2n+1,n} \left(\frac{q}{2}\right)^{2n} \right\} + (-1)^{\frac{x-r}{q}} t_r,$$

where $$t_r = \frac{q^{2n+1}}{2n+2} U_{2n+2}\left(\frac{r}{q}\right);$$

(ii) $x^{2n} - (x-q)^{2n} + \ldots + (-1)^{\frac{x-r}{q}} r^{2n}$

$$= \tfrac{1}{2} \left\{ (x^2 + qx)^n - q_{2n,1} \left(\frac{q}{2}\right)^2 (x^2 + qx)^{n-1} \right.$$

$$\left. + q_{2n,2} \left(\frac{q}{2}\right)^4 (x^2 + qx)^{n-2} - \ldots + (-1)^{n-1} q_{2n,n-1} \left(\frac{q}{2}\right)^{2n-2} (x^2 + qx) \right\} + (-1)^{\frac{x-r}{q}} s_r,$$

where $$s_r = \frac{q^{2n}}{2n+1} U_{2n+1}\left(\frac{r}{q}\right).$$

The coefficients $p_{2n+1,r}$ and $q_{2n,r}$ are formed from the quantities $E_1, E_2, \ldots, E_r$ in exactly the same way as $P_{2n+1,r}$ and $Q_{2n,r}$ are formed from $\beta_1, \beta_2, \ldots, \beta_r$, *i.e.*

$$p_{2n+1,r} = (-1)^r \{n_{n-r} - (n-1)_{n-r}(2n+1)_2 E_1 \\ + \ldots + (-1)^r (n-r)_{n-r}(2n+1)_{2r} E_r\},$$

$$q_{2n,r} = (-1)^r \{n_{n-r} - (n-1)_{n-r}(2n)_2 E_1 \\ + \ldots + (-1)^r (n-r)_{n-r}(2n)_{2r} E_r\}.*$$

The values of the constants t_r and s_r may be obtained most simply by putting $x = -\frac{1}{2}q$ in the first formula, and $x = 0$ or $-q$ in the second, using the values of the series given in § 10.

§ 37. It may be observed that the formulæ in the two preceding sections may be easily derived from the formulæ in § 28 (p. 180) of the paper cited in § 35 by putting $\frac{x}{q}$ for x and multiplying by q^{2n-1}, q^{2n}, or q^{2n+1} (as explained in § 4 of the present paper). We thus obtain the part of the expressions which involves $x^2 + qx$. The values of the constants T_r, S_r, t_r, s_r may then be obtained, as above, by putting $x = 0$ or $-q$, and $-\frac{1}{2}q$.

The same remark applies also to the formulæ in §§ 27 and 28, which may be similarly derived from §§ 3 and 15 of the paper in Vol. XXX.

Expressions for the series $(mq-p)^n + (mq-q-p)^n + \ldots + p^n$, &c. in terms of m, §§ 38–44.

§ 38. It is convenient to have formulæ which express such series as

$$1^n + 5^n + \ldots + (4m-3)^n,$$

$$3^n + 7^n + \ldots + (4m-1)^n, \text{ \&c.}$$

in terms of m. Such expressions are afforded by the general formula of § 24; for, putting

$$x = qm - p, \quad r = q - p,$$

* The coefficients $q_{2n,r}$ have, of course, no connection with the quantity q. I have retained the same notation for the coefficients as in the previous paper (Vol. XXX., pp. 166–204).

we find

$$(qm-p)^n+(qm-q-p)^n+\ldots+(q-p)^n$$
$$=\frac{q^n}{n+1}\left\{m^{n+1}-(n+1)_1 V_1\left(\frac{p}{q}\right)m^n+(n+1)_2 V_2\left(\frac{p}{q}\right)m^{n-1}-\ldots\right.$$
$$\left.+(-1)^n(n+1)_n V_n\left(\frac{p}{q}\right)m\right\}.$$

No constant is required, for (§ 24)

$$C_r=\frac{q^n}{n+1}\left\{V_{n+1}\left(\frac{q-p}{q}\right)-V_{n+1}\left(\frac{q-p}{q}\right)\right\}=0.$$

We may also express the right-hand side of the equation in the form

$$q^n\left\{\frac{m^{n+1}}{n+1}-A_1\left(\frac{p}{q}\right)m^n+n_1A_2\left(\frac{p}{q}\right)m^{n-1}-n_2A_3\left(\frac{p}{q}\right)m^{n-2}+\ldots\right.$$
$$\left.+(-1)^n n_{n-1}A_n\left(\frac{p}{q}\right)m\right\},$$

where, as in § 26,

$$A_r(x)=\frac{V_r(x)}{r}.$$

§ 39. Similarly, from § 25, we find

$$(qm-p)^n-(qm-q-p)^n+\ldots+(-1)^{m-1}(q-p)^n$$
$$=\frac{q^n}{n+1}\left\{(n+1)_1 U_1\left(\frac{p}{q}\right)m^n-(n+1)_2 U_2\left(\frac{p}{q}\right)m^{n-1}+\ldots\right.$$
$$\left.+(-1)^{n-1}(n+1)_n U_n\left(\frac{p}{q}\right)m+(-1)^n U_{n+1}\left(\frac{p}{q}\right)\right\}+(-1)^{m-1}C_r,$$

where $C_r=\dfrac{q^n}{n+1}U_{n+1}\left(\dfrac{q-p}{p}\right)=(-1)^n\dfrac{q^n}{n+1}U_{n+1}\left(\dfrac{p}{q}\right).$

Thus if we regard the series as ending with the term in m there is no constant term when m is even; and when m is uneven, the constant term is $(-1)^n\dfrac{2q^n}{n+1}U_{n+1}\left(\dfrac{p}{q}\right)$. We may also express the right-hand side of the equation in the form

$$q^n\left\{A_1'\left(\frac{p}{q}\right)m^n-n_1A_2'\left(\frac{p}{q}\right)m^{n-1}+\ldots+(-1)^{n-1}n_{n-1}A_n'\left(\frac{p}{q}\right)m\right.$$
$$\left.+(-1)^nA'_{n+1}\left(\frac{p}{q}\right)+(-1)^{m-1}\times(-1)^nA'_{n+1}\left(\frac{p}{q}\right)\right\};$$

where, as in § 26,

$$A_r'(x) = \frac{U_r(x)}{r}.$$

§ 40. Putting $q-p$ for p in § 38, we have

$$\{qm-(q-p)\}^n+\ldots+p^n = \frac{q^n}{n+1}\left\{m^{n+1}+(n+1)_1 V_1\left(\frac{p}{q}\right)m^n\right.$$

$$\left.+(n+1)_2 V_2\left(\frac{p}{q}\right)m^{n-1}+\ldots+(n+1)_n V_n\left(\frac{p}{q}\right)m\right\},$$

in which all the terms are positive.

§ 41. Thus, if we denote by $S_p(m, n)$ the value of the series whose highest term is $(qm-p)^n$ and lowest $(q-p)^n$, so that $S_{q-p}(m, n)$ denotes the series whose highest term is $\{qm-(q-p)\}^n$ and lowest p^n, we see that $S_p(m, n)$ and $S_{q-p}(m, n)$ when expressed in powers of m differ from each other only in the signs of the alternate terms; so that the one expression is derivable from the other by merely changing the sign of m (and also the sign of the whole expression, if n be even), the formulæ connecting them being

$$S_{q-p}(m, n) = (-1)^{n+1}S_p(-m, n),$$

or, which is the same relation,

$$S_p(m, n) = (-1)^{n+1}S_{q-p}(-m, n).$$

§ 42. Since

$$V_n(0)+V_n\left(\frac{1}{q}\right)+\ldots+V_n\left(\frac{q-1}{q}\right) = \frac{1}{q^{n-1}}V_n(0),$$

we find from § 36, by putting $p=0, 1, 2, \ldots, q-1$ and adding, that

$$S_0(m, n)+S_1(m, n)+\ldots+S_{q-1}(m, n) = \frac{1}{n+1}\{(qm)^{n+1}$$

$$+\tfrac{1}{2}(n+1)_1(qm)^n+(n+1)_2 B_1(qm)^{n-1}-(n+1)_4 B_2(qm)^{n-2}+\ldots\},$$

the series being continued as far as the term involving m or m^2 according as n is even or uneven.

In both cases the right-hand side

$$=\frac{1}{n+1}\{V_{n+1}(qm+1)-V_{n+1}(1)\},$$

and it is evident that this is also the value of the left-hand side, which represents the value of the series

$$1^n+2^n+3^n+\ldots+(qm)^n.$$

There are also similar results in which the suffixes are multiples of any of the divisors of q. These formulæ may be obtained by the process employed in § 19.

§ 43. Similarly, if we denote the value of the series

$$(qm-p)^n-(qm-q-p)^n+\ldots+(-1)^{m-1}(q-p)^n$$

by $\sigma_p(m, n)$, we have

$$\sigma_{q-p}(m, n)=\{qm-(q-p)\}^n-\ldots+(-1)^{m-1}p^n$$

$$=\frac{q^n}{n+1}\left\{(n+1)_1\,U_1\left(\frac{p}{q}\right)m^n+(n+1)_2\,U_2\left(\frac{p}{q}\right)m^{n-1}+\ldots\right.$$

$$\left.+(n+1)_n\,U_n\left(\frac{p}{q}\right)m+U_{n+1}\left(\frac{p}{q}\right)\right\}+(-1)^{m-1}\frac{q^n}{n+1}\,U_{n+1}\left(\frac{p}{q}\right),$$

so that

$$\sigma_{q-p}(m, n)=(-1)^n\,\sigma_p(-m, n).$$

§ 44. Since, if q is uneven,

$$U_n(0)-U_n\left(\frac{1}{q}\right)+U_n\left(\frac{2}{q}\right)-\ldots+U_n\left(\frac{q-1}{q}\right)=\frac{1}{q^{n-1}}\,U_n(0),$$

we find by addition that, q being uneven,

$$\sigma_0(m, n)-\sigma_1(m, n)+\sigma_2(m, n)-\ldots+\sigma_{q-1}(m, n)$$

$$=\frac{1}{n+1}\{(n+1)_1\,U_1(0)(qm)^n-(n+1)_2\,U_2(0)(qm)^{n-2}$$

$$+\ldots+(-1)^{n-1}(n+1)_n\,U_n(0)\,qm$$

$$+(-1)^n\,U_{n+1}(0)\}+(-1)^{m-1}\times(-1)^n\frac{1}{n+1}\,U_{n+1}(0)$$

$$=\frac{1}{2(n+1)}\{(n+1)(qm)^n+(n+1)_2\,\alpha_1(qm)^{n-2}+(m+1)_4\,\alpha_2(qm)^{n-4}+\ldots\}.$$

There is no term independent of m except when m and n are both uneven, and in this case the constant

$$(-1)^{\frac{1}{2}(n+1)}\frac{\alpha_{\frac{1}{2}(n+1)}}{n+1}$$

is to be added.

The first expression on the right-hand side clearly

$$=\frac{1}{n+1}\{U_{n+1}(qm+1)+(-1)^{m-1}U_{n+1}(1)\},$$

which is also the value of the left-hand side, as the σ-expression represents the series

$$(qm)^n-(qm-1)^n+(qm-2)^n-\ldots+(-1)^{qm-1}1^n.$$

There are also other similar results corresponding to multiples of the divisors of q.

The cases $q=2, 3, 4, 6$, §§ 45–56.

§ 45. Putting $q=2$, $p=1$, in § 38, we have

$$(2m-1)^n+(2m-3)^n+\ldots+1^n$$
$$=\frac{2^n}{n+1}\left\{m^{n+1}-(n+1)_2\frac{\beta_1}{2^2}m^{n-1}+(n+1)_4\frac{\beta_2}{2^4}m^{n-2}-\ldots\right\},$$

the series being continued up to and including the term in m or m^2 according as m is even or uneven.

Substituting for the β's their numerical values, the right-hand side becomes

$$2^n\left\{\frac{m^{n+1}}{n+1}-\tfrac{1}{24}n_1m^{n-1}+\tfrac{7}{960}n_3m^{n-3}-\tfrac{31}{8064}n_5m^{n-5}+\ldots\right\}.$$

§ 46. Since

$$V_{2n+1}(\tfrac{1}{3})=(-1)^{n+1}\frac{(2n+1)I_n}{3^{2n+1}},\quad V_{2n}(\tfrac{1}{3})=(-1)^n\frac{(3^{2n}-3)B_n}{2.3^{2n}},$$

we find, by putting $q=3$, $p=1$,

$$(3m-1)^n+(3m-4)^n+\ldots+2^n=3^n\left\{\frac{m^{n+1}}{n+1}+\tfrac{1}{6}m^n-\tfrac{1}{36}n_1m^{n-1}\right.$$
$$\left.-\tfrac{1}{81}n_2m^{n-2}+\tfrac{13}{3240}n_3m^{n-3}+\tfrac{1}{243}n_4m^{n-4}-\ldots\right\}.$$

The terms are alternately positive and negative in pairs, and the series is to be continued up to and including the term in m.

Changing the signs of the alternate terms on the right-hand side we have (§ 41)

$$(3m-2)^n+(3m-3)^n+\ldots+1^n=3^n\left\{\frac{m^{n+1}}{n+1}-\tfrac{1}{6}m^n-\tfrac{1}{36}n_1m^{n-1}\right.$$
$$\left.+\tfrac{1}{81}n_2m^{n-2}+\ldots\right\}.$$

the terms after the first being negative and positive in pairs.

§ 47. By putting $q=4$, $p=1$, we find in the same way

$$(4m-1)^n+(4m-5)^n+\ldots+3^n=4^n\left\{\frac{m^{n+1}}{n+1}+\tfrac{1}{4}m^n-\tfrac{1}{96}m^{n-1}\right.$$
$$\left.-\tfrac{1}{64}n_2m^{n-2}+\tfrac{7}{15360}n_3m^{n-3}+\tfrac{5}{1024}n_4m^{n-4}-\ldots\right\},$$

the series being continued as before.

Also, changing the signs of the alternate terms,

$$(4m-3)^n+(4m-7)^n+\ldots+1^n=4^n\left\{\frac{m^{n+1}}{n+1}-\tfrac{1}{4}m^n-\tfrac{1}{96}m^{n-1}+\ldots\right\}.$$

§ 48. The corresponding formulæ for $q=6$ are:

$$(6m-1)^n+(6m-5)^n+\ldots+5^n=6^n\left\{\frac{m^{n+1}}{n+1}+\tfrac{1}{3}m^n+\tfrac{1}{72}n_1m^{n-1}\right.$$
$$\left.-\tfrac{5}{324}n_2m^{n-2}-\tfrac{91}{25920}n_3m^{n-3}+\tfrac{17}{3888}n_4m^{n-4}+\ldots\right\},$$

in which the terms after the first are positive and negative in pairs; and

$$(6m-5)^n+(6m-11)^n+\ldots+1^n=6^n\left\{\frac{m^{n+1}}{n+1}-\tfrac{1}{3}m^n+\tfrac{1}{72}n_1m^{n-1}+\ldots\right\},$$

in which the terms after the second are positive and negative in pairs. Both series are to be continued so as to include the term in m.

§ 49. The numerical coefficients in the series of the four preceding sections, which are the values of the quantities $V_n(\tfrac{1}{2})$, $V_n(\tfrac{1}{3})$, $V_n(\tfrac{1}{4})$, $V_n(\tfrac{1}{6})$ for $n=1, 2, \ldots, 5$, have been taken from pp. 305—307 of Vol. XXIX.*

* The quantities $V_n(\tfrac{1}{2})$, $V_n(\tfrac{1}{3})$, $V_n(\tfrac{1}{4})$, $V_n(\tfrac{1}{6})$ are there denoted by v_n, s_n, w_n, t_n respectively. In the general value of t_{2n+1} (Vol. XXIX., p. 307), the factor $(-1)^n$ should be $(-1)^{n+1}$, but the signs of the values of t_1, t_3, ... are correct.

§ 50. The following list contains the values of the series

$$(qm-p)^n+(qm-q-p)^n+\ldots+(q-p)^n$$

for the values 1, 2, 3, 4 of n, when q has the values 2, 3, 4, 6, and p is 1 or $q-1$ (*i.e.*, has the only values which are prime to q for these values of q).

The values contained in the list were derived from §§ 45–48:

$$(2m-1)+\ldots+1=m^2,$$
$$(2m-1)^2+\ldots+1^2=\tfrac{4}{3}m^3-\tfrac{1}{3}m,$$
$$(2m-1)^3+\ldots+1^3=2m^4-m^2,$$
$$(2m-1)^4+\ldots+1^4=\tfrac{16}{5}m^5-\tfrac{8}{3}m^3+\tfrac{7}{15}m,$$
$$(2m-1)^5+\ldots+1^5=\tfrac{16}{3}m^6-\tfrac{20}{3}m^4+\tfrac{7}{3}m^2,$$

$$(3m-2)+\ldots+1=\tfrac{3}{2}m^2-\tfrac{1}{2}m,$$
$$(3m-2)^2+\ldots+1^2=3m^3-\tfrac{3}{2}m^2-\tfrac{1}{2}m,$$
$$(3m-2)^3+\ldots+1^3=\tfrac{27}{4}m^4-\tfrac{9}{2}m^3-\tfrac{9}{4}m^2+m,$$
$$(3m-2)^4+\ldots+1^4=\tfrac{81}{5}m^5-\tfrac{27}{2}m^4-9m^3+6m^2+\tfrac{13}{10}m,$$
$$(3m-2)^5+\ldots+1^5=\tfrac{81}{2}m^6-\tfrac{81}{2}m^5-\tfrac{135}{4}m^3+30m^3+\tfrac{39}{4}m^2-5m,$$

$$(3m-1)+\ldots+2=\tfrac{3}{2}m^2+\tfrac{1}{2}m,$$
$$(3m-1)^2+\ldots+2^2=3m^3+\tfrac{3}{2}m^2-\tfrac{1}{2}m,$$
$$(3m-1)^3+\ldots+2^3=\tfrac{27}{4}m^4+\tfrac{9}{2}m^3-\tfrac{9}{4}m^2-m,$$
$$(3m-1)^4+\ldots+2^4=\tfrac{81}{5}m^5+\tfrac{27}{2}m^4-9m^3-6m^2+\tfrac{13}{10}m,$$
$$(3m-1)^5+\ldots+2^5=\tfrac{81}{2}m^6+\tfrac{81}{2}m^5-\tfrac{135}{4}m^4-30m^3+\tfrac{39}{4}m^2+5m,$$

$$(4m-3)+\ldots+1=2m^2-m,$$
$$(4m-3)^2+\ldots+1^2=\tfrac{16}{3}m^3-4m^2-\tfrac{1}{3}m,$$
$$(4m-3)^3+\ldots+1^3=16m^4-16m^3-2m^2+3m,$$
$$(4m-3)^4+\ldots+1^4=\tfrac{256}{5}m^5-64m^4-\tfrac{32}{3}m^3+24m^2+\tfrac{7}{15}m,$$
$$(4m-3)^5+\ldots+1^5=\tfrac{512}{3}m^6-256m^5-\tfrac{160}{3}m^4+160m^3+\tfrac{14}{3}m-25m,$$

$(4m-1)+\ldots+3=2m^2+m,$

$(4m-1)^2+\ldots+3^2=\frac{16}{3}m^3+4m^2-\frac{1}{3}m,$

$(4m-1)^3+\ldots+3^3=16m^4+16m^3-2m^2-3m,$

$(4m-1)^4+\ldots+3^4=\frac{256}{5}m^5+64m^4-\frac{32}{3}m^3-24m^2+\frac{7}{15}m,$

$(4m-1)^5+\ldots+3^5=\frac{512}{3}m^6+256m^5-\frac{160}{3}m^4-160m^3+\frac{14}{3}m^2+25m,$

$(6m-5)+\ldots+1=3m^2-2m,$

$(6m-5)^2+\ldots+1^2=12m^3-12m^2+m,$

$(6m-5)^3+\ldots+1^3=54m^4-72m^3+9m^2+10m,$

$(6m-5)^4+\ldots+1^4=\frac{1296}{5}m^5-432m^4+72m^3+120m^2-\frac{91}{5}m,$

$(6m-5)^5+\ldots+1^5=1296m^6-2592m^5+540m^4+1200m^3$
$-273m^2-170m,$

$(6m-1)+\ldots+5=3m^2+2m,$

$(6m-1)^2+\ldots+5^2=12m^3+12m^2+m,$

$(6m-1)^3+\ldots+5^3=54m^4+72m^3+9m^2-10m,$

$(6m-1)^4+\ldots+5^4=\frac{1296}{5}m^5+432m^4+72m^3-120m^2-\frac{91}{5}m,$

$(6m-1)^5+\ldots+5^5=1296m^6+2592m^5+540m^4-1200m^3$
$-273m^2+170m.$

§ 51. Putting $q=2, p=1$ in § 39, and using the values

$$U_{2n+1}(\tfrac{1}{2})=(-1)^n\frac{(2n+1)E_n}{2^{2n+1}},\quad U_{2n}(\tfrac{1}{2})=0,$$

we find, after reduction,

$$(2m-1)^n-(2m-3)^n+\ldots+(-1)^{m-1}1^n$$
$$=2^{n-1}\left\{m^n-n_2\frac{E_1}{2^2}m^{n-2}+n_4\frac{E_2}{2^4}-\ldots+(-1)^{\frac{1}{2}(n-1)}n_{n-1}\frac{E_{\frac{1}{2}(n-1)}}{2^n}m\right\},$$

if n is uneven, and

$$=2^{n-1}\left\{m^n-n_2\frac{E_1}{2^2}m^{n-2}+n_4\frac{E_2}{2^4}-\ldots+(-1)^{\frac{1}{2}n}n_n\frac{E_{\frac{1}{2}n}}{2^n}\right.$$
$$\left.+(-1)^{m-1}\times(-1)^{\frac{1}{2}n}n_n\frac{E_{\frac{1}{2}n}}{2^n}\right\},$$

if n is even: viz. if n is uneven the series ends with the term in m; if n is even the series ends with the constant term and this term is repeated, but with the addition of the factor $(-1)^{m-1}$.

§ 52. Putting $q=3$, $p=1$ and substituting for $U_{2n+1}(\frac{1}{3})$ and $U_{2n}(\frac{1}{3})$ their numerical values, we find

$$(3m-1)^n-(3m-4)^n+\ldots+(-1)^{m-1}2^n=3^n\{\tfrac{1}{2}m^n+\tfrac{1}{12}n_1m^{n-1}$$

$$-\tfrac{1}{9}n_2m^{n-2}-\tfrac{13}{216}n_3m^{n-3}+\tfrac{11}{81}n_4m^{n-4}+\tfrac{121}{972}n_5m^{n-5}-\ldots$$

$$+\text{ term in } m^0+(-1)^{m-1}\times\text{ preceding term}\},$$

i.e. the series is to be continued up to and including the constant term, and then a term $=(-1)^{m-1}\times$ this constant term is to be added.

Thus the whole part of the expression which is independent of m is

$$\{1+(-1)^{m-1}\}\times(-1)^n\frac{U_{n+1}(\frac{1}{3})}{n+1},$$

which $=\{1+(-1)^{m-1}\}\times(-1)^s\dfrac{H_s}{3^{2s+1}}$, if $n=2s$,

and

$$=\{1+(-1)^{m-1}\}\times(-1)^{s-1}\frac{(2^{2s}-1)(3^{2s}-3)B_s}{4s.3^{2s}}, \text{ if } n=2s-1,$$

§ 53. Changing the sign of m on the right-hand side of the formula, we find (§ 43)

$$(3m-2)^n-(3m-5)^n+\ldots+(-1)^{m-1}1^n$$

$$=3^n\{\tfrac{1}{2}m^n-\tfrac{1}{12}n_1m^{n-1}-\tfrac{1}{9}n_2m^{n-2}+\ldots\},$$

the signs of the terms after the first being negative and positive in pairs. If the sign of the constant term is changed the sign of the additional term is also to be changed, *i.e.* we may conveniently regard the two terms as a single term having the factor $1+(-1)^{m-1}$.

§ 54. Putting $q=4$, $p=1$ and substituting for $U_{2n+1}(\frac{1}{4})$ and $U_{2n}(\frac{1}{4})$ their values, we find

$$(4m-1)^n-(4m-5)^n+\ldots+(-1)^{m-1}3^n=2.4^{n-1}\left\{m^n+\frac{Q_1}{4}n_1m^{n-1}\right.$$

$$\left.-\frac{P_1}{4^2}n_2m^{n-2}-\frac{Q_2}{4^3}n_3m^{n-3}+\frac{P_2}{4^4}n_4m^{n-4}+\frac{Q_3}{4^5}n_5m^{n-5}-\ldots\right\},$$

the series being continued up to the term in m^0, which, as before, is to be multiplied by $1+(-1)^{m-1}$.

Putting for $Q_1, P_1, Q_2, \ldots$ their numerical values the right-hand side

$$=2.4^{n-1}\{m^n+\tfrac{1}{4}n_1m^{n-1}-\tfrac{3}{16}n_2m^{n-2}-\tfrac{11}{64}n_3m^{n-3}+\tfrac{57}{256}n_4m^{n-4}$$
$$+\tfrac{361}{1024}n_5m^{n-5}-\ldots\};$$

and, changing the signs of the alternate terms,

$$(4m-3)^n-(4m-7)^n+\ldots+(-1)^{m-1}1^n$$
$$=2.4^{n-1}\{m^n-\tfrac{1}{4}n_1m^{n-1}-\tfrac{3}{16}n_2m^{n-2}+\ldots\}.$$

§ 55. The corresponding formulæ for $q=6$ are

$$(6m-1)^n-(6m-7)^n+\ldots+(-1)^{m-1}5^n=6^n\{\tfrac{1}{2}m^n+\tfrac{1}{6}n_1m^{n-1}$$
$$-\tfrac{5}{72}n_2m^{n-2}-\tfrac{23}{216}n_3m^{n-3}+\tfrac{205}{2592}n_4m^{n-4}+\tfrac{1681}{7776}n_5m^{n-5}-\ldots\},$$

and

$$(6m-5)^n-(6m-11)^n+\ldots+(-1)^{m-1}1^n$$
$$=6^n\{\tfrac{1}{2}m^n-\tfrac{1}{6}n_1m^{n-1}-\tfrac{5}{72}n_2m^{n-2}+\ldots\},$$

the last term in m^0 being multiplied by $1+(-1)^{n-1}$.*

§ 56. The following list which corresponds to that in § 48 contains the values of the series

$$(qm-p)^n-(qm-q-p)^n+\ldots+(-1)^{m-1}(q-p)^n$$

for the values 1, 2, 3, 4 of n, and the values 2, 3, 4, 6 of q,

* The coefficients in the series in §§ 51—55, viz., $U_n(\frac{1}{2})$, $U_n(\frac{1}{3})$, $U_n(\frac{1}{4})$, $U_n(\frac{1}{6})$, occur also as coefficients in the summation-formulæ in Vol. XXIX. (pp. 312—314), where they are denoted by u_n, h_n, g_n, k_n respectively. The numerical values of u_n and g_n for the first few values of n were given on pp. 312, 313 of that volume.

p having the values 1 and $q-1$. The results are derived from the formulæ in §§ 51–55:

$$(2m-1) - \ldots + (-1)^{m-1}\, 1 = m,$$
$$(2m-1)^2 - \ldots + (-1)^{m-1}\, 1^2 = 2m^2 - \{1 + (-1)^{m-1}\}\, \tfrac{1}{2},$$
$$(2m-1)^3 - \ldots + (-1)^{m-1}\, 1^3 = 4m^3 - 3m,$$
$$(2m-1)^4 - \ldots + (-1)^{m-1}\, 1^4 = 8m^4 - 12m^2 + \{1 + (-1)^{m-1}\}\, \tfrac{5}{2},$$
$$(2m-1)^5 - \ldots + (-1)^{m-1}\, 1^5 = 16m^5 - 40m^3 + 25m,$$

$$(3m-2) - \ldots + (-1)^{m-1}\, 1 = \tfrac{3}{2}m - \{1 + (-1)^{m-1}\}\, \tfrac{1}{4},$$
$$(3m-2)^2 - \ldots + (-1)^{m-1}\, 1^2 = \tfrac{9}{2}m^2 - \tfrac{3}{2}m - \{1 + (-1)^{m-1}\}\, 1,$$
$$(3m-2)^3 - \ldots + (-1)^{m-1}\, 1^3 = \tfrac{27}{2}m^3 - \tfrac{27}{4}m^2 - 9m + \{1 + (-1)^{m-1}\}\, \tfrac{13}{8},$$
$$(3m-2)^4 - \ldots + (-1)^{m-1}\, 1^4 = \tfrac{81}{2}m^4 - 27m^3 - 54m^2 + \tfrac{39}{2}m + \{1 + (-1)^{m-1}\}\, 11,$$
$$(3m-2)^5 - \ldots + (-1)^{m-1}\, 1^5 = \tfrac{243}{2}m^5 - \tfrac{405}{4}m^4 - 270m^3 + \tfrac{585}{4}m^2 + 165m - \{1 + (-1)^{m-1}\}\, \tfrac{121}{4},$$

$$(3m-1) - \ldots + (-1)^{m-1}\, 2 = \tfrac{3}{2}m + \{1 + (-1)^{m-1}\}\, \tfrac{1}{4},$$
$$(3m-1)^2 - \ldots + (-1)^{m-1}\, 2^2 = \tfrac{9}{2}m^2 + \tfrac{3}{2}m - \{1 + (-1)^{m-1}\}\, 1,$$
$$(3m-1)^3 - \ldots + (-1)^{m-1}\, 2^3 = \tfrac{27}{2}m^3 + \tfrac{27}{2}m^2 - 9m - \{1 + (-1)^{m-1}\}\, \tfrac{13}{8},$$
$$(3m-1)^4 - \ldots + (-1)^{m-1}\, 2^4 = \tfrac{81}{2}m^4 + 27m^3 - 54m^2 - \tfrac{39}{2}m + \{1 + (-1)^{m-1}\}\, 11,$$
$$(3m-1)^5 - \ldots + (-1)^{m-1}\, 2^5 = \tfrac{243}{2}m^5 + \tfrac{405}{4}m^4 - 270m^3 - \tfrac{585}{4}m^2 + 165m + \{1 + (-1)^{m-1}\}\, \tfrac{121}{4}.$$

$$(4m-3) - \ldots + (-1)^{m-1} 1 = 2m - \{1 + (-1)^{m-1}\}\, \tfrac{1}{2},$$
$$(4m-3)^2 - \ldots + (-1)^{m-1} 1^2 = 8m^2 - 4m - \{1 + (-1)^{m-1}\}\, \tfrac{3}{2},$$
$$(4m-3)^3 - \ldots + (-1)^{m-1} 1^3 = 32m^3 - 24m^2 - 18m + \{1 + (-1)^{m-1}\}\, \tfrac{11}{2},$$
$$(4m-3)^4 - \ldots + (-1)^{m-1} 1^4 = 128m^4 - 128m^3 - 144m^2 + 88m + \{1 + (-1)^{m-1}\}\, \tfrac{57}{2},$$
$$(4m-3)^5 - \ldots + (-1)^{m-1} 1^5 = 512m^5 - 640m^4 - 960m^3 + 880m^2 + 570m + \{1 + (-1)^{m-1}\}\, \tfrac{361}{2},$$

$$(4m-1)\ - \ldots + (-1)^{m-1}3\ = 2m + \{1 + (-1)^{m-1}\}\tfrac{1}{2},$$

$$(4m-1)^2 - \ldots + (-1)^{m-1}3^2 = 8m^2 + 4m - \{1 + (-1)^{m-1}\}\tfrac{3}{2},$$

$$(4m-1)^3 - \ldots + (-1)^{m-1}3^3 = 32m^3 + 24m^2 - 18m - \{1 + (-1)^{m-1}\}\tfrac{11}{32},$$

$$(4m-1)^4 - \ldots + (-1)^{m-1}3^4 = 128m^4 + 128m^3 - 144m^2 - 88m + \{1 + (-1)^{m-1}\}\tfrac{57}{2},$$

$$(4m-1)^5 - \ldots + (-1)^{m-1}3^5 = 512m^5 + 640m^4 - 960m^3 - 880m^2 + 570m + \{1 + (-1)^{m-1}\}\tfrac{361}{2},$$

$$(6m-5)\ - \ldots + (-1)^{m-1}1\ = 3m - \{1 + (-1)^{m-1}\}1,$$

$$(6m-5)^2 - \ldots + (-1)^{m-1}1^2 = 18m^2 - 12m - \{1 + (-1)^{m-1}\}\tfrac{5}{2},$$

$$(6m-5)^3 - \ldots + (-1)^{m-1}1^3 = 108m^3 - 108m^2 - 45m + \{1 + (-1)^{m-1}\}23,$$

$$(6m-5)^4 - \ldots + (-1)^{m-1}1^4 = 648m^4 - 864m^3 - 540m^2 + 552m + \{1 + (-1)^{m-1}\}\tfrac{205}{2},$$

$$(6m-5)^5 - \ldots + (-1)^{m-1}1^5 = 3888m^5 - 6480m^4 - 5400m^3 + 8280m^2 + 3075m - \{1 + (-1)^{m-1}\}1681,$$

$$(6m-1)\ - \ldots + (-1)^{m-1}5\ = 3m + \{1 + (-1)^{m-1}\}1,$$

$$(6m-1)^2 - \ldots + (-1)^{m-1}5^2 = 18m^2 + 12m - \{1 + (-1)^{m-1}\}\tfrac{5}{2},$$

$$(6m-1)^3 - \ldots + (-1)^{m-1}5^3 = 108m^3 + 108m^2 - 45m - \{1 + (-1)^{m-1}\}23,$$

$$(6m-1)^4 - \ldots + (-1)^{m-1}5^4 = 648m^4 + 864m^3 - 540m^2 - 552m + \{1 + (-1)^{m-1}\}\tfrac{205}{2},$$

$$(6m-1)^5 - \ldots + (-1)^{m-1}5^5 = 3888m^5 + 6480m^4 - 5400m^3 - 8280m^2 + 3075m + \{1 + (-1)^{m-1}\}1681.$$

ON THE PRIMITIVE SUBSTITUTION GROUPS OF DEGREE TEN.*

By G. A. Miller.

EVERY solvable group whose order is a composite number contains a self-conjugate Abelian subgroup whose order exceeds unity.† If the solvable group is primitive this self-conjugate subgroup must be transitive,‡ and, since a non-regular transitive group cannot be Abelian, it must also be regular. All of its substitutions besides identity must therefore be transformed transitively by all the substitutions of the group. Hence they must all be of a prime order (p) and the degree of the group must be a power of p. From this it follows directly that every primitive group of degree 10 must contain at least one composite factor of composition. Since the alternating and the symmetric group are well known we shall not consider them in what follows.

Since each of the subgroups of order 25 that is contained in the symmetric group of degree 10 contains a cyclical substitution of degree and order 5, and since every primitive group that contains a cyclical substitution of degree and order p contains the alternating group of this degree whenever the degree of the primitive group exceeds $p+2$,§ it follows that the orders of the groups under consideration cannot be divisible by 25 or by 7. They must be divisible by 5 since the order of any transitive group is a multiple of its degree. We may therefore assume that the order of each of these groups is $5\alpha_1(5k+1)$, where $5\alpha_1$ is the number of substitutions of the group that transform one of the conjugate subgroups of order 5 into itself, according to Sylow's theorem.

It is evident that each of the groups of order 81 that is contained in the symmetric group of degree 10 contains a substitution of degree 3. The groups under consideration can therefore not contain any subgroup of order 3^β, unless $\beta < 4$.

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* A very brief outline of the construction of these groups by means of tentative processes was published by Professor Cole, *Quarterly Journal of Mathematics*, Vol. XXVII., 1895, pp. 42–44. In the present article no tentative processes are employed and each step is proved. The method employed is not only very much less laborious than that by trials, but it also exhibits a number of important properties of these groups and indicates how they are related to known groups.

† Jordan, *Traité des Substitutions*, 1870, p. 395. ‡ *Ibid*, p. 41.

§ Miller, *Bulletin of the American Mathematical Society*, Vol. IV., 1898, p. 141.

It remains yet to determine a maximum number of the substitutions in the subgroups of order 2^α that are contained in the required groups, *i.e.* it remains to determine a superior limit of α.

Since the degree of each of the substitutions that can occur in one of the required groups must exceed 4* when the substitution is not identity, each of the positive substitutions of order 2 that can occur in one of the given subgroups of order 2^α must be of the form *ab.cd.ef.gh*. Such a subgroup can therefore not contain two positive substitutions of order 2 having the same system of intransitivity. Hence it follows directly that α cannot exceed 2 when the degree of each of the transitive constituents of this subgroup is 2.

If one of the transitive constituents of the given subgroup of order 2^α is of degree 4 the order of the subgroup cannot exceed the order of the group formed by the other constituents; viz. 16. Finally if one of the transitive constituents of the given subgroup is of degree 8 this constituent must contain a subgroup of half its order that does not include any substitution besides identity whose degree is less than 6.

If this subgroup is intransitive it must contain just two systems of intransitivity, viz. the systems of imprimitivity of the entire constituent group, and hence it must be obtained by making a transitive group of degree 4 simply isomorphic to itself. In this case its order could not exceed 8. If it is transitive it must be imprimitive and contain an intransitive subgroup whose order could not exceed 8 as has just been proved. If this order is 8 the group must contain negative substitutions. Hence we have proved that α cannot exceed 4 when the group of order 2^α is positive, and that it cannot exceed 8 when this group contains negative substitutions. The order of each of the required primitive groups that contains only positive substitutions must therefore be of the following form

$$5\alpha_1(5k+1) \equiv 5 . 2^\alpha . 3^\beta, \quad \alpha = 0, 1, 2, 3, 4; \ \beta = 0, 1, 2, 3.$$

We can easily prove that $\alpha_1 > 1$, for if $\alpha_1 = 1$ the group would be of order $5(5k+1)$ and its $5k+1$ conjugate subgroup of order 5 would contain $40(5k+1)$ elements while the entire group would contain $45(5k+1)$ elements.† The $5k$ substitutions of the group whose orders are not equal to 5 or 1 would contain $45(5k+1) - 40(5k+1) = 5(5k+1)$ elements. Since

* Netto, *Theory of Substitutions*, 1892, p. 138.

† Frobenius, *Crelle Journal*, Vol. CI., p. 287; cf. Miller, *Bulletin of the American Mathematical Society*, Vol. II., 1895, p. 75.

the degree of each of these substitutions would have to exceed 5 their total number of elements could not be less than $30k$. This is impossible since k must exceed unity in order that the group may contain a composite factor of composition.

The order of the largest subgroup that transforms one of the given $5k+1$ conjugate subgroups of order 5 into itself cannot exceed 40 since the subgroup cannot contain more than ten substitutions that are commutative to every substitution of the given subgroup of order 5. Hence $\alpha_1 = 2^\gamma$ and the given identity reduces to

$$2^{\gamma-1}(5k+1) \equiv 2^{\alpha-1}.3^\beta, \quad \alpha = 1, 2, 3, 4; \ \beta = 0, 1, 2, 3.$$

From this we observe that $5k+1$ must be of one of the sixteen divisors of $2^3.3^3$. Since the required divisors are $\equiv 1$, mod. 5, and cannot terminate with 1, they must terminate with 6. It is at once seen that the following three are the only possible pairs of values of α and β; $\alpha = 1, \beta = 1$; $\alpha = 2$, $\beta = 2$; $\alpha = 3, \beta = 3$. We proceed to prove that the last pair need not be considered, or that there is no positive primitive group of the degree 10 that contains $2^3.3^3 - 216$ subgroups of order 5.

It has been proved that the order of such a group would be 2160 and that each of its subgroups of order 5 would be transformed into itself by just ten substitutions. The subgroup containing all the substitutions of the entire group that do not involve a given element would include all the positive substitutions in the transitive substitution group of degree 9 which is simply isomorphic to the holomorph* of the non-cyclical group of order 9. Since this holomorph contains a self-conjugate regular subgroup of order 9 each of its $3k+1$ conjugate transitive subgroups of order 27 must contain three distinct subgroups of order 3 and degree 6; $3k+1$ cannot be greater than $216 \div 27$ since all of these subgroups must be conjugate. Hence $k=0$ or 1. It is clear that the former value of k cannot be possible since the group of isomorphisms of the non-cyclical group of order 9 contains more than one subgroup of order 3. The given group of order 216 must therefore contain just twelve subgroups of order 3 and degree 6. In the group of order 2160 there would have to be $12.10 \div 4 = 30$ such subgroups, and some one of them would have to be transformed into itself by at least $2160 \div 30 = 72$ substitutions.

It is easy to see that such a group of order 72 could not

* Burnside, *Theory of Groups*, 1897, p. 228.

occur in the required primitive group since the constituent of degree 6, which involves the elements of the given subgroup, would have to be of order 72, and the only group of this order and degree could clearly not be made isomorphic to any group whose degree does not exceed 4 in such a way as to give a group that contains no substitution whose degree is less than 6. Hence it is impossible to construct a positive primitive group of degree 10 that contains 216 subgroups of order 5. Since no imprimitive group of this degree is of order 1080 or 2160, and a primitive group cannot contain an intransitive self-conjugate subgroup, it is also impossible to construct a positive and negative primitive group containing the given number of subgroups of order 5. Hence each of the required primitive groups must contain either six or thirty-six subgroups of order 5.

If it contains six subgroups of order 5 its order must be 60, 120, or 240, since we have proved that such a subgroup must be transformed into itself by 10, 20, or 40 substitutions of the group. The entire group must transform these six subgroups according to an isomorphic substitution group of degree 6 involving 60 as a factor of composition. Since a self-conjugate subgroup of a primitive group must be transitive this isomorphism must be simple. That is, every primitive group of degree 10 that contains just six subgroups of order 5 must be simply isomorphic either to $(abcdef)_{60}$ or to $(abcdef)_{120}$.* It is well known that each of these groups contains only one conjugate set of maximal subgroups whose orders are obtained by dividing the order of the group by 10. Hence, *there are two, and only two, primitive groups of degree* 10 *that contain just six subgroups of order* 5, *and they are simply isomorphic respectively to the alternating and the symmetric groups of degree* 5.

If such a group contains thirty-six subgroups of order 5 its order must be 360, 720, or 1440 for the reason given above. If it is of order 1440 it must involve negative substitutions since the subgroup that transforms one of the given groups of order 5 into itself must involve such substitutions. We shall first consider the case when the group contains 360 as a composite factor of composition. It must then contain a self-conjugate subgroup of order 360, since there is only one simple abstract group of this order, and it must be simply isomorphic to the group of isomorphisms whose order is 1440 of this simple group of order 360, or to a subgroup of

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* Cayley, *Quarterly Journal of Mathematics*, Vol. XXV., 1891, p. 79.

this group of isomorphisms that includes the group of congredient isomorphisms. It is known, as well as evident, that this group of isomorphisms contains only four such subgroups, viz. one of order 360 and three of order 720, and that the last three groups are distinct.* It remains to prove that each one of these four subgroups, as well as the entire group of isomorphisms of the alternating group of order 360, contains only one set of conjugate maximal subgroups whose orders are obtained by dividing the orders of the groups by 10.

Since all the subgroups of order 9 in the given group of order 360 are conjugate, and each of them is transformed into itself by thirty-six operators of the group which form a maximal subgroup, each of the given five groups must contain maximal subgroups whose orders are obtained by dividing the orders of the entire group by 10. From the fact that the given group of order 360 contains no other subgroup of order 36, it follows directly that the given five groups can contain only one such system of conjugate subgroups. Hence, *each of these five groups can be represented in one, and only one, way as a primitive substitution group of degree* 10.

It is not difficult to prove that none of the groups of order 360, 720, or 1440 could contain 60 as a composite factor of composition. For if such a group would contain this factor it would have to contain a self-conjugate subgroup of order 60. As this subgroup could not include all the substitutions of order 5 it could not be self-conjugate. Since there is no simple group of composite order, besides those of orders 60 and 360, whose order divides 720, the *given seven groups must be all the possible primitive groups of degree* 10 *that do not contain the alternating group of this degree.* If we add to these seven groups the alternating and the symmetric group of degree 10, we have the nine possible primitive groups of this degree.†

Since the three primitive groups of order 720 are subgroups of the group of isomorphisms of the alternating group of degree 6, one of them must be simply isomorphic to the symmetric of this degree, as is well known. In one of the others the operators of order 5 are transformed into each of their four powers, while in the other they are transformed into only two powers. In other words, one of these groups contains thirty-six Abelian groups of order 10, while the other does not contain any Abelian group of this order. These facts are seen from the quotient groups of the group of order 40

* Hölder, *Mathematische Annalen*, Vol. XLVI., 1895, p. 344.

† Cole, *Quarterly Journal of Mathematics*, Vol. XXVII., 1895, p. 44.

in the given group of order 1440. Each of the primitive groups given above, with the exception of the alternating, the symmetric, and the group of order 360, is included in the list of the imprimitive groups of degree 12.* The group of order 1440 is obtained by using as head of the imprimitive group the simple isomorphism of the symmetric group of order 720 to itself, in which substitutions of degree 3 in one system correspond to substitutions of degree 6 in the other.

We have seen that each one of the nine primitive groups of degree 10 contains only one composite factor of composition and that these factors are the orders of the alternating groups of degrees 5, 6, and 10. Two of the groups contain the factor $5! \div 2$, two the factor $10! \div 2$, while there are five that contain the factor $6! \div 2$. The given group of order 1440 is especially interesting, since it is the only non-symmetric group of isomorphisms of an alternating group.†

Cornell University,
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A SIMPLE PROOF OF THE REALITY OF THE ROOTS OF DISCRIMINATING DETERMINANT EQUATIONS, AND OF KINDRED FACTS.

By Prof. E. B. Elliott.

1. WHAT is generally regarded as the simplest proof of the reality of the roots of the 'discriminating' determinant equations associated with quadratic forms, and more generally of the symmetrical Lagrangian characteristic equations associated with small motions of conservative systems, is that of Salmon as generalised by Routh. This proof, starting as it does from a property of first and second minor determinants, and relying on the delicate argument of Sturm's theorem, though brief cannot be regarded as free

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* Miller, *Quarterly Journal of Mathematics*, Vol. XXVIII., 1896, pp. 219–231.

† Hölder, *Mathematische Annalen*, Vol. XLVI., 1895, p. 340; also Miller, *Bulletin of the American Mathematical Society*, Vol. I., 1895, p. 258.

from difficulty to the unskilled. It is, I think, not adequately realised that reasoning of the simplest and most elementary nature suffices to establish propositions of this character, when we pay attention to the linear equations whose consistency is expressed by the vanishing of a determinant.

Take for instance the 'discriminating cubic' equation, in which a, b, c, f, g, h are real,

$$\begin{vmatrix} a-\kappa, & h \;, & g \\ h \;, & b-\kappa, & f \\ g \;, & f \;, & c-\kappa \end{vmatrix} = 0.$$

This expresses the necessary and sufficient condition that κ be such as to make the equations

$$ax + hy + gz = \kappa x,$$

$$hx + by + fz = \kappa y,$$

$$gx + fy + cz = \kappa z$$

consistent for values of x, y, z not all zero. If possible let one of the values of κ be imaginary, equal to $k + ik'$, say, where k' does not vanish. Let x, y, z go with κ in satisfying the linear equations. Another root of the cubic will then be $\kappa' = k - ik'$ the conjugate imaginary to κ; and x', y', z', the results of replacing i by $-i$ in x, y, z, will go with it in satisfying the linear equations. Multiply the linear equations, with x, y, z, κ in them, by x', y', z' respectively, add, and use on the left the x', y', z', κ' linear equations to simplify the coefficients of x, y, z. There results

$$(\kappa' - \kappa)(xx' + yy' + zz') = 0.$$

Now $\kappa' - \kappa = -2ik'$ and does not vanish. Hence, if $x = \xi + i\xi'$, $y = \eta + i\eta'$, $z = \zeta + i\zeta'$,

$$\xi^2 + \xi'^2 + \eta^2 + \eta'^2 + \zeta^2 + \zeta'^2 = 0;$$

i.e. a sum of positive squares, not all zero, must vanish. This being an absurdity, it follows that the supposition of there being an imaginary root κ is at variance with fact.

It is only for brevity that this has been expressed for the cubic. It is clear that the reasoning applies equally to establish that there can be no imaginary root of a discriminating determinant equation of any order.

2. Take now the general Lagrange's symmetrical characteristic equation

$$\begin{vmatrix} a_{11}-\kappa b_{11}, & a_{12}-\kappa b_{12}, & a_{13}-\kappa b_{13}, & \dots \\ a_{12}-\kappa b_{12}, & a_{22}-\kappa b_{22}, & a_{23}-\kappa b_{23}, & \dots \\ a_{13}-\kappa b_{13}, & a_{23}-\kappa b_{23}, & a_{33}-\kappa b_{33}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0,$$

where there are n rows and columns, where the a's and b's are real, and where the latter are restricted in value by the inequalities which ensure that the quadratic form

$$\Sigma b_{rr}\xi_r^2 + 2\Sigma b_{rs}\xi_r\xi_s$$

is always of one sign, and never vanishes, for real values not all zero of $\xi_1, \xi_2, \dots, \xi_n$. (The restriction that it never vanishes may, as will be seen later, be removed).

The equation is the necessary and sufficient condition for κ to be such that the n linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots &= \kappa\,(b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + \dots), \\ a_{12}x_1 + a_{22}x_2 + a_{23}x_3 + \dots &= \kappa\,(b_{12}x_1 + b_{22}x_2 + b_{23}x_3 + \dots), \\ a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + \dots &= \kappa\,(b_{13}x_1 + b_{23}x_2 + b_{33}x_3 + \dots), \\ \dots\dots\dots & \dots\dots\dots \end{aligned}$$

be consistent for values of the x's not all zero.

If there be an imaginary root κ, and if $x_1, x_2, \dots, x_n$ go with it in satisfying these linear equations, there must be a second imaginary root κ', the conjugate imaginary to κ, and so unequal to κ; and $x_1', x_2', \dots, x_n'$, the results of replacing i by $-i$ in $x_1, x_2, \dots, x_n$, must go with κ' in satisfying the linear equations.

Now multiply the linear equations, with κ and $x_1, x_2, \dots, x_n$ in them, by $x_1', x_2', \dots, x_n'$ respectively, add, and use on the left the facts that

$$a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \dots = \kappa'\,(b_{11}x_1' + b_{12}x_2' + b_{13}x_3' + \dots),$$

&c. &c.

We thus get

$$(\kappa' - \kappa)\,\{\Sigma b_{rr}x_r x_r' + \Sigma b_{rs}\,(x_r x_s' + x_r' x_s)\} = 0,$$

in which, as $\kappa' \neq \kappa$, the second factor must be the vanishing one.

But, if $x_r = \xi_r + i\xi_r' \quad (r = 1, 2, \ldots, n),$

so that, also $x_r' = \xi_r - i\xi_r' \quad (r = 1, 2, \ldots, n),$

$$x_r x_r' = \xi_r^2 + \xi_r'^2, \text{ and } x_r x_s' + x_r' x_s = 2(\xi_r \xi_s + \xi_r' \xi_s').$$

Consequently the condition is

$$(\Sigma b_{rr}\xi_r^2 + 2\Sigma b_{rs}\xi_r\xi_s) + (\Sigma b_{rr}\xi_r'^2 + 2\Sigma b_{rs}\xi_r'\xi_s') = 0.$$

But here the left is, by initial information, the sum of two parts which are of the same sign and cannot vanish. The condition cannot then be satisfied, and the supposition of there being an imaginary root κ is untenable.

3. We notice, of course, that the same theorem as to the reality of all roots holds, not only when, as above, there are suitable restrictions on the b's, while the a's are any real quantities, but also when the restrictions are on the a's, *i.e.* when the b's are any real quantities and the a's are real quantities such that

$$\Sigma a_{rr}\xi_r^2 + 2\Sigma a_{rs}\xi_r\xi_s$$

is of one sign and different from zero for all real values not all zero of the ξ's. We have simply to divide through every constituent of the determinant by κ, and so the determinant itself by κ^n, and apply the same reasoning throughout using $\frac{1}{\kappa}$ instead of κ. A supposed imaginary root being, by its essence, not zero or infinite, no doubt can affect the reasoning.

4. We may now remove the requirement that

$$\Sigma b_{rr}\xi_r^2 + 2\Sigma b_{rs}\xi_r\xi_s,$$

in the one case, or

$$\Sigma a_{rr}\xi_r^2 + 2\Sigma a_{rs}\xi_r\xi_s,$$

in the other, be never zero for real values not all zero of the ξ's, leaving only the requirement that it be always of one sign when not zero.* Direct reasoning from the linear equations can easily be made to give the conclusion when, to take the first case, the quadratic form is of the simple shape

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* Examples of Lagrangian determinants thus admitted are those obtained by taking κ from some but not all of the constituents in the principal diagonal of a symmetric determinant, and the bordered determinants of Routh's *Advanced Rigid Dynamics*, § 64.

$\Sigma b_{rr}\xi_r^2$, where there will now be less than n terms in the summation, but for generality an infinitesimal argument is simpler.

To the quadratic form

$$\Sigma b_{rr}\xi_r^2 + 2\Sigma b_{rs}\xi_r\xi_s$$

of which we are now told that it can never change sign for real not all vanishing ξ's, but not that it can never vanish, add ε times a sufficient number of squares of linearly independent linear functions of $\xi_1, \xi_2, \ldots, \xi_n$ to ensure that the altered quadratic form

$$\Sigma b'_{rr}\xi_r^2 + 2\Sigma b'_{rs}\xi_r\xi_s$$

is never zero unless all the ξ's vanish, taking ε real, small, and of the sign belonging to the given form. Each accented b differs from the corresponding unaccented one by a finite (or vanishing) real multiple of ε. Call the determinant equation of § 2

$$f(\kappa) = 0.$$

The result of replacing in it unaccented by accented b's will be of the form

$$f(\kappa) + \varepsilon\phi_1(\kappa) + \varepsilon^2\phi_2(\kappa) + \ldots + \varepsilon^n\phi_n(\kappa) = 0,$$

where the coefficients in $\phi_1, \phi_2, \ldots, \phi_n$ are finite (or vanishing), and where none of these functions of κ is of degree exceeding n.

This altered equation has only real roots by § 2; and this remains true, however much we diminish ε. Now, if $f(\kappa)=0$ had an imaginary root, ε might be taken so small that the above altered equation would have a root as near to that imaginary root as we please, and so also an imaginary one. Hence $f(\kappa)=0$ has only real roots. The only exception is when $f(\kappa)$ is really free from κ, as for instance in the case of

$$\begin{vmatrix} a-\kappa, & -\kappa \\ -\kappa, & -a-\kappa \end{vmatrix},$$

when there is either no root at all, or identical satisfaction for any value of κ, according as the value free from κ of $f(\kappa)$ is not or is zero.

5. The whole conclusion is then that Lagrange's equation, as written in § 2, can only be satisfied by an imaginary value of κ when, either

(1) it is an identity satisfied by all values of κ, or

(2) both the quadratic forms

$$\Sigma a_{rr}\xi_r^2 + 2\Sigma a_{rs}\xi_r\xi_s,$$
$$\Sigma b_{rr}\xi_r^2 + 2\Sigma b_{rs}\xi_r\xi_s$$

can assume both signs for real values of $\xi_1, \xi_2, \ldots, \xi_n$.

It does not militate against there being only real roots for the linear expressions on the right in the linear equations of § 2 to be linearly connected, or for those on the left to be linearly connected, or for those on the right and those on the left to be linearly connected in different ways. But if those on the left and those on the right are linearly connected in the same way, *i.e.* if the linear equations themselves are linearly connected, the case (1) above is encountered, and Lagrange's equation is an identity.

6. Another class of equations to which the same method may be applied is the class

$$\begin{vmatrix} \kappa a_{11} \quad , & \kappa a_{12}+b_{12}, & \kappa a_{13}+b_{13}, & \ldots, & \kappa a_{1n}+b_{1n} \\ \kappa a_{12}-b_{12}, & \kappa a_{22} \quad , & \kappa a_{23}+b_{23}, & \ldots, & \kappa a_{2n}+b_{2n} \\ \kappa a_{13}-b_{13}, & \kappa a_{23}-b_{23}, & \kappa a_{33} \quad , & \ldots, & \kappa a_{3n}+b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \kappa a_{1n}-b_{1n}, & \kappa a_{2n}-b_{2n}, & \kappa a_{3n}-b_{3n}, & \ldots, & \kappa a_{nn} \end{vmatrix} = 0,$$

where the determinant is formed from a symmetrical matrix $| a |$ and a skew symmetrical matrix $| b |$. The a's and b's are real.

This equation is the necessary and sufficient condition for κ to be such, that the linear equations

$$b_{12}x_2 + b_{13}x_3 + \ldots = \kappa\,(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots),$$
$$-b_{12}x_1 + b_{22}x_2 + b_{23}x_3 + \ldots = \kappa\,(a_{12}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots),$$
$$-b_{13}x_1 - b_{23}x_2 \qquad + \ldots = \kappa\,(a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + \ldots)$$

be consistent for values of $x_1, x_2, \ldots, x_n$ not all zero.

Suppose there to be a real root of the determinant equation. The linear equations, with this value of κ, are satisfied by real values of $x_1, x_2, \ldots, x_n$. (In fact, if imaginary values satisfied them their real parts alone must). Now multiply them by the real $x_1, x_2, \ldots, x_n$ respectively, and add. We get

$$0 = \kappa\,\{\Sigma a_{rr}x_r^2 + 2\Sigma a_{rs}x_rx_s\}.$$

Hence, if the quadratic form $\Sigma a_{rr}x_r^2 + 2\Sigma a_{rs}x_r x_s$ is one which is of one sign and not zero for every set of real and not all vanishing values of $x_1, x_2, \ldots, x_n$, we must have

$$\kappa = 0.$$

In this case then the determinant equation for κ has no non-vanishing real root.*

Now, the quadratic form being as supposed, let κ be an imaginary root, and let $x_1, x_2, \ldots, x_n$ go with it. The conjugate imaginary κ' must also be a root; and with it go $x_1', x_2', \ldots, x_n'$, the results of replacing i by $-i$ in $x_1, x_2, \ldots, x_n$.

Multiply the linear equations, with $x_1, x_2, \ldots, x_n$ and κ in them, by $x_1', x_2', \ldots, x_n'$ respectively, add, and use the $x_1', x_2', \ldots, x_n', \kappa'$ equations on the left. We get

$$\kappa' \{-(a_{11}x_1' + a_{12}x_2' + a_{13}x_3' + \ldots) x_1 - \ldots\}$$
$$= \kappa \{(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots) x_1' + \ldots\},$$

i.e. $$(\kappa + \kappa') \{\Sigma a_{rr} x_r x_r' + \Sigma a_{rs} (x_r x_s' + x_r' x_s)\} = 0,$$

i.e. as in § 2,

$$(\kappa + \kappa') \{(\Sigma a_{rr}\xi_r^2 + 2\Sigma a_{rs}\xi_r\xi_s) + (\Sigma a_{rr}\xi_r'^2 + 2\Sigma a_{rs}\xi_r'\xi_s')\} = 0,$$

where the ξ's and ξ''s are real and do not all vanish. Now, as before, the two quadratic forms in the second bracket are of the same sign and do not vanish. Consequently

$$\kappa + \kappa' = 0.$$

The conjugate imaginaries κ, κ' have then no real part, but are pure imaginaries $\pm ik'$.

Hence, provided that the quadratic form $\Sigma a_{rr}\xi_r^2 + 2\Sigma a_{rs}\xi_r\xi_s$ is always of one sign and not zero for real values, not all zero, of $\xi_1, \xi_2, \ldots, \xi_n$, the determinant equation we are considering is one of the form

$$\kappa^{n-2m} (\kappa^2 + \alpha_1^2)(\kappa^2 + \alpha_2^2) \ldots (\kappa^2 + \alpha_m^2) = 0.$$

If the order n be odd, one root at least is zero. This is merely the well-known fact that a skew symmetric determinant of odd order vanishes.

7. As in § 4 we may here too remove the restriction that the quadratic form be incapable of vanishing, leaving only the one that it be never capable of changing sign for real values of $\xi_1, \xi_2, \ldots, \xi_n$. The number of finite and vanishing

* This proof that there is no non-vanishing real root holds when the matrix of the a's is not symmetrical, provided that $\Sigma a_{rr}x_r^2 + \Sigma (a_{rs} + a_{sr}) x_r x_s$ be one signed for real values of $x_1, x_2, \ldots, x_n$.

roots may in such a case fall below n—indeed will, for the coefficient of κ^n is the determinant $|a|$, and vanishes as the a linear forms cannot now be linearly independent. It may also become a mere identity satisfied by all values of κ, which will be the case when the a and the b linear forms are connected by any the same linear relation. But, excluding this last possibility, such roots as remain will be all zero or purely imaginary, *i.e.* will all have no real part. For, if $f(\kappa)=0$ be the equation, we can as earlier, by adding ϵ times a sufficient number of squares to make the quadratic incapable of vanishing except for zero values of the ξ's, find an equation

$$f(\kappa)+\epsilon\psi_1(\kappa)+\epsilon^2\psi_2(\kappa)+\ldots+\epsilon^n\psi_n(\kappa)=0,$$

whose roots have no real part, however small ϵ be; whence it follows that $f(\kappa)=0$ can have no root with a real part, since, if it had a root $p+iq$, the ϵ equation would have a root $p+\delta p+i(q+\delta q)$, where δp, δq could be made smaller than any quantity we choose to assign by diminishing ϵ sufficiently, *i.e.* a root with a real part $p+\delta p$ as near p as we please.

8. In § 6 divide by κ^n, and put κ for $\frac{1}{\kappa}$. It is immediately deduced that, if $\Sigma a_{rr}\xi_r^2+2\Sigma a_{rs}\xi_r\xi_s$ cannot assume both signs or vanish for real values, not all zero, of $\xi_1, \xi_2, \ldots, \xi_n$, the equation

$$\begin{vmatrix} a_{11}\quad, & a_{12}+\kappa b_{12}, & a_{13}+\kappa b_{13}, & \ldots, & a_{1n}+\kappa b_{1n} \\ a_{12}-\kappa b_{12}, & a_{22}\quad, & a_{23}+\kappa b_{23}, & \ldots, & a_{2n}+\kappa b_{2n} \\ a_{13}-\kappa b_{13}, & a_{23}-\kappa b_{23}, & a_{33}\quad, & \ldots, & a_{3n}+\kappa b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n}-\kappa b_{1n}, & a_{2n}-\kappa b_{2n}, & a_{3n}-\kappa b_{3n}, & \ldots, & a_{nn} \end{vmatrix}=0$$

is of the form

$$(\alpha_1^2\kappa^2+1)(\alpha_2^2\kappa^2+1)\ldots(\alpha_m^2\kappa^2+1)=0,$$

i.e. $$(\kappa^2+\beta_1^2)(\kappa^2+\beta_2^2)\ldots(\kappa^2+\beta_m^2)=0,$$

where $2m \leqq n$, so that all the roots are pure imaginaries.

If the quadratic form may vanish but never change sign, there can in like manner be only pure imaginary and zero roots, unless the equation be a mere identity.

One fact as to the determinant equation of the present article is clear even when we are told nothing whatever as to the quadratic form. Except for the possibility of zero roots, which cannot occur unless the determinant $|a|$ vanishes, roots must go in pairs κ, $-\kappa$. For to put $-\kappa$ for κ in the equation leaves it unaltered.

ON $1^n(x-1)^n+2^n(x-2)^n+\ldots+(x-1)^n1^n$ AND OTHER SIMILAR SERIES.

By J. W. L. Glaisher.

§ 1. THE principal object of the present note is to give a formula for the series

$$1^n(x-1)^n+2^n(x-2)^n+3^n(x-3)^n+\ldots+(x-1)^n1^n$$

in powers of x. The form of the expression is peculiar, there being a long gap next to the highest power of x.

§ 2. Denoting the above series by $S_{n,n}(x)$ and using $(n)_r$ to denote the number of combinations of n things taken r together, the formula is:

$$S_{n,n}(x) = -(n)_1\frac{B_n}{n}x+(n)_3\frac{B_{n-1}}{n-1}x^3-(n)_5\frac{B_{n-2}}{n-2}x^5+\ldots$$

$$+(-1)^{\frac{1}{2}(n+1)}(n)_n\frac{B_{\frac{1}{2}(n+1)}}{\frac{1}{2}(n+1)}x^n+\frac{(n!)^2}{(2n+1)!}x^{2n+1},$$

if n is uneven; and

$$=-(n)_1\frac{B_n}{n}x+(n)_3\frac{B_{n-1}}{n-1}x^3-(n)_5\frac{B_{n-2}}{n-2}x^5+\ldots$$

$$+(-1)^{\frac{1}{2}n}(n)_{n-1}\frac{B_{\frac{1}{2}(n+2)}}{\frac{1}{2}(n+2)}x^{n-1}+\frac{(n!)^2}{(2n+1)!}x^{2n+1},$$

if n is even.

Thus when n is uneven there is a gap of n terms before the term in x^{2n+1}, and when n is even the gap is of $n+1$ terms. It is also noticeable that the coefficient of the term in x^{2n+1} is so different in form from the earlier terms, which depend upon the Bernoullian numbers; and that these Bernoullian numbers do not begin with B_1, but extend from $B_{\frac{1}{2}(n+1)}$ or $B_{\frac{1}{2}(n+2)}$ to B_n. The suffixes of the Bernoullian numbers decrease as the powers of x increase, the highest Bernoullian number B_n occurring in the coefficient of x.

§ 3. The last term $\dfrac{(n!)^2}{(2n+1)!}$ is the same in the two cases; and so is the series involving Bernoullian numbers, except that the number of terms, viz. $\dfrac{n+1}{2}$ when n is uneven and $\dfrac{n}{2}$ when

n is even, and the form of the last term when expressed by means of n, are different. The formula may therefore be written

$$S_{n,n}(x)=\frac{(n!)^2}{(2n+1)!}x^{2n+1}-(n)_1\frac{B_n}{n}x+(n)_3\frac{B_{n-1}}{n-1}x^3+(n)_5\frac{B_{n-2}}{n-2}x^5+\&c.,$$

the last term being

$$(-1)^{\frac{1}{2}(n+1)}(n)_n\frac{B_{\frac{1}{2}(n+1)}}{\frac{1}{2}(n+1)}x^n \text{ or } (-1)^{\frac{1}{2}n}(n)_{n-1}\frac{B_{\frac{1}{2}(n+2)}}{\frac{1}{2}(n+2)}x^{n-1},$$

according as n is uneven or even.

§ 4. Putting $n=1, 2, 3, \ldots$, in the formula, we find

$$S_{1,1}(x)=-\tfrac{1}{6}x+\tfrac{1}{6}x^3,$$
$$S_{2,2}(x)=-\tfrac{1}{30}x+\tfrac{1}{30}x^5,$$
$$S_{3,3}(x)=-\tfrac{1}{42}x+\tfrac{1}{60}x^3+\tfrac{1}{140}x^7,$$
$$S_{4,4}(x)=-\tfrac{1}{30}x+\tfrac{2}{63}x^3+\tfrac{1}{630}x^9,$$
$$S_{5,5}(x)=-\tfrac{5}{66}x+\tfrac{1}{12}x^3-\tfrac{1}{126}x^5+\tfrac{1}{2772}x^{11},$$
$$S_{6,6}(x)=-\tfrac{691}{2730}x+\tfrac{10}{33}x^3-\tfrac{1}{20}x^5+\tfrac{1}{12012}x^{13},$$
$$S_{7,7}(x)=-\tfrac{7}{6}x+\tfrac{691}{468}x^3-\tfrac{7}{22}x^5+\tfrac{1}{120}x^7+\tfrac{1}{51480}x^{15},$$
$$S_{8,8}(x)=-\tfrac{3617}{510}x+\tfrac{28}{3}x^3-\tfrac{1382}{585}x^5+\tfrac{4}{33}x^7+\tfrac{1}{218790}x^{17},$$

&c. &c.

§ 5. Reducing the coefficients to a common denominator, and writing the terms in the reverse order, these formulæ are:

$$S_{1,1}(x)=\frac{x^3-x}{6},$$

$$S_{2,2}(x)=\frac{x^5-x}{30},$$

$$S_{3,3}(x)=\frac{3x^7+7x^3-10x}{420},$$

$$S_{4,4}(x)=\frac{x^9+20x^3-21x}{630},$$

$$S_{5,5}(x)=\frac{x^{11}-22x^5+231x^3-210x}{2772},$$

$$S_{6,6}(x) = \frac{5x^{13} - 3003x^5 + 18200x^3 - 15202x}{60060},$$

$$S_{7,7}(x) = \frac{x^{15} + 429x^7 - 16380x^5 + 76010x^3 - 60060x}{51480},$$

$$S_{8,8}(x) = \frac{x^{17} + 26520x^7 - 516868x^5 + 2042040x^3 - 1551693x}{218790},$$

&c. &c.

§ 6. The isolated term in x^{2n+1} does not occur in the corresponding series in which the terms are alternately positive and negative; viz. supposing x even, and putting

$$\Sigma_{n,n}(x) = 1^n (x-1)^n - 2^n (x-2)^n + 3^n (x-3)^n - \ldots + (x-1)^n 1^n,$$

we have

$$\Sigma_{n,n}(x) = (n)_1 \frac{(2^{2n}-1)B_n}{n} x - (n)_3 \frac{(2^{2n-2}-1)B_{n-1}}{n-1} x^3 + (n)_5 \frac{(2^{2n-4}-1)B_{n-2}}{n-2}$$

$$- \ldots + (-1)^{\frac{1}{2}(n-1)} (n)_n \frac{(2^{n+1}-1)B_{\frac{1}{2}(n+1)}}{\frac{1}{2}(n+1)} x^n$$

$$\text{or } (-1)^{\frac{1}{2}n-1} (n)_{n-1} \frac{(2^{n+2}-1)B_{\frac{1}{2}(n+2)}}{\frac{1}{2}(n+2)} x^{n-1},^*$$

the first or second form of the last term being taken according as n is uneven or even. The denominators in these terms can contain only powers of 2.

As particular cases,

$$\Sigma_{1,1}(x) = \tfrac{1}{2}x,$$

$$\Sigma_{2,2}(x) = \tfrac{1}{2}x,$$

$$\Sigma_{3,3}(x) = \tfrac{3}{2}x - \tfrac{1}{4}x^3 = -\frac{x^3 - 6x}{4},$$

$$\Sigma_{4,4}(x) = \tfrac{17}{2}x - 2x^3 = -\frac{4x^3 - 17x}{2},$$

$$\Sigma_{5,5}(x) = \tfrac{155}{2}x - \tfrac{85}{4}x^3 + \tfrac{1}{2}x^5 = \frac{2x^5 - 85x^3 + 310x}{4},$$

$$\Sigma_{6,6}(x) = \tfrac{2073}{2}x - 310x^3 + \tfrac{51}{4}x^5 = \frac{51x^5 - 1240x^3 + 4146x}{4},$$

&c. &c.

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* This formula may be derived from that in § 2 by putting $\frac{1}{2}x$ for x, multiplying by 2^{2n+1}, and subtracting the original formula.

§ 7. An isolated term in x^{2n+1} occurs in the value of the series

$$1^n(x-1)^n + 3^n(x-3)^n + 5^n(x-5)^n + \ldots + (x-1)^n 1^n,$$

in which x is supposed even; for, denoting the series by $S_{n,n}(x, 2)$, we have, x being even,

$$S_{n,n}(x,2) = \tfrac{1}{2}\frac{(n!)^2}{(2n+1)!}x^{2n+1} + (n)_1\frac{(2^{2n-1}-1)B_n}{n}x - (n)_3\frac{(2^{2n-3}-1)B_{n-1}}{n-1}x^3$$

$$+ \ldots + (-1)^{\frac{1}{2}(n-1)}(n)_n\frac{(2^n-1)B_{\frac{1}{2}(n+1)}}{\frac{1}{2}(n+1)}x^n$$

$$\text{or } (-1)^{\frac{1}{2}n-1}(n)_{n-1}\frac{(2^{n+1}-1)B_{\frac{1}{2}(n+2)}}{\frac{1}{2}(n+2)}x^{n-1}.*$$

As before, the first or second form of the last term is to be taken according as n is uneven or even.

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* This formula, and that referred to in § 9 in which all the terms are positive, are derivable from the original formula in § 2; and, in general, if $a, b, c, \ldots$ are any divisors of x such that their product $abc\ldots$ is also a divisor of x, we may derive from the formula in § 2 the value of the series $1^n(x-1)^n + 2^n(x-2)^n + \ldots + (x-1)^n 1^n$, from which all terms have been omitted which are divisible by a or b or c .... The terms in the value of this series involve $\dfrac{(a^{2r-1}-1)(b^{2r-1}-1)(c^{2r-1}-1)\ldots B_r}{r}$. The method of proof is exactly similar to that employed in the case of the series $1^n + 2^n + \ldots + x^n$ in the *Messenger*, Vol. XXVIII., pp. 38—42, and the result bears a similar relation to the original formula, viz. if $[S_{n,n}(x)]_{a,b,c,\ldots}$ denotes the terms that are left in the series $S_{n,n}(x)$ when all those which are divisible by a or b or c ... have been removed, and if β_r denotes $(a^{2r-1}-1)(b^{2r-1}-1)(c^{2r-1}-1)\ldots B_r$, then

$$[S_{n,n}(x)]_{a,b,c,\ldots} = \frac{(n!)^2}{(2n+1)!}x^{2n+1}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots$$

$$+(-1)^{s-1}\left\{(n)_1\frac{\beta_n}{n} - (n)_3\frac{\beta_{n-1}}{n-1} + (n)_5\frac{\beta_{n-2}}{n-2} - \ldots\right\},$$

where s is the number of the divisors $a, b, c, \ldots$ and the series in brackets differs from that in the original formula for $S_{n,n}(x)$ only by the substitution of β's for B's.

If $a, b, c, \ldots$ are all the prime factors of x, so that $x = a^\alpha b^\beta c^\gamma \ldots$, then $[S_{n,n}(x)]_{a,b,c,\ldots}$ consists of all the terms in the series $S_{n,n}(x)$ which are prime to x, and the formula for the sum of these terms may be written

$$\frac{(n!)^2}{(2n+1)!}x^{2n+1}\Pi\left(1-\frac{1}{a}\right) - (n)_1\frac{B_n}{n}x\Pi(1-a^{2n-1}) + (n)_3\frac{B_{n-1}}{n-1}x^3\Pi(1-a^{2n-3}) - \&c.,$$

which differs from the value of $S_{n,n}(x)$ only by attaching to each power, x^r, of x the factor $\Pi(1-a^{2n-r})$, $=(1-a^{2n-r})(1-b^{2n-r})(1-c^{2n-r})$, $a, b, c, \ldots$ being the prime factors of x.

This result corresponds to Thacker's formula for the sum of those terms of the series $1^n + 2^n + \ldots + x^n$ which are prime to x (*Messenger*, *loc. cit*,, p. 41).

As particular cases:

$$\Sigma_{1,1}(x,\,2) = \tfrac{1}{6}x + \tfrac{1}{12}x^3 = \frac{x^3+2x}{12},$$

$$\Sigma_{2,2}(x,\,2) = \tfrac{7}{30}x + \tfrac{1}{60}x^5 = \frac{x^5+14x}{60},$$

$$\Sigma_{3,3}(x,\,2) = \tfrac{31}{42}x - \tfrac{7}{60}x^3 + \tfrac{3}{840}x^7 = \frac{3x^7 - 98x^3 + 620x}{840},$$

$$\Sigma_{4,4}(x,\,2) = \tfrac{127}{30}x - \tfrac{62}{63}x^3 + \tfrac{1}{1260}x^9 = \frac{2x^9 - 2480x^3 + 10668x}{2520},$$

&c. &c.

§ 8. When the terms in this series are alternately positive and negative (x being of the form $4h+2$), there is no isolated term, and the formula contains even, instead of uneven, powers of x. The Eulerian numbers occur in the coefficients instead of the Bernoullian numbers, so that these coefficients are always integral.

Denoting the series

$$1^n (x-1)^n - 3^n (x-3)^n + 5^n (x-5)^n - \ldots + (x-1)^n\, 1^n$$

by $\Sigma_{n,n}(x,\,2)$, we have, x being the form $4h+2$,

$$\Sigma_{n,n}(x,\,2) = E_n - (n)_2\, E_{n-1}x^2 + (n)_4\, E_{n-2}x^4 - \ldots$$
$$+ (-1)^{\frac{1}{2}(n-1)}\,(n)_{n-1}\, E_{\frac{1}{2}(n+1)}\, x^{n-1} \text{ or } (-1)^{\frac{1}{2}n}\,(n)_n\, E_{\frac{1}{2}n}\, x^n.$$

As particular cases:

$$\Sigma_{1,1}(x,\,2) = 1,$$
$$\Sigma_{2,2}(x,\,2) = 5 - x^2,$$
$$\Sigma_{3,3}(x,\,2) = 61 - 15x^2,$$
$$\Sigma_{4,4}(x,\,2) = 1385 - 366x^2 + 5x^4,$$
$$\Sigma_{5,5}(x,\,2) = 50521 - 13850x^2 + 305x^4,$$

&c. &c.

§ 9. The series

$$1^n (x-1)^n + 4^n (x-4)^n + 7^n (x-7)^n + \ldots + (x-2)^n\, 2^n,$$

x being a multiple of 3, contains the term $\frac{1}{3}\dfrac{(n)!}{(2n+1)!}x^{2n+1}$, and the coefficients in the series involve $\dfrac{(3^{2r-1}-1)B_r}{r}$. When the terms are alternately positive and negative, and x is of the form $6h+3$, the formula resembles that in § 8, the Eulerian numbers being replaced by certain coefficients H_n which are also integers.*

§ 10. The results stated in this paper have been derived from the following general formulæ in which ζ_r denotes $q^r A_r\left(\frac{p}{q}\right)$ and η_r denotes $q^r A_r'\left(\frac{p}{q}\right)$.†

(i) If x is a multiple of q, $=qr$ say,

$$p^n(x-p)^n+(p+q)^n(x-p-q)^n+(p+2q)^n(x-p-2q)^n+\dots$$
$$+(x-2q+p)^n(2q-p)^n+(x-q+p)^n(q-p)^n$$
$$=\frac{1}{q}\frac{(n!)^2}{(2n+1)!}x^{2n+1}+(-1)^n\frac{2}{q}\{(n)_1\zeta_{2n}x+(n)_3\zeta_{2n-2}x^3+\dots$$
$$+(n)_n\zeta_{n-1}x^n \text{ or } (n)_{n-1}\zeta_{n-2}x^{n-1}\}.$$

(ii) If x is an even multiple of q, $=2qr$ say,

$$p^n(x-p)^n-(p+q)^n(x-p-q)^n+\dots$$
$$+(x-2q+p)^n(2q-p)^n-(x-q+p)^n(q-p)^n$$
$$=(-1)^{n-1}\frac{2}{q}\{(n)_1\eta_{2n}x+(n)_2\eta_{2n-2}+\dots+(n)_n\eta_{n+1}x^n \text{ or } (n)_{n-1}\eta_{n-2}x^{n-1}\}.$$

(iii) If x is an uneven multiple of q, $=q(2r+1)$ say,

$$p^n(x-p)^n-(p+q)^n(x-p-q)^n+\dots$$
$$-(x-2q+p)^n(2q-p)^n+(x-q+p)^n(q-p)^n$$
$$=(-1)^n\frac{2}{q}\{\eta_{2n+1}+(n)_2\eta_{2n-1}x^2+\dots+(n)_n\eta_{n+1}x^n \text{ or } (n)_{n-1}\eta_{n+2}x^{n-1}\}.‡$$

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* The coefficients H_n are defined in the *Quarterly Journal*, Vol. XXIX., p. 47. When x is of the form $6h$ the terms involve $\dfrac{(2^{2r}-1)(3^{2r-1}-1)B_r}{r}$. In this case the formula is deducible from that in § 2.

† The functions $A_r(x)$ and $A_r'(x)$ are defined in the *Quarterly Journal*, Vol. XXIX., pp. 18 and 93.

‡ These formulæ occur in Part III. of a paper on the Bernoullian function which has not yet been printed. Parts I. and II. appeared in the *Quarterly Journal*, Vol. XXIX., pp. 1–168.

§ 11. The formula in § 2 is the case $p=1$, $q=1$ of (i) and that in § 7 is the case $p=1$, $q=2$. The formula in § 8 (involving Eulerian numbers) is the case $p=1$, $q=2$ of (iii). The formulæ referred to in § 9 are the cases $p=1$, $q=3$ of (i), (ii), and (iii).

I have not included more cases of the general formulæ in this note as my object was merely to draw attention to the result contained in § 2. The series

$$1^n (x-1)^n + 2^n (x-2)^n + \ldots + (x-1)^n 1^n$$

is so simple in form that it would seem unlikely that its value in powers of x should not have been given before, but, though I have had this formula by me for some time, I have not succeeded in finding it elsewhere; nor have I found any of the other results of the same class which are derivable from the general formulæ by giving particular values to p and q.

NOTE ON ALGEBRAIC EQUATIONS IN WHICH THE TERMS OF HIGHER DEGREES HAVE SMALL COEFFICIENTS.

By W. B. Morton, M.A.

THE following point arose in the course of a physical investigation. A quadratic equation is supplemented by a series of terms of higher degree, of which the coefficients are very small. The resulting equation, which we may write in the form

$$0 = a + 2bx + cx^2 + \phi(x) \ldots\ldots\ldots\ldots\ldots (1),$$

will have two roots differing from the roots of the unaltered quadratic by small quantities of the order of the coefficients in ϕ. It is required to find the condition that these displaced roots should be equal.

It is clear that an approximation to this condition may be obtained as follows. Distribute the small terms in ϕ over the large coefficients a, b, c, writing the equation in the form

$$0 = (a+\alpha) + 2(b+\beta)x + (c+\gamma)x^2 \ldots\ldots\ldots (2),$$

where α, β, γ are functions of x satisfying the identity

$$\alpha + 2\beta x + \gamma x^2 = \phi.$$

Now put in α, β, γ for x the value $-\frac{b}{c}$ or $-\frac{a}{b}$ and treat the equation as a quadratic with constant coefficients. We thus get the condition in the form

$$(b+\beta_0)^2 = (a+\alpha_0)(c+\gamma_0) \ldots\ldots\ldots\ldots (3),$$

$\alpha_0, \beta_0, \gamma_0$ being the results of the substitution in α, β, γ.

The question arises as to what partition of the small terms in ϕ will yield, on this method, the best approximation.

To answer this we find directly an expression for the required condition involving ascending powers of the small quantities in ϕ.

Differentiating (1), we have

$$0 = 2b + 2cx + \phi'.$$

Put in this $x = -\frac{b}{c} + \theta$, then, using Taylor's series, we have

$$0 = 2c\theta + \phi_0' + \phi_0''\theta + \tfrac{1}{2}\phi_0'''\theta^2 + \ldots,$$

where
$$\phi_0 = \phi\left(-\frac{b}{c}\right).$$

Solving this for θ, we find

$$\theta = -\frac{\phi_0'}{2c} + \frac{\phi_0'\phi_0''}{4c^2} - \frac{\phi_0'}{16c^3}(2\phi_0''^2 + \phi_0'\phi_0''') + \&c. \ldots (4).$$

Putting $-\frac{b}{c} + \theta$ for x in equation (1) gives the condition in the form

$$a = \frac{b^2}{c} - \phi_0 + \frac{\phi_0'^2}{4c} - \frac{\phi_0'^2\phi_0''}{8c^2} + \frac{\phi_0'^2}{48c^3}(\phi_0'\phi_0''' + 3\phi_0''^2) + \text{etc. } (5).$$

Now equation (3) gives us

$$a = -\alpha_0 + \frac{(b+\beta_0)^2}{c+\gamma_0}$$

$$= \frac{b^2}{c} - \left(\alpha_0 - \frac{2b}{c}\beta_0 + \frac{b^2}{c^2}\gamma_0\right) + \frac{1}{4c}\left(2\beta_0 - \frac{2b}{c}\gamma_0\right)^2$$

$$- \frac{1}{8c^2}\left(2\beta_0 - \frac{2b}{c}\gamma_0\right)^2 2\gamma_0 + \ldots (6).$$

Since $\alpha_0 - \frac{2b}{c}\beta_0 + \frac{b^2}{c}\gamma_0 = \phi_0$, the terms of the first order in equations (5) and (6) are identical, *i.e.* any partition of ϕ will give the criterion correct to this extent. We see further that if α, β, γ are taken to satisfy the three equations

$$\left.\begin{aligned} \alpha + 2\beta x + \gamma x^2 &= \phi \\ 2\beta + 2\gamma x &= \pm\,\phi' \\ 2\gamma &= \phi'' \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(7),$$

we shall have the coincidence carried to the third order.

If we carry out the same process, using the value $-\frac{a}{b}$ for x, we shall arrive at the set

$$\left.\begin{aligned} \alpha + 2\beta x + \gamma x^2 &= \phi \\ 2\alpha + 2\beta x &= \pm\,(2\phi - x\phi') \\ 2\alpha &= 2\phi - 2x\phi' + x^2\phi'' \end{aligned}\right\} \quad\ldots\ldots\ldots(8).$$

It will be seen that if we take the upper sign in the second equation of each set, the two sets give the same values for α, β, γ, but different values are got if the lower signs be taken. Therefore we have one grouping of the terms in ϕ which gives third order correctness when $-\frac{b}{c}$ is put for x, another when $-\frac{a}{b}$ is used, and a third which will do for either value. In the last-mentioned case the second and third of equations (7) are got by differentiating the first, as if α, β, γ were constants. The arrangement of equation (1) corresponding to these values is

$$0 = (a + \phi - x\phi' + \tfrac{1}{2}x^2\phi'') + 2\,(b + \tfrac{1}{2}\phi' - \tfrac{1}{2}x\phi'')\,x + (c + \tfrac{1}{2}\phi'')\,x^2 \quad\ldots\ldots\ldots\ldots(9).$$

If $\phi \equiv dx^3 + ex^4 + fx^5 + \ldots$, this gives

$$\begin{aligned} 0 &= a + 2bx + cx^2 + (dx^3 + ex^4 + fx^5) + \ldots \\ &= (a + dx^3 + 3ex^4 + 6fx^5 + \ldots) \\ &\quad + (2b - 3dx^2 - 8ex^4 - 15fx^5 - \ldots)\,x \\ &\quad + (c + 3dx + 6ex^2 + 10fx^3 + \ldots)\,x^2 \quad\ldots\ldots\ldots(10). \end{aligned}$$

The following extension to an equation of degree n is readily suggested. Let the n^{ic} with the added small terms of higher degree be written

$$0 = (a_n, a_{n-1}, \ldots, a_0)\,(1x)^n + \phi \quad\ldots\ldots\ldots\ldots(11),$$

and suppose we want the conditions that the n displaced roots should be equal.

Write the equation in the form

$$0 = (a_n + \alpha_n, a_{n-1} + \alpha_{n-1}, \ldots, a_0 + \alpha_0)(1x)^n \ldots\ldots(12),$$

where the α's are functions which satisfy the equations

$$\left.\begin{aligned}(\alpha_n, \alpha_{n-1}, \ldots, \alpha_0)(1x)^n &= \phi \\ n(\alpha_{n-1}, \ldots, \alpha_0)(1x)^{n-1} &= \phi' \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ n!\,\alpha_0 &= \phi^n\end{aligned}\right\}\ldots\ldots\ldots\ldots(13).$$

Put in the α's any one of the approximate values for x, viz. $-\frac{a_n}{a_{n-1}}, -\frac{a_{n-1}}{a_{n-2}}$, &c., then we can prove that the conditions may be written $\frac{a_n + \alpha_n}{a_{n-1} + \alpha_{n-1}} = \frac{a_{n-1} + \alpha_{n-1}}{a_{n-2} + \alpha_{n-2}} =$ &c. correctly to the third order of small quantities.

Assuming equations (13) we can write the $(n-1)$ successive differentials of (11) in the form

$$\left.\begin{aligned}&0 = (a_{n-1} + \alpha_{n-1}, a_{n-2} + \alpha_{n-2}, \ldots, a_0 + \alpha_0)(1x)^{n-1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &0 = (a_1 + \alpha_1) + (a_0 + \alpha_0)x\end{aligned}\right\}\ldots(14).$$

From these we have the system of equations to be satisfied by the repeated root

$$\left.\begin{aligned}0 &= (a_1 + \alpha_1) + (a_0 + \alpha_0)x \\ 0 &= (a_2 + \alpha_2) + (a_1 + \alpha_1)x \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 &= (a_n + \alpha_n) + (a_{n-1} + \alpha_{n-1})x\end{aligned}\right\}\ldots\ldots\ldots(15).$$

Let these be satisfied by $x = x_0 + \theta$, where x_0 is any one of the values $-\frac{a_n}{a_{n-1}}$, &c.

Taking the r^{th} equation of set (15), and using Taylor's theorem

$$\begin{aligned}0 &= a_r + \alpha_r + \theta\alpha_r' + \tfrac{1}{2}\theta^2\alpha_r'' + \ldots \\ &\quad + (x_0 + \theta)\{a_{r-1} + \alpha_{r-1} + \theta\alpha'_{r-1} + \tfrac{1}{2}\theta^2\alpha''_{r-1} + \ldots\} \\ &= a_r + \alpha_r + x_0(a_{r-1} + \alpha_{r-1}) + \theta\{a_{r-1} + \alpha_{r-1} + \alpha_r' + x_0\alpha'_{r-1}\} \\ &\quad + \tfrac{1}{2}\theta^2\{\alpha_r'' + x_0\alpha''_{r-1} + 2\alpha'_{r-1}\} + \ldots \ldots\ldots\ldots\ldots(16),\end{aligned}$$

in which, in the α's, the value x_0 is now put for x.

Consider now equations (13). On subtracting the second from the differential of the first, and so on, they give

$$\left.\begin{array}{r}(\alpha_n', \alpha'_{n-1}, \ldots, \alpha_0')(1x)^n = 0\\ (\alpha'_{n-1}, \ldots, \alpha_0')(1x)^{n-1} = 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \alpha_1' + \alpha_0' x = 0\end{array}\right\}\ldots\ldots\ldots\ldots(17),$$

which are equivalent to

$$\left.\begin{array}{r}\alpha_1' + \alpha_0' x = 0\\ \alpha_2' + \alpha_1' x = 0\\ \ldots\ldots\ \ldots\ldots\\ \alpha_n' + \alpha'_{n-1} x = 0\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(18),$$

therefore $\alpha_r' + x\alpha'_{r-1} = 0,$

and $\alpha_r'' + x\alpha''_{r-1} + \alpha'_{r-1} = 0.$

Therefore equation (16) reduces to

$$0 = a_r + \alpha_r + x_0(a_{r-1} + \alpha_{r-1}) + \theta(a_{r-1} + \alpha_{r-1}) + \tfrac{1}{2}\theta^2\alpha'_{r-1} + \&c.,$$

or $$x_0 + \theta = -\frac{a_r + \alpha_r}{a_{r-1} + \alpha_{r-1}} - \frac{\theta^2}{2}\frac{\alpha'_{r-1}}{a_{r-1}} + \text{higher orders.}$$

Equating this to the value got from the next equation, we have

$$\frac{a_r + \alpha_r}{a_{r-1} + \alpha_{r-1}} = \frac{a_{r+1} + \alpha_{r+1}}{a_r + \alpha_r} + \frac{\theta^2}{2}\left\{\frac{\alpha_r'}{a_r} - \frac{\alpha'_{r-1}}{a_{r-1}}\right\}.$$

But $\alpha_r' = -x\alpha'_{r-1}$; therefore the bracket following $\frac{\theta^2}{2}$

$$= -\frac{\alpha'_{r-1}}{a_r}\left\{x_0 + \frac{a_r}{a_{r-1}}\right\},$$

Now x_0 differs from $-\frac{a_r}{a_{r-1}}$ by a quantity of the same order as θ or α.

Therefore, in writing the condition as

$$\frac{a_r + \alpha_r}{a_{r-1} + \alpha_{r-1}} = \frac{a_{r+1} + \alpha_{r+1}}{a_r + \alpha_r},$$

we are omitting terms of the fourth and higher orders.

Queen's College,
Belfast,
7th April, 1899.

CORRECTION OF AN ERROR IN A PREVIOUS PAPER (pp. 36—49).

By G. W. Walker, B.A., A.R.C.Sc.

Mr. T. J. Bromwich, of St. John's College, has kindly called my attention to a numerical slip which occurs in my paper "On the scattering of electro-magnetic waves by a sphere," pp. 36–49 of the present volume.

The true values of the approximation to the functions on pp. 46, 47 ought to be

$$S_n(\lambda a) = (-)^n \frac{(\lambda a)^n}{1.3\ldots 2n+1}\left\{1 - \tfrac{1}{2}\frac{(\lambda a)^2}{2n+3}\right\},$$

$$f_n(\lambda a) = (-)^n \frac{1}{1.3\ldots 2n+1}\left\{1 - \tfrac{1}{2}\frac{(\lambda a)^2}{(2n+3)}\right\},$$

$$f_n(\kappa a) = (-)^n \frac{1.3\ldots(2n-1)}{(\kappa a)^{2n+1}}\left\{1 + \frac{(\kappa a)^2}{2.2n-1}\right\},$$

$$S_n(\kappa a) = (-)^n \frac{(\kappa a)^n}{1.3\ldots 2n+1}\left\{1 - \frac{\kappa^2 a^2}{2.2n+3}\right\},$$

$$\frac{\partial}{\partial a} a\, S_n(\kappa a) = (-)^n \frac{(\kappa a)^n}{1.3\ldots 2n+1}\left\{n+1 - \frac{n+3}{2}.\frac{\kappa^2 a^2}{2n+3}\right\},$$

$$\frac{1}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\lambda a) = (-)^n \frac{1}{1.3\ldots 2n+1}\left\{n+1 - \frac{n+3}{2}.\frac{\lambda^2 a^2}{2n+3}\right\},$$

$$\frac{1}{a^n}\frac{\partial}{\partial a} a^{n+1} f_n(\kappa a) = (-)^{n+1} \frac{1.3'\ldots 2n-1}{(\kappa a)^{2n+1}}\left\{n + \frac{n-2}{2.2n-1}\kappa^2 a^2\right\}.$$

The value of A_n' is unaltered, while A_n involves the factor $(\lambda^2 - \kappa^2)\, a^2$ in place of $(\lambda - \kappa)\, a$.

Thus A_n involves κa to a power *two* higher than A_n', instead of *one* higher as in my paper. The conclusions are thus unaltered.

Trinity College,
Cambridge,
12th October, 1899.

A CONGRUENCE THEOREM RELATING TO THE BERNOULLIAN NUMBERS.

By J. W. L. Glaisher.

§ 1. THE object of the present paper is to prove the following theorem relating to the Bernoullian numbers:—if B_n be the n^{th} Bernoullian number, and if p be any uneven prime and $j=\frac{p-1}{2}$, then

$$\frac{B_n}{n} \equiv (-1)^{tj} \frac{B_{n-tj}}{n-tj}, \text{ mod. } p,$$

t being any positive integer such that $n-tj$ is positive.

This theorem was enunciated in a paper* in Vol. XXIX. of the *Messenger* (p. 60), and the mode of proof was indicated on pp. 61–63. On account of its fundamental character I have wished to give the proof in a form which should be complete in itself, and not dependent upon any results quoted from another paper. Most of the investigations required in the proof have been given elsewhere as will be seen from the references in the notes.

In a second paper, having the same title, and which will appear in the same volume of the *Messenger*, I have considered the theorem and its consequences in some detail, and also its relation to Staudt's theorem.

Recurring formula for $\frac{(a^{2n}-1)B_n}{2n}$.

§ 2. Defining the Bernoullian numbers by the usual equation

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + \frac{B_1}{2!}x - \frac{B_2}{4!}x^3 + \frac{B_3}{6!}x^5 - \&\text{c.},$$

we have, a being any quantity,

$$\frac{1}{e^x-1} - \frac{a}{e^{ax}-1} = \frac{a-1}{2} - \frac{(a^2-1)B_1}{2!}x + \frac{(a^4-1)B_2}{4!}x^3 - \&\text{c.}$$

* "Fundamental theorems relating to the Bernoullian numbers," pp. 49–63.

Now let

$$\psi_n(a) = \frac{(a^{2n}-1)\,B_n}{2n}, \quad n > 0,$$

and for convenience put

$$\psi_0(a) = \frac{a-1}{2}.$$

Then

$$\frac{1}{e^x-1} - \frac{a}{e^{ax}-1} = \psi_0(a) - \frac{\psi_1(a)}{1!}x + \frac{\psi_2(a)}{3!}x^3 - \frac{\psi_3(a)}{5!}x^5 + \&c.$$

§ 3. Now, supposing a to be a positive integer,

$$\frac{1}{e^x-1} - \frac{a}{e^{ax}-1} = \frac{e^{(a-1)x} + e^{(a-2)x} + \ldots + e^x + 1 - a}{e^{ax}-1}.$$

The numerator

$$= (e^x-1)\,\{e^{(a-2)x} + 2e^{(a-3)x} + 3e^{(a-4)x} + \ldots + (a-2)\,e^x + a - 1\},$$

and therefore

$$\frac{e^{(a-2)x} + 2e^{(a-3)x} + 3e^{(a-4)x} + \ldots + (a-2)\,e^x + a - 1}{e^{(a-1)x} + e^{(a-2)x} + e^{(a-3)x} + \ldots + e^x + 1}$$

$$= \psi_0(a) - \frac{\psi_1(a)}{1!}x + \frac{\psi_2(a)}{3!}x^3 - \frac{\psi_3(a)}{5!}x^5 + \&c.$$

§ 4. Now let

$$S_r(a) = 1^r + 2^r + 3^r + \ldots + (a-1)^r,$$

then we have

$$e^{(a-2)x} + 2e^{(a-3)x} + 3e^{(a-4)x} + \ldots + (a-2)\,e + a - 1$$

$$= \left\{a + S_1(a)\,x + \frac{S_2(a)}{2!}x^2 + \frac{S_3(a)}{3!}x^3 + \&c.\right\}$$

$$\times \left\{\psi_0(a) - \frac{\psi_1(a)}{1!}x + \frac{\psi_2(a)}{3!}x^3 - \frac{\psi_3(a)}{5!}x^5 + \&c.\right\};$$

whence, equating the coefficients of x^{2n-1}, and writing for $\psi_0(a)$ its value $\frac{a-1}{2}$, we obtain the recurring equation

$$a\psi_n(a) - (2n-1)_2 S_2(a)\psi_{n-1}(a) + (2n-1)_4 S_4(a)\psi_{n-2}(a) - \dots + (-1)^{n-1}(2n-1)_{2n-2} S_{2n-2}(a)\psi_1(a)$$

$$= (-1)^n\left[(a-2)^{2n-1} + 2(a-3)^{2n-1} + 3(a-4)^{2n-1} + \dots + (a-2)1^{2n-1} - \frac{a-1}{2}\{1^{2n-1} + 2^{2n-1} + 3^{2n-1} + \dots + (a-1)^{2n-1}\}\right],*$$

in which $(n)_r$ denotes the number of combinations of n things taken r together.

§5. In this recurring equation the coefficients $(2n-1)_2$, $(2n-1)_4$, ... are all integers, as also are $S_2(a)$, $S_4(a)$, ..., which are sums of powers. The right-hand side is also an integer if a is uneven, but may have the denominator 2 when a is even.

By putting $n = 1, 2, 3, \dots$, we may calculate the values of $\psi_1(a)$, $\psi_2(a)$, $\psi_3(a)$, ..., and it is evident that $\psi_n(a)$ so calculated can contain only powers of a in the denominator besides the factor 2 which may occur when a is even. Thus $\psi_n(a)$ must be of the form $\frac{\text{integer}}{a^\mu 2^\nu}$; and, since the coefficient of $\psi_n(a)$ in the recurring formula is a, we see that only one new a can enter into the denominator with each new equation, so that μ cannot be greater than n; also $\nu = 0$ if a is uneven, and may be 0 or 1 if a is uneven.†

§6. It will now be shown, by means of the recurring formula, that

$$\psi_n(a) \equiv (-1)^{tj}\psi_{n-tj}, \text{ mod. } p,$$

where p is any prime and $j = \frac{p-1}{2}$. No simplification is produced by the particular form of the coefficients in the recurring formula for $\psi_n(a)$, and it is therefore convenient to

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* This recurring formula was obtained (by the same process as in the text) in the *Messenger*, Vol. XXIX., pp. 50–52. Recurring formulæ for $\psi_n(a)$ are also given, *loc. cit.*, pp. 56–60, but they are not available for proving the theorem $\psi_n(a) \equiv (-1)^{tj}\psi_{n-tj}(a)$, mod. p, by the method used in this paper. Other recurring formula which could be so used are given in § 20 of the present paper.

† The fact that $\psi_n(a)$ is an integer, except for powers of a and 2, affords a proof of Sylvester's theorem (enunciated by him without proof) that if n contains p^i as a factor, but does not contain $\frac{p-1}{2}$ as a factor, then the denominator of B_n is divisible by p^i (*Messenger*, *loc. cit.*, pp. 49–50, 52–53). In the statement of Sylvester's theorem *loc. cit.*, p. 49 n should be replaced by $2n$ both in the text and in Sylvester's own enunciation, quoted in the note, *i.e.* the condition is that $2n$ (not n) should not contain $p-1$.

prove a more general proposition by means of a recurring formula which includes that for $\psi_n(a)$. This proof occupies §§ 7-16.

Proof that $X_{2r+1} \equiv X_{2r+1-t(p-1)}$, mod. p.

§ 7. Let X_1, X_3, X_5, ... be quantities connected by the recurring equation

$$\lambda X_{2n+1} + (2n+1)_2 b_2 X_{2n-1} + (2n+1)_4 b_4 X_{2n-3} + \dots + (2n+1)_{2n} b_{2n} X_1 = c_{2n+1},$$

which is supposed to hold good for all positive and integral values of n.

Let λ be any constant, and suppose that b_{2r} and c_{2r+1} are quantities which satisfy respectively the congruences

$$b_{2r} \equiv b_{2r-(p-1)}, \text{ mod. } p,$$

$$c_{2r+1} \equiv c_{2r+1-(p-1)}, \text{ mod. } p,$$

for all prime values of p, and for all values of r such that the suffix on the right-hand side is > 0.

It will be shown that the X's satisfy the congruence

$$X_{2r+1} \equiv X_{2r+1-(p-1)}, \text{ mod. } p,$$

for all prime values of p and for all values of r greater than $\frac{p-3}{2}$.

§ 8. It follows from the congruences satisfied by the b's and c's that

$$b_{2r} \equiv b_{2r-t(p-1)}, \text{ mod. } p,$$

$$c_{2r+1} \equiv c_{2r+1-t(p-1)}, \text{ mod. } p,$$

where t is any positive integer, such that the suffixes are positive. (No meaning is assigned to any quantity with negative suffix).

§ 9. In proving the result

$$X_{2r+1} \equiv X_{2r+1-(p-1)}, \text{ mod. } p,$$

it is convenient to introduce the quantities X_0, X_2, X_4, ... which are all supposed to be zero, and also the zero quantities b_1, b_3, b_5, ... and c_2, c_4, c_6, ...

The recurring relation then becomes

$$\lambda X_{2n+1} + (2n+1)_1 b_1 X_{2n} + (2n+1)_2 b_2 X_{2n-1} + \dots + (2n+1)_{2n} b_{2n} X_1 + (2n+1)_{2n+1} b_{2n+1} X_0 = c_{2n+1};$$

and since b_{2r+1} and c_{2r} are zero for all values of r, we have

$$b_r \equiv b_{r-t(p-1)}, \text{ mod. } p,$$

$$c_r \equiv c_{r-t(p-1)}, \text{ mod. } p,$$

for all positive and integral values of r, subject only to the condition that r is greater than $t(p-1)$.

The result to be proved is that

$$X_r \equiv X_{r-(p-1)}, \text{ mod. } p,$$

for all values of r greater than $p-1$, which is equivalent to

$$X_r \equiv X_{r-t(p-1)}, \text{ mod. } p,$$

for all positive and integral values of r subject only to the condition that r is greater than $t(p-1)$.

It will now be assumed that this congruence holds good for the values $p, p+1, p+2, \ldots, 2n$ of r, and by means of the above recurring equation it will be shown that it holds good also for $r = 2n+1$.

§ 10. The proof mainly depends upon the following theorem,* in which, as throughout this paper, $(n)_r$ denotes the number of combinations of n things taken r together:— if p is any prime, and q and s are less than p, then

$$(kp+q)_{gp+s} \equiv (k)_g \times (q)_s, \text{ mod. } p, \text{ if } s \leqq q,$$

and $$\equiv 0, \text{ mod. } p, \text{ if } s > q.$$

§ 11. Let $2n+1 = kp+q$, p being any uneven prime, and $q<p$. Then writing, for brevity, m for $kp+q$ in certain cases, we have, from the recurring equation,

$$\begin{aligned} c_{2n+1} - \lambda X_{2n+1} &= b_{2n+1}X_0 + (2n+1)_1 b_{2n} X_1 + (2n+1)_2 b_{2n-1} X_2 \\ &\quad + \ldots + (2n+1)_{2n-1} b_2 X_{2n-1} + (2n+1)_{2n} b_1 X_{2n} \\ &= b_m X_0 + (m)_1 b_{2n} X_1 + \ldots + (m)_{p-1} b_{m-p+1} X_{p-1} \\ &\quad + (m)_p b_{m-p} X_p + (m)_{p+1} b_{m-p-1} X_{p+1} + \ldots + (m)_{2p-1} b_{m-2p+1} X_{2p-1} \\ &\quad + (m)_{2p} b_{m-2p} X_{2p} + \ldots + (m)_{3p-1} b_{m-3p+1} X_{3p-1} \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (m)_{(k-1)p} b_{p+q} X_{(k-1)p} + \ldots + (m)_{kp-1} b_{q+1} X_{kp-1} \\ &\quad + (m)_{kp} b_q X_{kp} + \ldots + (m)_{kp+q-1} b_1 X_{kp+q-1}, \end{aligned}$$

* *Quarterly Journal,* Vol. XXX., p. 152.

in which each of the first k lines contains p terms, and the last, or $(k+1)^{\text{th}}$, line contains q terms.

§ 12. Reducing the coefficients, and also the b's and X's, on the right-hand side, by the formulæ

$$\begin{aligned}(m)_{gp+s} &\equiv (k)_g \times (q)_s, \text{ mod. } p, \text{ if } s \leqq q,\\ &\equiv 0, \text{ mod. } p, \text{ if } s > q,\\ b_{gp+s} &\equiv b_{g+s}, \text{ mod. } p,\\ X_{gp+s} &\equiv X_{g+s}, \text{ mod. } p,\end{aligned}$$

the equation gives

$$\begin{aligned} c_{kp+q} - \lambda X_{kp+q} \equiv\ & b_{k+q}X_0 + (q)_1\, b_{k+q-1}X_1 + \ldots + (q)_q\, b_kX_q \\ &+ (k)_1\,\{b_{k+q-1}X_1 + (q)_1\, b_{k+q-2}X_2 + \ldots + (q)_q\, b_{k-1}X_{q+1}\} \\ &+ (k)_2\,\{b_{k+q-2}X_2 + (q)_1\, b_{k+q-3}X_3 + \ldots + (q)_q\, b_{k-2}X_{q+2}\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ (k)_{k-1}\,\{b_{q+1}X_{k-1} + (q)_1\, b_qX_k + \ldots + (q)_q\, b_1X_{k+q-1}\} \\ &+ (k)_k\,\{b_qX_k + (q)_1\, b_{q-1}X_{k+1} + \ldots + (q)_{q-1}\, b_1X_{k+q-1}\}, \text{ mod. } p. \end{aligned}$$

Collecting the coefficients of $X_0, X_1, X_2, \ldots$, the expression on the right-hand side

$$\begin{aligned} &= b_{k+q}X_0 \\ &+ \{(k)_1 + (q)_1\}\, b_{k+q-1}X_1 \\ &+ \{(k)_2 + (k)_1\,(q)_1 + (q)_2\}\, b_{k+q-2}X_2 \\ &+ \{(k)_3 + (k)_2\,(q)_1 + (k)_1\,(q)_2 + (q)_3\}\, b_{k+q-3}X_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \{(k)_k + (k)_{k-1}\,(q)_1 + (k)_{k-2}\,(q)_2 + \ldots + (q)_k{}^{*}\}\, b_qX_k \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \{(k)_k\,(q)_{q-1} + (k)_{k-1}\,(q)_q\}\, b_1X_{k+q-1}. \end{aligned}$$

§ 13. The coefficients of $b_{k+q}X_0$, $b_{k+q-1}X_1$, $b_{k+q-2}X_2$, ..., in this expression are the same as those of $x^0, x^1, x^2, \ldots$ in the product

$$\{1 + (k)_1\, x + (k)_2\, x^2 + \ldots\} \times \{1 + (q)_1\, x + (q)_2\, x^2 + \ldots\},$$

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* This term is not reached unless $k \leqq q$. We may regard the coefficient of $b_{k+q-t}X_t$ as being always $(k)_t + (k)_{t-1}\,(q)_1 + \ldots + (q)_{q-1}\,(k)_1 + (q)_t$, if we suppose that $(n)_r$ denotes zero when $r > n$.

that is, in the product

$$(1+x)^k \times (1+x)^q = (1+x)^{k+q}.$$

Thus the coefficient of $b_{k+q-r}X_r$ is $(k+q)_r$, and the congruence becomes

$$c_{kp+q} - \lambda X_{kp+q} \equiv b_{k+q}X_0 + (k+q)_1 b_{k+q-1}X_1 + (k+q)_2 b_{k+q-2}X_2 + \ldots$$
$$+ (k+q)_{k+q-1} b_1 X_{k+q-1}, \text{ mod. } p.$$

The recurring formula shows that the expression on the right-hand side $= c_{k+q} - \lambda X_{k+q}$, and therefore we have

$$c_{kp+q} - \lambda X_{kp+q} \equiv c_{k+q} - \lambda X_{k+q}, \text{ mod. } p.$$

Now $\quad c_{kp+q} \equiv c_{k+q}$, mod. p,

and therefore, dividing out by λ,

$$X_{kp+q} \equiv X_{k+q}, \text{ mod. } p,$$

so that the congruence

$$X_r \equiv X_{r-(p-1)}, \text{ mod. } p,$$

is true also for $r = 2n+1$.

§ 14. Now this congruence is certainly true for $r = p$, for, putting $2n+1 = p$ in the recurring relation (§ 9), we have

$$c_p - \lambda X_p = b_p X_0 + (p)_1 b_{p-1} X_1 + (p)_2 b_{p-2} X_2 + \ldots$$
$$+ (p)_{p-2} b_2 X_{p-2} + (p)_p b_1 X_{p-1},$$

and, since $X_0 = 0$ and $X_{p-1} = 0$, the right-hand side $\equiv 0$, mod. p, so that

$$c_p - \lambda X_p \equiv 0, \text{ mod. } p.$$

Putting $n = 0$ in the recurring formula, we have

$$c_1 - \lambda X_1 = 0,$$

and therefore $\quad c_p - \lambda X_p \equiv c_1 - \lambda X_1$, mod. p,

whence, since $c_p \equiv c_1$, mod. p, we find

$$X_p \equiv X_1, \text{ mod. } p.$$

Thus the theorem is true for $r=p$, and therefore for $r=p+1$, and for $r=p+2, \ldots$, and so on; that is, it is true for all values of $r>p-1$.

§ 15. It remains now to consider what values of p are excluded in the previous investigation. It is clear that p must not be a divisor of the numerator or denominator of λ, as otherwise we could not divide out by λ in § 13, nor may it be a divisor of a denominator of any of the X's. Now in calculating the X's from the recurring formula we see that their denominators can contain only powers of the numerator of λ and numbers occurring in the denominators of the b's and c's. Thus p may be any uneven prime which is not a divisor of the numerator or denominator of λ, or of any denominator of the b's and c's.

When the b's and c's are integers, the only condition is that p must not be a divisor of the numerator or denominator of λ.

§ 16. With respect to the b's and c's, it is evident that the required congruences (§ 7)

$$b_{2r} \equiv b_{2r-(p-1)}, \quad c_{2r+1} \equiv c_{2r+1-(p-1)}, \text{ mod. } p,$$

are satisfied, if they are powers, or sums of powers each multiplied by a constant, viz. if

$$b_{2r} = A\alpha^{2r} + B\beta^{2r} + C\gamma^{2r} + \ldots,$$

$$c_{2r+1} = A_1\alpha_1^{2r+1} + B_1\beta_1^{2r+1} + C_1\gamma_1^{2r+1} + \ldots,$$

where $A, B, \ldots, \alpha, \beta, \ldots, A_1, B_1, \ldots, \alpha_1, \beta_1, \ldots$ are arbitrary quantities. It is not necessary that these quantities should be integers, but when they are fractional p must not be a divisor of any of the denominators. The numerators need not be prime to p, for, *e.g.*

$$A\alpha^{2r} \equiv A\alpha^{2r-(p-1)}, \text{ mod. } p,$$

whether A or α is divisible by p or not, and the presence of terms $\equiv 0$, mod. p does not interfere with the investigation.*

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* The proof in §§ 7–16 that $X_{2n+1} \equiv X_{2n+1-t(p-1)}$, mod p, is contained in a paper "On a congruence theorem having reference to an extensive class of coefficients," communicated to the *London Mathematical Society*, on June 8, 1899, where the same method is applied also to two other general recurring series.

Application to $\psi_n(a)$.

§ 17. Comparing the recurring formula in § 7 with that for $\psi_n(a)$ found in § 4, we see that the latter is included in the former by putting

$$\lambda = a,$$

$$X_{2r+1} = (-1)^{r+1}\psi_{r+1}(a),$$

$$b_{2r} = S_{2r}(a),$$

$$c_{2r+1} = (a-2)^{2r+1} + 2(a-3)^{2r+1} + 3(a-4)^{2r+1} + \ldots + (a-2)\,1^{2r+1}$$

$$-\frac{a-1}{2}\{1^{2r+1} + 2^{2r+1} + 3^{2r+1} + \ldots + (a-1)^{2r+1}\}.$$

Thus b_{2r} is always integral, and c_{2r+1} is integral when a is uneven, but may have the denominator 2 when a is even.

Therefore, by the theorem proved, if p is any uneven prime which is not a divisor of a, then

$$(-1)^n\psi_n(a) \equiv (-1)^{n-tj}\psi_{n-tj}(a), \text{ mod. } p,$$

where $j = \dfrac{p-1}{2}$.

Writing for the ψ's their values, this congruence is

$$\frac{(a^{2n}-1)B_n}{n} \equiv \frac{(a^{2n-2tj}-1)B_{n-tj}}{n-tj}, \text{ mod. } p,$$

which is true for every uneven prime p which is not a divisor of a.

Now, for all values of a,

$$a^{2n} - 1 \equiv a^{2n-t(p-1)} - 1, \text{ mod. } p,$$

and if we take a to be a primitive root of p, neither side of this congruence is divisible by p unless $p-1$ is a divisor of $2n$, in which case, by Staudt's theorem, p is a factor of the denominator of B_n, or, as we may conveniently say, a Staudt factor of B_n.

Thus, if p is not a Staudt factor of B_n, we may divide out by the factors $a^{2n}-1$ and $a^{2n-t(p-1)}-1$, thus obtaining the theorem

$$\frac{B_n}{n} \equiv (-1)^{tj}\frac{B_{n-tj}}{n-tj}, \text{ mod. } p,$$

which holds good for all values of $p > 2$ except the Staudt factors of B_n, that is except those values of p which make $j = \frac{p-1}{2}$ a divisor of n.

§ 18. As mentioned in § 1 the discussion of the theorem has formed the subject of another paper, and I therefore omit all consideration of it here, the object of this paper being merely to give the whole proof in a complete form. It may be noticed, however, that the theorem enables us to assign the residue of the numerator of B_n with respect to any prime p which is not a Staudt factor of B_n, while Staudt's theorem itself assigns the residue when p is a Staudt factor, so that for all values of p we can assign the residue of the numerator of B_n, the denominator being supposed to be known by Staudt's theorem.

Other recurring formulæ for $\psi_n(a)$.

§ 19. The recurring formula in § 7 would result from the equation

$$\frac{c_1 x + \frac{c_3}{3!} x^3 + \frac{c_5}{5!} x^5 + \&c.}{\lambda + \frac{b_2}{2!} x^2 + \frac{b_4}{4!} x^4 + \&c.} = X_1 x + \frac{X_3}{3!} x^3 + \frac{X_5}{5!} x^5 + \&c.,$$

and it would also result from the equation

$$\frac{c_1 x + \frac{\gamma_2}{2!} x^2 + \frac{c_3}{3!} x^3 + \frac{\gamma_4}{4!} x^4 + \&c.}{\lambda + \frac{\beta_1}{1!} x + \frac{b_2}{2!} x^2 + \frac{\beta_3}{3!} x^3 + \frac{b_4}{4!} x^4 + \&c.} = X_1 x + \frac{X_3}{3!} x^3 + \frac{X_5}{5!} x^5 + \&c.,$$

the γ's and β's being any quantities whatever provided that we know that the left-hand side is expansible in positive uneven powers of x.

The equation in § 3 belongs to the second form, for it may be written

$$\frac{\frac{1}{2}(a-1)\{e^{(a-1)x} + e^{(a-2)x} + \ldots + 1\} - e^{(a-2)x} - 2e^{(a-3)x} - \ldots - (a-2)e^{x} - (a-1)}{e^{(a-1)x} + e^{(a-2)x} + \ldots + e^{x} + 1}$$

$$= \psi_1(a)\, x - \frac{\psi_2(a)}{3!} x^3 + \frac{\psi_3(a)}{5!} x^5 - \&c.$$

§ 20. If we write the left-hand side in the form

$$\frac{(a-1)\sinh\frac{1}{2}(a-1)x+(a-3)\sinh\frac{1}{2}(a-3)x+\ldots}{2\cosh\frac{1}{2}(a-1)x+2\cosh\frac{1}{2}(a-3)x+\ldots},^*$$

where in the denominator cosh 0, which occurs when a is uneven, is to be replaced by $\frac{1}{2}$, we obtain the following recurring formula:

(i) if a is uneven, and

$$\sigma_r(a)=1^r+2^r+3^r+\ldots+\left(\frac{a-1}{2}\right)^r,$$

then

$$a\psi_n(a)-2(2n-1)_2\,\sigma_2(a)\,\psi_{n-1}(a)+2(2n-1)_4\,\sigma_4(a)\,\psi_{n-2}(a)-\ldots$$
$$+(-1)^{n-1}2(2n-1)_{2n-2}\,\sigma_{2n-2}(a)\,\psi_1(a)=(-1)^{n-1}2\sigma_{2n}(a);$$

(ii) if a is even, and

$$\Sigma_r(a)=1^r+3^r+5^r+\ldots+(a-1)^r,$$

then

$$a\psi_n(a)-(2n-1)_2\frac{\Sigma_2(a)}{2}\psi_{n-1}(a)+(2n-1)_4\frac{\Sigma_4(a)}{2^3}\psi_{n-2}(a)-\ldots$$
$$+(-1)^{n-1}(2n-1)_{2n-2}\frac{\Sigma_{2n-2}(a)}{2^{2n-1}}\psi_1(a)=(-1)^{n-1}\frac{\Sigma_{2n}(a)}{2^{2n-1}}.$$

These formulæ are included in the general recurring formula of § 7, and therefore, like the formula of § 4, they show that $\psi_n(a)\equiv(-1)^{\upsilon}\psi_{n-\upsilon}(a)$, mod. p, p being any uneven prime which is not a divisor of a.

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* This form of the equation may be easily deduced from

$$\tfrac{1}{2}x\coth\tfrac{1}{2}x=1+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\frac{B_3}{6!}x^6-\&c.,$$

for this equation gives at once

$$\tfrac{1}{2}a\coth\tfrac{1}{2}ax-\tfrac{1}{2}\coth\tfrac{1}{2}x=\psi_1(a)\,x-\frac{\psi_2(a)}{3!}x^3+\frac{\psi_5(a)}{5!}x^5-\&c.,$$

and the left-hand side

$$=\frac{d}{dx}\log\frac{\sinh\frac{1}{2}ax}{\sinh\frac{1}{2}x}=\frac{d}{dx}\log\{2\cosh\tfrac{1}{2}(a-1)x+2\cosh\tfrac{1}{2}(a-3)x+\ldots\}.$$

THE THEORY OF THE G-FUNCTION.

By E. W. BARNES, B.A., Fellow of Trinity College, Cambridge.

§ 1. THE function which it is proposed to call the G-function is substantially the simplest transcendental integral function with zeros at the points

$$z = -n\omega, \quad n = 1, 2, \ldots, \infty;$$

the zero at the point $z = -n\omega$ being of order n.

It is a natural extension of the Gamma function, and possesses many properties analogous to those of the latter.

It satisfies one of the simplest difference equations with transcendental non-periodic coefficients,

$$f(z+1) = \Gamma(z) f(z).$$

It possesses a transformation theorem; it admits an asymptotic expansion at infinity valid for all values of the variable except those near or on the negative part of the real axis; and all asymptotic expansions connected with it involve merely the known Bernoullian functions and numbers.

Its logarithm admits of expression as a contour integral capable of several interesting modifications; the function $\int^z \Gamma(z)\, dz$ can be expressed in terms of it; it enters into the expression of wide classes of elementary indefinite integrals whose evaluation has been effected for particular values of the variable in terms of recognised analytic functions.

And finally it cannot be expressed as the solution of a differential equation, whose coefficients are not functions of the same or a more general nature.

The present paper attempts to establish as briefly as possible all the chief propositions in the theory of the function.

It follows naturally a theory of the Gamma function which I have recently developed.*

§ 2. The G-function possesses a short bibliography. About forty years ago Kinkelin† defined a function $G(z)$ satisfying the relation

$$G(z+1) = z^z\, G(z).$$

* Barnes, "Theory of the Gamma Function," *Messenger of Mathematics*, Vol. XXIX., pp. 64 et seq.

† Kinkelin, "Ueber eine mit der Gamma function verwandte Transcendente und deren auwendung auf die Integral-rechnung," *Crelle*, Bd. LVII., pp. 122–158.

The function would in the notation of this paper be written

$$\exp. [z \log \Gamma(z) - \log G(z+1)].$$

Hölder* has considered the function

$$e^{\int_0^x \pi x \cot \pi x \, dx},$$

which, in our notation, would be written

$$(2\pi)^x \frac{G(1-x)}{G(1+x)}.$$

The papers of both Kinkelin and Hölder are concerned more with this quotient than with the G-functions proper.

More recently Alexeiewsky has published a paper† containing many of the properties of these functions. The paper is written in Russian and is difficult to obtain. My knowledge of it is derived from indications in the *Jahrbuch über die Fortschritte der Mathematik*,‡ and a synopsis communicated by Lie to the *Leipzig Berichte.*§

Alexeiewsky's procedure differs from that here adopted in that his results are obtained by applying difference formulæ to the subject of integration of the definite integral which expresses $\log \Gamma(x)$.

Recently Dr. Glaisher|| has considered the function

$$\operatorname{ilg} x = \int_0^x \log \Gamma(\mu)\, d\mu,$$

which may, in our notation, be written

$$e^{\operatorname{ilg} x} = \frac{\{\Gamma(x)\}^{x-1}}{G(x)} (2\pi)^{\frac{x}{2}} e^{-\frac{x.x-1}{2}}.$$

§ 3. The G-function we define by the difference equation

$$G(z+1) = \Gamma(z)\, G(z),$$

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* Hölder, "Ueber eine transcendente Function," *Göttingen Nächrichten*, 1886, pp. 514–522.

† Alexeiewsky, "Ueber die Functionen welche der Gamma Functionen ähnlich sind." *Annales de l'Imp. Univ. de Charkow*, 1889.

‡ Vol. XXII., p. 439.

§ Alexeiewsky, "Ueber eine Classe von Functionen die der Gamma Function analog sind," *Leipzig Berichte*, 1894, Vol. XLVI., pp. 268–275.

|| Glaisher, "On products and series involving prime numbers only," *Quarterly Journal of Mathematics*, Vol. XXVI., pp. 1–174.

coupled with the relation $G(1) = 1$ together with a certain prescription at infinity, which will appear in the course of the analysis which immediately follows.

We have

$$\log G(z+1) = \log \Gamma(z) + \log G(z).$$

And hence, on repetition,

$$\log G(z+n+1) = \log(z+n-1) + 2\log(z+n-2) + \ldots + n\log z + (n+1)\log\Gamma(z) + \log G(z).$$

Thus

$$-\log G(z+1) = -\log G(z+n+1) + n\log\Gamma(z) + n\log z + (n-1)\log(z+1) + \ldots + \log(z+n-1),$$

and hence

$$-\log G(z+1) = -\log G(z+n+1) + n\log\Gamma(z) + n\log z.(z+1)\ldots(z+n-1)$$
$$-\sum_{k=1}^{n-1}\left[k\log\left(1+\frac{z}{k}\right) - z + \frac{z^2}{2k}\right] - \log 1^1.2^2\ldots(n-1)^{n-1} - (n-1)z + \frac{z^2}{z}\sum_{k=1}^{n-1}\frac{1}{k};$$

or, as it is more convenient to write it,

$$-\log G(z+1) + \log G(z+n+2) = (n+1)\log\Gamma(z) + (n+1)\log z(z+1)\ldots(z+n)$$
$$-\log\prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right] - \log 1^1.2^2\ldots n^n - nz + \frac{z^2}{2}\sum_{k=1}^{n}\frac{1}{k}.$$

Now, when n is a large positive integer, it is known* that

$$\log z.(z+1)\ldots(z+n) = -\log\Gamma(z) + \tfrac{1}{2}\log 2\pi + (n+\tfrac{1}{2}+z)\log n - n$$

+ terms which vanish with n infinite.

And by a theorem due originally to Kinkelin,†

$$\log 1^1.2^2\ldots n^n = \log A + \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n - \frac{n^2}{4} + \text{ terms}$$

which vanish with n infinite,

* *v.* Theory of the Gamma Function, § 30.

† *Crelle*, Bd. LVII., pp. 133 and 134. The constant A in the form quoted is due to Dr. Glaisher, who has considered it in several papers, cf. *Messenger*, Vol. VI., pp. 71, et. seq.; VII., pp. 43, et seq.; XXIII., pp. 145, et seq.; XXIV., pp. 1, et seq. An independent proof of the theorem is given later in this paper (§ 25).

where A is a definite constant whose numerical value is

$$1{\cdot}28242713\ldots.$$

Hence we have

$$-\log G(z+1)+\log G(z+n+2)$$

$$=-\log \prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right]+\frac{n+1}{2}\log 2\pi$$

$$-n.(n+1)-nz-\log A+\frac{n^2}{4}+\frac{z^2+z}{2}+\tfrac{1}{12}+\gamma\frac{z^2}{2}$$

$$+\left\{n^2+\frac{3n}{2}+\tfrac{1}{2}+(n+1)z\right\}\log n-\left\{\frac{n^2}{2}+\frac{n}{2}+\tfrac{1}{12}-\frac{z^2}{2}\right\}\log n$$

+ terms which vanish when n is infinite.

We may now assume that

$$\log G(z+1)=\underset{n=\infty}{\text{Lt.}}\left[\log\prod_{k=1}^{n}\left\{\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right\}\right.$$

$$\left.+\frac{z}{2}\log 2\pi-\frac{z.z+1}{2}-\gamma\frac{z^2}{2}\right],$$

provided this value is such that $G(z)$ satisfies the difference equation

$$G(z+1)=\Gamma(z)\,G(z),$$

and then it will have been established that this value of $\log G(z+1)$ is such that

$$\underset{n=\infty}{\text{Lt.}}\log G(z+n+2)=\frac{n+1+z}{2}\log 2\pi$$

$$+\left[\frac{n^2}{2}+n+\tfrac{5}{12}+\frac{z^2}{2}+(n+1)z\right]\log n-\frac{3n^2}{4}-n-nz-\log A+\tfrac{1}{12}$$

$$+\text{ terms which vanish with }\frac{1}{n},$$

which is the prescription at infinity spoken of at the beginning of the present section.

§ 4. We proceed now to shew that

$$G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{z.z-1}{2}} e^{-\gamma\frac{(z-1)^2}{2}} \prod_{k=1}^{\infty}\left[\left(1+\frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

does satisfy the difference equation

$$f(z+1) = \Gamma(z) f(z).$$

It is evident by Weierstrass' factor theorem that the product for $G(z)$ as written is absolutely convergent for all finite values of $|z|$.

Again

$$\prod_{k=1}^{n}\left[\left(1+\frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

$$= z\frac{\prod_{k=1}^{n}\left[\left(1+\frac{z}{k}\right)^{k-1}\left(\frac{k}{k+1}\right)^{k+1} e^{1-z+\frac{(z-1)^2}{2k}}\right]}{\left(1+\frac{z}{n}\right)^{n+1}\left(\frac{n}{n+1}\right)^{n+1}}.$$

Hence

$$\frac{G(z+1)}{G(z)} = (2\pi)^{\frac{1}{2}} e^{-z-\gamma z+\frac{\gamma}{2}} \underset{k=\infty}{\text{Lt.}} \left\{\frac{\left(1+\frac{z}{k}\right)^{k+1}\left(\frac{k}{k+1}\right)^{k+1}}{\prod_{r=1}^{k}\left[\left(1+\frac{z}{r}\right)\left(\frac{r}{r+1}\right)^{r+1} e^{1+\frac{(z-1)^2-z^2}{2r}}\right]}\right\}$$

$$= (2\pi)^{\frac{1}{2}} e^{-z+\frac{\gamma}{2}} \underset{k=\infty}{\text{Lt.}} \frac{\Gamma(z)\, e^{z-1}}{\prod_{r=1}^{k}\left[\left(\frac{r}{r+1}\right)^{r+1} e^{1+\frac{1}{2r}}\right]}$$

$$= (2\pi)^{\frac{1}{2}}\Gamma(z)\, e^{\frac{\gamma}{2}-1} \underset{k=\infty}{\text{Lt.}} \frac{(k+1)^{k+1}}{k!\, e^{k+\frac{1}{2}(\gamma+\log k)}}$$

$$= \Gamma(z)\,(2\pi)^{\frac{1}{2}} e^{\frac{\gamma}{2}-1} \underset{k=\infty}{\text{Lt.}} \frac{k^k(k+1)\,e}{(2\pi)^{\frac{1}{2}} k^{k+\frac{1}{2}} e^{+\frac{\gamma}{2}} k^{\frac{1}{2}}}$$

by Stirling's theorem,

$$= \Gamma(z).$$

We thus see that

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{z.z+1}{2}-\gamma\frac{z^2}{2}} \prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right]$$

is a solution of the difference equation

$$G(z+1) = \Gamma(z)\, G(z),$$

and that it is such that, when x is large, real, and positive,

$$\log G(x+a+1) = \frac{x+a}{2}\log 2\pi - \log A - \tfrac{1}{12} - \frac{3x^2}{4} - ax$$

$$+ \left(\frac{x^2}{2} - \tfrac{1}{12} + \frac{a^2}{2} + ax\right)\log x + \text{terms which vanish with } \frac{1}{x}.$$

We may write this in the form

$$G(x+a+1) = \frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{x+a}{2}}\, x^{\frac{(x+a)^2}{2} - \frac{1}{12}}\, e^{\frac{3}{4}x^2 - ax + \frac{(\ldots)}{x} + \frac{(\ldots)}{x^2} + \ldots}$$

which, when $a = 0$, agrees with the approximation of Alexeiewsky.

As a *Corollary*, note that it has been proved incidentally that $G(z+1)$ is the only solution of the difference equation

$$f(z+1) = \Gamma(z) f(z),$$

which is infinite when z is real, positive, and infinite, like

$$\frac{(2\pi)^{\frac{z}{2}}}{A}\, z^{\frac{z^2}{2} - \frac{1}{12}}\, e^{-\frac{3z^2}{4} + \frac{1}{12}}.$$

The general solution is evidently

$$f(z) = G(z)\ [\text{simply periodic function of } z \text{ of period unity}].$$

§ 5. We proceed now to express $G(z)$ as an infinite product of simple Gamma functions.

Take again the fundamental difference equation

$$G(z+1) = \Gamma(z)\, G(z),$$

then, an accent being used to denote differentiation with regard to z,

$$\frac{G'(z+1)}{G(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + \frac{G'(z)}{G(z)},$$

whence

$$\frac{G'(z+n+2)}{G(z+n+2)} = \frac{\Gamma'(z+n+1)}{\Gamma(z+n+1)} + \frac{\Gamma'(z+n)}{\Gamma(z+n)} + \frac{\Gamma'(z+n-1)}{\Gamma(z+n-1)} + \dots$$
$$+ \frac{\Gamma'(z+1)}{\Gamma(z+1)} + \frac{G'(z+1)}{G(z+1)}.$$

Now for sufficiently small values of $|z|$ we have by Taylor's theorem

$$\frac{\Gamma'(z+1+k)}{\Gamma(z+1+k)} = \frac{\Gamma'(1+k)}{\Gamma(1+k)} + z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}$$
$$+ \frac{z^2}{2!}\frac{d^2}{dk^2}\frac{\Gamma'(1+k)}{\Gamma(1+k)} + \dots,$$

provided the coefficients in the expansion be finite.

But, from the definition of $\Gamma(z)$,

$$-\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \gamma + \sum_{m=1}^{\infty}\left(\frac{1}{z+m} - \frac{1}{m}\right),$$

and therefore, when $r > 1$,

$$\frac{d^r}{dk^r}\frac{\Gamma'(1+k)}{\Gamma(1+k)} = (-)^{r-1} r! \sum_{m=1}^{\infty}\frac{1}{(m+k)^{r+1}},$$

so that

$$\frac{\Gamma'(z+1+k)}{\Gamma(z+1+k)} - \frac{\Gamma'(1+k)}{\Gamma(1+k)} - z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}$$
$$= \sum_{r=2}^{\infty}\left[(-)^{r-1}\sum_{m=1}^{\infty}\frac{1}{(k+m)^{r+1}}z^r\right].$$

And thus

$$\sum_{k=0}^{n}\left[\frac{\Gamma'(z+1+k)}{\Gamma(z+1+k)} - \frac{\Gamma'(1+k)}{\Gamma(1+k)} - z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right]$$
$$= \sum_{k=0}^{n}\sum_{r=2}^{\infty}(-)^{r-1}\left\{\sum_{m=1}^{\infty}\frac{1}{(k+m)^{r+1}}\right\}z^r$$
$$= \sum_{r=2}^{\infty}(-)^{r-1}\left\{\sum_{k=0}^{n}\sum_{m=1}^{\infty}\frac{1}{(k+m)^{r+1}}\right\}z^r,$$

where we are not re-arranging a double series, if n be not actually infinite.

Now it is an immediate deduction from Eisenstein's theorem* that

$$\operatorname*{Lt.}_{n=\infty} \sum_{k=0}^{n} \sum_{m=1}^{\infty} \frac{1}{(k+m)^{r+1}}$$

is finite when $r \geqq 2$.

And hence

$$\operatorname*{Lt.}_{n=\infty} \sum_{k=0}^{n} \left[\frac{\Gamma'(z+1+k)}{\Gamma(z+1+k)} - \frac{\Gamma'(1+k)}{\Gamma(1+k)} - z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right]$$

is finite for values of $|z|$ less than unity.

We have now the relation

$$-\frac{G'(z+1)}{G(z+1)} = \sum_{k=0}^{n} \left[\frac{\Gamma'(z+1+k)}{\Gamma(z+1+k)} - \frac{\Gamma'(1+k)}{\Gamma(1+k)} - z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right]$$

$$-\frac{G'(z+n+2)}{G(z+n+2)} + \sum_{k=0}^{n}\left[\frac{\Gamma'(1+k)}{\Gamma(1+k)} + z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right].$$

And hence for sufficiently small values of $|z|$ we may take

$$-\frac{G'(z+1)}{G(z+1)} = \alpha + 2\beta z$$

$$+\frac{d}{dz}\log\prod_{k=0}^{\infty}\left[\frac{\Gamma(z+1+k)}{\Gamma(1+k)} e^{-z\frac{\Gamma'(1+k)}{\Gamma(1+k)} + \frac{z^2}{2}\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}}\right],$$

provided

$$\operatorname*{Lt.}_{n=\infty} \frac{G'(z+n+2)}{G(z+n+2)} = -\alpha - 2\beta z + \sum_{k=0}^{n}\left[\frac{\Gamma'(1+k)}{\Gamma(1+k)} + z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right],$$

and provided therefore that the expression on the right-hand side tends to the same value when n is infinite as that obtained in § 4, which yields

$$\operatorname*{Lt.}_{n=\infty} \frac{G'(z+n+2)}{G(z+n+2)} = (n+1+z)\log n + \tfrac{1}{2}\log 2\pi - n.$$

If, then, we can prove that, for suitable values of α and β,

$$\operatorname*{Lt.}_{n=\infty} -\alpha - 2\beta z + \sum_{k=0}^{n}\left[\frac{\Gamma'(1+k)}{\Gamma(1+k)} + z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right]$$

$$= (n+1+z)\log n + \tfrac{1}{2}\log 2\pi - n,$$

* Forsyth, *Theory of Functions*, § 56, p. 87.

we shall have shown that, for such values of α and β,

$$-\frac{G'(z+1)}{G(z+1)}=\alpha+2\beta z$$
$$+\frac{d}{dz}\prod_{k=0}^{\infty}\left[\frac{\Gamma(z+1+k)}{\Gamma(1+k)}e^{-z\frac{\Gamma'(1+k)}{\Gamma(1+k)}-\frac{z^2}{2}\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}}\right],$$

for we have seen in § 4 Cor. that $\frac{G'(z+1)}{G(z+1)}$ is the only solution of

$$f(z+1)=f(z)+\frac{\Gamma'(z)}{\Gamma(z)},$$

which has such a limit at infinity.

For this purpose we must interpolate the Lemmas of the two following sections.

§ 6. Put, in conformity with Gauss' notation, $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$. Then we will show that, when n is a large positive integer,

$$\sum_{r=1}^{n}\psi(r)=n\log n-n+\tfrac{1}{2}+\text{ terms vanishing with }\frac{1}{n}.$$

It is inexpedient to use the Maclaurin sum-formula, inasmuch as not only does it rest on an unsatisfactory basis, but the determination of the constant cannot be effected by any process more simple than the following procedure.

We have from the difference equation for $\Gamma(z)$

$$\psi(z+1)=\frac{1}{z}+\psi(z).$$

And hence

$$\psi(m)=\tfrac{1}{1}+\tfrac{1}{2}+\ldots+\frac{1}{m-1}+\psi(1),$$

so that $\sum_{r=1}^{n}\psi(r)=n\psi(1)+\frac{n-1}{1}+\frac{n-2}{2}+\ldots+\frac{1}{n-1}$.

Thus

$$\sum_{r=1}^{n}\psi(r)=n\left[\psi(1)+\tfrac{1}{1}+\tfrac{1}{2}+\ldots+\frac{1}{n-1}\right]-\tfrac{1}{1}-\tfrac{2}{2}-\ldots-\frac{n-1}{n-1}$$
$$=n\left[\psi(1)+\sum_{r=1}^{n-1}\frac{1}{r}\right]-n+1.$$

Thus, when n is a large positive integer,

$$\sum_{r=1}^{n} \psi(r) = n \left[\psi(1) + \gamma + \log(n-1) + \frac{1}{2(n-1)} \right.$$

$$\left. + \text{ terms vanishing with } \frac{1}{n^2} \right] - n + 1.$$

Now $\psi(1) = -\gamma$, as is readily seen from the product for $\Gamma(z)$. Hence we have finally

$$\sum_{r=1}^{n} \psi(r) = n \log n - n + \tfrac{1}{2} + \text{terms vanishing with } \frac{1}{n}.$$

§ 7. We will next show that, when n is a large positive integer,

$$\sum_{r=1}^{n} \psi'(r) = \log n + 1 + \gamma + \text{terms which vanish with } \frac{1}{n},$$

where $\psi'(r) = \dfrac{d^2}{dr^2} \log \Gamma(r)$.

From the fundamental difference equation,

$$\psi'(z+1) = -\frac{1}{z^2} + \psi'(z),$$

and hence by repeated application

$$\psi'(m) = -\left[\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(m-1)^2} \right] + \psi'(1).$$

Hence

$$\sum_{r=1}^{n} \psi'(r) = n\psi'(1) - \frac{n-1}{1^2} - \frac{n-2}{2^2} - \ldots - \frac{1}{(n-1)^2}.$$

Now from the product expression for $\Gamma(z)$ it may be readily deduced that

$$\psi'(1) = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \ldots$$

$$= \frac{\pi^2}{6}.$$

Substituting this value, we find

$$\sum_{r=1}^{n}\psi'(r)=n\frac{\pi^2}{6}-n\left[\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{(n-1)^2}\right]+\frac{1}{1^2}+\frac{2}{2^2}+\ldots+\frac{n-1}{(n-1)^2}$$

$$=n\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots\text{ ad inf.}\right]+\gamma+\log(n-1)$$

$$+\text{ terms vanishing with }\frac{1}{n}.$$

But on differentiating the expression obtained previously [*Gamma function*, § 27, *Cor.*], and making $a=0$, $\omega=1$, it is readily seen that

$$\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots=\frac{1}{n}+\frac{1}{2n^2}+\text{ terms vanishing with }\frac{1}{n^3}.$$

Finally then

$$\sum_{r=1}^{n}\psi'(r)=\log n+1+\gamma+\text{ terms vanishing with }\frac{1}{n}.$$

§ 8. Return now to the completion of the investigation of § 5.

We were required to prove that, for suitable values of α and β, when n is a large positive integer,

$$-\alpha-2\beta z+\sum_{k=0}^{n}\left[\frac{\Gamma'(1+k)}{\Gamma(1+k)}+z\frac{d}{dk}\frac{\Gamma'(1+k)}{\Gamma(1+k)}\right]$$

$$=(n+1+z)\log n+\tfrac{1}{2}\log 2\pi-n+\text{ terms which vanish with }\frac{1}{n}.$$

The left-hand side of this equality may be written

$$-\alpha-2\beta z+\sum_{r=1}^{n+1}[\psi(r)+z\psi'(r)],$$

whose value by §§ 6 and 7 is

$$(n+1)\log n-n+z(\log n+1+\gamma-2\beta)+\tfrac{1}{2}-\alpha$$

$$+\text{ terms which vanish with }\frac{1}{n}.$$

But if we put

$$\alpha=\tfrac{1}{2}-\tfrac{1}{2}\log 2\pi,$$

$$\beta=\frac{1+\gamma}{2},$$

this becomes

$$(n+1+z)\log n + \tfrac{1}{2}\log 2\pi - n + \text{terms which vanish with } \frac{1}{n},$$

which is the expression required.

Substitute now these values of α and β, and we obtain for $-\dfrac{G'(z+1)}{G(z+1)}$ the expression

$$\tfrac{1}{2} - \tfrac{1}{2}\log 2\pi + (1+\gamma)z + \frac{d}{dz}\log \prod_{k=0}^{\infty}\left[\frac{\Gamma(z+1+k)}{\Gamma(1+k)}\, e^{-z\psi(1+k)-\frac{z^2}{2}\psi'(1+k)}\right],$$

which we have proved convergent for values of $|z|$ less than unity.

Integrate now with respect to z, obtain the value of the constant of integration by making $z=0$, and we have finally

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{z}{2}-\frac{1+\gamma}{2}z^2} \prod_{k=1}^{\infty}\left[\frac{\Gamma(k)}{\Gamma(z+k)}\, e^{+z\psi(k)+\frac{z^2}{2}\psi'(k)}\right].$$

As this is perhaps the first time in analysis when an infinite product occurs, the non-exponential term of whose primary factor involves a transcendental function, it merits detailed attention.

In Weierstrass' fundamental investigation* the primary factor is of the form

$$u_r = \left(1 - \frac{z}{a_r}\right) e^{q_r},$$

where q_r is a rational integral function of z. And the first step of the proof proceeds by taking the expansion

$$\log u_r = q_r - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{z}{a_r}\right)^n.$$

where the infinite series converges provided $|z| < |a_r|$.

But in the case of the expression which has been obtained the primary factor is

$$u_k = \left[\frac{\Gamma(k)}{\Gamma(z+k)}\, e^{z\psi(k)+\frac{z^2}{2!}\psi'(k)}\right],$$

* *Vide* Forsyth, *Theory of Functions*, §§ 50 et seq.

And the employment of a similar expansion gives

$$\log u_k = \sum_{n=3}^{\infty} \frac{(-)^n (z)^n}{n} \left[\sum_{m=0}^{\infty} \frac{1}{(m+k)^n}\right],$$

a series which does not in general converge unless $|z| < 1$.

It is precisely under such a limitation that we have so far proved the validity of the product. We proceed to prove it true for all finite values of $|z|$ by an actual transformation.*

§ 9. For this purpose we shall first show that the product

$$P(z) = \Gamma(z)\,(2\pi)^{\frac{z}{2}} e^{(\gamma-\frac{1}{2})z - \frac{z^2}{2}\left(\frac{\pi^2}{6}+1+\gamma\right)} z \times \prod_{0}^{\infty}{}_{m} \prod_{0}^{\infty}{}'_{n} \left[\left(1+\frac{z}{m+n}\right) e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}}\right],$$

which by Weierstrass' factor-theorem is absolutely convergent for all finite values of $|z|$, and by an immediate Corollary to Eisenstein's theorem to a limit independent of all ratios of the infinite limiting values of m and n, can by actual transformation be thrown into the equally convergent form

$$(2\pi)^{\frac{z}{2}} e^{-\frac{z}{2}-\frac{1+\gamma}{2}z^2} \prod_{k=1}^{\infty} \left[\frac{\Gamma(k)}{\Gamma(z+k)} e^{z\psi(k)+\frac{z^2}{2}\psi'(k)}\right].$$

For since $P(z)$ is a product convergent as has been stated, its terms may be grouped as we please without altering its value or convergency.

Hence

$$P(z) = \Gamma(z)\,(2\pi)^{\frac{z}{2}} e^{(\gamma-\frac{1}{2})z - \frac{z^2}{2}\left(\frac{\pi^2}{6}+1+\gamma\right)} . z \prod_{m=1}^{\infty} \left[\left(1+\frac{z}{m}\right) e^{-\frac{z}{m}+\frac{z^2}{2m^2}}\right]$$

$$\times \prod_{m=0}^{\infty} \prod_{n=1}^{\infty} \left(\frac{\left(1+\frac{z+m}{n}\right) e^{-\frac{z+m}{n}}}{\left(1+\frac{m}{n}\right) e^{-\frac{m}{n}}} . e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}+\frac{z}{n}}\right)$$

$$= (2\pi)^{\frac{z}{2}} e^{(\gamma-\frac{1}{2})z - \frac{z^2}{2}\left(\frac{\pi^2}{6}+1+\gamma\right)} e^{-\gamma z + \frac{z^2}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}}$$

$$\times \prod_{m=0}^{\infty} \left[\frac{\Gamma(1+m)}{\Gamma(z+m+1)} e^{-\gamma z - z\sum_{n=1}^{\infty}\frac{1}{n+m} + \frac{z^2}{2}\sum_{n=1}^{\infty}\frac{1}{(n+m)^2} + z\sum_{n=1}^{\infty}\frac{1}{n}}\right].$$

* I propose to develop elsewhere the extension of Weierstrass' method of proof to such generalised primary factors.

Now $$-\psi(1+m)=\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{m+n}-\frac{1}{n}\right],$$

$$\psi'(1+m)=\sum_{n=1}^{\infty}\frac{1}{(n+m)^2}.$$

$$P(z)=(2\pi)^{\frac{z}{2}}e^{-\frac{z}{2}-\frac{z^2}{2}(1+\gamma)}\prod_{m=0}^{\infty}\left[\frac{\Gamma(1+m)}{\Gamma(z+m+1)}e^{z\psi(1+m)+\frac{z^2}{2}\psi'(1+m)}\right].$$

We thus see that the latter product is convergent for all finite values of $|z|$, and hence that $P(z)=G(z+1)$.

In conclusion then $G(z+1)$ can be expressed in either of the three forms

$$(2\pi)^{\frac{z}{2}}e^{-\frac{z.z+1}{2}-\gamma\frac{z^2}{2}}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right],$$

$$(2\pi)^{\frac{z}{2}}e^{-\frac{z.z+1}{2}-\gamma\frac{z^2}{2}}\prod_{k=1}^{\infty}\left[\frac{\Gamma(k)}{\Gamma(z+k)}e^{z\psi(k)+\frac{z^2}{2}\psi'(k)}\right],$$

$$(2\pi)^{\frac{z}{2}}e^{(\gamma-\frac{1}{2})z-\frac{z^2}{2}\left(\frac{\pi^2}{6}+1+\gamma\right)}\Gamma(z).z$$
$$\times\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}{}'\left[\left(1+\frac{z}{m+n}\right)e^{-\frac{z}{m+n}+\frac{z^2}{2(m+n)^2}}\right],$$

and each form is a product convergent for all finite values of $|z|$.

§ 10. It is natural in investigating the properties of $G(z)$ to attempt at the outset to combine the difference equation

$$G(z+1)=\Gamma(z)\,G(z),$$

with the relation

$$\Gamma(z)\,\Gamma(1-z)=\frac{\pi}{\sin z\pi}.$$

Substituting $-z$ for z, we have

$$G(1-z)=\Gamma(-z)\,G(-z).$$

Wherefore

$$G(z+1)\,G(-z+2)=\Gamma(z)\,\Gamma(1-z)\,G(z)\,G(1-z),$$

so that if we write

$$\phi(z)=\frac{G(1+z)}{G(1-z)},$$

we have the difference equation

$$\phi(z)=\frac{\pi}{\sin z\pi}\,\phi(z-1).$$

The most general solution of this equation is

$$e^{\frac{\pi i}{2}(z^2-z)}\left(\frac{\pi}{\sin z\pi}\right)^z$$

multiplied by a simply periodic function of z of period unity.

We proceed to represent the particular solution which is the value of $\dfrac{G(1+z)}{G(1-z)}$ as a definite integral.

Take the first product-expression for $G(z)$ and we have

$$\phi(z)=\frac{G(1+z)}{G(1-z)}=\frac{(2\pi)^{\frac{z}{2}}e^{-z\frac{z+1}{2}-\gamma\frac{z^2}{2}}\prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right]}{(2\pi)^{-\frac{z}{2}}e^{-z\frac{z-1}{2}-\gamma\frac{z^2}{2}}\prod_{k=1}^{\infty}\left[\left(1-\frac{z}{k}\right)^k e^{z+\frac{z^2}{2k}}\right]}$$

$$=(2\pi)^z e^{-z}\prod_{k=1}^{\infty}\left[\left(\frac{1+\frac{z}{k}}{1-\frac{z}{k}}\right)^k e^{-2z}\right].$$

Differentiating with respect to z,

$$\frac{d}{dz}\log\phi(z)=\log 2\pi-1+\sum_{k=1}^{\infty}\left[\frac{k}{z+k}-\frac{k}{z-k}-z\right]$$

$$=\log 2\pi-1-2\sum_{k=1}^{\infty}\left[\frac{k^2}{z^2-k^2}+1\right]$$

$$=\log 2\pi-1-2\sum_{k=1}^{\infty}\left[\frac{z^2}{z^2-k^2}\right],$$

so that, using the partial fraction expansion for $\pi z \cot \pi z$ obtained in trigonometry,

$$\frac{d}{dz} \log \phi(z) = \log 2\pi - \pi z \cot \pi z.$$

Integrate and determine the constant of integration by the fact that $\phi(0) = 1$, and we have

$$\log \phi(z) = z \log 2\pi - \int_0^z \pi z \cot \pi z \, dz.$$

Thus $$\log \frac{G(1-z)}{G(1+z)} = \int_0^z \pi z \cot \pi z \, dz - z \log 2\pi,$$

a relation originally due to Kinkelin.

From the expression of the integral just written in terms of $\log \frac{G(1-z)}{G(1+z)}$, many others may be expressed in terms of a similar function of quotients of G functions.

We have seen that

$$\int_0^z \frac{\pi z}{\tan \pi z} dz = z \log 2\pi + \log \frac{G(1-z)}{G(1+z)},$$

and we deduce

$$\int_0^z \log \sin \pi z \, dz = z \log \frac{\sin \pi z}{2\pi} + \log \frac{G(1-z)}{G(1+z)},$$

$$\int_0^z \log \cos \pi z \, dz = (z - \tfrac{1}{2}) \log \frac{\cos \pi z}{2\pi} - \tfrac{1}{2} \log z - \log \Gamma(\tfrac{1}{2} - z) + \log \frac{G(\frac{1}{2} + z)}{G(\frac{1}{2} - z)},$$

$$\int_0^z \pi z \tan \pi z \, dz = -\tfrac{1}{2} \log \frac{\cos \pi z}{\pi} - z \log 2\pi - \log \Gamma(\tfrac{1}{2} - z) + \log \frac{G(\frac{1}{2} + z)}{G(\frac{1}{2} - z)}.$$

As has been shown by Kinkelin, the following integrals can be readily evaluated in a similar manner:

$$\int_0^z \log \tan \pi z \, dz, \quad \int_0^z \frac{\sin \pi z}{\pi z} dz, \quad \int_0^z \frac{z^2 dz}{\sin^2 \pi z}, \quad \int_0^z \frac{z^2 dz}{\cos^2 \pi z},$$

$$\int_0^z \frac{z^2 \cos \pi z \, dz}{\sin^2 \pi z}, \quad \int_0^z \log(\cos bz - \cos az) \, dz, \quad \int_0^z \frac{\log z \, dz}{1 + a^2 z^2},$$

$$\int_0^{\frac{1}{2}\pi} \log(1 + 2a \cos z + a^2) \, dz, \quad \int_0^z \frac{\tan^{-1} az}{z} dz, \quad \text{etc., etc.}$$

The results are obtained by elementary transformations which can be readily discovered when necessity arises.

As an example of the application of such formulæ to the evaluation of definite integrals, consider

$$\int_0^1 \log \sin \pi z \, dz.$$

By the first derived formula above, its value is

$$-\log 2\pi + \underset{z=1}{\mathrm{Lt.}} \left[\log \frac{G(1-z)}{G(z)} \sin \pi z \right]$$

$$= -\log 2\pi + \log \pi.$$

And so

$$\int_0^1 \log \sin \pi z \, dz = -\log 2,$$

a result usually given in the text-books.*

And it is evident that by assigning to the upper limit of any of the preceding integrals values which cause the quotient of the G functions to reduce to known constants, we may obtain the values of a large series of definite integrals.

§ 11. From the identity just obtained, which may be written

$$G(1-z) = G(1+z) \frac{1}{(2\pi)^z} e^{\int_0^z \pi z \cot \pi z \, dz},$$

we may at once deduce the behaviour of $G(1+z)$ as z approaches infinity along the negative direction of the real axis.

For when z approaches infinity along the positive direction of the real axis, we have seen (§ 4) that $G(1+z)$ behaves like

$$\frac{e^{\frac{1}{12}}}{A} (2\pi)^{\frac{z}{2}} z^{\frac{z^2}{2} - \frac{1}{12}} e^{-\frac{3}{4}z^2}.$$

And hence as z approaches infinity in the negative direction of the real axis, $G(1+z)$ behaves like

$$\frac{e^{\frac{1}{12}}}{A} (2\pi)^{\frac{}{2}} (-z)^{\frac{z^2}{2} - \frac{1}{12}} e^{-\frac{3}{4}z^2 + \int_0^{-z} \pi z \cot \pi z \, dz}.$$

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* See, for example, Williamson, *Int. Calc.*, 6th Edition, § 118.

In the former case $G(1+z)$ tends uniformly to an infinite limit, in the latter the nature of the ultimate line of essential singularity is given by the expression just written.

§ 12. After successfully expressing $\log \dfrac{G(z+1)}{G(z-1)}$ as an indefinite integral of a simply periodic function, it is natural to enquire whether $\log G(z+1)$ can be expressed as a similar integral of a Gamma function. This is in fact the case, and we shall show that

$$\log G(z+1) = z \log \Gamma(z+1) - \int_0^z \log \Gamma(1+z)\, dz$$
$$- \frac{z.z+1}{2} + \frac{z}{2} \log 2\pi.$$

From the first product expression for $G(z+1)$, we derive

$$\frac{d}{dz} \log G(z+1) = \tfrac{1}{2} \log 2\pi - (z+\tfrac{1}{2}) - \gamma z + \sum_{k=1}^{\infty} \left[\frac{k}{z+k} - 1 + \frac{z}{k}\right]$$
$$= \tfrac{1}{2} \log 2\pi - (z+\tfrac{1}{2}) - \gamma z - z \sum_{k=1}^{\infty} \left[\frac{1}{z+k} - \frac{1}{k}\right]$$
$$= \tfrac{1}{2} \log 2\pi - z - \tfrac{1}{2} + z \frac{d}{dz} \log \Gamma(z+1).$$

Integrate now with respect to z, taking zero as the lower limit, and we obtain the formula stated.

This proposition we shall usually refer to as Alexeiewsky's theorem—it having been originally given by him.

§ 13. Suppose that we put $z = 1$ in Alexeiewsky's theorem—we obtain

$$\int_0^1 \log \Gamma(1+z)\, dz = \log \Gamma(2) - \log G(2) - 1 + \tfrac{1}{2} \log 2\pi$$
$$= \tfrac{1}{2} \log 2\pi - 1,$$

which is Raabe's formula for the case $a = 1$ in the known theory of the Gamma function.*

It is now an obvious investigation to find $\int_0^z \log \Gamma(a+z)\, dz$.

* Vide, *Theory of the Gamma Function*, § 8.

Put $z+a$ for $z+1$ in the first product expression for $G(z+1)$; then we have on logarithmic differentiation,

$$\frac{d}{dz}\log G(z+a) = \tfrac{1}{2}\log 2\pi - (z+a-\tfrac{1}{2}) - \gamma(z+a-1)$$

$$- (z+a-1)\sum_{k=1}^{\infty}\left[\frac{1}{z+a-1+k} - \frac{1}{k}\right]$$

$$= \tfrac{1}{2}\log 2\pi - (z+a-\tfrac{1}{2}) + (z+a-1)\frac{d}{dz}\log\Gamma(z+a).$$

Integrate between the limits 0 and z, and we find

$$\int_0^z \log\Gamma(z+a)\,dz = \frac{z}{2}\log 2\pi - \frac{z\,.\,(z+2a-1)}{2}$$

$$+ (z+a-1)\log\Gamma(z+a) - \log\frac{G(z+a)}{G(a)} - (a-1)\log\Gamma(a),$$

the natural extension of Alexeiewsky's theorem.

§ 14. We will now give the asymptotic expansion of

$$\log G(z+a+1),$$

when z is any point in the domain of infinity, but not in the vicinity or on the negative branch of the real axis, and a is any quantity such that $|a|$ is finite.

It has been shown (*Theory of the Gamma Function*, § 41) that, under the restrictions of a and z just enumerated,

$$\log\Gamma(z+a) = (z+a-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi$$

$$+ \sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{2n\,.\,(2n-1)\,z^{2n-1}} + \sum_{n=1}^{\infty}\frac{(-)^{n-1}S_n(a)}{az^n},$$

where $S_n(a)$ is the nth Bernoullian function of a.

Hence integrating with respect to z, we obtain the asymptotic expansion

$$\int^z \log\Gamma(z+a)\,dz = C + \left(\frac{z^2}{2} + az - \frac{z}{2}\right)\log z - \int^z\left(\frac{z}{2} + a - \tfrac{1}{2}\right)dz$$

$$- \frac{z^2}{2} + \frac{z}{2}\log 2\pi + S_1(a)\log z + \tfrac{1}{12}\log z$$

$$+ \sum_{n=2}^{\infty}\frac{(-)^n B_n}{2n\,.\,(2n-1)\,(2n-2)\,z^{2n-2}} + \sum_{n=2}^{\infty}\frac{(-)^n S_n(a)}{(n-1)\,nz^{n-1}},$$

where C is a constant with respect to z.

Substitute now the value of $\int_0^z \log\Gamma(z+a)\,dz$ given in § 13, namely

$$\int_0^z \log\Gamma(z+a)\,dz = \frac{z}{2}\log 2\pi - \frac{z\,.\,z+2a-1}{2}$$
$$+(z+a-1)\log\Gamma(z+a)-(a-1)\log\Gamma(a)-\log\frac{G(z+a)}{G(a)}.$$

Then we obtain the asymptotic expansion

$$-\log\frac{G(z+a)}{G(a)}+(z+a-1)\log\Gamma(z+a)-(a-1)\log\Gamma(a)$$
$$-\frac{z\,.\,z+2a-1}{2}+\frac{z}{2}\log 2\pi$$
$$=C+\left(\frac{z^2}{2}+az-\frac{z}{2}\right)\log z-\frac{3z^2}{4}-az+\frac{z}{2}+\frac{z}{2}\log 2\pi$$
$$+\left(\frac{1}{12}+\frac{a\,.\,a-1}{2}\right)\log z$$
$$+\sum_{n=1}^{\infty}\frac{(-)^{n+1}B_{n+1}}{2n+2\,.\,2n+1\,.\,2nz^{2n}}+\sum_{n=1}^{\infty}\frac{(-)^{n+1}S_{n+1}(a)}{(n+1)\,nz^n},$$

on substituting the values

$$B_1=\tfrac{1}{6},$$
$$S_1(a)=\frac{a^2}{2}-\frac{a}{2}.$$

Hence, utilising the asymptotic expansion for $\log\Gamma(z+a)$, we find

$$\log G(z+a)=C-\frac{z\,.\,z+2a-1}{2}+\frac{3z^2}{4}+az-\frac{z}{2}$$
$$-\left(\frac{z^2}{2}+az-\frac{z}{2}\right)\log z-\left(\frac{1}{12}+\frac{a\,.\,a-1}{2}\right)\log z$$
$$+\sum_{n=1}^{\infty}\frac{(-)^{n}B_{n+1}}{(2n+2)\,.\,(2n+1)\,.\,2nz^{2n}}+\sum_{n=1}^{\infty}\frac{(-)^{n}S_{n+1}(a)}{(n+1)\,.\,nz^n}$$
$$+(z+a-1)(z+a-\tfrac{1}{2})\log z-z(z+a-1)+\frac{z+a-1}{2}\log 2\pi$$
$$+(z+a-1)\sum_{n=1}^{\infty}\frac{(-)^{n-1}B_n}{2n\,.\,(2n-1)\,.\,z^{2n-1}}+(z+a-1)\sum_{n=1}^{\infty}\frac{(-)^{n-1}S_n(a)}{nz^n},$$

where C is still a constant with respect to z.

Increase now a into $a+1$, then we find on reduction

$$\log G(z+a+1) = C - \frac{3z^2}{4} - az + \frac{z+a}{2}\log 2\pi + \left(\frac{(z+a)^2}{2} - \frac{1}{12}\right)\log z$$

$$+ a\left[\sum_{n=1}^{\infty} \frac{(-)^{n-1}B_n}{2n \,.\, 2n-1 \,.\, z^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-)^{n-1}S_n(a+1)}{nz^n}\right]$$

$$+ \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{2n+2 \,.\, 2n \,.\, z^{2n}} + \sum_{n=1}^{\infty} \frac{(-)^n S_{n+1}(a+1)}{nz^n},$$

C being constant with respect to z.

But in § 4 we obtained the asymptotic equality

$$\log G(z+a+1) = \tfrac{1}{12} - \log A - \frac{3z^2}{4} - az + \frac{z+a}{2}\log 2\pi + \left(\frac{(z+a)^2}{2} - \frac{1}{12}\right)\log z$$

$$+ \text{ terms which vanish with } \frac{1}{z}.$$

We see then that

$$C = \tfrac{1}{12} - \log A.$$

We thus have finally, for all values of z near infinity except those which are in the vicinity of the negative direction of the axis of x and for all values of a such that $|a|$ is finite, the asymptotic expansion

$$G(z+a+1) = \frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{z+a}{2}} z^{\frac{(z+a)^2}{2} - \frac{1}{12}} e^{-\frac{3z^2}{4} - az + \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{(2n+z) \,.\, 2nz^{2n}}}$$

$$\times e^{\sum_{n=1}^{\infty} \frac{(-)^{n-1}B_n a}{2n \,.\, (2n-1) \,.\, z^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-)^n}{nz^n}\{S_{n+1}(a+1) - aS_n(a+1)\}}.$$

In the course of this proof it has been assumed that we have the right to integrate an asymptotic series. The general theory of asymptotic series lies outside the present investigations.

For the validity of the process in question it is sufficient to refer to a theorem of Poincaré.*

§ 15. Some deductions from the asymptotic expansion just obtained are of interest.

Putting $a = 0$, we obtain the asymptotic approximations,

$$\log G(z+1) = \tfrac{1}{12} - \log A + \frac{z}{2}\log 2\pi + \left(\frac{z^2}{2} - \tfrac{1}{12}\right)\log z - \frac{3z^2}{4} + \sum_{s=1}^{\infty} \frac{(-)^s B_{s+1}}{2s.(2s+2).z^{2s}}.$$

It proves convenient to write, in accordance with a more general notation,

$${}_2S_n(a \mid 1, 1) = (a-1)\, S_n(a) - S_{n+1}(a) + a\,\frac{S'_{n+1}(0 \mid 1)}{n+1},$$

so that

$${}_2S_n(a \mid 1, 1) = (a-1)\, S_n(a) - S_{n+1}(a) + a \begin{cases} 0, & n \text{ even} \\ \dfrac{(-)^{\frac{n-1}{2}} B_{\frac{n+1}{2}}}{n+1}, & n \text{ odd.} \end{cases}$$

And then the asymptotic expansion for $\log G(z+a)$ takes the form

$$\log G(z+a) = \tfrac{1}{12} - \log A + \frac{z+a-1}{2}\log 2\pi + \left(\frac{(z+a-1)^2}{2} - \tfrac{1}{12}\right)\log z - \frac{3z^2}{4} - (a-1)\,z$$

$$+ \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{nz^n}\, {}_2S_n(a \mid 1, 1) + \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{(2n+2).2nz^{2n}} + \sum_{n=1}^{\infty} \frac{(-)^n B_n}{2n.(2n-1)\, z^{2n-1}}.$$

There is a very simple expansion for $\log \dfrac{G(z+a)}{G(z)}$ which may be readily deduced.

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* Poincaré, *Mécanique Celeste*, T. II., § 122, p. 13. Vide also *Acta Math.*, T. VIII., pp. 295 et seq.

For, putting $a = 0$ in the expansion just obtained, we have the asymptotic equality,

$$\log G(z) = \tfrac{1}{12} - \log A + \frac{z-1}{2}\log 2\pi + \left(\frac{z^2}{2} - z + \tfrac{5}{12}\right)\log z - \tfrac{3}{4}z^2 + z$$

$$+ \sum_{n=1}^{\infty} \frac{(-)^n B_n}{2n\,.\,(2n-1)\,.\,z^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{2n\,.\,2n+z\,.\,z^{2n}}\,.$$

And hence

$$\log \frac{G(z+a)}{G(z)} = \frac{a}{2}\log 2\pi + \left[\frac{a^2}{2} - az\right]\log z$$

$$- az + \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{nz^n}\,{}_2S_n(a \mid 1, 1).$$

We thus see that there is an exact analogy between the asymptotic expansions for the G and Γ functions. It is noteworthy that the coefficients in the case of the G-functions can all be expressed in terms of simple Bernoullian numbers and functions.

§ 16. It is well known that in the theory of simple Gamma functions Raabe's formula expresses $\int_0^1 \log \Gamma(z+a)\,dz$ in simple terms.

We now proceed to Alexeiewsky's Analogue of Raabe's formula, and we will prove that

$$\int_0^1 \log G(z+a)\,dz = a\log\Gamma(a) - \log G(a+1)$$

$$- \frac{a\,.\,a-1}{2} + \frac{a}{2}\log 2\pi + C,$$

where C is a constant whose value is given by

$$C = \tfrac{1}{4}\log\left[\frac{e^{\frac{1}{3}}}{2\pi A^8}\right],$$

where A is Dr. Glaisher's constant.

Let $$f(a) = \int_0^1 \log G(z+a)\,dz.$$

Then differentiating with regard to a,

$$\frac{d}{da} f(a) = \int_0^1 \frac{G'(z+a)}{G(z+a)} dz$$

$$= \log \left[\frac{G(1+a)}{G(a)}\right],$$

so that by the fundamental difference equations for the G-functions,

$$\frac{d}{da} f(a) = \log \Gamma(a).$$

Now, by § 12,

$$\int_0^a \log \Gamma(a)\, da = a \log \Gamma(a) - \log G(a+1) - \frac{a \,.\, a-1}{2} + \frac{a}{2} \log 2\pi,$$

hence

$$\int_0^1 \log G(z+a)\, dz = a \log \Gamma(a) - \log G(a+1) - \frac{a \,.\, a-1}{2} + \frac{a}{2} \log 2\pi + C,$$

where C is an absolute constant.

Putting $a = 1$, we see that

$$C = \int_0^1 \log G(z+1)\, dz - \tfrac{1}{2} \log 2\pi.$$

Now, by the fundamental product expression for $G(z+1)$, namely

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{z.z+1}{2} - \gamma \frac{z^2}{2}} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}}\right],$$

we have at once

$$\int_0^1 \log G(z+1)\, dz$$

$$= \int_0^1 \left[\frac{z}{2} \log 2\pi - \frac{z \,.\, z+1}{2} - \gamma \frac{z^2}{2} + \sum_{n=1}^{\infty} \left\{k \log\left(1+\frac{z}{k}\right) - z + \frac{z^2}{2k}\right\}\right]$$

$$= \tfrac{1}{4} \log 2\pi - \tfrac{5}{12} - \frac{\gamma}{6} + \sum_{k=1}^{\infty} \left[k \log\left(1+\frac{1}{k}\right) - k + k^2 \log\left(1+\frac{1}{k}\right) - \tfrac{1}{2} + \frac{1}{6k}\right]$$

$$= \tfrac{1}{4} \log 2\pi - \tfrac{5}{12} - \frac{\gamma}{6} + \sum_{k=1}^{\infty} \left[k \,.\, (k+1) \log\left(1+\frac{1}{k}\right) - k - \tfrac{1}{2} + \frac{1}{6k}\right].$$

Now

$$\sum_{k=1}^{\infty} k.(k+1)\log\left(1+\frac{1}{k}\right)$$

$$= \sum_{k=1}^{\infty} [k.\overline{k+1}\log(k+1) - k.(k+1)\log k]$$

$$= \underset{n=\infty}{\text{Lt.}} \left[-2\sum_{k=1}^{n} k\log k + n.(n+1)\log(n+1)\right].$$

Hence

$$\int_0^1 \log G(z+1)\,dz$$

$$= \tfrac{1}{4}\log 2\pi - \tfrac{5}{12} - \frac{\gamma}{6} + \underset{n=\infty}{\text{Lt.}} \left[-2\sum_{k=1}^{n} k\log k\right.$$

$$\left. + n.(n+1)\log(n+1) - \frac{n.\overline{n+1}}{2} - \frac{n}{2} + \tfrac{1}{6}\sum_{n=1}^{n}\frac{1}{k}\right]$$

$$= \tfrac{1}{4}\log 2\pi - \tfrac{5}{12} - \frac{\gamma}{6} + \left[-2\log A - (n^2+n+\tfrac{1}{6})\log n + \tfrac{1}{2}n^2\right.$$

$$+ (n^2+n)\log n + n + 1 - \frac{1}{2} - \frac{n^2}{2} - n + \frac{\gamma}{6} - \tfrac{1}{6}\log n$$

$$\left. + \text{ terms which vanish with } \frac{1}{n}\right]$$

$$= \tfrac{1}{4}\log 2\pi + \tfrac{1}{12} - 2\log A.$$

Thus

$$C = -\tfrac{1}{4}\log 2\pi + \tfrac{1}{12} - 2\log A = \tfrac{1}{4}\log\left[\frac{e^{\frac{1}{3}}}{2\pi A^8}\right],$$

the result required.

§ 17. It is now possible to express Dr. Glaisher's constant A in terms of $G(\frac{1}{2})$, and as a *Corollary* to deduce Alexeiewsky's value for the constant C just investigated.

We will show that

$$A = \frac{2^{\frac{1}{36}} e^{\frac{1}{12}}}{\pi^{\frac{1}{6}} G^{\frac{2}{3}}(\frac{1}{2})}.$$

From the fundamental product expression

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-z\frac{z+1}{2} - \gamma\frac{z^2}{2}} \prod_{k=1}^{\infty} \left[\left(1+\frac{z}{k}\right)^k e^{-z+\frac{z^2}{2k}} \right],$$

we obtain, on making $z = \frac{1}{2}$, and remembering that

$$G(z+1) = \Gamma(z)\, G(z) \text{ and } \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

$$\log[\sqrt{(\pi)}\, G(\tfrac{1}{2})] = \tfrac{1}{4}\log 2\pi - \tfrac{3}{8} - \frac{\gamma}{8} + \sum_{k=1}^{\infty} \left[k\log(2k+1) - k\log 2k - \tfrac{1}{2} + \frac{1}{8k} \right],$$

$$= \tfrac{1}{4}\log 2\pi - \frac{3+\gamma}{8} + \tfrac{1}{2}\sum_{k=1}^{\infty} [(2k+1)\log(2k+1) + 2k\log 2k - 4k\log 2k - \log(2k+1) - 1 + \tfrac{1}{4}k].$$

The summation just written

$$= \operatorname*{Lt.}_{n=\infty} \Big[\sum_{k=1}^{2n+1} k\log k - 4\sum_{k=1}^{n} k\log k - \sum_{k=1}^{2n+1}\log k + \sum_{k=1}^{n}\log k + \tfrac{1}{4}\sum_{k=1}^{n}\frac{1}{k} - 4\,\frac{n.n+1}{2}\log 2 + n\log 2 - n \Big],$$

$$= \operatorname*{Lt.}_{n=\infty} \Big[\log A + \left\{ \frac{(2n+1)^2}{2} + \frac{2n+1}{2} + \tfrac{1}{12} \right\} \log(2n+1) - \tfrac{1}{4}(2n+1)^2 - 4\,.\,\frac{n.n+1}{2}\log 2 - 4\log A$$

$$- 4\left(\frac{n^2}{2} + \frac{n}{2} + \tfrac{1}{12} \right)\log n + n^2 - \log 2\pi - (2n+1+\tfrac{1}{2})\log(2n+1)$$

$$+ (2n+1) + n\log 2 + \log 2\pi + (n+\tfrac{1}{2})\log n - 2n + \tfrac{1}{4}(\gamma + \log n)\Big],$$

which on reduction becomes

$$-3\log A - \tfrac{5}{12}\log 2 + 1 + \frac{\gamma}{4}.$$

Hence

$$\log[(2\pi)^{\frac{1}{2}} G(\tfrac{1}{2})] = \tfrac{1}{4}\log 2\pi - \tfrac{3}{8} - \tfrac{3}{2}\log A - \tfrac{5}{24}\log 2 + \tfrac{1}{2},$$

so that $\log G(\frac{1}{2}) = -\frac{1}{4}\log\pi + \frac{1}{8} - \frac{3}{2}\log A + \frac{1}{24}\log 2,$

and therefore finally

$$A = \frac{2^{\frac{1}{36}} e^{\frac{1}{12}}}{\pi^{\frac{1}{6}} G^{\frac{2}{3}}(\frac{1}{2})}.$$

Now, in § 16, we saw that

$$C = \frac{1}{4}\log\left[\frac{e^{\frac{1}{3}}}{2\pi A^8}\right].$$

Hence, substituting the value of A,

$$C = \frac{1}{12}\log\left[\frac{G^{16}(\frac{1}{2})\pi}{2^{\frac{11}{3}}e}\right],$$

which is the expression given by Alexeiewsky.*

§ 18. It is now easy to deduce Dr. Glaisher's formula† for A, namely,

$$A = 2^{\frac{7}{36}}\pi^{-\frac{1}{6}}e^{\frac{1}{3}+\frac{2}{3}\int_0^{\frac{1}{2}}\log\Gamma(1+x)dx},$$

Make $z = \frac{1}{2}$ in Alexeiewsky's theorem, (§ 12), and we find

$$\int_0^{\frac{1}{2}}\log\Gamma(1+z)\,dz = \frac{1}{2}\log\Gamma(\tfrac{3}{2}) - \log G(\tfrac{3}{2}) - \tfrac{3}{8} + \tfrac{1}{4}\log 2\pi,$$

so that, since $\qquad G(\frac{3}{2}) = \sqrt{(\pi)}\,G(\frac{1}{2}),$

$$G(\tfrac{1}{2}) = e^{-\frac{3}{8}}.2^{\frac{1}{4}}.e^{-\int_0^{\frac{1}{2}}\log\Gamma(1+x)dx},$$

and therefore, by § 17,

$$A = \frac{2^{\frac{1}{36}}e^{\frac{1}{12}}}{\pi^{\frac{1}{6}}}e^{\frac{1}{4}}2^{\frac{1}{6}}e^{\frac{2}{3}\int_0^{\frac{1}{2}}\log\Gamma(1+x)dx},$$

$$= 2^{\frac{7}{36}}\pi^{-\frac{1}{6}}e^{\frac{1}{3}+\frac{2}{3}\int_0^{\frac{1}{2}}\log\Gamma(1+x)dx},$$

the expression required.

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* *Fortschritte der Mathematik*, Bd. XXII., p. 439.

† Glaisher, "On the product $1^1.2^2\ldots n^n$," *Messenger*, Vol. VII., p. 47.

It is interesting to note the analogy between the identities

$$\Gamma(\tfrac{1}{2}) = \pi^{\frac{1}{2}},$$
$$G(\tfrac{1}{2}) = A^{-\frac{3}{2}}\pi^{-\frac{1}{4}}e^{\frac{1}{8}}2^{\frac{1}{24}}.$$

This analogy gives an additional argument to those already cited by Dr. Glaisher* for regarding A as a definite constant of analysis.

§ 19. We proceed now to the analogue of Gauss' Theorem in the theory of simple Gamma functions.

It will be proved that

$$\prod_{r=0}^{n-1}\prod_{s=0}^{n-1} G\left(z+\frac{r+s}{n}\right) = K(2\pi)^{\frac{n.n-1}{2}z} n^{-\frac{n^2z^2}{2}+nz} G(nz),$$

where K is a constant whose value is given by either of the expressions

$$\frac{1}{A^{n^2-1}} e^{\frac{n^2-1}{12}} (2\pi)^{-\frac{n-1}{2}} n^{-\frac{5}{12}},$$

or $$G^{\frac{2(n^2-1)}{3}}(\tfrac{1}{2})\,\pi^{\frac{(n-1).(n-2)}{6}}\,2^{-\frac{(n-1).(n+1)}{4}}\,n^{-\frac{5}{12}}.$$

Let $$f(z) = G(nz),$$

then $$\frac{f(z+1)}{f(z)} = \Gamma(nz+n-1)\,\Gamma(nz+n-2)\ldots\Gamma(nz).$$

Now, by Gauss' theorem,

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\ldots\Gamma\left(z+\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-zn}\Gamma(nz).$$

Hence

$$\frac{f(z+1)}{f(z)} = \prod_{r=0}^{n-1}\Bigg[(2\pi)^{-\frac{n-1}{2}} n^{n\left(z+\frac{r}{n}\right)-\frac{1}{2}}$$
$$\times\,\Gamma\left(z+\frac{r}{n}\right)\Gamma\left(z+\frac{r+1}{n}\right)\ldots\Gamma\left(z+\frac{r+n-1}{n}\right)\Bigg].$$

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* Glaisher, *Messenger*, Vol. XXIV., p. 16.

And therefore

$$f(z+1)=(2\pi)^{-\frac{n(n-1)}{2}}\, n^{n^2z+\frac{n^2-2n}{2}} \prod_{r=0}^{n-1}\prod_{s=0}^{n-1} \Gamma\left(z+\frac{r+s}{n}\right) f(z).$$

A solution of this difference equation is

$$(2\pi)^{-\frac{n(n-1)}{2}z}\, n^{\frac{n^2z^2}{2}-nz} \prod_{r=0}^{n-1}\prod_{s=0}^{n-1} G\left(z+\frac{r+s}{n}\right),$$

and the general solution is equal to this expression multiplied by a simply periodic function of period unity.

We proceed to show that $G(nz)$ is equal to the expression just written multiplied by the constant K, whose value has been given above; in other words the simply periodic function reduces to this constant.

When $|z|$ is very large and z is not real and negative, we have seen (§ 15) that

$$\text{Lt. } G(z+1)=\frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{z}{2}} z^{\frac{z^2}{2}-\frac{1}{12}} e^{-\frac{3z^2}{4}+\text{ terms vanishing with } \frac{1}{z}}.$$

And hence, under similar circumstances,

$$\text{Lt. } G(z)=\frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{z-1}{2}} z^{\frac{z^2}{2}-z+\frac{5}{12}} e^{-\frac{3z^2}{4}+z+\text{ vanishing terms}}.$$

So that, writing $z+\frac{r}{n}$ for z,

$$\text{Lt. } G\left(z+\frac{r}{n}\right)=\frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{z}{2}+\frac{r}{2n}-\frac{1}{2}}\left(z+\frac{r}{n}\right)^{\frac{z^2}{2}+z\left(\frac{r}{n}-1\right)+\frac{r^2}{2n^2}-\frac{r}{n}+\frac{5}{12}}$$

$$\times e^{-\frac{3z^2}{4}-\frac{3zr}{n}-\frac{3r^2}{4n^2}+z+\frac{r}{n}+\frac{(\dots)}{z}+\dots}.$$

Now, under such conditions for z,

$$\text{Lt. }\left(1+\frac{\alpha}{z}\right)^z=e^{\alpha-\frac{\alpha^2}{2z}+\dots}.$$

And hence

$$\text{Lt. } G\left(z+\frac{r}{n}\right)=\frac{e^{\frac{1}{12}}}{A}(2\pi)^{\frac{z}{2}+\frac{r}{2n}-\frac{1}{2}}.\, z^{\frac{z^2}{2}+2\left(\frac{r}{n}-1\right)+\frac{z^2}{2n^2}-\frac{r}{n}+\frac{5}{12}}$$

$$\times e^{-\frac{3z^2}{4}+z\left(1-\frac{r}{n}\right)+\frac{(\dots)}{z}+\dots}.$$

Give now r all values from 0 to $(n-1)$, and we obtain, on multiplying the resulting expansions,

$$\text{Lt.}\, G(z)\, G\left(z+\frac{1}{n}\right) \dots G\left(z+\frac{n-1}{n}\right)$$

$$= \frac{e^{\frac{n}{12}}}{A^n} (2\pi)^{\frac{nz}{2}-\frac{n+1}{4}} . z^{\frac{nz^2}{2} - z\frac{n+1}{2} + \frac{n}{12} + \frac{1}{4} + \frac{1}{12n}} . e^{-\frac{3n}{4}z^2 + \frac{n+1}{2}z + \frac{(\dots)}{2}\dots}.$$

Substitute now $z+\frac{s}{n}$ for z, and, as before, multiply the expansions which result from assigning to s all values from 0 to $(n-1)$, and we obtain, on reduction,

$$\text{Lt.} \prod_{r=0}^{n-1} \prod_{s=0}^{n-1} G\left(z+\frac{r+s}{n}\right)$$

$$= \frac{e^{\frac{n^2}{12}}}{A^{n^2}} (2\pi)^{\frac{n^2z-n}{2}} z^{\frac{n^2z^2}{2} - nz + \frac{5}{12}} e^{-\frac{3n^2}{4}z^2 + zn + \frac{(\dots)}{z} + \dots}.$$

But from § 15, under such circumstances,

$$\text{Lt.}\, G(nz) = \frac{e^{\frac{1}{12}}}{A} (2\pi)^{\frac{nz-1}{2}} (nz)^{\frac{n^2z^2}{2} - nz + \frac{5}{12}} e^{-\frac{3}{4}n^2z^2 + nz + \frac{(\dots)}{z}\dots}.$$

Thus we see that, when $|z|$ is large, but z not real and negative,

$$\text{Lt.}\, (2\pi)^{-\frac{n.n-1}{2}z}\, n^{\frac{n^2z^2}{2} - nz} \prod_{r=0}^{n-1} \prod_{s=0}^{n-1} G\left(z+\frac{r+s}{n}\right)$$

$$= \text{Lt.}\, KG(nz), \text{ where } K = \left(\frac{e^{\frac{1}{12}}}{A}\right)^{n^2-1} (2\pi)^{-\frac{n-1}{2}} n^{-\frac{5}{12}},$$

exponentials being neglected which involve powers of $\frac{1}{z}$.

But we have seen that the two sides of this equation can only differ by a function of $e^{2\pi i z}$.

This function is therefore unity; and hence

$$KG(nz) = (2\pi)^{-\frac{n.n-1}{2}z}\, n^{\frac{n^2z^2}{2} - nz} \prod_{r=0}^{n-1} \prod_{s=0}^{n-1} G\left(z+\frac{r+s}{n}\right),$$

an equality which holds for all values of z.

We have shown that

$$K = \frac{1}{A^{n^2-1}} e^{\frac{n^2-1}{12}} (2\pi)^{-\frac{n-1}{2}} n^{-\frac{5}{12}};$$

and hence, using the identity

$$A = \frac{2^{\frac{1}{36}} e^{\frac{1}{12}}}{\pi^{\frac{1}{6}} G^{\frac{2}{3}}(\frac{1}{2})},$$

K may equally be written $\{G(\frac{1}{2})\}^{\frac{2}{3}(n^2-1)} \pi^{\frac{n-1\,.\,n-2}{6}} 2^{-\frac{(n-1).(n+1)}{4}} n^{-\frac{5}{12}}$.

§ 20. The analogue of Euler's theorem may now be written down at once.

Make $z=0$ in the result just obtained, and observe that from the product expression for $G(z)$

$$\underset{z=0}{\text{Lt.}} \frac{G(nz)}{G(z)} = \underset{z=0}{\text{Lt.}} \frac{\Gamma(z)}{\Gamma(nz)} = n;$$

then we find

$$\prod_{r=0}^{n-1} \prod_{s=0}^{n-1}{}' G\left(\frac{r+s}{n}\right) = Kn,$$

the accent denoting that the term in the product for which $\left.\begin{matrix} r=0 \\ s=0 \end{matrix}\right\}$ must be omitted.

$$\text{Thus} \quad \prod_{r=0}^{n-1} \prod_{s=0}^{n-1}{}' G\left(\frac{r+s}{n}\right) = \frac{n^{\frac{7}{12}}}{A^{n^2-1}} e^{\frac{n^2-1}{12}} (2\pi)^{-\frac{n-1}{2}},$$

the analogue requred.

It is worth noting that, when $n=1$, each side reduces to unity—a result which furnishes some verification of the accuracy of the algebra.

The results of §§ 19 and 20 have previously been given for the ilg function by Dr. Glaisher.*

§ 21. We conclude at this point the investigation of the elementary properties of the G function which are immediately deducible from its product-expression.

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* Glaisher, *Quarterly Journal of Mathematics*, Vol. XXVIII., pp. 69–72.

It is evident that we might now construct an extended G function satisfying the difference equation

$$f(z+1) = G(z) f(z),$$

and corresponding to Dr. Glaisher's function $\text{ilg}_2 z$.

The function and all higher functions possess properties exactly analogous to those already investigated, for instance

$$\int_0^z G(z+a)\, dz$$

can be expressed in terms of the next higher G function. But the consideration of such functions, and the properties which they possess and the constants to which they give rise, is for the present reserved.

Contour integrals and asymptotic approximations connected with the G function.

§ 22. The G function has been defined as an uniform integral function whose sole finite zeros are at the points

$$z = -n\omega; \quad n = 0, 1, 2, \ldots, \infty,$$

the zero at the point $-n\omega$ being of order $(n+1)$. Or more simply $G(z+1)$ is a uniform transcendental integral function having its sole finite zeros at the points

$$z = -n\omega; \quad n = 1, 2, \ldots, \infty,$$

the zero at the point $-n\omega$ being of order n.

We seek now to express $\log G(z+1)$ as a contour integral, and we propose to investigate the allied theory of the asymptotic representation of expressions of the type

$$\sum_{n=1}^{m} \frac{n}{(a+n\omega)^s},$$

when m is large and s has any values real or complex, the many-valued function $(a+n\omega)^s$ having its principal value with respect to the axis of $-\omega$.

The investigation which follows might easily be extended to series of the type

$$\sum_{n=1}^{m} \frac{n^k}{(a+n\omega)^s},$$

where k is any positive integer, and from it might be deduced contour integrals to express higher G functions with poles at the points $z = -n\omega$, $n = 1, 2, 3, \ldots, \infty$ of respective order n^k.

The interesting fact about all these expansions is that there is no necessity to introduce as coefficients functions other than the Bernoullian functions and constants of a and ω.

The predominant terms of the approximations to particular cases of some of the series which follow have recently been obtained from a different point of view by Dr. Glaisher.

§ 23. We first proceed to obtain an asymptotic expansion for

$$\sum_{n=1}^{m-1} \frac{n}{(a+n\omega)^s},$$

when m is large, and a, s, and ω have any complex values.

We have previously considered* the extended Riemann ζ function defined, when $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is positive, by the equality

$$\zeta(s, a, \omega) = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-az}}{1-e^{-\omega z}} (-z)^{s-1}\, dz,$$

the integral being taken round a contour extending along the axis of ω^{-1} from $+\infty$ round the origin and back again to $+\infty$, and enclosing the origin but no other pole of the subject of integration. That value of $(-z)^{s-1} = e^{(s-1)\log(-z)}$ is to be taken which is such that $\log(-z)$ is made uniform by a cross-cut along the axis of the contour of the integral, and is real when z is real and negative. And we obtained (§ 35) an expression when m is a large positive integer, which may be written in the form

$$\zeta(s, a, \omega) = \sum_{n=0}^{m-1} \frac{1}{(a+n\omega)^s} - \frac{1}{1-s}\frac{1}{\omega^s m^{s-1}} - \frac{1}{(m\omega)^s}\left(\frac{a}{\omega} - \frac{1}{2}\right)$$

$$+ \sum_{r=1}^{\infty} (-)^{r+1} \binom{s+r-1}{r} \left\{\frac{S_r(a/\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^{s+r}}\right\}.$$

This asymptotic equality serves to define $\zeta(s, a, \omega)$ for all values of s, a, ω.

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* *Theory of the Gamma Function*, Part III.

Differentiate this result with regard to ω and we find

$$\sum_{n=1}^{m-1} \frac{sn}{(a+n\omega)^{s+1}} = -\frac{\partial}{\partial\omega}\zeta(s, a, \omega) + \frac{s}{1-s}\frac{1}{\omega^{s+1}m^{s-1}}$$

$$+ \frac{m}{(m\omega)^{s+1}}\left\{\frac{(s+1)a}{\omega} - \frac{s}{2}\right\}$$

$$+ \frac{s}{\omega}\sum_{r=1}^{\infty}(-)^r\binom{s+r-1}{r}\left\{\frac{S_r(a/\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^{s+r}}\right\}$$

$$+ \sum_{r=1}^{\infty}\frac{(-)^{r+1}}{m^{r+s}\omega^s}\binom{s+r-1}{r}\frac{\partial}{\partial\omega}S_r\left(\frac{a}{\omega}\right),$$

for (*Gamma Function*, § 16) $\frac{S_r(a/\omega)}{\omega^r} = S_r\left(\frac{a}{\omega}\right)$, while $\frac{{}_1B_{r+1}(\omega)}{\omega^r}$ is (§ 15) independent of ω.

Now $$\frac{\partial}{\partial\omega}S_r\left(\frac{a}{\omega}\right) = -\frac{a}{\omega^2}\frac{d}{d\left(\frac{a}{\omega}\right)}S_r\left(\frac{a}{\omega}\right)$$

$$= -\frac{ar}{\omega^2}\left\{S_{r-1}\left(\frac{a}{\omega}\right) + {}_1B_r(1)\right\} \quad \text{if } r > 1$$

$$= -\frac{ar}{\omega^{r+1}}\{S_{r-1}(a/\omega) + {}_1B_r(\omega)\},$$

and, if $r = 1$,

$$\frac{\partial}{\partial\omega}S_1\left(\frac{a}{\omega}\right) = \frac{\partial}{\partial\omega}\left(\frac{a^2}{2\omega^2} - \frac{a}{2\omega}\right)$$

$$= -\frac{1}{\omega}\left(\frac{a^2}{\omega^2} - \frac{a}{2\omega}\right).$$

Hence, if $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is positive,

$$\sum_{n=1}^{m-1}\frac{sn}{(a+n\omega)^{s+1}} = \frac{i\Gamma(1-s)}{2\pi}\int\frac{ze^{-(a+\omega)z}}{(1-e^{-\omega z})^2}(-z)^{s-1}dz + \frac{s}{1-s}\frac{1}{\omega^{s+1}m^{s-1}}$$

$$+ \frac{1}{(m\omega)^s}\left[\frac{(s+1).a}{\omega^2} - \frac{s}{2\omega}\right] - \frac{s}{(m\omega)^{s+1}}\left(\frac{a^2}{\omega^2} - \frac{a}{2\omega}\right)$$

$$+ \frac{s}{\omega}\sum_{r=1}^{\infty}(-)^r\binom{s+r-1}{r}\left[\frac{S_r(a/\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^{s+r}}\right]$$

$$+ \frac{s}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}}{(m\omega)^{s+r+1}}\binom{s+r}{r}a\left[S_r(a/\omega) + {}_1B_{r+1}(\omega)\right].$$

Divide by s, change s into $s-1$, and we get

$$\sum_{n=1}^{m-1} \frac{n}{(a+n\omega)^s} = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}(-z)^{s-1}dz + \frac{1}{2-s} \cdot \frac{1}{\omega^s m^{s-2}}$$

$$+ \frac{as}{\omega^2 . s-1} \frac{1}{(m\omega)^{s-1}} - \frac{1}{2(m\omega)^{s-1}.\omega} - \frac{a^2}{\omega^2.(m\omega)^s} + \frac{a}{2\omega(m\omega)^s}$$

$$+ \frac{1}{\omega}\sum_{r=1}^{\infty}(-)^r \binom{s+r-2}{r} \frac{S_r(a/\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^{s+r-1}}$$

$$+ \frac{1}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}}{(m\omega)^{s+r}} \binom{s+r-1}{r} a\left[S_r(a/\omega) + {}_1B_{r+1}(\omega)\right],$$

the expansion required.

It is evident that this expansion is valid if $\mathfrak{R}\left(\frac{a}{\omega}+1\right)$ is positive, as may be rigorously established by an investigation similar to that undertaken for the fundamental expansion in the *Theory of the Gamma Function* (§ 23).

From Abel's investigation of the binomial theorem for a complex index, we see with the given prescription of $(a+m\omega)^s$ that

$$\frac{m}{(a+m\omega)^s} = \frac{1}{m^{s-1}\omega^s}\left[1 + \sum_{r=1}^{\infty}(-)^r \binom{s+r-1}{r} \frac{a^r}{(m\omega)^r}\right]$$

$$= \frac{1}{\omega.(m\omega)^{s-1}} - \frac{a}{\omega(m\omega)^s} + \frac{1}{\omega}\sum_{r=1}^{\infty}(-)^r \binom{s+r-2}{r} \frac{a^r}{(m\omega)^{r+s-1}}$$

$$+ \frac{1}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}}{(m\omega)^{s+r}} \binom{s+r-1}{r} . a . a^r,$$

and therefore we may write this expression in the form

$$\sum_{n=1}^{m} \frac{n}{(a+n\omega)^s} = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}(-z)^{s-1}dz + \frac{1}{2-s} \cdot \frac{1}{\omega^s m^{s-2}}$$

$$+ \left(\frac{as}{\omega^2 . s-1} + \frac{1}{2\omega}\right) \frac{1}{(m\omega)^{s-1}} - \left(\frac{a^2}{\omega^2} + \frac{a}{2\omega}\right)\frac{1}{(m\omega)^s}$$

$$+ \frac{1}{\omega}\sum_{r=1}^{\infty}(-)^r \binom{s+r-2}{r} \frac{S_r(a+\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^{s+r-1}}$$

$$+ \frac{1}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}}{(m\omega)^{s+r}} \binom{s+r-1}{r} a\left[S_r(a+\omega) + {}_1B_{r+1}(\omega)\right].$$

Put now

$$\zeta_2(s, a/\omega) = \frac{i\Gamma(1-s)}{2\pi} \int \frac{e^{-az}}{(1-e^{-\omega z})^2} (-z)^{s-1} dz,$$

and we have finally, changing $a + \omega$ into a,

$$\sum_{n=0}^{m-1} \frac{n+1}{(a+n\omega)^s} = \zeta_2(s, a/\omega) + \frac{1}{(2-s).\omega^2} \cdot \frac{1}{(m\omega)^{s-2}}$$

$$+ \left[\frac{as}{\omega^2 s - 1} - \frac{s+1}{2\omega.(s-1)}\right] \frac{1}{(m\omega)^{s-1}} - \left[\frac{a^2}{\omega^2} - \frac{3a}{2\omega} + \tfrac{1}{2}\right] \frac{1}{(m\omega)^s}$$

$$+ \frac{a-\omega}{\omega} \sum_{r=1}^{\infty} \frac{(-)^{r+1}}{(m\omega)^{s+r}} \binom{s+r-1}{r} [S_r(a/\omega) + {}_1B_{r+1}(\omega)]$$

$$+ \frac{1}{\omega} \sum_{r=1}^{\infty} \frac{(-)^r}{(m\omega)^{s+r-1}} \binom{s+r-2}{r} [S_k(a/\omega) + {}_1B_{r+1}(\omega)].$$

This asymptotic expansion may be used to define $\zeta_2(s, a/\omega)$ for all values of s, a and ω. It can be readily seen that the function so defined does not depend on m, and with this definition we see that, for all values of s, a, ω, we have

$$\zeta_2(s+1, a+\omega/\omega) = -\frac{1}{s} \frac{\partial}{\partial\omega} \zeta(s, a, \omega).$$

In case any doubts should arise as to the legitimacy of differentiation under the sign of integration, it is sufficient to remark that this formula can be established *de novo* by a process similar to that employed to establish the result from which we started. The method given has the advantage of brevity.

§ 24. It is proposed to call $\zeta_2(s, a/\omega)$ the second extended Riemann ζ function with two equal parameters ω.

It is evidently infinite when $s = 2$.

Putting $s = 2$ in the formula of § 23, we obtain at once

$$\underset{s=2}{\text{Lt.}} \left[\zeta_2(s, a/\omega) - \frac{1}{(s-2)\omega^2}\right]$$

$$= \underset{m=\infty}{\text{Lt.}} \left[\sum_{n=1}^{m} \frac{n}{(a-\omega+n\omega)^2} - \frac{1}{n\omega^2}\right] + \frac{\gamma}{\omega^2} - \frac{1}{\omega^2}\log\omega$$

$$= \frac{1}{\omega^2}\left\{-1 - \frac{d^2}{d\left(\frac{a}{\omega}\right)^2} \log G\left(\frac{a}{\omega}\right)\right\} - \frac{1}{\omega^2}\log\omega$$

$$= -\frac{1}{\omega^2} - \frac{d^2}{da^2} \log G\left(\frac{a}{\omega}\right) - \frac{1}{\omega^2}\log\omega.$$

It is evident that, when s is a positive integer > 2,

$$\zeta_2(s, a/\omega) = \frac{(-)^{s-1}}{(s-1)!}\frac{d^s}{da^s}\log G\left(\frac{a}{\omega}\right).$$

Since $\zeta_2(s+1, a+\omega/\omega) = -\frac{1}{s}\frac{\partial}{\partial\omega}\zeta(s, a, \omega),$

we see that when s is a negative integer

$$\zeta_2(s+1, a+\omega/\omega) = \frac{1}{s}\frac{\partial}{\partial\omega}[S_{-s}(a/\omega) + {}_1B_{1-s}(\omega)]$$

(*Gamma Function*, § 27).

Now

$$\frac{\partial}{\partial\omega}S_{-s}(a/\omega) = \frac{\partial}{\partial\omega}\omega^{-s}S_{-s}\left(\frac{a}{\omega}\right)$$

$$= -s.\omega^{-s-1}.S_{-s}\left(\frac{a}{\omega}\right) + \omega^{-s}\left(-\frac{a}{\omega^2}\right)\frac{\partial}{\partial\left(\frac{a}{\omega}\right)}S_{-s}\left(\frac{a}{\omega}\right)$$

$$= -\frac{s}{\omega}S_{-s}(a/\omega) + \frac{as}{\omega}[S_{-s-1}(a/\omega) + {}_1B_{-s}(\omega)], \text{ when } -s > 1.$$

Thus, when s is a negative integer greater than 0,

$$\zeta_2(s, a+\omega/\omega) = -\frac{1}{\omega}[S_{-s+1}(a/\omega) - aS_{-s}(a/\omega)] + \frac{a}{\omega}{}_1B_{-s+1}(\omega) - \frac{1}{\omega}{}_1B_{2-s}(\omega).$$

And therefore

$$\zeta_2(s, a/\omega) = {}_2S_{-s}(a/\omega, \omega) + {}_2B_{1-s}(\omega, \omega),$$

where, in conformity with § 15, we put

$${}_2S_n(a/\omega, \omega) = \frac{a-\omega}{\omega}S_n(a/\omega) - \frac{1}{\omega}S_{n+1}(a/\omega) + \frac{a}{\omega}{}_1B_{n+1}(\omega),$$

and $${}_2B_n(\omega, \omega) = -{}_1B_n(\omega) - \frac{1}{\omega}{}_1B_{n+1}(\omega).$$

To find the value of $\zeta_2(0, a/\omega)$ put $s = \epsilon$, where ϵ is very small, in the asymptotic expansion of the preceding section, and expand in powers of ε, retaining only the absolute term and the first power. We find, when $\mathfrak{R}\left(\frac{a}{\omega}\right)$ is positive,

$$\frac{m.m+1}{2} - \varepsilon \sum_{n=1}^{m} n \log(a+n\omega)$$

$$= -\frac{i}{2\pi}\int \frac{e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}\{1+\varepsilon\log(-z)+\ldots\}(1+\gamma\varepsilon+\ldots)\frac{dz}{z}$$

$$+\frac{m^2}{2}\left(1+\frac{\varepsilon}{2}+\ldots\right)\{1-\epsilon\log(\omega m)-\ldots)-m\omega\left[\frac{a\epsilon}{\omega^2}-\frac{1-\epsilon\log(\omega m)+..}{2\omega}\right]$$

$$-\left(\frac{a^2}{\omega^2}+\frac{a}{2\omega}\right)\{1-\varepsilon\log(\omega m)+..\}\frac{\varepsilon}{\omega}\sum_{r=2}^{\infty}\frac{(-)^{r-1}}{r.(r-1)}\frac{S_r(a+\omega)+{}_1B_{r+1}(\omega)}{(m\omega)^{r-1}}$$

$$+\frac{1}{\omega}[S_1(a+\omega)+{}_1B_2(\omega)][1-\varepsilon\log(\omega m)+\ldots](1-\varepsilon)$$

$$+\frac{\epsilon}{\omega}\sum_{r=1}^{\infty}\frac{a(-)^{r+1}}{r(m\omega)^r}[S_r(a+\omega)+{}_1B_{r+1}(\omega)],$$

and we obtain an asymptotic identity true for all values of s, a, ω when the integral is replaced by

$$\zeta_2(0, a+\omega/\omega)+\varepsilon\left[\frac{\partial}{\partial s}\zeta_2(s, a+\omega/\omega)\right]_{s=0}+\ldots.$$

Equating the absolute terms, we find

$$\frac{m.m+1}{2} = \zeta_2(0, a+\omega)+\frac{m^2}{2}+\frac{m}{2}-\left(\frac{a^2}{\omega^2}+\frac{a}{2\omega}\right)$$
$$+\frac{1}{\omega}\left\{a+\frac{2\omega}{a^2}-\frac{a}{2}+\frac{\omega}{12}\right\},$$

for $${}_1B_2(\omega)=\frac{S_2'(0/\omega)}{2}=\omega\frac{B_1}{2}=\frac{\omega}{12}.$$

Thus $$\zeta_2(0, a/\omega)=\left[\frac{a^2}{2\omega^2}-\frac{a}{\omega}+\tfrac{5}{12}\right]={}_2S_1'(a/\omega, \omega),$$

in accordance with our previous notation, the accent denoting differentiation with regard to a.

In order to complete the table of values of $\zeta_2(s, a/\omega)$ for integral values of s, we must investigate its value as s approaches unity.

Put $s=1$ in the formula of § 23, and we find

$$\operatorname*{Lt.}_{s=1}\left[\zeta_2(s, a/\omega)+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\frac{1}{s-1}\right]$$

$$=\operatorname*{Lt.}_{m=\infty}\left[\sum_{n=1}^{m}\frac{n}{a+(n-1)\,\omega}+\frac{m}{\omega}+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\log m\omega-\left(\frac{a}{\omega^2}-\frac{1}{2\omega}\right)\right]$$

$$=\frac{1}{\omega}\operatorname*{Lt.}_{m=\infty}\frac{d}{d\left(\dfrac{a}{\omega}\right)}\left(\log\prod_{n=1}^{m}\left\{\left(1+\frac{\dfrac{a}{\omega}-1}{n}\right)^n-m\frac{a}{\omega}+\left(\frac{a}{\omega}-1\right)^2\frac{1}{2n}\right\}\right.$$

$$\left.-\frac{\gamma}{2}\left(\frac{a}{\omega}-\tfrac{1}{2}\right)^2-\left(\frac{a^3}{2\omega^2}-\frac{a}{2\omega}\right)\right)+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\log\omega$$

$$=\frac{1}{\omega}\frac{d}{d\left(\dfrac{a}{\omega}\right)}\log G\left(\frac{a}{\omega}\right)-\tfrac{1}{2}\log 2\pi+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\log\omega.$$

Thus we have

$$\operatorname*{Lt.}_{s=1}\left[\zeta_2(s, a/\omega)+\frac{{}_2S_1^{(2)}(a/\omega,\omega)}{s-1}\right]$$

$$=\frac{d}{da}\log G\left(\frac{a}{\omega}\right)-\frac{1}{2\omega}\log 2\pi+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\log\omega.$$

Tabulating our results, we see that for integral values of s,

$$\zeta_2(s, a/\omega)=\frac{(-)^{s-1}}{(s-1)!}\frac{d^s}{da^s}\log G\left(\frac{a}{\omega}\right)\quad\text{when } s>2,$$

$$\operatorname{Lt.}\left[\zeta_2(s, a/\omega)-\frac{1}{(s-2)\,\omega^2}\right]=-\frac{d^2}{da^2}\log G\left(\frac{a}{\omega}\right)-\frac{1}{\omega^2}\log\omega-\frac{1}{\omega^2}$$

when $s=2$,

$$\operatorname{Lt.}\left[\zeta_2(s, a/\omega)+\frac{{}_2S_1^{(2)}(a/\omega,\omega)}{s-1}\right]=\frac{d}{da}\log G\left(\frac{a}{\omega}\right)$$

$$+\left(\frac{a}{\omega^2}-\frac{1}{\omega}\right)\log\omega-\frac{1}{2\omega}\log 2\pi\quad\text{when } s=1,$$

$\zeta_2(s, a/\omega)={}_2S_1^{(1)}(a/\omega,\omega)$ when $s=0$,

$\zeta_2(s, a/\omega)={}_2S_{-s}(a/\omega,\omega)+{}_2B_{1-s}(\omega,\omega)$ when $s<0$.

§ 25. Equate next the terms involving ϵ in the identity of the previous paragraph. We proceed to show that we shall obtain an asymptotic equality which can be quickly used to give a contour integral for $\log G(z+1)$.

We find at once, if $a+\omega$ be positive with respect to ω,

$$-\sum_{n=1}^{m} n\log(a+n\omega) = -\frac{i}{2\pi}\int \frac{e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}\,[\log(-z)+\gamma]\,\frac{dz}{z}$$

$$+\frac{m^2}{2}(\tfrac{1}{2}-\log m\omega) - m\omega\left(\frac{a}{\omega^2}+\frac{1}{2\omega}\log m\omega\right) + \left(\frac{a^2}{\omega^2}+\frac{a}{2\omega}\right)\log\omega m$$

$$+\sum_{r=1}^{\infty}\frac{(-)^r}{r.r+1}\,\frac{S_{r+1}(a+\omega)+{}_1B_{r+2}(\omega)}{(m\omega)^r}$$

$$-\frac{1}{\omega}[1+\log m\omega]\left[\frac{a^2}{2\omega}+\frac{a}{2}+\frac{\omega}{12}\right]$$

$$+\frac{1}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}a}{r\,(m\omega)^r}[S_r(a+\omega)+{}_1B_{r+1}(\omega)],$$

and this asymptotic equality holds for all values of a and ω if the integral is replaced by

$$\left[\frac{\partial}{\partial s}\zeta_2(s,\,a+\omega/\omega)\right]_{s=0}.$$

In the contour integral the axis of ω^{-1} is a cross-cut for $\log(-z)$, and that value of the logarithm is to be taken which is such that the imaginary part of the initial value of $\log(-\omega^{-1})$ lies between 0 and $-2\pi i$.

Thus, when $\mathfrak{R}\left(\dfrac{a}{\omega}+1\right)$ is positive, and the contour of the integral is taken along the axis of ω^{-1},

$$-\log\prod_{n=1}^{m}(a+n\omega)^n = -\frac{i}{2\pi}\int\frac{e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}\log(-z)\,\frac{dz}{z}$$

$$+\gamma\left\{\frac{a^2}{2\omega^2}-\frac{1}{12}\right\} - \left[\frac{m^2}{2}+\frac{m}{2}-\frac{a^2}{2\omega^2}+\frac{1}{12}\right]\log\omega m$$

$$+\left[\frac{m^2}{4}-\frac{ma}{\omega}-\frac{a^2}{2\omega^2}-\frac{a}{2\omega}-\frac{1}{12}\right]$$

$$+\sum_{r=1}^{\infty}\frac{(-)^r}{r.(r+1)}\,\frac{S_{r+1}(a+\omega)+{}_1B_{r+2}(\omega)}{(m\omega)^r}$$

$$+\frac{a}{\omega}\sum_{r=1}^{\infty}\frac{(-)^{r+1}}{r}\,\frac{S_r(a+\omega)+{}_1B_{r+1}(\omega)}{(m\omega)^r}\quad\text{.........(1)},$$

which is important as being the asymptotic expansion of the logarithm of the product

$$\prod_{n=1}^{m} (a + n\omega)^n.$$

Make now $a = 0$, $\omega = 1$, and we find, when m is a large positive integer, and the integral is taken along a contour having for axis the positive half of the real axis,

$$\log 1^1.2^2....m^m = \frac{i}{2\pi}\int \frac{e^{-z}}{(1-e^{-z})^2}\log(-z)\frac{dz}{z} + \gamma\left(-\frac{a^2}{2} + \tfrac{1}{12}\right)$$

$$+ \left[\frac{m^2}{2} + \frac{m}{2} + \tfrac{1}{12}\right]\log m - \frac{m^2}{4} + \tfrac{1}{12} + \sum_{r=1}^{\infty}\frac{(-)^{r-1}{}_1B_{r-2}(1)}{r.(r+1).m^r}.$$

This is the result which has been previously cited (§ 3). The dominant terms have been previously obtained in a different way by Dr. Glaisher.

We note that his constant A may be expressed as a contour integral by the relation

$$\log A = \frac{i}{2\pi}\int \frac{e^{-z}}{(1-e^{-z})^2}\log(-z)\frac{dz}{z} - \gamma\left(\frac{a^2}{2} - \tfrac{1}{12}\right) + \tfrac{1}{12},$$

the axis of the contour being the positive half of the real axis.

By subtracting the identity (1) from the asymptotic identity which results from making $a = 0$ in it, we obtain, if $(\omega + a)$ be positive with respect to ω, and the axis of $\dfrac{1}{\omega}$ is the axis of the contour of the integral

$$\log \prod_{n=1}^{m}\left(1 + \frac{a}{n\omega}\right)^n = -\frac{i}{2\pi}\int \frac{e^{-\omega z} - e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}\log(-z)\frac{dz}{z}$$

$$-\frac{\gamma a^2}{2\omega^2} - \frac{a^2}{2\omega^2}\log \omega m + \frac{ma}{\omega} + \frac{a^2}{2\omega^2} + \frac{a}{2\omega}$$

$$+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{r.r+1}\,\frac{S_{r+1}(a+\omega)}{(m\omega)^r} + \frac{a}{\omega}\sum_{r=1}^{\infty}\frac{(-)^r}{r}\,\frac{S_r(a+\omega) + {}_1B_{r+1}(\omega)}{(m\omega)^r}.$$

Hence

$$\log \prod_{n=1}^{m} \left[\left(1+\frac{a}{n\omega}\right)^n e^{-\frac{a}{\omega}+\frac{a^2}{2n\omega^2}}\right]$$

$$= -\frac{i}{2\pi}\int \frac{e^{-\omega k}-e^{-(a+\omega)z}}{(1-e^{-\omega z})^2}\log(-z)\frac{dz}{z} - \frac{a^2}{2\omega^2}\log\omega + \frac{a^2}{2\omega^2} + \frac{a}{2\omega}$$

$$+\frac{a^2}{2\omega^2}\left[-\frac{1}{m}+\sum_{r=1}^{\infty}\frac{(-)^r {}_1B_r(\omega)}{m^{r+1}\omega^{r-1}}\right]+\sum_{r=1}^{\infty}\frac{(-)^{r-1}}{r.r+1}\cdot\frac{S_{r+1}(a+\omega)}{(m\omega)^r}$$

$$+\frac{a}{\omega}\sum_{r=1}^{\infty}\frac{(-)^r}{r}\frac{S_r(a+\omega)+{}_1B_{r+1}(\omega)}{(m\omega)^r},$$

which is the asymptotic expansion of the first m terms of the product which represents $\log G\left(1+\frac{a}{\omega}\right)$.

Make now $m=\infty$, and remember that

$$\log G(a+1) = \log \prod_{n=1}^{\infty}\left[\left(1+\frac{a}{n}\right)^n e^{-a+\frac{a^2}{2n}}\right]$$

$$+\frac{a}{2}\log 2\pi - \frac{a.(a+1)}{2} - \gamma\frac{a^2}{2}.$$

Then we have, if $\mathfrak{R}(a+1)$ be positive, and the contour of the integral is taken along the positive half of the real axis,

$$\log G(a+1) = -\frac{i}{2\pi}\int\frac{e^{-z}-e^{-(a+1)z}}{(1-e^{-z})^2}\log(-z)\frac{dz}{z} - \frac{\gamma a^2}{2} + \frac{a}{2}\log 2\pi,$$

which gives the required representation of $\log G(a+1)$ as a contour integral.

§ 26. Before attempting to reduce this integral to the form given by Alexeiewsky, it is desirable to recall a couple of formulæ in definite integrals.

We have previously shown (*Gamma Function*, § 28), that

$$\gamma = \int_0^1 \frac{1-e^{-z}-e^{-\frac{1}{z}}}{z}dz,$$

and hence, when ϵ is a small real quantity,

$$\int_\epsilon^\infty \frac{e^{-z}dz}{z} = +\log\epsilon - \gamma + \text{terms vanishing with } \epsilon.$$

Again following Stern,* it is easy to show that

$$\int_0^\infty \left[\frac{1}{e^z-1}+\tfrac{1}{2}e^{-z}-\frac{1}{z}\right]\frac{dz}{z}=-\tfrac{1}{2}\log 2\pi.$$

For from the theorem just stated,

$$\int_0^\infty \frac{dz}{z}\left(e^{-z}-e^{-2z}\right)=\log 2.$$

Again from the formula,

$$\log\Gamma(a)=\int_0^\infty \left[(a-1)\,e^{-z}-\frac{e^{-z}-e^{-az}}{1-e^{-z}}\right]\frac{dz}{z},$$

we have, putting $a=\frac{3}{2}$,

$$\log\frac{\sqrt{\pi}}{2}=\int_0^\infty \frac{e^{-z}dz}{z}\left[\tfrac{1}{2}-\frac{1-e^{-\frac{z}{2}}}{1-e^{-z}}\right]$$

$$=\int_0 \frac{e^{-z}da}{z}\left(\tfrac{1}{2}-\frac{1}{1+e^{-\frac{z}{2}}}\right).$$

And hence

$$\tfrac{1}{2}\log\frac{\pi}{2}=\int_0^\infty \frac{e^{-z}dz}{z}\left(\tfrac{1}{2}-\frac{1}{e^z+1}\right)$$

$$=\int_0^\infty \frac{dz}{z}\left(\frac{1}{e^z+1}-\frac{e^{-z}}{2}\right).$$

Now $\displaystyle\int_0^\infty \left[\frac{1}{e^z-1}+\tfrac{1}{2}e^{-z}-\frac{1}{z}\right]\frac{dz}{z}$ is finite,

and therefore is equal to

$$\int_0^\infty \left(\frac{1}{e^{2z}-1}+\tfrac{1}{2}e^{-2z}-\frac{1}{2z}\right)\frac{dz}{z}$$

$$=\int_0^\infty \left(\frac{1}{2\,(e^z-1)}-\frac{1}{2\,(e^z+1)}+\tfrac{1}{2}e^{-2z}-\frac{1}{2z}\right)\frac{dz}{z}$$

$$=\int_0^\infty \left[\frac{1}{2\,(e^z-1)}+\tfrac{1}{4}e^{-z}-\frac{1}{2z}\right]\frac{dz}{z}-\tfrac{1}{4}\log\frac{\pi}{2}-\tfrac{1}{2}\log 2,$$

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* Stern, *Göttingen Studien*, 1847. Vide also Bierens de Haan, *Intégrales Definies* (iv), p. 196.

so that

$$\int_0^\infty \left(\frac{1}{e^z-1} + \tfrac{1}{2}e^{-z} - \frac{1}{z}\right)\frac{dz}{z} = -\tfrac{1}{2}\log 2\pi.$$

We immediately deduce that

$$\int_\epsilon^\infty \left[\frac{1}{e^z-1} + \tfrac{1}{2}e^{-z} - \frac{1}{z}\right]\frac{dz}{z} = -\tfrac{1}{2}\log 2\pi$$

+ terms which vanish with ε.

It is evident that this formula would result immediately from one previously obtained [*Gamma Function*, § 32],

$$-\log \rho_1(\omega) = \int_0^\infty \left[\frac{e^{-\omega z}}{1-e^{-\omega z}} - \frac{S_1^{(2)}(\omega/\omega)}{z} + e^{-z}S_1'(\omega/\omega)\right]\frac{dz}{z},$$

on making $\omega = 1$.

The investigation which has just been given is perhaps worthy of retention as a verification of results which were obtained in the former paper.

§ 27. Take now the formula

$$\log G(a+1) = -\frac{i}{2\pi}\int \frac{e^{-z}-e^{-(a+1)z}}{(1-e^{-z})^2}\log(-z)\frac{dz}{z} - \frac{\gamma a^2}{2} + \frac{a}{2}\log 2\pi,$$

and reduce the contour of the integral to a straight line from $+\infty$ to ϵ, a circle of radius ϵ round the origin, and a straight line back again from ϵ to ∞, ϵ being a small real positive quantity.

The value of $\log(-z)$ when z is at any point of the real axis falls short of its value when z is at the same point after having passed round the origin by $2\pi i$.

Thus we have

$$\log G(a+1) = \int_\epsilon^\infty \frac{e^{-z}-e^{-(a+1)z}}{(1-e^{-z})^2}\frac{dz}{z} - \frac{\gamma a^2}{2} + \frac{a}{2}\log 2\pi$$

$$- \frac{i}{2\pi}\int \frac{e^{-z}-e^{-(a+1)z}}{(1-e^{-z})^2}\log(-z)\frac{dz}{z},$$

the last integral being taken round a small circle of radius ϵ surrounding the origin.

Put $z = \epsilon e^{i\theta}$, and we obtain as the value of this last integral, neglecting higher powers of ϵ,

$$-\frac{i}{2\pi}\int_0^{2\pi} i d\theta \left[\log \epsilon - \pi i + i\theta\right] \frac{\left[a\epsilon e^{i\theta} - \frac{1}{2}(a^2+2a)\epsilon^2 e^{2i\theta}\right]}{\epsilon^2 e^{2i\theta}} \times \left[1 + \epsilon e^{i\theta} + \ldots\right]$$

$$= \frac{i}{2\pi}\int_0^{2\pi} d\theta \left[\frac{a}{\epsilon} e^{-i\theta} - \frac{a^2}{2}\right]\left[\log \epsilon - \pi i + i\theta\right] + \text{terms vanishing with } \epsilon$$

$$= -\frac{a^2}{2}\log \epsilon - \frac{a}{\epsilon} + \text{terms vanishing with } \epsilon.$$

Thus

$$\log G(a+1) = \int_\epsilon^\infty \frac{e^{-z} - e^{-(a+1)z}}{(1-e^{-z})^2}\frac{dz}{z} - \frac{\gamma a^2}{2} + \frac{a}{2}\log 2\pi - \frac{a^2}{2}\log \epsilon - \frac{a}{\epsilon}$$

$$= \int_\epsilon^\infty \frac{e^{-z}dz}{z}\left[\frac{a.a-1}{2} - \frac{a}{1-e^{-z}} + \frac{1-e^{-az}}{(1-e^{-z})^2}\right]$$

$$+ a\left[\int_\epsilon^\infty \left(\frac{1}{e^z-1} + \tfrac{1}{2}e^{-z} - \frac{1}{z}\right)\frac{dz}{z} + \tfrac{1}{2}\log 2\pi\right]$$

$$- \frac{a^2}{2}\left[\int_\epsilon^\infty \frac{e^{-z}dz}{z} + \gamma + \log \epsilon\right] + a\left[\int_\epsilon^\infty \frac{dz}{z^2} - \frac{1}{\epsilon}\right]$$

$$+ \text{terms vanishing with } \epsilon,$$

$$= \int_\epsilon^\infty \frac{e^{-z}dz}{z}\left[\frac{a.a-1}{2} - \frac{a}{1-e^{-z}} + \frac{1-e^{-az}}{(1-e^{-z})^2}\right]$$

$$+ \text{terms vanishing with } \epsilon.$$

Make now $\epsilon = 0$, and we have finally

$$\log G(a+1) = \int_0^\infty \frac{e^{-z}dz}{z}\left[\frac{a.a-1}{2} - \frac{a}{1-e^{-z}} + \frac{1-e^{-az}}{(1-e^{-z})^2}\right]$$

the expression for $\log G(a+1)$ as a definite integral given by Alexeiewsky.

§ 28. It is easy to obtain another form for this integral. For we have seen that

$$\log G(a+1) = \int_\epsilon^\infty \frac{e^{-z} - e^{-(a+1)z}}{(1-e^{-z})^2}\frac{dz}{z} - \frac{\gamma a^2}{2} + \frac{a}{2}\log 2\pi - \frac{a^2}{2}\log \epsilon - \frac{a}{\epsilon}$$
$$+ \text{terms vanishing with } \epsilon.$$

Now

$$\int_\epsilon^\infty \frac{e^{-z}dz}{(1-e^{-z})^2} = -\left[\frac{1}{1-e^{-z}}\right]_\epsilon^\infty = \frac{1}{\epsilon} - \tfrac{1}{2} + \text{terms vanishing with } \epsilon,$$

and

$$\int_\epsilon^\infty \frac{e^{-z}zdz}{(1-e^{-z})^2} = \underset{n=\infty}{\text{Lt.}} \left\{-\left[\frac{z}{1-e^{-z}}\right]_\epsilon^n + \left[\log(e^z-1)\right]_\epsilon^n\right\}$$

$$= 1 - \log\epsilon + \text{terms vanishing with } \epsilon.$$

Hence

$$\log G(a+1) = \int_0^\infty \frac{e^{-z}dz}{z(1-e^{-z})^2}\left[1 - az - \frac{a^2z^2}{2} - e^{-az}\right]$$

$$- \frac{a^2}{2}(1+\gamma) + \frac{a}{2}\log\frac{2\pi}{e}.$$

§ 29. It is worth noting that the asymptotic approximations obtained in § 25 may be utilised to verify, for the case when z is a large integer, the asymptotic approximation for $G(z+a+1)$ obtained in § 15. For we have obtained an asymptotic approximation for

$$\prod_{n+1}^{m-1} (a+n\omega)^n.$$

Putting $\omega = 1$, we have the product

$$(a+1)^1(a+2)^2 \ldots \{a+(m-1)\omega\}^{m-1},$$

which is equal to

$$\frac{\Gamma^m(a+m)\,G(a+1)}{G(a+m+1)}.$$

The verification now proceeds exactly as in the similar case of the simple Gamma functions.*

Finally, for the sake of analogy with the more general transcendent which will be called the double Gamma function, we would indicate another notation which in many ways simplifies the formulæ which have been obtained in the preceding paragraphs.

The new function $\Gamma_2(z/\omega)$ is defined by the relation

$$\Gamma_2^{-1}(z/\omega) = G\left(\frac{z}{\omega}\right)(2\pi)^{-\frac{z}{2\omega}}\,\omega^{\frac{(z-\omega)^2}{2\omega^2}}.$$

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* Vide *Theory of the Gamma Function*, § 34.

It may be at once seen, from the equation

$$G(z+1) = \Gamma(z)\, G(z),$$

that we derive the equation

$$\frac{\Gamma_2^{-1}(z+\omega)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z/\omega)}{\sqrt{\left(\frac{2\pi}{\omega}\right)}}.$$

If we put

$$\psi_2^{(1)}(z/\omega) = \frac{d}{dz}\log\Gamma_2(z/\omega),$$

$$\psi_2^{(2)}(z/\omega) = \frac{d^2}{dz^2}\log\Gamma_2(z/\omega),$$

and so on, we see that the table of § 24 may be written

$$\zeta_2(s, a/\omega) = \frac{(-)^s}{(s-1)!}\psi_2^{(s)}(a), \text{ when } s > 2,$$

$$= \frac{1}{(s-2)\,\omega^2} + \psi_2^{(2)}(a) - \frac{1}{\omega^2}, \text{ when } s = 2,$$

$$= -\frac{{}_2S_1^{(2)}(a/\omega)}{s-1} - \psi_2^{(1)}(a), \text{ when } s = 1;$$

so that, for integral values of s, $\zeta_2(s, a/\omega)$ is expressed in more simple form by the new notation.

So, also, it may be readily seen that we have more simple integral expressions than those which have been investigated. Thus

$$\log\Gamma_2(a/\omega) = \int_0^\infty \frac{dz}{z}\left\{\frac{e^{-az}-1}{(1-e^{-\omega z})^2} + 1 - e^{-z} + \frac{{}_2S_0'(a/\omega) - {}_2S_0'(0/\omega)}{z}\right.$$

$$\left. - e^{-z}\,{}_2S_0(a/\omega)\right\},$$

and a series of formulæ which arise from the equality of the parameters in the more general theory.

The non-existence of a differential equation for $G(z)$ whose coefficients are more simple functions.

§ 30. We conclude this paper by showing that just as $\Gamma(z)$ will not satisfy a differential equation whose coefficients are algebraic, or simply or doubly periodic functions of z, so $G(z)$ will not satisfy an equation whose coefficients are

algebraic functions of z,
periodic functions of z,
simple Gamma functions of z,
or finite combinations of such functions.

The result is one which may be extended to all the higher G functions which were mentioned in § 21.

The proof is the natural modification of the one already given for the case of simple Gamma functions.* We shall therefore indicate it as briefly as possible.

In the first place it may be proved exactly as formerly (§ 45) that if the theorem is true for the logarithmic differential of $G(z)$ it is true for $G(z)$. We shall then confine ourselves to the consideration of the function

$$\phi(z) = \frac{d}{dz} \log G(z).$$

§ 31. Since $G(z)$ satisfies the difference equation

$$G(z+1) = \Gamma(z)\, G(z),$$

it is evident that $\phi(z)$ will satisfy the equation

$$\phi(z+1) = \phi(z) + \psi(z),$$

where, as before, we write $\psi(z) = \frac{d}{dz} \log \Gamma(z)$.

Suppose that $y = \phi(x)$ satisfies the differential equation

$$f(x, y, y', \ldots, y^{(n)}) = 0,$$

so transformed that it is rational and integral in the quantity y and its derivatives.

Let the terms of class s be symbolically

$$R_0(x)\, Q_s^0(y),\ R_1(x)\, Q_s^1(y),\ \ldots,\ R_k(x)\, Q_s^k(y),$$

the terms of class $(s-1)$ being

$$S_0(x)\, Q^0_{s-1}(y),\ \ldots,\ S_l(x)\, Q^l_{s-1}(y),$$

and the functions $R(x)$ and $S(x)$ being holomorphic.

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* Vide *Theory of the Gamma Function*, Part V.

If $\phi(x)$ satisfies the differential equation, $\phi(x)+\psi(x)$ will satisfy the equation in which $x+1$ is written for x.

Divide the equations respectively by $R_0(x)$ and $R_0(x+1)$ and subtract one from the other.

Then we obtain the equation

$$\frac{R_1(x+1)}{R_0(x+1)} Q_s'[\phi(x)+\psi(x)]+\ldots+\frac{R_k(x+1)}{R_0(x+1)} Q_s^k[\phi(x)+\psi(x)]$$

$$-\left\{\frac{R_1(x)}{R_0(x)} Q_s'[\phi(x)]+\ldots+\frac{R_k(x)}{R_0(x)} Q_s^k[\phi(x)]\right\}$$

$$+Q_s^0[\phi(x)+\psi(x)]-Q_s^0[\phi(x)]+\text{terms of lower class}=0.$$

But $$Q_s^0[\phi(x)+\psi(x)]-Q_s^0[\phi(x)]$$

consists solely of terms of lower class than s.

Hence either the equation which has been obtained vanishes identically, or we can reduce the equation for y to one in which there are fewer terms of class s.

The equation cannot vanish identically unless the coefficients of the various terms of class s all vanish, which gives

$$\frac{R_1(x+1)}{R_0(x+1)}-\frac{R_1(x)}{R_0(x)}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots=0,$$

$$\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)}=0,$$

so that the ratios $\dfrac{R_1(x)}{R_0(x)}, \ldots, \dfrac{R_k(x)}{R_0(x)}$ are simply periodic functions of period unity.

The equation for y can thus always be reduced to one of the form

$$R(x)[p_0(x)\,Q_s^0(y)+\ldots+p_k(x)\,Q_s^k(y)]$$

$$+S_0(x)\,Q_{s-1}^0(y)+\ldots+S_l(x)\,Q_{s-1}^l(y)$$

$$+\text{terms of lower class}=0,$$

when all the coefficients are holomorphic functions, and, in addition, the functions $p(x)$ are simply periodic of period unity.

Divide the equation by $R(x)$ and subtract it from the equation which results from changing x into $(x+1)$.

We obtain

$$p_0(x)[Q_s^0\{\phi(x)+\psi(x)\}-Q_s^0\{\phi(x)\}]+\dots$$
$$+p_k(x)[Q_s^k\{\phi(x)+\psi(x)\}-Q_s^k\{\phi(x)\}]$$
$$+\frac{S_0(x+1)}{R(x+1)}[Q^0_{s-1}\{\phi(x)+\psi(x)\}]-\frac{S_0(x)}{R(x)}Q^0_{s-1}\{\phi(x)\}+\dots$$
$$+\frac{S_l(x+1)}{R(x+1)}Q^l_{s-1}\{\phi(x)+\psi(x)\}-\frac{S_l(x)}{R(x)}Q^l_{s-1}\{\phi(x)\}$$
$$+\text{ terms of lower class}=0\dots\dots(1).$$

This is an equation of class $(s-1)$, unless it vanishes identically.

Now, in general,

$$Q_s^0\{\phi(x)+\psi(x)\}-Q_s^0\{\phi(x)\}$$
$$=\sum_{k=0}^{m_1}[Q_1^{0,k}\psi(x)\,Q_{s-1}^{0,k}\phi(x)]+\sum_{k=0}^{m_2}[Q_2^{0,k}\psi\{(x)\}\,Q_{s-2}^{0,k}\{\phi(x)]+\dots$$
$$+\sum_{k=0}^{m_s}Q_s^{0,k}[\psi(x)],$$

and in particular cases any of the terms on the right-hand side may disappear provided all do not disappear.

The equation (1) will not vanish identically unless

$$\left.\begin{aligned}\frac{S_0(x+1)}{R(x+1)}-\frac{S_0(x)}{R(x)}&=\sum_{k=0}^{n_0}p_k(x)\,Q_1^k\{\psi(x)\}\\ &\dots\dots\dots\dots\\ \frac{S_l(x+1)}{R(x+1)}-\frac{S_l(x)}{R(x)}&=\sum_{k=0}^{n_l}p_k(x)\,Q_1^k\{\psi(x)\}\end{aligned}\right\}.$$

Any number or all the terms on the right-hand side of this set of equations may vanish, but in the latter case the original equation can be reduced to the form

$$R(x)[p_{1,0}(x)\,Q_s^0(y)+\dots+p_{1,k}(x)\,Q_s^k(y)$$
$$+p_{2,0}(x)\,Q^0_{s-1}(y)+\dots+p_{2,l}(x)\,Q^l_{s-1}(y)]$$
$$+T_0(x)\,Q^0_{s-2}(y)+\dots+T_m(x)\,Q^m_{s-2}(y)$$
$$+\text{ terms of lower class}=0\dots\dots(2),$$

the p's being simply periodic functions of period unity and all the coefficients being holomorphic.

Either, then, the original equation can be reduced to this form, or at least one of the ratios $\frac{S_0(x)}{R(x)}, \ldots, \frac{S_2(x)}{R(x)}$ is composed of an additive number of solutions of equations of the type

$$f(x+1) - f(x) = p(x)\,\psi'(x).$$

But a solution of this equation is $p(x)\,\phi'(x)$, and the most general solution is $p(x)\,\phi'(x) + \beta(x)$, where $\beta(x)$ is an arbitrary simply periodic function of x of period unity.

Thus one of the ratios $\frac{S_0(x)}{R(x)}, \ldots, \frac{S_2(x)}{R(x)}$ must be a function generated from the G function.

Thus either the original equation implicitly contains the G function or its derivatives among its coefficients, or it is reducible to the form (2).

Continue our former procedure and we see that either at least one of the ratios $\frac{T_0(x)}{R(x)}, \ldots, \frac{T_m(x)}{R(x)}$ is composed of an additive number of solutions of equations of the type

$$f(x+1) - f(x) = p(x)\, Q_2\{\psi(x)\},$$

or the equation is reducible to one in which the ratios of terms of the three highest classes are simply periodic functions of x of period unity.

A solution of the difference equation just written is again a function generated from the G function.

The successive repetitions of the argument are now evident. Ultimately we reduce the equation to one in which either all the coefficients are simply periodic functions (which is absurd) or to ones in which the last term is generated from the G function.

The proposition then is finally established. The G function cannot satisfy a differential equation whose coefficients are not generated from the function itself.

Extensions of the method of proof just employed to more general classes of functions lie beyond the range of the present paper.

ON THE EXPANSIONS IN POWERS OF ARC OF THE COORDINATES OF POINTS ON A CURVE IN EUCLIDEAN SPACE OF MANY DIMENSIONS.

By H. W. Richmond, M.A., King's College, Cambridge.

THE first few terms of the series which express the coordinates of points of a plane or twisted curve in powers of the lengths of the arc are a familiar feature in various text-books on Differential Calculus or Geometry of three dimensions; and the far more interesting investigation of the relations that connect the coefficients of consecutive terms of the expansions, and the means by which the coefficients may be successively calculated, is beautifully treated by Professor G. B. Mathews in a short paper published in Vol. XXVI. of this *Journal* (p. 27). In consideration of the importance of these series in ordinary geometry, and the attention now paid to hypergeometry, an extension of Professor Mathews' results to Euclidean space of many dimensions may be of use, apart from the particular matter in which I needed it. It will not be necessary to stipulate any fixed number of dimensions for the following reasons: firstly, since a space of n dimensions may be determined which contains any $n+1$ points, and in particular $n+1$ consecutive points of the given curve, it follows that the expansions up to the n^{th} power of the arc are the same for a curve which lies wholly in a space of n dimensions as for the more general curve which does not; and secondly, because, from the expansions applicable to a curve which lies in n-dimensional space, those applicable to a curve contained in space of fewer dimensions may be deduced by much the same method which derives the formulæ for a plane curve from those belonging to a curve in space, viz. making the torsion equal to zero. No limit will therefore be imposed on the number of dimensions, and all the spaces of one, two, three, four, ..., n dimensions—(which will be represented by the symbols S_1, S_2, S_3, S_4, ..., S_n; the S_1 being also called a straight line)—are to be imagined immersed in and surrounded by space of an unlimited number of dimensions.

At any point O of a curve there is as a rule:

a determinate S_1 containing two consecutive points of the curve,

"	"	S_2	"	three	"	"	"	"
"	"	S_3	"	four	"	"	"	"
....								
"	"	S_n	"	$n+1$	"	"	"	"

and so on. These are called the tangent S_1 or tangent line, the tangent S_2, the tangent S_3, ..., of the curve at O.

Let Ox_1 be the tangent line (or tangent S_1) at O; draw now

in the tangent		S_2,	Ox_2 perpendicular to the tangent				S_1,
,,	,,	S_3,	Ox_3	,,	,,	,,	S_2,
,,	,,	S_4,	Ox_4	,,	,,	,,	S_3,

....

and so on. Ox_1, Ox_2, Ox_3, ... are a set of mutually perpendicular straight lines through O which may be called the principal axes of the curve at O, and are such that the first r of them lie in and determine the tangent S_r at O. If $(x_1, x_2, x_3, \ldots)$, the coordinates of a point of the curve referred to this coordinate-system, be severally expanded in powers of s the length of the arc, no power of s lower than the r^{th} will be present in x_r; the expansions will in fact take the forms

$$x_1 = s + \text{higher powers},$$

$$x_2 = b_2 s^2 \div 2! + \text{higher powers},$$

$$x_3 = b_3 s^3 \div 3! + \quad ,, \qquad ,,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_n = b_n s^n \div n! + \quad ,, \qquad ,,$$

Let P be a point of the curve situated at a small arcual distance σ from O. The curve has at P, just as at O, a tangent line, a tangent S_2, a tangent S_3, ..., and a complete system of mutually perpendicular principal axes; and in the limit, when P moves up to O, the tangent S_r's at O and P lie in the tangent S_{r+1} of O. Let κ_r denote the limit of the ratio of the angle between the tangent S_r's at O and P to σ the length of the arc OP, and let κ_r be called the measure of the r^{th} curvature at O: (in space of three dimensions κ_1 would be called the curvature and κ_2 the torsion). Now, if no powers of σ above the first are considered, the tangent S_r at O is

$$x_{r+1} = 0, \; x_{r+2} = 0, \; \ldots;$$

while the tangent S_r at P is found to be

$$x_{r+1} = \sigma . b_{r+1} x_r \div b_r, \; x_{r+2} = 0, \; \ldots :$$

the small angle between them is $\sigma b_{r+1} \div b_r$; and therefore by definition

$$b_{r+1} = \kappa_r b_r.$$

We therefore arrive at the results

$$x_1 = s + \text{higher powers},$$
$$x_2 = \kappa_1 s^2 \div 2! + \text{higher powers},$$
$$x_3 = \kappa_1\kappa_2 s^3 \div 3! + \quad ,, \quad ,,$$
$$x_4 = \kappa_1\kappa_2\kappa_3 s^4 \div 4! + \quad ,, \quad ,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The leading coefficients of these series depend only on the values of the curvatures at O; later coefficients will depend also on the values at O of the successive differentials of the curvatures with respect to the arc. In order to calculate them it is necessary to find, as far as the first power of σ, the cosines of the angles between each of the principal axes at P and O, *i.e.* to form a table of the direction cosines of the principal axes of P referred to those of O. Let

$$Ox_1,\ Ox_2,\ Ox_3,\ \ldots;\quad Px_1',\ Px_2',\ Px_3',\ \ldots$$

be the two systems of principal axes, and let, for the moment, the symbol (l, m) stand for the cosine of the angle between the directions Ox_l and Px_m'. Clearly all those cosines in which l and m are unequal are small quantities containing at least the first power of σ as a factor; while those for which l and m are equal differ from unity by a small quantity of order σ^2. Again, since the tangent S_r of P lies in the limit in the tangent S_{r+1} of O, when powers of σ above the first are disregarded, and by the same reasoning the tangent S_r of O lies in the tangent S_{r+1} of P, all the direction cosines in which l and m differ by more than unity will vanish if σ^2 be neglected. Finally, then, it is to be expected that, if no powers of σ above the first are retained, all direction cosines for which l and m are equal will be unity; all those for which l and m differ by one will be multiples of σ, and all others will vanish.

Assume that the series we are seeking are

$$\left.\begin{aligned} x_1 &= s + \Sigma\,({}^1a_r)\,s^r \div r\,!; && (r = 2, 3, 4, \ldots)\\ x_2 &= \kappa_1 s^2 \div 2! + \Sigma\,({}^2a_r)\,s^r \div r\,!; && (r = 3, 4, 5, \ldots)\\ x_3 &= \kappa_1\kappa_2 s^3 \div 3! + \Sigma\,({}^3a_r)\,s^r \div r\,!; && (r = 4, 5, 6, \ldots)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\}\ldots(1),$$

the direction cosines of Px_1' are proportional to the differentials of those series, σ being put in place of s after differentiation, *i.e.* to

$$1 + {}^1a_2\sigma,\ \kappa_1\sigma,\ 0,\ 0,\ \ldots,$$

since σ^2 is to be disregarded. Dividing by the square root of the sum of the squares, we find for the direction cosines of Px_1'

$$1,\ \kappa_1\sigma,\ 0,\ 0,\ \dots .$$

Again, the direction cosines of any line in the tangent S_2 at P are proportional to

$$A\frac{dx_1}{ds}+B\frac{d^2x_1}{ds^2},\ A\frac{dx_2}{ds}+B\frac{d^2x_2}{ds^2},\ A\frac{dx_3}{ds}+B\frac{dx_3}{ds^2},\ \dots,$$

and those of any line in the tangent S_3 at P to

$$A\frac{dx_1}{ds}+B\frac{d^2x_1}{ds^2}+C\frac{d^3x_1}{ds^3},\ \text{\&c., \&c.},$$

σ being substituted for s after the differentiations are performed, and $\sigma^2, \sigma^3, \sigma^4, \dots$ dropped. The values of the direction cosines are now obtained with little difficulty, and prove to be as follows:

	Ox_1,	Ox_2,	Ox_3,	Ox_4,	Ox_5, ...
Px_1'	1,	$\kappa_1\sigma$,	0,	0,	0, ...
Px_2'	$-\kappa_1\sigma$,	1,	$\kappa_2\sigma$,	0,	0, ...
Px_3'	0,	$-\kappa_2\sigma$,	1,	$\kappa_3\sigma$,	0, ...
Px_4'	0,	0,	$-\kappa_3\sigma$,	1,	$\kappa_4\sigma$, ...
.....	......	......	......	......	......

To apply these results to the solution of the problem now under consideration, I observe that the coefficients in equations (1) are functions of the values of the curvatures at O, and their differentials with respect to s the arc. Precisely the same formulæ must express the coordinates of a point referred to the principal axes of the curve at P, provided we replace the coefficients in the series by their values at P. If then we write

$$(\kappa_r)+\sigma\times\frac{d}{ds}(\kappa_r)\ \text{in place of}\ \kappa_r,$$

$$({}^m a_n)+\sigma\times\frac{d}{ds}({}^m a_n)\ \text{in place of}\ ({}^m a_n)$$

in the equations (1), we obtain the values $(X_1, X_2, X_3, \dots)$ of the coordinates of a point of the curve at arcual distince s from P referred to the principal axes at P. The coordinates of this point referred to the principal axes at O are easily found by the help of the table of direction cosines above, and the fact that the coordinates of P referred to the principal axes at O are $(\sigma, 0, 0, 0, \ldots)$: viz.,

$$x_1 = X_1 + \sigma - \kappa_1 \sigma X_2,$$
$$x_2 = \kappa_1 \sigma . X_1 + X_2 - \kappa_2 \sigma X_3,$$
$$x_3 = \kappa_2 \sigma X_2 + X_3 - \kappa_3 \sigma X_4,$$
$$x_4 = \kappa_3 \sigma X_3 + X_4 - \kappa_4 \sigma X_5,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

it being always understood that σ^2 is neglected. But this same point is on the curve at arcual distance $s + \sigma$ from O; its coordinates $(x_1, x_2, x_3, \ldots)$ may therefore be equally well derived from equations (1) by the substitution of $s + \sigma$ for s. Equating the two expressions for these coordinates, we arrive at the following series of relations by means of which the successive coefficients are calculated:

$$({}^1a_{n+1}) = \frac{d}{ds}({}^1a_n) - \kappa_1({}^2a_n),$$

$$({}^2a_{n+1}) = \frac{d}{ds}({}^2a_n) - \kappa_2({}^3a_n) + \kappa_1({}^1a_n),$$

$$({}^3a_{n+1}) = \frac{d}{ds}({}^3a_n) - \kappa_3({}^4a_n) + \kappa_2({}^2a_n),$$

$$({}^4a_{n+1}) = \frac{d}{ds}({}^4a_n) - \kappa_4({}^5a_n) + \kappa_3({}^3a_n),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

The expansions as far as s^5 for any curve in Euclidean space of unlimited dimensions are found to be

$$x_1 = s - \kappa_1^2(s^3/3\,!) - 3\kappa_1\kappa_1'(s^4/4\,!)$$
$$+ (\kappa_1^4 + \kappa_1^2\kappa_2^2 - 3\kappa_1'^2 - 4\kappa_1\kappa_1'')(s^5/5\,!) + \ldots,$$

$$x_2 = \kappa_1(s^2/2\,!) + \kappa_1'(s^3/3\,!) + (\kappa_1'' - \kappa_1\kappa_2^2 - \kappa_1^3)(s^4/4\,!)$$
$$+ (\kappa_1''' - 6\kappa_1^2\kappa_1' - 3\kappa_2^2\kappa_1' - 3\kappa_1\kappa_2\kappa_2')(s^5/5\,!) + \ldots,$$

$$x_3 = \kappa_1\kappa_2(s^3/3\,!) + (2\kappa_2\kappa_1' + \kappa_1\kappa_2')(s^4/4\,!)$$
$$+ (3\kappa_2\kappa_1'' + 3\kappa_1'\kappa_2' + \kappa_1\kappa_2'' - \kappa_1^3\kappa_2 - \kappa_2^3\kappa_1 - \kappa_1\kappa_2\kappa_3^2)(s^5/5\,!) + \ldots,$$

$$x_4 = \kappa_1\kappa_2\kappa_3(s^4/4\,!) + (3\kappa_1'\kappa_2\kappa_3 + 2\kappa_1\kappa_2'\kappa_3 + \kappa_1\kappa_2\kappa_3')(s^5/5\,!) + \ldots,$$

$$x_5 = \kappa_1\kappa_2\kappa_3\kappa_4(s^5/5\,!) + \ldots,$$

dashes signifying differentiations with respect to s, the length of the arc, and the functions which form the coefficients in these series all referring to the values of the curvatures and their differentials at O. If we make κ_3 and κ_4 vanish, we obtain formulæ of ordinary space of three dimensions, which will be seen to be in agreement with those given by Professor Mathews. The only difference between the foregoing method and that of Professor Mathews is of small importance, viz., that the idea of obtaining the principal axes of the curve at P from those at O by means of rotations is here avoided.

Besides establishing these expansions, we have placed ourselves in a position to apply to any Euclidean Hyperspace the artifices used by Dr. Routh in his well-known paper in Vol. VII. of this *Journal* (p. 42). If $(x_1, x_2, \ldots)$ be the coordinates of some point of interest in connexion with the point O of the curve, referred to the principal axes at O, and

$$(x_1 + \delta x_1,\ x_2 + \delta x_2,\ \ldots)$$

be the coordinates of the point similarly related to P, referred to the principal axes at P, the squared distance of the two points is

$$(\delta x_1 + \sigma - \kappa_1 \sigma x_2)^2 + (\delta x_2 + \kappa_1 \sigma x_1 - \kappa_2 \sigma x_3)^2 + (\delta x_3 + \kappa_2 \sigma x_2 - \kappa_3 \sigma x_4)^2$$
$$+ (\delta x_4 + \kappa_3 \sigma x_3 - \kappa_4 \sigma x_5)^2 + \ldots,$$

and, if $(l_1, l_2, l_3, \ldots)$ and $(l + \delta l_1, l_2 + \delta l_2, \ldots)$ be direction cosines of two lines analogously related to the curve, the square of the small angle between them is

$$(\delta l_1 - \kappa_1 \sigma l_2)^2 + (\delta l_2 + \kappa_1 \sigma l_1 - \kappa_2 \sigma l_3)^2 + (\delta l_3 + \kappa_2 \sigma l_2 - \kappa_3 \sigma l_4)^2 + \ldots .$$

King's College, Cambridge,
November 26th, 1899.

ON THE RESIDUES OF THE SUMS OF PRODUCTS OF THE FIRST $p-1$ NUMBERS, AND THEIR POWERS, TO MODULUS p^2 OR p^3.

By J. W. L. Glaisher.

§ 1. In a paper at the beginning of the present volume* certain results were obtained with respect to the divisibility of the sums of the products of the first n numbers taken r together. The object of the present paper is to supplement these results by others which I have since obtained. These new results assign the residues of the sums of products when $n=p-1$ or p, p being any uneven prime, to the next higher power of the modulus, viz., to mod. p^2 or p^3. With the help of these residues I have been able also to assign the residues of the sums of products of the squares and cubes of the numbers 1, 2, ..., $p-1$.

§ 2. Let p be any uneven prime, and let A_r denote the sum of the products of the first $p-1$ numbers taken r together, then

$$(x+1)(x+2)\dots(x+p-1)=x^{p-1}+A_1x^{p-2}+A_2x^{p-3}+\dots+A_{p-2}x+A_{p-1}.\dagger$$

Putting $x+1$ for x, multiplying by $x+1$ and equating coefficients as on p. 2 (§ 3) of this volume,‡ we obtain the equations

$$(p)_2=A_1,$$

$$(p)_3+(p-1)_2A_1=2A_2,$$

$$(p)_4+(p-1)_3A_1+(p-2)_2A_2=3A_3,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$(p)_{p-1}+(p-1)_{p-2}A_1+(p-2)_{p-3}A_2+\dots+(3)_2A_{p-3}=(p-2)A_{p-2},$$

$$1+A_1+A_2+A_3+\dots+A_{p-2}=(p-1)A_{p-1},$$

where $(n)_r$ is used to denote the number of combinations of n things taken r together.

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* "Congruences relating to the sums of products of the first n numbers and to other sums of products," pp. 1—35.

† The A's are the same as in the previous paper, except that in this paper n is always equal to p, an uneven prime.

‡ As the previous paper is in this volume, it will be sufficient to give page and section, as the number of the page will always show whether the reference is to the previous or to the present paper.

§ 3. These equations show (§ 3, pp. 2, 3) that A_1, A_2, A_3, ..., A_{p-2}, and $A_{p-1}+1$, are all divisible by p.

Let

$$\alpha_r = \frac{A_r}{p}, \quad (r = 1, 2, \ldots, p-2),$$

and let

$$\alpha_{p-1} = \frac{A_{p-1}+1}{p}.$$

Thus all the α's are integers. Replacing A's by α's, the preceding system of equations becomes

$$\frac{p-1}{2} = \alpha_1,$$

$$\frac{(p-1)(p-2)}{3!} + \frac{(p-1)(p-2)}{2!}\alpha_1 = 2\alpha_2,$$

$$\frac{(p-1)(p-2)(p-3)}{4!} + \frac{(p-1)(p-2)(p-3)}{3!}\alpha_1 + \frac{(p-2)(p-3)}{2!}\alpha_2 = 3\alpha_3,$$

....,

$$\frac{(p-1)\ldots 2}{(p-1)!} + \frac{(p-1)\ldots 1}{(p-2)!}\alpha_1 + \frac{(p-2)\ldots 2}{(p-3)!}\alpha_2 + \ldots + \frac{3.2}{2!}\alpha_{p-3} = (p-2)\alpha_{p-2},$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{p-2} = -\alpha_{p-1} + A_{p-1},$$

whence

$$-\tfrac{1}{2} \equiv \alpha_1, \text{ mod. } p,$$

$$\tfrac{1}{3} + \alpha_1 \equiv 2\alpha_2, \text{ mod. } p,$$

$$-\tfrac{1}{4} - \alpha_1 + 3\alpha_2 \equiv 3\alpha_3, \text{ mod. } p,$$

$$\tfrac{1}{5} + \alpha_1 - 4\alpha_2 + \frac{3.4}{1.2}\alpha_3 \equiv 4\alpha_4, \text{ mod. } p,$$

....,

$$-\frac{1}{p-1} - \alpha_1 + (p-2)\alpha_2 - (p-2)_2\alpha_3 + \ldots + (p-2)_{p-4}\alpha_{p-3} \equiv (p-2)\alpha_{p-2}, \text{ mod. } p,$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{p-2} \equiv -\alpha_{p-1} - 1, \text{ mod. } p.$$

The general formula, n being $<p$, is

$$\frac{1}{n}+\alpha_1-(n-1)_1\alpha_2+(n-1)_2\alpha_3-\ldots+(-1)^{n-1}(n-1)_{n-3}\alpha_{n-2}$$
$$\equiv(-1)^{n-1}(n-1)\,\alpha_{n-1},\ \text{mod. } p,$$

or, as it may be written, by multiplying by n,

$$1+(n)_1\,\alpha_1-(n)_2\,2\alpha_2+(n)_3\,3\alpha_3-\ldots+(-1)^{n-1}(n)_{n-2}(n-2)\alpha_{n-2}$$
$$+(-1)^n(n)_{n-1}(n-1)\alpha_{n-1}\equiv 0,\ \text{mod. } p.$$

§ 4. Now, let

$$\frac{x}{e^x-1}=1+V_1x+\frac{V_2}{2\,!}x^2+\frac{V_3}{3\,!}x^3+\frac{V_4}{4\,!}x^4+\&c.,$$

so that $\qquad V_1=-\tfrac{1}{2},\quad V_{2n+1}=0,\ (n>0),$

$$V_{2n}=(-1)^{n-1}B_n,\text{*}$$

where B_n is the nth Bernoullian number.

By multiplying up by e^x-1 and equating coefficients, we obtain the recurring formula

$$1+(n)_1\,V_1+(n)_2\,V_2+(n)_3\,V_3+\ldots+(n)_{n-1}\,V_{n-1}=0.$$

§ 5. Comparing this equation with the congruence at the end of the preceding section, we find that, r being $<p-1$,

$$r\alpha_r\equiv(-1)^{r-1}\,V_r,\ \text{mod. } p,$$

that is, $\qquad \dfrac{A_r}{p}\equiv(-1)^{r-1}\dfrac{V_r}{r},\ \text{mod. } p,$

and therefore

$$\frac{A_1}{p}\equiv-\tfrac{1}{2},\ \text{mod. } p,$$

$$\frac{A_{2t+1}}{p}\equiv 0,\ \text{mod. } p,\quad (t>0),$$

$$\frac{A_{2t}}{p}\equiv(-1)^t\frac{B_t}{2t},\ \text{mod. } p.$$

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* This V-notation for the Bernoullian numbers is in many cases very convenient, especially in simplifying general formulæ. See *Quarterly Journal*, Vol. XXIX., p. 116.

The first of these congruences is obvious, since $A_1 = \frac{1}{2}p(p-1)$; the second, viz. that $A_{2t+1} \equiv 0$, mod. p^2, $t > 0$, was obtained in the previous paper (§§ 16, 33, pp. 11, 17). The third result is, I believe, new.

§ 6. Putting $n=p$ in the recurring formula in § 4, we have

$$1 + (p)_1 V_1 + (p)_2 V_2 + (p)_3 V_3 + \ldots + (p)_{p-1} V_{p-1} = 0.$$

Now from Staudt's theorem we know that p occurs in the denominator of V_{p-1}, and not in the denominator of any p with lower suffix. This can also be seen from the recurring formula in § 4.

Dividing throughout by p, we have therefore

$$V_1 - \frac{V_2}{2} + \frac{V_3}{3} - \ldots + \frac{V_{p-2}}{p-2} \equiv - V_{p-1} - \frac{1}{p}, \text{ mod. } p.$$

The left-hand side of this congruence cannot have p as a factor of its denominator: the same must therefore be true of the right-hand side.

This can also be shown independently; for, by Staudt's theorem,

$$B_{\frac{1}{2}(p-1)} = \text{integer} + (-1)^{\frac{1}{2}(p-1)} \left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{p} + \ldots\right\},$$

so that

$$V_{p-1} + \frac{1}{p} = \text{integer} - \frac{1}{2} - \frac{1}{3} - \ldots,$$

the term $\frac{1}{p}$ not occurring on the right-hand side.

§ 7. Comparing the congruence in the preceding section with

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{p-1} \equiv - \alpha_{p-1} - 1, \text{ mod. } p \text{ (§ 3)},$$

we see that

$$\alpha_{p-1} + 1 \equiv V_{p-1} + \frac{1}{p}, \text{ mod. } p;$$

whence

$$\frac{A_{p-1}+1}{p} \equiv -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}, \text{ mod. } p.$$

The fact that $(p-1)! \equiv -1$, mod. p, constitutes Wilson's theorem; and the result just obtained shows that

$$(p-1)! \equiv -1 + \left\{-1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}\right\} p, \text{ mod. } p^2.$$

§8. It was shown in the previous paper (§ 19, p. 11) that, if r is uneven and > 1 and $< p-1$,

$$A_r \equiv \frac{p(p-r)}{2} A_{r-1}, \text{ mod. } p^3,\text{*}$$

and from § 5 of the present paper we have, r being uneven,

$$A_{r-1} \equiv (-1)^{\frac{1}{2}(r-1)} \frac{B_{\frac{1}{2}(r-1)}}{r-1} p, \text{ mod. } p^2;$$

whence

$$A_r \equiv (-1)^{\frac{1}{2}(r-1)} \frac{p^2(p-r)}{2} \frac{B_{\frac{1}{2}(r-1)}}{r-1}, \text{ mod. } p^3,$$

giving

$$A_r \equiv (-1)^{\frac{1}{2}(r+1)} \frac{rB_{\frac{1}{2}(r-1)}}{2(r-1)} p^2, \text{ mod. } p^3;$$

or, putting $r = 2t+1$, $(t > 0)$,

$$A_{2t+1} \equiv (-1)^{t+1} \frac{(2t+1) B_t}{4t} p^2, \text{ mod. } p^3.$$

§9. It has thus been proved that

$$\frac{A_{2t}}{p} \equiv (-1)^t \frac{B_t}{2t}, \text{ mod. } p,$$

$$\frac{A_{2t+1}}{p^2} \equiv (-1)^{t+1} \frac{(2t+1) B_t}{4t}, \text{ mod. } p, \quad (t > 0),$$

$$\frac{A_{p-1}+1}{p} \equiv -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}, \text{ mod. } p.$$

We may conveniently write this last equation

$$A_{p-1} \equiv -1 + Jp, \text{ mod. } p^2.$$

where $$J = -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}.$$

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* It was shown also that when $r > 3$ the modulus is p^4.

§ 10. From these results we can deduce corresponding formulæ giving the residues of $S_r(1, 2, ..., p)$; for, evidently (§ 22, p. 12), n being any positive integer,

$$S_r(1, 2, ..., n) = S_r(1, 2, ..., n-1) + nS_{r-1}(1, 2, ..., n-1),$$

and therefore

$$S_r(1, 2, ..., p) = A_r + pA_{r-1};$$

whence

$$\frac{S_{2t}(1, 2, ..., p)}{p} \equiv \frac{A_{2t}}{p}, \text{ mod. } p,^*$$

and

$$\frac{S_{2t+1}(1, 2, ..., p)}{p^2} \equiv \frac{A_{2t+1}}{p^2} + \frac{A_{2t}}{p}, \text{ mod. } p, \quad (t > 0)$$

$$\equiv (-1)^{t+1} \frac{(2t-1)B_t}{4t}, \text{ mod. } p, \quad (t > 0).$$

These formulæ hold good so long as the suffiix does not exceed $p-2$. For the case in which the suffix is $p-1$, we have

$$S_{p-1}(1, 2, ..., p) = (p-1)! + pA_{p-2};$$

whence, since $A_{p-2} \equiv 0$, mod. p^2,

$$\frac{S_{p-1}(1, 2, ..., p) + 1}{p} \equiv \frac{(p-1)! + 1}{p}, \text{ mod. } p^2$$

$$\equiv -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}, \text{ mod. } p.$$

§ 11. For the sake of reference the formulæ obtained in the preceding sections are collected together in the following list, in which $S_r(1, 2, ..., n)$ denotes the sum of the products of the quantities 1, 2, 3, ..., n taken r together:

(i) $$\frac{A_{2t}}{p} = \frac{S_{2t}(1, 2, ..., p-1)}{p} \equiv (-1)^t \frac{B_t}{2t}, \text{ mod } p,$$

(ii) $$\frac{A_{2t+1}}{p^2} = \frac{S_{2t+1}(1, 2, ..., p-1)}{p^2} \equiv (-1)^{t+1} \frac{(2t+1)B_t}{4t},$$

mod. p, $(t > 0)$,

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* The modulus is p^2 if $t > 1$.

(iii) $\dfrac{A_{p-1}+1}{p}=\dfrac{(p-1)!+1}{p}\equiv J$, mod. p,

(iv) $\dfrac{S_{2t}(1, 2, \ldots, p)}{p}\equiv(-1)^t\dfrac{B_t}{2t}$, mod. p,

(v) $\dfrac{S_{2t+1}(1, 2, \ldots, p)}{p^2}\equiv(-1)^{t+1}\dfrac{(2t-1)B_t}{4t}$, mod. p, $(t>0)$,

(vi) $\dfrac{S_{p-1}(1, 2, \ldots, p)+1}{p}\equiv J$, mod. p,

where $$J=-1+(-1)^{\frac{1}{2}(p+1)}B_{\frac{1}{2}(p-1)}+\frac{1}{p},$$

I have verified formulæ (i), (ii), (iv), (v) for $p=5, 7, 11, 13, 17$ for all values of t, (iii) for values of p up to $p=23$, and (vi) up to $p=19$, by means of the tables of $S_r(1, 2, 3, \ldots, n)$ on pp. 26–28, which extend to $n=22$.

§ 12. It is perhaps worth while to give the numerical values of the residues for the first few values of the suffixes. These residues were used in the verifications.*

Denoting, as throughout the paper, $S_r(1, 2, \ldots, p-1)$ by A_r, and, for the moment $S_r(1, 2, \ldots, p)$ by S_r, we have

$$\begin{aligned}
A_2 &\equiv S_2 \equiv -\tfrac{1}{12}p, && \text{mod. } p^2,\\
A_4 &\equiv S_4 \equiv \tfrac{1}{120}p, && \quad ,,\\
A_6 &\equiv S_6 \equiv -\tfrac{1}{252}p, && \quad ,,\\
A_8 &\equiv S_8 \equiv \tfrac{1}{240}p, && \quad ,,\\
A_{10} &\equiv S_{10} \equiv -\tfrac{1}{132}p && \quad ,,\\
A_{12} &\equiv S_{12} \equiv \tfrac{691}{32760}p, && \quad ,,\\
A_{14} &\equiv S_{14} \equiv -\tfrac{1}{12}p, && \quad ,,\\
&\&c. && \&c.,
\end{aligned}$$

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* The residues of the quantity $J=-1+(-1)^{\frac{1}{2}(p+1)}B_{\frac{1}{2}(p-1)}+\frac{1}{p}$ are: for $p=5$, $J\equiv 0$, mod. 5; for $p=7$, $J\equiv 5$, mod. 7; for $p=11$, $J\equiv 1$, mod. 11; for $p=13$, $J\equiv 0$, mod. 13; for $p=17$, $J\equiv 5$, mod. 17; for $p=19$, $J\equiv 2$, mod. 19; for $p=23$, $J\equiv 8$, mod. 23.

and

$A_3 \equiv \frac{1}{8}p^2,$	$S_3 \equiv \frac{1}{24}p^2,$	mod. p^3,
$A_5 \equiv -\frac{1}{48}p^2,$	$S_5 \equiv -\frac{1}{80}p^2,$	" ,
$A_7 \equiv \frac{1}{72}p^2,$	$S_7 \equiv \frac{5}{504}p^2,$	" ,
$A_9 \equiv -\frac{3}{160}p^2,$	$S_9 \equiv -\frac{7}{480}p^2,$	" ,
$A_{11} \equiv \frac{1}{24}p^2,$	$S_{11} \equiv \frac{3}{88}p^2,$	" ,
$A_{13} \equiv -\frac{691}{5040}p^2,$	$S_{13} \equiv -\frac{7601}{65520}p^2,$	" ,
$A_{15} \equiv \frac{5}{8}p^2,$	$S_{15} \equiv \frac{13}{24}p^2,$	" ,
&c.	&c.	

§ 13. The expressions for $S_r(1, 2, \ldots, n-1)$ and for $S_r(1, 2, \ldots, n)$ in terms of n were given on pp. 24 and 25 for $r = 1, 2, \ldots, 7$; and it is easy to verify that the coefficients of the terms of lowest dimensions in n in these expressions are the same as the coefficients of the residues in the preceding lists. This must necessarily be the case, for, *e.g.* the residue of $\frac{A_4}{p}$, mod. p, could not be $\frac{1}{120}$ for all values of p, unless $\frac{1}{120}$ were the coefficient of n in the expression for $S_4(1, 2, \ldots, n-1)$. Thus the residues of

$$\frac{S_{2t}(1, 2, \ldots, p-1)}{p}, \text{ \&c.}$$

in (i), (ii), (iv), (v) of § 11 are the values which would be obtained by putting $n = 0$ in the expressions for

$$\frac{S_{2t}(1, 2, \ldots, n-1)}{n}, \text{ \&c.}$$

in terms of n.

We therefore find that, if n be any positive integer, and if $S_r(1, 2, \ldots, n-1)$ be expressed in terms of n as in § 46 (p. 24), then the lowest term is $(-1)^t \frac{B_t}{2t} n$, when $n = 2t$, and is $(-1)^{t+1} \frac{(2t+1)B_t}{4t} n^2$, when $n = 2t+1$, $(t > 0)$.

Similarly, if $S_r(1, 2, \ldots, n)$ be expressed in terms of n as in § 48 (p. 26), the lowest term is $(-1)^t \frac{B_t}{2t} n$, when $n = 2t$, and is

$$(-1)^{t+1} \frac{(2t-1) B_t}{4t} n^2, \text{ when } n = 2t+1, \quad (t > 0).*$$

§ 14. The first three formulæ of § 11, viz.

$$A_{2t} \equiv (-1)^t \frac{B_t}{2t} p, \text{ mod. } p^2,$$

$$A_{2t+1} \equiv (-1)^{t+1} \frac{(2t+1) B_t}{4t} p^2, \text{ mod. } p^3, \quad (t > 0),$$

$$A_{p-1} \equiv -1 + Jp, \text{ mod. } p^2,$$

where
$$J = -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p},$$

will be applied in the following sections to obtain some similar results relating to other quantities.

Residues of $1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots + \frac{1}{(p-1)^n}$.

§ 15. In the previous paper proofs were given of Wolstenholme's theorems (i) that the numerator of

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1}$$

is divisible by p^2, and (ii) that the numerator of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2}$$

is divisible by p (§§ 1, 4, 5, pp. 1, 3).

It will now be shown that similar results hold good with respect to the quantity H_n, where

$$H_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots + \frac{1}{(p-1)^n},$$

and that we can assign the residue of H_n with respect to p^2 or p^3.

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* These results might also be established independently by a method corresponding to that by which the residues were obtained.

§ 16. We have (§ 2, p. 321),

$$(x-1)(x-2)\ldots(x-p+1)=x^{p-1}-A_1x^{p-2}+A_2x^{p-3}-\ldots-A_{p-2}x+A_{p-1},$$

and therefore $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p-1}$ are the roots of the equation

$$A_{p-1}x^{p-1}-A_{p-2}x^{p-2}+\ldots+A_2x^2-A_1x+1=0.$$

The quantity

$$H_n=1+\frac{1}{2^n}+\frac{1}{3^n}+\ldots+\frac{1}{(p-1)^n}$$

is the sum of the n^{th} powers of the roots of this equation, and therefore, by Newton's rule,

$$A_{p-1}H_1\ \ -A_{p-2}=0,$$

$$A_{p-1}H_2\ \ -A_{p-2}H_1\ \ +2A_{p-3}=0,$$

$$A_{p-1}H_3\ \ -A_{p-2}H_2\ \ +A_{p-3}H_1-3A_{p-4}=0,$$

....,

$$A_{p-1}H_{p-2}-A_{p-2}H_{p-3}+A_{p-3}H_{p-4}+\ldots+A_2H_1-(p-2)A_1=0,$$

$$A_{p-1}H_{p-1}-A_{p-2}H_{p-2}+A_{p-3}H_{p-3}+\ldots+A_2H_2-A_1H_1+p-1=0.$$

§ 17. Since A_{p-2} is divisible by p^2, the first equation shows that

$$H_1\equiv -A_{p-2}, \text{ mod. } p^3.$$

Thus H_1 is divisible by p^2; and therefore, since A_{p-3} is divisible by p, the second equation shows that

$$H_2\equiv 2A_{p-3}, \text{ mod. } p^2.$$

Thus H_2 is divisible by p; and therefore from the third equation

$$H_3\equiv -3A_{p-4}, \text{ mod. } p^3.$$

Similarly, $\quad H_4\equiv 4A_{p-5}, \quad \text{mod. } p^2,$

and so on, the modulus being alternately p^3 and p^2, up to

$$H_{p-3}\equiv (p-3)A_2, \text{ mod. } p^2.$$

§ 18. The next equation gives

$$A_{p-1}H_{p-2}-(p-2)A_1\equiv 0, \text{ mod. } p^3;$$

whence, since $A_1 = \frac{1}{2}p(p-1)$,

$$H_{p-2} \equiv \frac{(p-2)(p^2-p)}{2(-1+Jp)}, \text{ mod. } p^3$$

$$\equiv \tfrac{1}{2}(p-2)(p-p^2)(1+Jp), \text{ mod. } p^3,$$

and therefore

$$H_{p-2} \equiv -p - (J - \tfrac{3}{2})p^2, \text{ mod. } p^3.$$

The final equation gives

$$A_{p-1}H_{p-1} + p - 1 \equiv 0, \text{ mod. } p^2;$$

whence $$H_{p-1} \equiv \frac{1-p}{-1+Jp}, \text{ mod. } p,$$

and therefore

$$H_{p-1} \equiv -1 - (J-1)p, \text{ mod. } p^2.$$

The value of J (§ 11) is

$$J = -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}.$$

§ 19. Thus, if $2t < p-1$,

$$H_{2t} \equiv 2tA_{p-2t-1}, \text{ mod. } p^2;$$

and therefore, by § 11, putting $h = \frac{1}{2}(p-1)$,

$$H_{2t} \equiv 2t(-1)^{h-t}\frac{B_{h-t}}{p-2t-1}p, \text{ mod. } p^2$$

$$\equiv (-1)^{h-t+1}\frac{2tB_{h-t}}{2t+1}p, \text{ mod. } p^2.$$

Also, if $2t-1 < p-2$ ($t=1$ included),

$$H_{2t-1} \equiv -(2t-1)A_{p-2t},$$

$$\equiv -(2t-1).(-1)^{h-t+1}\frac{(p-2t)B_{h-t}}{2(p-2t-1)}p^2, \text{ mod. } p^3,$$

$$\equiv (-1)^{h-t}\frac{(2t-1)tB_{h-t}}{2t+1}p^2, \text{ mod. } p^3.$$

The residues of H_{p-1} and H_{p-2} in terms of Bernoullian numbers were given in the preceding section.

§ 20. Continuing the equations given by Newton's rule from that in which the highest suffix of H is p to that in which it is $2p-2$, we have

$$A_{p-1}H_p \;-A_{p-2}H_{p-1}+A_{p-3}H_{p-2}-\ldots+A_2H_3-A_1H_2+H_1=0,$$

$$A_{p-1}H_{p+1}-A_{p-2}H_p \;+A_{p-3}H_{p-1}-\ldots+A_2H_4-A_1H_3+H_2=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$A_{p-1}H_{2p-3}-A_{p-2}H_{2p-4}+A_{p-3}H_{2p-5}-\ldots+A_2H_p \;-A_1H_{p-1}+H_{p-2}=0,$$

$$A_{p-1}H_{2p-2}-A_{p-2}H_{2p-3}+A_{p-3}H_{2p-4}-\ldots+A_2H_{p+1}-A_2H_p \;+H_{p-1}=0.$$

Now, with respect to the A's and H's with uneven suffixes, we know that

$$A_3,\, A_5,\, \ldots,\, A_{p-2},\, H_1,\, H_3,\, \ldots,\, H_{p-4}$$

are divisible by p^2, but that A_1 and H_{p-2} are divisible only by p; and, with respect to the A's and H's with even suffixes, we know that

$$A_2,\, A_4,\, \ldots,\, A_{p-3},\, H_2,\, H_4,\, \ldots,\, H_{p-3}$$

are divisible by p, but that

$$A_{p-1} \equiv -1,\ \text{mod.}\ p,\quad H_{p-1} \equiv -1,\ \text{mod.}\ p.$$

The first $p-3$ of the above equations therefore show that

$$H_p,\, H_{p+2},\, \ldots,\, H_{2p-5}$$

are divisible by p^2, and that

$$H_{p+1},\, H_{p+3},\, \ldots,\, H_{2p-4}$$

are divisible by p.

The last equation but one shows that

$$A_{p-1}H_{2p-3} - A_1H_{p-1} + H_{p-2} \equiv 0,\ \text{mod.}\ p^2,$$

whence, since

$$A_{p-1} \equiv -1,\ \text{mod.}\ p,\ A_1 \equiv -\tfrac{1}{2}p,\ \text{mod.}\ p^2,$$

and $\quad H_{p-1} \equiv -1,\ \text{mod.}\ p,\ H_{p-2} \equiv -p,\ \text{mod.}\ p^2$ (§ 18),

we find $\quad H_{2p-3} \equiv -\tfrac{3}{2}p,\ \text{mod.}\ p^2.$

The last equation shows that

$$H_{2p-2} \equiv H_{p-1} \equiv -1,\ \text{mod.}\ p.$$

§ 21. In general, when $n>p-1$, the H's are connected by the recurring relation

$$A_{p-1}H_n-A_{p-2}H_{n-1}+A_{p-3}H_{n-2}-\ldots+A_2H_{n-p+3}-A_1H_{n-p+2}+H_{n-p+1}=0,$$

and, by proceeding exactly as in the last section, we see that generally, r being any positive integer,

$$H_{r(p-1)+1},\ H_{r(p-1)+3},\ \ldots,\ H_{r(p-1)+p-4}$$

are divisible by p^2, that

$$H_{r(p-1)+2},\ H_{r(p-1)+4},\ \ldots,\ H_{r(p-1)+p-3}$$

are divisible by p, and that

$$H_{r(p-1)+p-2}\equiv-\frac{r+2}{2}p,\ \text{mod. } p^2,$$

and $$H_{r(p-1)}\equiv-1,\ \text{mod. } p.$$

§ 22. I now proceed to determine the residues of H_n to the next higher power of the modulus, *i.e.*, to determine the residue of H_n with respect to p^3 when n is uneven, and with respect to p^2 when n is even.

Let $n=rq+s$, where $q=p-1$ and $s<q$. Then, if $r>0$, the recurring equation for H_n is

$$A_{p-1}H_{rq+s}-A_{p-2}H_{rq+s-1}+\ldots+A_2H_{(r-1)q+s+2}-A_1H_{(r-1)q+s+1}+H_{(r-1)q+s}=0.$$

Now, among the A's,

$$A_{p-2},\ A_{p-4},\ \ldots,\ A_3 \text{ are divisible by } p^2,$$
$$A_{p-3},\ A_{p-5},\ \ldots,\ A_2\ \ ,,\quad ,,\quad ,,\ p,$$

the anomalous A's being

$$A_1\equiv-\tfrac{1}{2}p,\ \text{mod. } p^2,\quad A_{p-1}\equiv-1,\ \text{mod. } p,$$

and, among the H's, we know that H_n is divisible by p^2, if n is uneven, and by p if n is even, the exceptional cases being

$$H_{rq+p-2}\equiv-\frac{r+2}{2}p,\ \text{mod. } p^2,$$

$$H_{rq}\equiv-1,\ \text{mod. } p.$$

§ 23. Since $q=p-1$ is even, rq is even, and when s is even and >0 and $<p-1$, the only terms in the recurring relation for H_{rq+s} which are not necessarily divisible by p^2 are

$$A_{p-1}H_{rq+s}+A_{p-s-1}H_{rq}+H_{(r-1)q+s}.$$

This expression is therefore $\equiv 0$, mod. p^2, so that

$$H_{rq+s} \equiv H_{(r-1)q+s} - A_{p-s-1}, \text{ mod. } p^2,$$

whence, putting $r = 1, 2, \ldots, r$, and adding,

$$H_{rq+s} \equiv H_s - rA_{p-s-1}, \text{ mod. } p^2.$$

Now, (§ 17), s being even and $< p - 1$,

$$H_s \equiv sA_{p-s-1}, \text{ mod. } p^2,$$

and therefore $\quad H_{rq+s} \equiv (s-r)\,A_{p-s-1}, \text{ mod. } p^2.$

This formula holds good for the even values $2, 4, \ldots, p-3$ of s.

§ 24. When s is uneven and $< p - 2$, there are five terms in the recurring formula which are not necessarily divisible by p^3, and we have

$$A_{p-1}H_{rq+s} - A_{p-s-1}H_{rq} + A_{p-s-2}H_{(r-1)q+p-2} - A_1H_{(r-1)q+s+1} + H_{(r-1)q+s} \equiv 0, \text{ mod. } p^3.$$

Now $\quad A_{p-1} \equiv -1, \text{ mod. } p,\quad H_{rq} \equiv -1, \text{ mod. } p,$

$$H_{(r-1)q+p-2} \equiv -\tfrac{1}{2}(r+1)p, \text{ mod. } p^2,$$

$$A_1 \equiv -\tfrac{1}{2}p, \text{ mod. } p^2,$$

and therefore

$$H_{rq+s} - H_{(r-1)q+s} \equiv A_{p-s-1} - \tfrac{1}{2}(r+1)\,pA_{p-s-2} + \tfrac{1}{2}pH_{(r-1)q+s+1}, \text{ mod. } p^3.$$

Since s is uneven, $s+1$ is even, and therefore, from the preceding section,

$$H_{(r-1)q+s+1} \equiv (s-r+2)\,A_{p-s-2}, \text{ mod. } p^3.$$

Thus we have

$$H_{rq+s} - H_{(r-1)q+s} \equiv A_{p-s-1} - \tfrac{1}{2}(2r-s-1)\,pA_{p-s-2}, \text{ mod. } p^3.$$

Now, from the formula quoted in § 8, viz., that if r be uneven and > 1 and $< p - 1$,

$$A_r \equiv \frac{p\,(p-r)}{2}A_{r-1}, \text{ mod. } p^3,$$

we find $\quad A_{p-s-1} \equiv \tfrac{1}{2}(s+1)pA_{p-s-2}, \text{ mod. } p^3,$

when s has any of the values $1, 3, \ldots, p-4$.

Therefore

$$H_{rq+s} - H_{(r-1)q+s} \equiv (s-r+1)\,pA_{p-s-2}, \text{ mod. } p^3,$$

whence, putting $r = 1, 2, \ldots, r$ and adding, we find

$$H_{rq+s} - H_s \equiv \left\{sr - \frac{r(r-1)}{2}\right\} pA_{p-s-2}, \text{ mod. } p^3.$$

Now, by § 17, if s is uneven and $< p-2$,

$$H_s \equiv -sA_{p-s-1}, \text{ mod. } p^3,$$

and, as just shown,

$$A_{p-s-1} \equiv \tfrac{1}{2}(s+1) pA_{p-s-2}, \text{ mod. } p^3,$$

so that
$$H_s \equiv -\frac{s(s+1)}{2} pA_{p-s-2}, \text{ mod. } p^3.$$

Therefore

$$H_{rq+s} \equiv \left\{-\frac{s(s+1)}{2} + sr - \frac{r(r-1)}{2}\right\} pA_{p-s-2}, \text{ mod. } p^3;$$

that is

$$H_{rq+s} \equiv -\tfrac{1}{2}(s-r)(s-r+1) pA_{p-s-2}, \text{ mod. } p^3.$$

This formula holds good for the uneven values $1, 3, \ldots, p-4$ of s.

§ 25. It remains to consider the cases of $s = 0$ and $s = p-2$. In the first case $n = rq$, and we have

$$A_{p-1}H_{rq} + H_{(r-1)q} \equiv 0, \text{ mod. } p^2,$$

Now, (§ 11), $A_{p-1} \equiv -1 + Jp$, mod. p^2,

where
$$J = -1 + (-1)^{\frac{1}{2}(p+1)} B_{\frac{1}{2}(p-1)} + \frac{1}{p}.$$

Therefore $H_{rq} \equiv (1 + Jp) H_{(r-1)q}$, mod. p^2,

whence, putting $r = 2, 3, \ldots, r$,

$$H_{rq} \equiv (1 + rJp) H_q, \text{ mod. } p^2.$$

Now, (§ 18), $H_q \equiv -1 - (J-1)p$, mod. p^2,

and therefore $H_{rq} \equiv -1 - (rJ-1)p$, mod. p^2.

§ 26. For $n = rq + p - 2$, the relation is

$$A_{p-1}H_{rq+p-2} - A_1 H_{rq} + H_{(r-1)q+p-2} \equiv 0, \text{ mod. } p^3,$$

that is

$$A_{p-1}H_{rq+p-2} \equiv \tfrac{1}{2}(p^2 - p) H_{rq} - H_{(r-1)q+p-2}, \text{ mod. } p^3;$$

whence

$$H_{rq+p-2} - H_{(r-1)q+p-2} \equiv \tfrac{1}{2}(p - p^2) H_{rq} + Jp\, H_{rq+p-2}, \text{ mod. } p^3$$

$$\equiv \tfrac{1}{2}(p^2 - p)\{1 + (rJ-1)p\} - Jp \times \tfrac{1}{2}(r+2)p, \text{ mod. } p^3, \quad (\S\, 21).$$

Reducing the right-hand side, we have

$$H_{rq+p-2} - H_{(r-1)q+p-2} \equiv -\tfrac{1}{2}p - \{(r+1)J - 1\}p^2, \text{ mod. } p^3;$$

whence, putting $r = 1, 2, \ldots, r$, and adding,

$$H_{rq+p-2} - H_{p-2} \equiv -\tfrac{1}{2}rp - \left\{\frac{r(r+3)}{2}J - r\right\}p^2, \text{ mod. } p^3.$$

Substituting for H_{p-2} its value $-p - (J - \tfrac{3}{2})p^2$, (§ 18), we have, finally,

$$H_{rq+p-2} \equiv -\frac{r+2}{2}p - \left\{\frac{(r+1)(r+2)}{2}J - \frac{2r+3}{2}\right\}p^2, \text{ mod. } p^3.$$

§ 27. It has thus been shown that

(i) if $s = 2, 4, 6, \ldots, p-3$,

$$H_{rq+s} \equiv (s-r) A_{p-s-1}, \text{ mod. } p^2, \quad (\S\, 23),$$

(ii) if $s = 1, 3, 5, \ldots, p-4$,

$$H_{rq+s} \equiv -\tfrac{1}{2}(s-r)(s-r+1)p\, A_{p-s-2}, \text{ mod. } p^3, \quad (\S\, 24),$$

(iii) $H_{rq} \equiv -1 - (rJ-1)p$, mod. p^2, (§ 25),

(iv) $$H_{rq+p-2} \equiv -\frac{r+2}{2}p - \left\{\frac{(r+1)(r+2)}{2}J - \frac{2r+3}{2}\right\}p^2, \text{ mod. } p^3, \quad (\S\, 26),$$

where $q = p - 1$. In all the formulæ r may have the value zero.

The residues of the A's were expressed by means of Bernoullian numbers in § 11, viz.

$$A_{p-s-1}\ (s \text{ even}) \equiv (-1)^{\frac{1}{2}(p-s-1)} \frac{B_{\frac{1}{2}(p-s-1)}}{p-s-1}p, \text{ mod. } p^2$$

$$\equiv (-1)^{\frac{1}{2}(p-s+1)} \frac{B_{\frac{1}{2}(p-s-1)}}{s+1}p, \text{ mod. } p^2,$$

$$A_{p-s-2}\ (s \text{ uneven}) \equiv (-1)^{\frac{1}{2}(p-s-2)} \frac{B_{\frac{1}{2}(p-s-2)}}{p-s-2}p, \text{ mod. } p^2$$

$$\equiv (-1)^{\frac{1}{2}(p-s)} \frac{B_{\frac{1}{2}(p-s-2)}}{s+2}p, \text{ mod. } p^2.$$

The value of J was given in § 25.

§ 28. It may be remarked that, if in (ii) the right-hand side be expressed in terms of A_{p-s-1}, the formula becomes

$$\text{(ii)}\quad H_{rq+s} \equiv -\frac{(s-r)(s-r+1)}{s+1}\, p\, A_{p-s-1},\ \text{mod. } p^3,$$

and if we express the right-hand side in (iii) and (iv) in terms of A_{p-1}, these formulæ become

$$\text{(iii)}\quad H_{rq} \equiv p-1-r(A_{p-1}+1),\ \text{mod. } p^2,$$

$$\text{(iv)}\quad H_{rq+p-2} \equiv -\tfrac{1}{2}\{(r+2)p+(r+1)(r+2)(A_{p-1}+1)p - (2r+3)p^2\},\ \text{mod. } p^3.$$

§ 29. If we denote by H_n' the numerator of H_n, *i.e.* if H_n' denote the numerator of

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots+\frac{1}{(p-1)^n},$$

then

$$H_n' \equiv (-1+Jp)^n H_n,$$

the modulus being p^2 or p^3 according as n is even or uneven.

Thus we find

(i) if $s = 2, 4, 6, \dots, p-3$,

$$H'_{rq+s} \equiv (s-r)A_{p-s-1},\ \text{mod. } p^2,$$

(ii) if $s = 1, 3, 5, \dots, p-4$,

$$H'_{rq+s} \equiv \tfrac{1}{2}(s-r)(s-r+1)\, p\, A_{p-s-2},\ \text{mod. } p^{\cdot},$$

$$\text{(iii)}\quad H'_{rq} \equiv -1-(2rJ-1)p,\ \text{mod. } p^2,$$

$$\text{(iv)}\quad H'_{rq+p-2} \equiv \tfrac{1}{2}[(r+2)p+(2r+3)\{(r+2)J-1\}p^2],\ \text{mod. } p^3.$$

§ 30. I have verified these formulæ for $p=5$ as far as H'_{20}, and for $p=7$ as far as H'_{16}. The details of the formulæ and verifications in these cases are as follows:

For $p=5$, (i) and (ii) give

$$H'_{4r+2} \equiv (2-r)A_2,\ \text{mod. } 5^2;\quad H'_{4r+1} \equiv \tfrac{1}{2}(r-1)(r-2)5A_2,\ \text{mod. } 5^3,$$

whence, since $A_2 = 35$,

$$H'_{4r+2} \equiv (4-2r)\,5 \equiv (3r+4)\,5,\ \text{mod. } 5^2,$$

$$H'_{4r+1} \equiv (r-1)(r-2)\,5^2,\ \text{mod. } 5^3.$$

These formulæ were verified for $r = 0, 1, 2, 3, 4$.

When $p=5$, $J=0$, so that this value of p does not afford a good verification of (iii) and (iv). The formula (iii) gives

$$H'_{4r} \equiv 4, \text{ mod. } 5,$$

which was verified for $r = 1, 2, 3, 4, 5$; and (iv) gives

$$H'_{4r+3} \equiv \tfrac{1}{2}\{(r+2)\,5 - (2r+3)\,5^2\}, \text{ mod. } 5^3,$$

whence, since $\qquad \tfrac{1}{2} \equiv 3 + 2.5, \text{ mod. } 5^2,$

$$H'_{4r+3} \equiv (3r+1)\,5 + (r+1)\,5^2, \text{ mod. } 5^3.$$

This formula was verified for $r = 0, 1, 2, 3, 4$.

§ 31. For $p = 7$, (i) and (ii) give

$$H'_{6r+2} \equiv (2-r)\,A_4,\quad H'_{6r+4} \equiv (4-r)\,A_2, \text{ mod. } 7^2,$$

$$H'_{6r+1} \equiv \tfrac{1}{2}(r-1)(r-2)\,7A_4,\quad H'_{6r+3} \equiv \tfrac{1}{2}(r-3)(r-4)\,7A_2, \text{ mod. } 7^3,$$

whence, since $\qquad A_4 = 1624 \equiv 1.7, \text{ mod. } 7^2,$

and $\qquad A_2 = 175 \equiv 4.7, \text{ mod. } 7^2,$

we find $\qquad H'_{6r+2} \equiv (2-r)\;7 \equiv (6r+2)\,7, \text{ mod. } 7^2,$

$$H'_{6r+4} \equiv (2-4r)\,7 \equiv (3r+2)\,7, \text{ mod. } 7^2,$$

$$H'_{6r+1} \equiv 4\,(r-1)(r-2)\,7^2 \equiv (4r^2+2r+1)\,7^2, \text{ mod. } 7^3,$$

$$H'_{6r+3} \equiv 2\,(r-3)(r-4)\,7^2 \equiv (2r^2+3)\,7^2, \text{ mod. } 7^3.$$

These formulæ were verified for $r = 0, 1, 2$.*

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* It may be convenient to give the values of the H's used in these verifications.

For $p = 5$,

$H_1' = 50$, $H_2' = 820$, $H_3' = 16\,280$, $H_4' = 357\,904$, $H_5' = 8\,252\,000$, $H_6' = 194\,397\,760$,
$H_7' = 4\,624\,680\,320$, $H_8' = 110\,523\,752\,704$, $H_9' = 2\,647\,111\,616\,000$,
$H_{10}' = 63\,466\,432\,537\,600$, $H_{11}' = 1\,522\,433\,104\,271\,360$, $H_{12}' = 36\,529\,334\,432\,763\,904$,
$H_{13}' = 876\,595\,894\,487\,244\,800$, $H_{14}' = 21\,037\,008\,518\,043\,811\,840$,
$H_{15}' = 504\,872\,725\,632\,177\,766\,400$, $H_{16}' = 12\,116\,759\,959\,500\,088\,213\,504$,
$H_{17}' = 290\,800\,015\,862\,515\,770\,982\,400$, $H_{18}' = 6\,979\,173\,721\,033\,620\,729\,364\,480$,
$H_{19}' = 167\,499\,849\,534\,749\,071\,510\,077\,440$, $H_{20}' = 4\,019\,992\,552\,757\,173\,784\,232\,853\,504$.

For $p = 7$,

$H_1' = 1\,764$, $H_2' = 773\,136$, $H_3' = 444\,273\,984$, $H_4' = 290\,539\,581\,696$,
$H_5' = 200\,610\,400\,564\,224$, $H_6' = 141\,727\,869\,124\,448\,256$,
$H_7' = 101\,143\,400\,834\,944\,548\,864$, $H_8' = 72\,514\,862\,031\,522\,895\,036\,416$,
$H_9' = 52\,103\,129\,720\,841\,632\,885\,243\,904$, $H_{10}' = 37\,476\,298\,202\,061\,058\,687\,475\,122\,176$,
$H_{11}' = 26\,969\,446\,338\,598\,136\,973\,236\,417\,593\,344$,
$H_{12}' = 19\,413\,186\,108\,695\,429\,557\,310\,844\,133\,441\,536$,
$H_{13}' = 13\,975\,769\,975\,383\,353\,963\,783\,001\,555\,215\,581\,184$,
$H_{14}' = 10\,061\,935\,961\,230\,787\,737\,319\,694\,943\,951\,043\,690\,496$,
$H_{15}' = 7\,244\,371\,787\,185\,418\,312\,915\,680\,974\,328\,950\,291\,431\,424$,
$H_{16}' = 5\,215\,867\,854\,041\,746\,999\,797\,068\,420\,111\,368\,841\,966\,125\,056$.

When $p=7$, $J=5$, so that (iii) gives

$$H'_{6r} \equiv 6 + 4r.7, \text{ mod. } 7^2,$$

which was verified for $r=1$ and 2. From (iv) we have

$$H'_{6r+5} \equiv \tfrac{1}{2}\left[(r+2)\,7 + \{5\,(r+2)(2r+3) - (2r+3)\}\,7^2\right], \text{ mod. } 7^3,$$

whence, since $\qquad \tfrac{1}{2} \equiv 4 + 3.7, \text{ mod. } 7^2,$

we find $\quad H'_{6r+5} \equiv (4r+1)\,7 + (5r^2+2r+3)\,7^2, \text{ mod. } 7^3.$

This formula was verified for $r=1$ and 2.

Residues of $S_r\{1^2, 2^2, \ldots, (p-1)^2\}$.

§ 32. It was shown in § 58 (p. 34) of the previous paper that $S_r\{1^2, 2^2, \ldots, (p-1)^2\}$, the sum of the products of the numbers $1^2, 2^2, \ldots, (p-1)^2$ taken r together, is divisible by p for all values of r from 1 to $p-2$ inclusive, except $r=\frac{1}{2}(p-1)$, in which case $S_r\{1^2, 2^2, \ldots, (p-1)^2\} \equiv (-1)^{\frac{1}{2}(p-1)}\,2$, mod. p. These results will now be extended to modulus p^2.

§ 33. The numbers $1, 2, \ldots, p-1$ are the roots of the equation

(i) $\quad A_0x^{p-1} - A_1x^{p-2} + A_2x^{p-3} - \ldots - A_{p-2}x + A_{p-1} = 0.$*

The equation whose roots are the squares of the roots of this equation is

$$\{A_0x^{\frac{1}{2}(p-1)} + A_2{}^{\frac{1}{2}(p-3)} + \ldots + A_{p-1}\}^2$$
$$- \{A_1x^{\frac{1}{2}(p-2)} + A_3x^{\frac{1}{2}(p-4)} + \ldots + A_{p-2}x^{\frac{1}{2}}\}^2 = 0.$$

Writing this equation in the form

(ii) $\quad x^{p-1} - \beta_1x^{p-2} + \beta_2x^{p-3} - \ldots - \beta_{p-2}x + \beta_{p-1} = 0,$

so that $\qquad \beta_r = S_r\{1^2, 2^2, \ldots, (p-1)^2\},$

we find that $\beta_1, \beta_2, \ldots$ are given in terms of the A's by the following equations, in which h is used to denote $\frac{1}{2}(p-1)$.

$$-\beta_1 = 2A_0A_2 - A_1A_1,$$
$$\beta_2 = 2A_0A_4 - 2A_1A_3 + A_2A_2,$$
$$-\beta_3 = 2A_0A_6 - 2A_1A_5 + 2A_2A_4 - A_3A_3,$$

....

* It is convenient to introduce the quantity A_0, the value of which is unity.

$$(-1)^{h-1}\beta_{h-1} = 2A_0A_{p-3} - 2A_1A_{p-4} + \ldots + (-1)^{h-1}A_{h-1}A_{h-1},$$

$$(-1)^{h}\beta_{h} = 2A_0A_{p-1} - 2A_1A_{p-2} + \ldots + (-1)^{h}A_{h}A_{h},$$

$$(-1)^{h+1}\beta_{h+1} = 2A_2A_{p-1} - 2A_3A_{p-2} + \ldots + (-1)^{h+1}A_{h+1}A_{h+1},$$

....

$$-\beta_{p-2} = 2A_{p-3}A_{p-1} - A_{p-2}A_{p-2},$$

$$\beta_{p-1} = A_{p-1}A_{p-1}.$$

Now A_0 and A_{p-1} are not divisible by p, A_1 and A_{2t} are divisible by p only, and A_{2t+1} is divisible by p^2.

These equations therefore show that

$$(-1)^r\beta_r \equiv 2A_0A_{2r}, \text{ mod. } p^2, \quad (r = 1, 2, \ldots, h-1),$$

$$(-1)^r\beta_r \equiv 2A_{p-1}A_{2r-p+1}, \text{ mod. } p^2, \quad (r = h+1, h+2, \ldots, p-2),$$

$$(-1)^h\beta_h \equiv 2A_0A_{p-1}, \text{ mod. } p^2, \quad \beta_{p-1} = (A_{p-1})^2;$$

so that, from $r=1$ to $r=\frac{1}{2}(p-3)$ inclusive,

$$\beta_r \equiv (-1)^r 2A_{2r} \equiv \frac{B_r}{r}p, \text{ mod. } p^2;$$

from $r=\frac{1}{2}(p+1)$ to $r=p-2$,

$$\beta_r \equiv (-1)^{r-1} 2A_{2r-p+1} \equiv (-1)^{h-1}\frac{B_{r-h}}{r-h}p, \text{ mod. } p^2;$$

and
$$\beta_h \equiv (-1)^{h-1} 2(1-Jp), \text{ mod. } p^2,$$

$$\beta_{p-1} \equiv 1 - 2Jp, \text{ mod. } p^2.$$

§ 34. The preceding determination of the residues may be conveniently arranged in the following condensed form, which corresponds, as will be seen, to the method of investigation adopted in the case of the cubes and higher powers of the numbers $1, 2, \ldots, p-1$.

Let $a_r = (-1)^r A_r$ and $b_r = (-1)^r\beta_r$, so that the original and the transformed equations become

$$\text{(i)} \quad a_0x^{p-1} + a_1x^{p-2} + a_2x^{p-3} + \ldots + a_{p-2}x + a_{p-1} = 0,$$

and
$$\text{(ii)} \quad x^{p-1} + b_1x^{p-2} + b_2x^{p-3} + \ldots + b_{p-2}x + b_{p-1} = 0.$$

The object of this change is to make all the terms positive in both equations.

If the equation (i) is $f(x)=0$, equation (ii) is

$$f(x^{\frac{1}{2}})f(-x^{\frac{1}{2}})=0,$$

and it may be written

$$\text{(iii)}\quad (a_0+a_2+\ldots+a_{p-1})^2-(a_1+a_3+\ldots+a_{p-2})^2=0,$$

the coefficients alone being expressed, as the powers are not required,* since the value of b_r is derivable from (iii) by merely selecting those terms in which the sum of the suffixes is $2r$.

In order to obtain the residue of b_r, we need only retain the term or terms in b_r which are divisible only by the lowest power of p, the modulus being the power of p by which all the other terms in b_r are divisible. We know that a_0, a_{p-1} are not divisible by p, that a_1, a_{2t} are divisible by p only, and that a_{2t+1} is divisible by p^2. It is evident therefore that every term in the second term, $(a_1+a_3+\ldots+a_{p-1})^3$, in (iii) is divisible by p^3, so that we need only take account of the first term.

We thus find

$$b_r\equiv 2a_0a_{2r},\ \text{mod. } p^2,\ (r=1,\ 2,\ \ldots,\ h),$$

$$b_r\equiv 2a_{p-1}a_{2r-p+1},\ \text{mod. } p^2,\ (r=h+1,\ h+2,\ \ldots,\ p-2),$$

or, as we may conveniently write these formulæ,

$$b_s \equiv 2a_0a_{2s},\quad \text{mod. } p^2,\ (s=1,\ 2,\ \ldots,\ h-1),$$

$$b_{h+s}\equiv 2a_{p-1}a_{2s},\ \text{mod. } p^2,\ (\quad ,, \qquad ,, \quad),$$

with $\qquad b_h\equiv 2a_{p-1},\ \text{mod. } p^2;\ \ b_{p-1}=(a_{p-1})^2.$

Passing to β's and A's,† these formulæ give

$$\beta_s\equiv(-1)^{h-1}\beta_{h+s}\equiv(-1)^s\,2A_{2s},\ \text{mod. } p^2,$$

$$\beta_h\equiv(-1)^h\,2A_{p-1},\ \beta_{p-1}\equiv(A_{p-1})^2,\ \text{mod. } p^2.$$

---

* The power of x to which the coefficient a_r is attached is $x^{\frac{1}{2}(p-1-r)}$.

† Except for the fact that the A's are the actual sums of products, and that their values have been tabulated on pp. 26–28, it would be scarcely worth while to replace the a's by A's. The residues of the a's are given (§ 14) by the formulæ

$$a_r\ (r\text{ even})\equiv(-1)^{\frac{1}{2}r}\,\frac{B_{\frac{1}{2}r}}{r}\,p,\ \ \text{mod. } p^2,$$

$$a_r\ (r\text{ uneven})\equiv(-1)^{\frac{1}{2}(r-1)}\,\frac{rB_{\frac{1}{2}(r-1)}}{2(r-1)}\,p^2,\ \ \text{mod } p^3,$$

$$a_{p-1}\equiv -1+Jp,\ \ \text{mod. } p^2.$$

§ 35. I have verified these formulæ for $p=7$. In this case $h=3$, $A=175\equiv 4.7$, mod. 7^2, $A_4=1624\equiv 1.7$, mod. 7^2, $J=5$, and the formulæ give

$$\beta_1\equiv\beta_4\equiv -\,2A_2\equiv 6.7,\ \text{mod. } 7^2,$$

$$\beta_2\equiv\beta_5\equiv\ \ 2A_4\equiv 2.7,\ \text{mod. } 7^2,$$

$$\beta_3\equiv 2\,(1-5.7)\equiv 2+4.7,\ \text{mod. } 7^2,$$

$$\beta_6\equiv\ 1-2.5.7\ \equiv 1+4.7,\ \text{mod. } 7^2.$$

The calculated values of the β's are

$$\beta_1=91,\ \beta_2=3003,\ \beta_3=44473,\ \ \beta_4=296296,$$
$$\beta_5=773136,\ \ \beta_6=518400,$$

which give the same residues as those obtained from the formulæ.

Residues of $S_r\{1^3, 2^3, \dots, (p-1)^3\}$.

§ 36. Let

$$\gamma_r=S_r\{1^3, 2^3, \dots, (p-1)^3\},$$

and let $c_r=(-1)^r\gamma_r$, so that the equation whose roots are $1^3, 2^3, \dots, (p-1)^3$ is

$$\text{(ii)}\quad x^{p-1}+c_1x^{p-2}+c_2x^{p-3}+\dots+c_{p-2}x+c_{p-1}=0.$$

Now the equation whose roots are the cubes of the roots of (i), viz. $f(x)=0$, is $f(x^{\frac{1}{3}})\,f(\omega_1x^{\frac{1}{3}})\,f(\omega_2x^{\frac{1}{3}})=0$, where 1, ω_1, ω_2 are the cube roots of unity. Writing only the coefficients as in § 34, this equation is

$$\text{(iii)}\qquad \alpha_0^{\,3}+\alpha_1^{\,3}+\alpha_2^{\,3}-3\alpha_0\alpha_1\alpha_2=0,$$

where

$$\alpha_0=a_0+a_3+a_6+\dots,$$
$$\alpha_1=a_1+a_4+a_7+\dots,$$
$$\alpha_2=a_2+a_5+a_8+\dots\,.$$

To obtain c_r we have to select those terms from (iii) in which the sum of the suffixes is $3r$, and (as before) to obtain the residue we require only those terms in c_{3r} which are divisible only by the lowest power of p, the modulus being the power of p by which all the other terms in c_{3r} are divisible.

Three cases arise, depending upon the residue when p is divided by 3, viz.

I. $p-1=3h$; II. $p-1=3h+1$; III. $p-1=3h+2$.

§ 37. *Case I.* $p=3h+1$, h being therefore even. We have

$$\alpha_0 = a_0 + a_3 + a_6 + \ldots + a_{p-1},$$

$$\alpha_1 = a_1 + a_4 + a_7 + \ldots + a_{p-3},$$

$$\alpha_2 = a_2 + a_5 + a_8 + \ldots + a_{p-2}.$$

Now a_0 and a_{p-1} are the only two non-divisible p's; a_1, a_{2t} are divisible by p only, and a_{2t+1} by p^2. Therefore every term in α_1^3 and α_2^3 is divisible by p^3, and we need only consider the equation

$$\alpha_0^3 - 3\alpha_0\alpha_1\alpha_2 = 0.$$

Omitting numerical coefficients the terms in α_0^3 which are not divisible by p^2 are

$$a_0^2 a_6,\quad a_0^2 a_{12},\quad \ldots,\quad a_0^2 a_{p-1},$$

$$a_0 a_{p-1} a_6,\quad a_0 a_{p-1} a_{12},\quad \ldots,\quad a_0 a_{p-1} a_{p-7},$$

$$a_{p-1}^2 a_0,\quad a_{p-1}^2 a_6,\quad a_{p-1}^2 a_{12},\quad \ldots,\quad a_{p-1}^3.$$

Every term in α_1 and α_2 is divisible by p, and therefore every term in the product $\alpha_0\alpha_1\alpha_2$ must be divisible by p^2 at least.

The sums of the suffixes in the terms not divisible by p^2 are, in the first line, 6, 12, ..., $p-1$; in the second, $p+5$, $p+11$, ..., $2p-8$; in the third, $2p-2$, $2p+4$, $2p+10$, ..., $3p-3$. These are the values of $3r$, and the corresponding values of r are therefore 2, 4, ..., h; $h+2$, $h+4$, ..., $2h-2$; $2h$, $2h+2$, ..., $3h$.

The numerical coefficient of a term of the form $a_l^2 a_m$ is 3, and of a term of the form $a_l a_m a_n$ is 6.

We thus find

$$\begin{aligned} c_r &\equiv 3a_0^2 a_{3r}, \text{ mod. } p^2, && (r=2, 4, \ldots, h) \\ &\equiv 6a_0 a_{p-1} a_{3r-3h}, \text{ mod. } p^2, && (r=h+2, h+4, \ldots, 2h-2) \\ &\equiv 3a_{p-1}^2 a_{3r-6h}, \text{ mod. } p^2, && (r=2h, 2h+2, \ldots, 3h-2); \end{aligned}$$

or, as we may write these formulæ,

$$c_s \equiv 3a_0^2 a_{3s}, \text{ mod. } p^2, \quad (s = 2, 4, \ldots, h-2),$$

$$c_{h+s} \equiv 6a_0 a_{p-1} a_{3s}, \quad ,, \quad , \quad (\quad ,, \quad\quad ,, \quad),$$

$$c_{2h+s} \equiv 3a^2_{p-1} a_{3s}, \quad ,, \quad , \quad (\quad ,, \quad\quad ,, \quad)$$

with

$$c_p \equiv 3a_0^2 a_{p-1}, \quad c_{2h} \equiv 3a_0 a^2_{p-1}, \quad c_{3h} \equiv a^3_{p-1}, \text{ mod. } p^2.$$

§ 38. Selecting now the terms divisible by p^2 and not by p^3, and putting the corresponding values of r in the same line, we have, from α_0^3,

$$a_0^2 a_3, \; a_0^2 a_9, \; \ldots, \; a_0^2 a_{p-4}, \quad (r = 1, 3, \ldots, h-1),$$

$$a_0 a_{p-1} a_3, \; a_0 a_{p-1} a_9, \; \ldots, \; a_0 a_{p-1} a_{p-4}, \quad (r = h+1, h+3, \ldots, 2h-1),$$

$$a^2_{p-1} a_3, \; a^2_{p-1} a_9, \; \ldots, \; a^2_{p-1} a_{p-4}, \quad (r = 2h+1, 2h+3, \ldots, 3h-1);$$

and, from $\alpha_0 \alpha_1 \alpha_3$,

$$a_0 a_1 a_2, \; a_0 a_1 a_8, \; \ldots, \; a_0 a_1 a_{p-5}, \quad (r = 1, 3, \ldots, h-1),$$

$$a_{p-1} a_1 a_2, \; a_{p-1} a_1 a_8, \; \ldots, \; a_{p-1} a_1 a_{p-5}, \quad (r = h+1, h+3, \ldots, 2h-1).$$

The coefficients of all the terms in $\alpha_0 \alpha_1 \alpha_3$ are unity; and we find

$$c_s \equiv 3a_0^2 a_{3s} - 3a_0 a_1 a_{3s-1}, \quad \text{mod. } p^3, \quad (s = 1, 3, \ldots, h-1),$$

$$c_{h+s} \equiv 6a_0 a_{p-1} a_{3s} - 3a_{p-1} a_1 a_{3s-1}, \quad ,, \quad , \quad (\quad ,, \quad\quad ,, \quad),$$

$$c_{2h+s} \equiv 3a^2_{p-1} a_{3s}, \quad ,, \quad , \quad (\quad ,, \quad\quad ,, \quad).$$

§ 39. *Case II.* $p = 3h + 2$, h being therefore uneven. We have

$$\alpha_0 = a_0 + a_3 + a_6 + \ldots + a_{p-2},$$

$$\alpha_1 = a_1 + a_4 + a_7 + \ldots + a_{p-1},$$

$$\alpha_2 = a_2 + a_5 + a_8 + \ldots + a_{p-3}.$$

In Case I. the two non-divisible coefficients a_0 and a_{p-1} occurred in the same expression α_0, but in this case a_{p-1} occurs in a different expression α_1. Every term in α_2 is divisible by p, and therefore the term α_2^3 in the equation (iii) can contribute no term that is to be retained. Selecting the terms which are not divisible by p^2, we obtain, from α_0^3,

$$a_0^2a_6,\ a_0^2a_{12},\ \ldots,\ a_0^2a_{p-5},\quad (r=2,\ 4,\ \ldots,\ h-1),$$

from α_1^3, the term $a^2_{p-1}a_1$ $(r=2h+1)$ and

$$a^2_{p-1}a_4,\ a^2_{p-1}a_{10},\ \ldots,\ a^2_{p-1}a_{p-7},\quad (r=2h+2,\ 2h+4,\ \ldots,\ 3h-1),$$

and from $\alpha_0\alpha_1\alpha_2$

$$a_0a_{p-1}a_2,\ a_0a_{p-1}a_8,\ \ldots,\ a_0a_{p-1}a_{p-3},\quad (r=h+1,\ h+3,\ \ldots,\ 2h).$$

We thus find

$$c_s \equiv 3a_0^2a_{3s},\qquad \text{mod. } p^2,\ (s=2,\ 4,\ \ldots,\ h-1),$$

$$c_{h+s} \equiv -3a_0a_{p-1}a_{3s-1},\qquad ,,\ ,\ (s=1,\ 3,\ \ldots,\ h),$$

$$c_{2h+s} \equiv 3a^2_{p-1}a_{3s-2},\qquad ,,\ ,\ (s=2,\ 4,\ \ldots,\ h-1),$$

with $$c_{2h+1} \equiv 3a^2_{p-1}a_1,\ \text{mod. } p^2.$$

§ 40. Selecting now the terms divisible by p^2, but not by p^3, we have, from α_0^3,

$$a_0^2a_3,\ a_0^2a_9,\ \ldots,\ a_0^2a_{p-2},\quad (r=1,\ 3,\ \ldots,\ h),$$

from α_1^3,

$$a_{p-1}a_1a_4,\ a_{p-1}a_1a_{10},\ \ldots,\ a_{p-1}a_1a_{p-7}\ (r=h+2,\ h+4,\ \ldots,\ 2h-1),$$

$$a^2_{p-1}a_7,\ a^2_{p-1}a_{13},\ \ldots,\ a^2_{p-1}a_{p-4},\ (r=2h+3,\ 2h+6,\ \ldots,\ 3h),$$

and, from $\alpha_0\alpha_1\alpha_2$,

$$a_0a_1a_2,\ a_0a_1a_8,\ \ldots,\ a_0a_1a_{p-3},\ (r=1,\ 3,\ \ldots,\ h),$$

$$a_0a_{p-1}a_5,\ a_0a_{p-1}a_{11},\ \ldots,\ a_0a_{p-1}a_{p-6}\ (r=h+2,\ h+4,\ \ldots,\ 2h-1);$$

whence we find

$$c_s \equiv 3a_0^2a_{3s} - 3a_0a_1a_{3s-1},\ \text{mod. } p^3,\ (s=1,\ 3,\ \ldots,\ h),$$

$$c_{h+s} \equiv 6a_{p-1}a_1a_{3s-2} - 3a_0a_{p-1}a_{3s-1}\quad ,,\ ,\ (s=2,\ 4,\ \ldots,\ h-1),$$

$$c_{2h+s} \equiv 3a^2_{p-1}a_{3s-2}\quad ,,\ ,\ (s=3,\ 5,\ \ldots,\ h).$$

§ 41. *Case III.* $p=3h+3$, h being therefore even. We have

$$\alpha_0 = a_0 + a_3 + a_6 + \ldots + a_{p-3},$$

$$\alpha_1 = a_1 + a_4 + a_7 + \ldots + a_{p-2},$$

$$\alpha_2 = a_2 + a_5 + a_8 + \ldots + a_{p-1}.$$

In this Case the non-divisible coefficient a_{p-1} occurs in α_2. Every term in α_1 is divisible by p, and therefore α_1^3 contributes no term to be retained.

The terms not divisible by p^2 are, from α_0^3,

$$a_0^2a_6, a_0^2a_{12}, \ldots, a_0^2a_{p-3}, (r=2, 4, \ldots, h),$$

from α_2^3,

$$a^2_{p-1}a_2, a^2_{p-1}a_8, \ldots, a^2_{p-1}a_{p-7}, (r=2h+2, 2h+4, \ldots, 3h);$$

and, from $\alpha_0\alpha_1\alpha_2$, the term $a_0a_{p-1}a_1$ $(r=h+1)$ and

$$a_0a_{p-1}a_4, a_0a_{p-1}a_{10}, \ldots, a_0a_{p-1}a_{p-5}, (r=h+2, h+4, \ldots, 2h),$$

giving

$$\begin{aligned} c_s &\equiv 3a_0^2a_{3s}, \quad \text{mod. } p^2, (s=2, 4, \ldots, h), \\ c_{h+s} &\equiv -3a_0a_{p-1}a_{3s-2}, \quad ,, \quad , (,, \quad ,,), \\ c_{2h+s} &\equiv 3a^2_{p-1}a_{3s-4}, \quad ,, \quad , (,, \quad ,,), \end{aligned}$$

and

$$c_{h+1} \equiv -3a_0a_{p-1}a_1, \text{ mod. } p^2.$$

§ 42. The terms divisible by p^2, but not by p^3, are, from α_0^3,

$$a_0^2a_3, a_0^2a_9, \ldots, a_0^2a_{p-6}, (r=1, 3, \ldots, h-1),$$

from α_2^3,

$$a^2_{p-1}a_5, a^2_{p-1}a_{11}, \ldots, a^2_{p-1}a_{p-4}, (r=2h+3, 2h+5, \ldots, 3h+1),$$

and, from $\alpha_0\alpha_1\alpha_2$,

$$a_0a_1a_2, a_0a_1a_8, \ldots, a_0a_1a_{p-7}, (r=1, 3, \ldots, h-1),$$

$$a_0a_{p-1}a_7, a_0a_{p-1}a_{13}, \ldots, a_0a_{p-1}a_{p-2}, (r=h+3, h+5, \ldots, 2h+1),$$

$$a_1a_{p-1}a_6, a_1a_{p-1}a_{12}, \ldots, a_1a_{p-1}a_{p-8}, (r=h+3, h+5, \ldots, 2h+1),$$

giving

$$\begin{aligned} c_s &\equiv 3a_0^2a_{3s} - 3a_0a_1a_{3s-1}, \quad \text{mod. } p^3, (s=1, 3, \ldots, h-1), \\ c_{h+s} &\equiv -3a_0a_{p-1}a_{3s-2} - 3a_1a_{p-1}a_{3s-3}, \quad ,, \quad , (s=3, 5, \ldots, h+1), \\ c_{2h+s} &\equiv 3a^2_{p-1}a_{3s-4}, \quad ,, \quad , (s=3, 5, \ldots, h+1). \end{aligned}$$

§ 43. Collecting the results obtained in the last seven sections, and replacing the c's by γ's and the a's by A's and putting $a_0=1$, $A_1\equiv-\frac{1}{2}p$, mod. p^2, $A_{p-1}\equiv-1$, mod. p, we obtain the following list of formulæ in which

$$\gamma_r = S_r\{1^3, 2^3, 3^3, \ldots, (p-1)^3\}.$$

Case I. $p-1=3h$, so that $h=\frac{1}{3}(p-1)$ and is even.

$$\gamma_s \equiv 3A_{3s}, \text{ mod. } p^2, \ (s=2, 4, \ldots, h-1),$$
$$\gamma_{h+s} \equiv -6A_{3s}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad),$$
$$\gamma_{2h+s} \equiv 3A_{3s}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad),$$
$$\gamma_h \equiv 3A_{p-1} \text{ mod. } p^2; \ \gamma_{2h} \equiv 3A^2_{p-1}, \text{ mod. } p^2,$$
$$\gamma_s \equiv 3A_{3s} + \tfrac{3}{2}pA_{3s-1}, \text{ mod. } p^3, \ (s=1, 3, \ldots, h-1),$$
$$\gamma_{h+s} \equiv -6A_{3s} - \tfrac{3}{2}pA_{3s-1}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad),$$
$$\gamma_{2h+s} \equiv 3A_{3s}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad).$$

Case II. $p-1=3h+1$, so that $h=\frac{1}{3}(p-2)$ and is uneven.

$$\gamma_s \equiv 3A_{3s}, \quad \text{mod. } p^2, \ (s=2, 4, \ldots, h-1),$$
$$\gamma_{h+s} \equiv 3A_{3s-1} \quad ,, \quad, (s=1, 3, \ldots, h),$$
$$\gamma_{2h+s} \equiv 3A_{3s-2}, \quad ,, \quad, (s=2, 4, \ldots, h),$$
$$\gamma_{2h+1} \equiv -\tfrac{3}{2}p, \text{ mod. } p^2,$$
$$\gamma_s \equiv 3A_{3s} + \tfrac{3}{2}pA_{3s-1}, \text{ mod. } p^3, \ (s=1, 3, \ldots, h),$$
$$\gamma_{h+s} \equiv 3A_{3s-1} + 6pA_{3s-2}, \quad ,, \quad, (s=2, 4, \ldots, h-1),$$
$$\gamma_{2h+s} \equiv 3A_{3s-2} \quad ,, \quad, (s=3, 5, \ldots, h).$$

Case III. $p-1=3h+2$, so that $h=\frac{1}{3}(p-3)$ and is even.

$$\gamma_s \equiv 3A_{3s}, \quad \text{mod. } p^2, \ (s=2, 4, \ldots, h),$$
$$\gamma_{h+s} \equiv 3A_{3s-2}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad),$$
$$\gamma_{2h+s} \equiv 3A_{3s-4} \quad ,, \quad, (\quad ,, \quad\quad ,, \quad),$$
$$\gamma_{h+1} \equiv \tfrac{3}{2}p, \text{ mod. } p^2,$$
$$\gamma_s \equiv 3A_{3s} + \tfrac{3}{2}pA_{3s-1}, \text{ mod. } p^3, \ (s=1, 3, \ldots, h-1),$$
$$\gamma_{h+s} \equiv 3A_{3s-2} - \tfrac{3}{2}pA_{3s-3} \quad ,, \quad, (s=3, 5, \ldots, h+1),$$
$$\gamma_{2h+s} \equiv 3A_{3s-4}, \quad ,, \quad, (\quad ,, \quad\quad ,, \quad).$$

In all three cases

$$\gamma_{3h} = \gamma_{p-1} = A^3_{p-1}.$$

§44. Whenever in the preceding list the right-hand member of the congruence consists of two terms, we may combine the two into a single term by means of the formula (§ 8)

$$A_r \ (r \text{ uneven}) \equiv \tfrac{1}{2}p\,(p-r)\,A_r, \text{ mod. } p^3.$$

Thus, for example, we may write the second group in Case I.,

$$\gamma_s \equiv -\tfrac{3}{2}\,(3s-1)\,pA_{3s-1}, \text{ mod. } p^3, \ (s=1, 3, \ldots, h-1),$$

$$\gamma_{h+s} \equiv \ \tfrac{3}{2}\,(6s-1)\,pA_{3s-1}, \quad ,, \quad , (\quad ,, \quad\quad ,, \quad),$$

$$\gamma_{2h+s} \equiv -\tfrac{9}{2}pA_{3s-1}, \quad ,, \quad , (\quad ,, \quad\quad ,, \quad).$$

§ 45. I have verified the formulæ for $p=7$, but the verification afforded by so small a value of p is not a very good one. This value of p falls under Case I., and the formulæ are

$$\gamma_2 \equiv 3A_6, \text{ mod. } 7^2, \ \ \gamma_4 \equiv 3A_6^{\,2}, \text{ mod. } 7^2,$$

$$\gamma_1 \equiv \ \ 3A_3 + \tfrac{3}{2}.7.A_2, \text{ mod. } 7^3,$$

$$\gamma_3 \equiv -6A_3 - \tfrac{3}{2}.7.A_2, \text{ mod. } 7^3,$$

$$\gamma_5 \equiv \ \ 3A_3, \text{ mod. } 7^3.$$

For $p=7$,

$$A_6 = 720 \equiv 6+4.7, \text{ mod. } 7^2; \ A_3 = 735 \equiv 1.7^2, \text{ mod. } 7^3;$$

$$A_2 = 175 \equiv 4.7, \text{ mod. } 7^2; \ \text{ and } \tfrac{1}{2} \equiv 4, \text{ mod. } 7.$$

Thus the first two congruences give

$$\gamma_2 \equiv 4, \text{ mod. } 7^2, \quad \gamma_4 \equiv 3+5.7, \text{ mod. } 7^2,$$

and the others give

$$\gamma_1 \equiv (3+12.4)\,7^2 \equiv 2.7^2, \text{ mod. } 7^3,$$

$$\gamma_3 \equiv (-6+2.4)\,7^2 \equiv 2.7^2, \text{ mod. } 7^3,$$

$$\gamma_5 \equiv 3.7^2, \text{ mod. } 7^3.$$

The calculated values of the γ's are

$$\gamma_1 = 441; \ \gamma_2 = 63655, \ \gamma_3 = 3587535,$$

$$\gamma_4 = 74550304, \ \gamma_5 = 444273984,$$

which give the same residues as the formulæ.

Residues of $S_r\,\{1^n, 2^n, \ldots, (p-1)^n\}$.

§ 46. In a similar manner we may determine the residues of

$$S_r = S_r\,\{1^4, 2^4, \ldots, (p-1)^4\}.$$

If we put $d_r = (-1)^r \delta_r$, and

$$\alpha_0 = a_0 + a_4 + a_8 + \ldots,$$
$$\alpha_1 = a_1 + a_5 + a_9 + \ldots,$$
$$\alpha_2 = a_2 + a_6 + a_{10} + \ldots,$$
$$\alpha_3 = a_3 + a_7 + a_{11} + \ldots,$$

then the equation

$$\text{(ii)} \quad x^{p-1} + d_1 x^{p-2} + d_2 x^{p-3} + \ldots + d_{p-2} x + d_{p-1} = 0$$

is the same as

$$\text{(iii)} \quad \alpha_0^4 - \alpha_1^4 + \alpha_2^4 - \alpha_3^4 - 2\alpha_0^2\alpha_2^2 + 2\alpha_1^2\alpha_3^2$$
$$- 4\alpha_0^2\alpha_1\alpha_3 + 4\alpha_1^2\alpha_0\alpha_2 - 4\alpha_2^2\alpha_1\alpha_3 + 4\alpha_3^2\alpha_0\alpha_2 = 0,$$

in which, as in §§ 34 and 36, the coefficients alone are written.

There are four cases according as

$$p - 1 = 4h,\ 4h+1,\ 4h+2,\ 4h+3,$$

which have to be treated independently in the same way as the three cases of $n = 3$.

§ 47. For $n = 5$ there are five cases; and in general, for the determination of the residues of

$$\lambda_r = S_r \{1^n, 2^n, \ldots, (p-1)^n\},$$

we have to consider n cases. The expressions for $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are given by

$$\alpha_0 = a_0 + a_n + a_{2n} + \ldots,$$
$$\alpha_1 = a_1 + a_{n+1} + a_{2n+1} + \ldots,$$
$$\alpha_2 = a_2 + a_{n+2} + a_{2n+2} + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\alpha_{n-1} = a_{n-1} + a_{2n-1} + a_{3n-1} + \ldots,$$

and the equation (iii) is

$$\prod_{k=1}^{k=n} (a_0 + \omega_k \alpha_1 + \omega_k^2 \alpha_2 + \ldots + \omega_k^{n-1} \alpha_{n-1}) = 0,$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are the n^{th} roots of unity.

The product on the left-hand side of (iii) may also be expressed, as is well known, by the cyclical determinant formed from the elements $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$.*

* The values of this determinant for $n = 5$ and $n = 6$ were given in the *Quarterly Journal*, Vol. XV., p. 354 and Vol. XVI., p. 33.

§ 48. A complete determination of the residues of λ_r would necessarily be intricate, if only on account of the number of cases, and I therefore confine myself to the determination of the values of r for which λ_r is not divisible by p.

There are two cases according as a_{p-1} occurs, or does not occur, in α_0 (*i.e.* in the same α as a_0). In the first case $p-1$ is a multiple of n, $=nh$ say, and, in the second, $p-1=nh+t$, t not being zero.

Let $$l_r = (-1)^r \lambda_r.$$

In the first case, in which $p-1=nh$, the only non-divisible terms are derived from $\alpha_0{}^n$, and are therefore given by $(a_0+a_{p-1})^n$. We thus find

$$l_h \equiv (n)_1\, a_0^{n-1} a_{p-1}, \qquad \text{mod. } p^2,$$
$$l_{2h} \equiv (n)_2\, a_0^{n-2} (a_{p-1})^2, \qquad ,, \quad ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$l_{sh} \equiv (n)_s\, a_0^{n-s} (a_{p-1})^s, \qquad ,, \quad ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$l_{(n-1)h} \equiv (n)_{n-1}\, a_0 (a_{p-1})^{n-1}, \qquad ,, \quad .$$

It is evident that the modulus is p^2, *i.e.* that all the other terms must be divisible by p^2; for the sum of the suffixes must be a multiple of $p-1$, and if the suffixes are not all 0's and $(p-1)$'s there must be at least two suffixes, *i.e.* the term must be divisible by p^2.

§ 49. In the second case, in which $p-1=nh+t$, a_{p-1} occurs in α_t, and, in the equation (iii), viz.

$$\Pi\,(\alpha_0 + \omega_k \alpha_1 + \omega_k^2 \alpha_2 + \ldots + \omega_k^{n-1} \alpha_{n-1}) = 0,$$

the only α-terms which can produce terms not divisible by p are given by

$$\Pi\,(\alpha_0 + \omega_k^t \alpha_t).$$

The non-divisible terms are therefore

$$\text{(iv)} \quad (a_0 + \omega_1^t a_{p-1})\,(a_1 + \omega_2^t a_{p-1}) \ldots (a_3 + \omega_n^t a_{p-1}),$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are the n^{th} roots of unity.

Now it can be shown that the equation whose roots are $\omega_1^t, \omega_2^t, \ldots, \omega_n^t$ is

$$(x^q - 1)^\delta = 0,$$

where δ is the greatest common measure of n and t and $q = \dfrac{n}{\delta}$.

Therefore $(x-\omega_1^{\iota})(x-\omega_2^{\iota})\dots(x-\omega_n^{\iota})=0$

is the same equation as $(x^q-1)^{\delta}=0$, and, by putting $x=-\dfrac{a_0}{a_{p-1}}$, we see that the expression (iv) is equal to

$$\{a_0^{\,q}-(-a_{p-1})^q\}^{\delta}.$$

Thus the values of r which correspond to the non-divisible terms are

$$\frac{q(p-1)}{n},\ \frac{2q(p-1)}{n},\ \dots,\ \frac{(\delta-1)q(p-1)}{n},$$

that is $\quad \dfrac{p-1}{\delta},\ 2\dfrac{p-1}{\delta},\ \dots,\ (\delta-1)\dfrac{p-1}{\delta};$

and, putting $\quad \dfrac{p-1}{\delta}=g,$

we find $\quad l_{sg}\equiv(-1)^{s(g+1)}(\delta)_s\, a_0^{\,n-sg}(a_{p-1})^{sg}$, mod. p^2,

s having the values $1, 2, \dots, \delta-1$.

§ 50. Passing from l's and a's to λ's and A's, we have thus obtained the formulæ:

I. $p-1=nh$,

$$\lambda_{sh}\equiv(-1)^{sh}(n)_s(A_{p-1})^s,\ \text{mod. } p^2,$$

where s has the values $1, 2, \dots, n-1$.

II. $p-1=nh+t$,

$$\lambda_{sg}\equiv(-1)^{s(g+1+q)}(\delta)_s(A_{p-1})^{sq},\ \text{mod. } p^2,$$

where δ is the greatest common measure of n and t,

$$g=\frac{p-1}{\delta},\quad q=\frac{n}{\delta},$$

and s has the values $1, 2, \dots, \delta-1$.

Case I. is included in case II., for, when $t=0$, $\delta=n$, $g=h$, and $q=1$.

§ 51. Since

$$A_{p-1}\equiv-1+Jp,\ \text{mod. } p^2,$$

we have finally

$$\lambda_{sg} \equiv (-1)^{s(\sigma+1)} (\delta)_s (1 - sqJp), \text{ mod. } p^2,$$

where δ, s, and q are as in the preceding section and s has the values $1, 2, \ldots, \delta - 1$.

This formula holds good also when $t = 0$, in which case, as just mentioned, $g = h$, $\delta = n$, $q = 1$.

§ 52. It has therefore been shown that λ_r is always divisible by p, except for the values $r = g, 2g, \ldots, (\delta - 1) g$, where $p - 1 = nh + t$, δ is the greatest common measure of n and t, and $q = \dfrac{p-1}{\delta}$. If, then, n is a prime there are no non-divisible values of r, unless $p - 1$ is a multiple of n; and, when n is not a prime, there are no non-divisible values of r if the remainder t is prime to n. When t is not prime to n the number of non-divisible values is equal to the greatest common measure of n and t diminished by 1.*

§ 53. These formulæ may be partially verified in the cases of $n = 4$, 5, 6 by means of the expanded cyclical determinant. This value for $n = 4$ was given in § 46 (p. 348); the values for $n = 5$ and 6 were referred to in the note on p. 349.

There is of course always the term α_0^n which gives rise to Case I. (§ 48), viz., to the non-divisible values in the case in which $p - 1$ is a multiple of n.

The other terms in the expanded determinant which give rise to non-divisible terms are those which involve a_0 and only one other α. These terms are

for $n = 4$, $\quad -2\alpha_0^2\alpha_2^2$,

for $n = 5$, $\quad$ none,

for $n = 6$, $\quad -3\alpha_0^4\alpha_3^2 + 3\alpha_0^2\alpha_3^4 + 2\alpha_0^3\alpha_2^3 + 2\alpha_0^3\alpha_4^3$.

The single term in the case $n = 4$ shows that if $p - 1 = 4h + 1$ or $4h + 3$, there is no value of r for which l_r is not divisible by p, and that if $p - 1 = 4h + 2$, there is one such value of r, viz., $r = \dfrac{2(p-1)}{4} = \dfrac{p-1}{2}$, and we have

$$l_{\frac{1}{2}(p-1)} = -2(a_{p-1})^2, \text{ mod. } p^2,$$

* This diminution by 1 is caused by the omission of the value $r = p - 1$ which has been excluded throughout the whole investigation; since $\lambda_{p-1} = (A_{p-1})^n$ identically.

which agrees with the formula in § 49, for $\delta=2$, $g=\frac{1}{2}(p-1)$, $q=2$.

For $n=5$ there is no term involving α_0 with only one other α, which is in accordance with § 52.

For $n=6$ the terms show that for $p-1=6p+1$ or $6h+5$ there is no non-divisible value of r, and that

when $\quad p-1=6h+2,\ l_{\frac{1}{2}(p-1)} \equiv \ 2(a_{p-1})^3$, mod. p^2,

" $\quad p-1=6h+4,\ l_{\frac{1}{2}(p-1)} \equiv \ 2(a_{p-1})^3$, " "

" $\quad p-1=6h+3,\ l_{\frac{1}{3}(p-1)} \equiv -3(a_{p-1})^2$, " "

and $\quad l_{\frac{2}{3}(p-1)} \equiv \ 3(a_{p-1})^4$, " ".

In the first two cases $\delta=2$, $g=\frac{1}{2}(p-1)$, $q=3$; and in the third $\delta=3$, $g=\frac{1}{3}(p-1)$, $q=2$; so that the results agree with those given by the formula in § 49.

§ 54. Since

$$S_r(1^n, 2^n, \ldots, p^n) = S_r\{1^n, 2^n, \ldots, (p-1)^n\} + p^n S_{r-1}\{1^n, 2^n, \ldots, (p-1)^n\},$$

it is evident that when $n>1$ the residues of $S_r(1^n, 2^n, \ldots, p^n)$ are the same as those of $S_r\{1^n, 2^n, \ldots, (p-1)^n\}$ which have been obtained, the modulus being the same, viz., p^2 or p^3 as the case may be.*

It may be remarked that H_n', the numerator of

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\ldots+\frac{1}{(p-1)^n},$$

is $S_{p-2}\{1^n, 2^n, \ldots, (p-1)^n\}$. The residue of this quantity has therefore been assigned in § 29 for all positive values of n.

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* For $n=2$ the modulus was p^2 for all the residues (§ 33), so that there is no exception in this case.

THE QUARTIC SURFACES WITH FOURTEEN, FIFTEEN, AND SIXTEEN NODES.

By C. M. Jessop.

THE sixteen nodal quartic surface is discussed by Kummer in the well-known papers in the *Monatsberichte der Akademie zu Berlin*, 1864, and in the *Abhandlungen der Akademie*, 1866. I have investigated the results stated without proof in the latter paper by a preliminary discussion of the fifteen nodal surface and an application of a result therein obtained to the sixteen nodal surface. The forms arrived at for the latter surface are seen to be similar to those of Kummer, but in a somewhat more symmetrical shape.

The quartic surface $\sqrt{(xx')} + \sqrt{(yy')} + \sqrt{(zz')} = 0$ has as singular tangent planes

$$x = 0,\ x' = 0,\ y = 0,\ y' = 0,\ z = 0,\ z' = 0,$$

and has as nodes

$$(xyz),\ (xy'z),\ (xyz'),\ (xy'z'),\ (x'yz),\ (x'y'z),\ (x'yz'),\ (x'y'z'),$$

together with

$$x = 0,\ x' = 0,\ yy' = zz';\ \ y = 0,\ y' = 0,\ zz' = xx';$$
$$z = 0,\ z' = 0,\ xx' = yy'.$$

We shall investigate under what conditions other singular tangent planes and nodes can occur.

The surface may be written in the form

$$(xx' + yy' - zz' - 2kxy)^2 = 4xy\,\{x'y' + k^2xy - k\,(xx' + yy' - zz')\}$$
$$= 4xy\,\{(kx - y')\,(ky - x') + kzz'\}.$$

If the expression $(kx - y')\,(ky - x') + kzz'$ is the product of linear factors, we shall obtain another pair of singular tangent planes.

Now the condition that this should be the case is that quantities α, β can be found such that

$$z' \equiv k\alpha\beta z + \alpha\,(kx - y') + \beta\,(ky - x'),$$

and then the given expression becomes

$$(k\beta z + kx - y')\,(k\alpha z + ky - x').$$

Hence we have as the required condition

$$k\alpha x - \beta x' + k\beta y - \alpha y' + k\alpha\beta z - z' \equiv 0,$$

or generally

$$\left.\begin{array}{l} Ax + A'x' + By + B'y' + Cz + C'z' \equiv 0 \\ \text{with the conditions} \quad AA' = BB' = CC' \end{array}\right\} \ldots\text{(I.)}.$$

The two singular tangent planes are then

$$Cz + Ax + B'y' = 0,$$

$$Cz + By + A'x' = 0;$$

and from the symmetrical form of the equation I. we see that there will also be the singular tangent planes

$$C'z' + Ax + By = 0,$$

$$Ax + By + Cz = 0.$$

It is easily seen on trial that no other singular tangent planes will arise from this method of treating the equation of the surface.

From the form of the identity I. we see that the four planes thus arrived at meet in a point whose coordinates are given by the equations

$$Ax = A'x', \quad By = B'y', \quad Cz = C'z',$$

and which is seen to be on the surface, and since the line of intersection of any two of the singular tangent planes meets the surface in two nodes, it follows that this last point is also a node.

Hence the given surface is fifteen nodal and has ten singular tangent planes if the condition I. is satisfied.

Each of the four new singular tangent planes passes through *one* of the set of nodes $x = 0$, $x' = 0$, $yy' = zz'$, &c., and each singular tangent plane contains six nodes.

We now consider under what conditions the surface is sixteen nodal.

The surface may be written in the form

$$(xx' + yy' - zz' - 2kxx')^2 = 4xx' \{k^2xx' - k(xx' + yy' - zz') + yy'\}$$
$$= 4xx' \{k(k-1)xx' + (1-k)yy' + kzz'\}.$$

It remains therefore to investigate under what conditions the expression $\alpha xx' + \beta yy' + \gamma zz'$ can be the product of two linear factors.

It is clear that for this to occur four of the eight nodes (xyz), &c., must lie in one plane and four in another, let the plane which contains $(x'yz)$, $(xy'z)$, (xyz'), $(x'y'z')$ be called t, then it must be possible to determine constants $p, \ldots, a, \ldots$, so that

$$\left.\begin{aligned} px' &\equiv by - cz + k_1 t \\ qy' &\equiv cz - ax + k_2 t \\ rz' &\equiv ax - by + k_3 t \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{(II.)},$$

with the condition $k_1 + k_2 + k_3 = 1$.

Now if $\alpha xx' + \beta yy' + \gamma zz' \equiv u \, . \, t$, it follows that, since (xyz) does not lie on t, we must have

$$\alpha : \beta : \gamma = pa : qb : rc,$$

$$u \equiv \frac{k_1\alpha}{p} x + \frac{k_2\beta}{q} y + \frac{k_3\gamma}{r} z \equiv k_1 ax + k_2 by + k_3 cz,$$

$$t \equiv px' + qy' + rz'.$$

Taking for α, β, γ the quantities $k(k-1)$, $1-k$, k, we easily see that it is necessary that

$$\frac{1}{pa} + \frac{1}{qb} + \frac{1}{rc} = 0 \ldots\ldots\ldots\ldots\ldots\text{(III.)}.$$

Again, multiplying the equations II. by λ, μ, ν, and adding, if λ, μ, ν satisfy the condition

$$k_1\lambda + k_2\mu + k_3\nu = 0 \ldots\ldots\ldots\ldots\ldots\text{(IV.)},$$

we obtain

$$\lambda px' + \mu qy' + \nu rz' - \lambda(by - cz) - \mu(cz - ax) - \nu(ax - by) \equiv 0.$$

If this be written in the form

$$Ax + A'x' + By + B'y' + Cz + C'z' \equiv 0,$$

the condition $AA' = BB' = CC'$ gives

$$\lambda pa(\mu - \nu) = \mu qb(\nu - \lambda) = \nu rc(\lambda - \mu) \ldots\ldots\text{(V.)}.$$

By aid of III. the equations V. are seen to reduce to *one* only. Thus from IV. and V. we obtain two sets of values for λ, μ, ν. That is, we find *two* independent relations of the form I. between the planes x, y, z, x', y', z'. Writing them in the form

$$\left.\begin{aligned} A_1x + A_1'x' + B_1y + B_1'y' + C_1z + C_1'z' &\equiv 0 \\ A_2x + A_2'x' + B_2y + B_2'y' + C_2z + C_2'z' &\equiv 0 \end{aligned}\right\} \ldots\text{(VI.)},$$

we have, by the same reasoning as that applied to the fifteen nodal quartic, the following as singular tangent planes

$$x,\ y,\ z,\ x',\ y',\ z',\ u,\ t,$$

$$A_1x+B_1y+C_1z,\ A_1'x'+B_1y+C_1z,\ A_1x+B_1'y'+C_1z,\ A_1x+B_1y+C_1'z',$$

$$A_2x+B_2y+C_2z,\ A_2'x'+B_2y+C_2z,\ A_2x+B_2'y'+C_2z,\ A_2x+B_2y+C_2'z',$$

or *sixteen* in all.

Moreover from VI. we see that the last two sets of four planes are each concurrent, and the two singular points thus obtained are in addition to the original fourteen nodes.

It is easy to see that through each singular point there pass six singular tangent planes, and on each singular tangent plane there lie six nodes, which is a known property of a Kummer's surface.

The coordinates of the two additional nodes are given by

$$A_1x = A_1'x',\ \&c.,\quad A_2x = A_2'x',\ \&c.$$

ON THE ALGEBRA OF DIFFERENCE-TABLES.

By Prof. J. D. Everett, F.R.S.

THE object of this paper is to explain a notation which yields automatically most of the formulæ applicable to ordinary tables of successive differences.

In the standard arrangement of such tables, the first column contains consecutive values of the function u or $f(x)$ for equal increments of x; the second column contains their first differences written opposite the intervals between the entries in the first column; the third column contains their second differences written opposite the intervals in the second column, and therefore opposite the numbers in the first column, and so on.

The operation Δ carries us from any entry in the table to the entry next below it on the right, so that u, Δu, $\Delta^2 u$, $\Delta^3 u$, &c. lie in one line sloping downwards to the right from u as a starting point. We shall in like manner use the symbol δ to denote the result of passing from any entry to the entry next *above* it on the right; so that u, δu, $\delta^2 u$, $\delta^3 u$, &c. will lie in one line sloping upwards to the right from u. This completely defines our notation. It is equivalent to defining

$$\Delta f(x) = f(x+h) - f(x);\quad \delta f(x) = f(x) - f(x-h);$$

the increment h being constant throughout the table. The operations Δ^{-1} and δ^{-1}, or $1/\Delta$ and $1/\delta$, carry us in the opposite directions to Δ and δ. The operation $\Delta^m\delta^n$ carries us (from any entry in the table) m steps down to the right and n steps up to the right; the order in which the steps are taken being a matter of indifference. We may, in fact, regard m and n as the coordinates of the point of arrival relative to the point of departure, the two lines along which the powers of Δ and δ are disposed being the axes of coordinates. It is obvious that, with arbitrary positive or negative integer values of m and n, the operator $\Delta^m\delta^n$ can carry us from any entry in the table to any other. As particular cases, $\Delta^n\delta^n$ or $(\Delta\delta)^n$ carries us $2n$ columns to the right, the point of arrival being at the same level as the point of departure; $\Delta^n\delta^{-n}$ or $(\Delta/\delta)^n$ carries us n places down the column in which the starting point lies; and $\delta^n\Delta^{-n}$ or $(\delta/\Delta)^n$ carries us n places up it.

$\delta^2\Delta^{-2}$		$\delta^3\Delta^{-1}$		δ^4
	$\delta^2\Delta^{-1}$		δ^3	
$\delta\Delta^{-1}$		δ^2		$\Delta\delta^3$
	δ		$\Delta\delta^2$	
1		$\Delta\delta$		$\Delta^2\delta^2$
	Δ		$\Delta^2\delta$	
$\Delta\delta^{-1}$		Δ^2		$\Delta^3\delta$
	$\Delta^2\delta^{-1}$		Δ^3	
$\Delta^2\delta^{-2}$		$\Delta^3\delta^{-1}$		Δ^4

These geographical relations are exhibited in the annexed scheme, which represents a fragment of a difference-table, divided into compartments for the purpose of more precisely indicating relative position.

Proceeding from left to right along any sloping line of adjacent entries, we have a geometrical progression with the common ratio Δ or δ according as the slope is downwards or upwards.

In any column, read downwards, the ratio is Δ/δ, or, read upwards, δ/Δ, and the sum of the indices of Δ and δ for any column expresses the order of the differences contained in the column. For the first column this sum is zero.

In travelling horizontally, the entries which stand at the same level are in alternate columns, and their ratio in going from left to right is $\Delta\delta$.

Automatic Law.

We now come to the gist of our subject. The position of $\Delta\delta$ opposite the space between Δ and δ shows that the operation $\Delta\delta$ is equivalent to the operation $\Delta - \delta$. The point which I wish to emphasise is that *the equation*,

$$\Delta\delta = \Delta - \delta \quad \text{.....................(1)}$$

contains the whole theory of difference-tables.

As elementary illustrations of its working, we have

$$\Delta^2 - \delta^2 = (\Delta - \delta)(\Delta + \delta) = \Delta\delta(\Delta + \delta) = \Delta^2\delta + \Delta\delta^2,$$

$$\Delta^2 + \delta^2 = (\Delta - \delta)^2 + 2\Delta\delta = \Delta^2\delta^2 + 2\Delta\delta.$$

Systematic Development of the Law.

Equation (1) divided by δ gives $\Delta = \frac{\Delta}{\delta} - 1$ ....(2),

" " Δ " $\delta = 1 - \frac{\delta}{\Delta}$ ....(3),

" " $\Delta\delta$ " $1 = \delta^{-1} - \Delta^{-1}$ ......(4).

All four equations express the same arithmetical fact—the fact that, in a group of three adjacent entries arranged like α, β, z in the margin, we

$$\begin{matrix} & \alpha & \\ b & & z \\ & \beta & \end{matrix}$$

have $z = \beta - \alpha$. In (1) the origin (which would be denoted by 1) is at b.

In (2) it has been displaced by δ and is at α.

In (3) " " Δ " β.

In (4) " " $\Delta\delta$ " z.

Every equation between our operators expresses an arithmetical fact—an arithmetical relation which must always hold between entries in certain specified relative positions; and when the equation has been divided by $\Delta^m\delta^n$ it expresses the same arithmetical fact as viewed from a different origin; the divisor being the specification of the new origin relative to the old. Transposition of a term from one side of the equation to the other involves no change either in the origin or in the arithmetical relation expressed. The most useful transformations of equation (1), by mere change of origin and transposition, are (2) and (3) together with

$$\frac{\Delta}{\delta} = 1 + \Delta \text{ ....} (4),$$

$$\frac{\delta}{\Delta} = 1 - \delta \text{ ....} (5).$$

From these last we deduce by the binomial theorem

$$\left(\frac{\Delta}{\delta}\right)^n = 1 + n\Delta + \frac{n(n-1)}{2}\Delta^2 + \frac{n(n-1)(n-2)}{2.3}\Delta^3 + \&c. ...(6),$$

$$\left(\frac{\delta}{\Delta}\right)^n = 1 - n\delta + \frac{n(n-1)}{2}\delta^2 - \frac{n(n-1)(n-2)}{2.3}\delta^3 + \&c. .....(7),$$

and, from (2) and (3),

$$\Delta^n = \left(\frac{\Delta}{\delta}\right)^n - n\left(\frac{\Delta}{\delta}\right)^{n-1} + \frac{n(n-1)}{2}\left(\frac{\Delta}{\delta}\right)^{n-2} - \&c. ...(8),$$

$$\delta^n = 1 - n\frac{\delta}{\Delta} + \frac{n(n-1)}{2}\left(\frac{\delta}{\Delta}\right)^2 - \&c. \text{ ....} (9).$$

Put $n = 3$. Then in the annexed scheme of entries

$$\begin{array}{ccccccc} & & & A_3, \\ & & A_2, & \\ & A_1, & & \\ A, & & & \\ & a, & & \\ B, & & \alpha, & \\ & b, & & z, \\ C, & & \beta, & \\ & c, & & \\ D, & & & \\ & D_1, & D_2, & \\ & & & D_3, \end{array}$$

(6) gives $\left(\frac{\Delta}{\delta}\right)^3 = 1 + 3\Delta + 3\Delta^2 + \Delta^3$,

(7) " $\left(\frac{\delta}{\Delta}\right)^3 = 1 - 3\delta + 3\delta^2 - \delta^3$,

(8) " $\Delta^3 = \left(\frac{\Delta}{\delta}\right)^3 - 3\left(\frac{\Delta}{\delta}\right)^2 + 3\left(\frac{\Delta}{\delta}\right) - 1$,

(9) " $\delta^3 = 1 - 3\frac{\delta}{\Delta} + 3\left(\frac{\delta}{\Delta}\right)^2 - \left(\frac{\delta}{\Delta}\right)^3$.

$$\left.\begin{array}{llll} \text{With } A \text{ as origin} & (6) \text{ asserts} & D = A + 3a + 3\alpha + z \\ \quad\text{,,} \quad D \quad \text{,,} & (7) \quad \text{,,} & A = D - 3c + 3\beta - z \\ \quad\text{,,} \quad A \quad \text{,,} & (8) \quad \text{,,} & z = D - 3C + 3B - A \end{array}\right\} ...(10),$$

which is also the assertion of (9) with D as origin.

(9) can be derived from (8) by dividing by $\left(\frac{\Delta}{\delta}\right)^3$, which is the specification of D from the origin A. It will be noticed

that in all these cases the entry at one corner of the triangle ADz is expressed in terms of the entries along the opposite side.

NEGATIVE EXPONENTS.

When n is a negative integer, or a positive or negative fraction, the equations will be interpretable unless they yield divergent series.

Putting $-n$ for n in (6) and (7), we have

$$\left(\frac{\Delta}{\delta}\right)^n = \left(\frac{\delta}{\Delta}\right)^{-n} = (1-\delta)^{-n} = 1 + n\delta + \frac{n(n+1)}{2}\delta^2 + \&c.\ldots(11),$$

$$\left(\frac{\delta}{\Delta}\right)^n = \left(\frac{\Delta}{\delta}\right)^{-n} = (1+\Delta)^{-n} = 1 - n\Delta + \frac{n(n+1)}{2}\Delta^2 - \&c.\ldots(12).$$

Thus we have

$$\left.\begin{array}{l} \left(\frac{\Delta}{\delta}\right)^3 = 1 + 3\delta + 6\delta^2 + 10\delta^3 + \&c. \\ \left(\frac{\delta}{\Delta}\right)^3 = 1 - 3\Delta + 6\Delta^2 - 10\Delta^3 + \&c. \\ \text{that is, in the scheme figured above,} \\ D = A + 3A_1 + 6A_2 + 10A_3 + \&c. \\ A = D - 3D_1 + 6D_2 - 10D_3 + \&c. \end{array}\right\}\ldots\ldots(13).$$

In most practical cases the series in (11), (12) converge rapidly, owing to the smallness of the higher orders of differences. As far as I am aware the theorems expressed by these equations are new.

Putting $n=1$, they give

$$\left.\begin{array}{l} \frac{\Delta}{\delta} = 1 + \delta + \delta^2 + \delta^3 + \&c. \\ \frac{\delta}{\Delta} = 1 - \Delta + \Delta^2 - \Delta^3 + \&c. \\ \text{that is, in the figured scheme,} \\ D = C + b + \alpha + \&c. \\ A = B - b + \beta - \&c. \end{array}\right\}\ldots\ldots\ldots\ldots(14).$$

Fractional Exponents.

When n is a fraction, (6), (7), (11), (12) become formulae for interpolation. For instance, to interpolate a term midway between A and B, we may put $n=\frac{1}{2}$ in (6) or (11) with A as origin, or in (7) or (12) with B as origin.

The amount to be added to A is, by (6),

$$\frac{1}{2}a - \frac{1}{8}\alpha + \frac{1}{16}z - \&c.$$

and by (11) it is

$$\frac{1}{2}A_1 + \frac{3}{8}A_2 + \frac{5}{16}A_3 + \&c. \quad \ldots\ldots\ldots\ldots\ldots(15).$$

The amount to be added to B to obtain the same result is, by (7),

$$-\frac{1}{2}a - \frac{1}{8}a_1 - \frac{1}{16}a_2 - \&c.$$

and, by (12),

$$-\frac{1}{2}b + \frac{3}{8}\beta - \&c. \quad \ldots\ldots\ldots\ldots(16).$$

We have thus four independent modes of computing the interpolated term. Four others can be obtained by putting $n=1\frac{1}{2}$ in the same formulæ and thus computing the amounts to be added to the term preceding A and the term succeeding B; four others by putting $n=2\frac{1}{2}$, and so on; but as n increases the convergence becomes less rapid, and increasing weight is given to higher orders of differences, which, with the data available in practice, are apt to be irregular and untrustworthy. As a test, I have calculated log 55·5 by (6), (7), (11), and (12), from a seven-place table of logarithms of two-figure numbers; differences being carried to the third order. Seven independent computations were made; in two of them n was $2\frac{1}{2}$ with log 58 as the starting point; and in no instance did the error exceed 1 in the 7th decimal place. I also employed 6 and 11 with $n=1000$ to compute log 102122 from log 101122 and three orders of differences, as given in Hutton's tables, p. 215, to 20 places. The two computations agreed to 9 places, and agreed with the eight-place value given at p. 190 of Hutton.

Powers of $\Delta\delta$.

The operator $\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}}$ has some interesting properties. We have

$$\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}} = \frac{\Delta\delta}{\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}}} = \frac{\Delta-\delta}{\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}}} = \left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}} - \left(\frac{\delta}{\Delta}\right)^{\frac{1}{2}} \ldots\ldots\ldots(17),$$

showing that, if A, B, C are three consecutive values of the tabulated function, the result of the operation when performed on B is the excess of the value interpolated midway between B and C above that interpolated midway between A and B.

We have also

$$\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}} = \left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}}\delta = \left(\frac{\delta}{\Delta}\right)^{\frac{1}{2}}\Delta \quad \text{..............(18)},$$

showing that the result is a difference of the first order interpolated midway between δ and Δ.

Squaring the first and last members of (17), we get

$$\Delta\delta = \left\{\left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}} - \left(\frac{\delta}{\Delta}\right)^{\frac{1}{2}}\right\}^2 = \frac{\Delta}{\delta} + \frac{\delta}{\Delta} - 2 \quad \text{......(19)},$$

a result which can be verified by multiplying together the values $\Delta = \frac{\Delta}{\delta} - 1$, $\delta = 1 - \frac{\delta}{\Delta}$.

When n is any positive integer, we have

$$(\Delta\delta)^n = \left\{\left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}} - \left(\frac{\delta}{\Delta}\right)^{\frac{1}{2}}\right\}^{2n} = \left\{\left(\frac{\Delta}{\delta}\right)^n + \left(\frac{\delta}{\Delta}\right)^n\right\}$$

$$-2n\left\{\left(\frac{\Delta}{\delta}\right)^{n-1} + \left(\frac{\delta}{\Delta}\right)^{n-1}\right\} + \frac{2n(2n-1)}{2}\left\{\left(\frac{\Delta}{\delta}\right)^{n-2} + \left(\frac{\delta}{\Delta}\right)^{n-2}\right\}$$

$$-\dots(-1)^n . \frac{2n(2n-1)\dots(n+1)}{1.2\dots n} \quad \text{.........(20)},$$

a result which is believed to be new.

The absolute term with which the series concludes is equal to *minus* the sum of the coefficients of the other terms, or to minus twice the sum of the coefficients of the bracketed pairs. This equality follows from the fact that the sum of the coefficients in the expansion of $(a-b)^{2n}$ is zero.

Using the symbol ϕ_m to denote $\left(\frac{\Delta}{\delta}\right)^m + \left(\frac{\delta}{\Delta}\right)^m - 2$, equation (20) reduces to

$$(\Delta\delta)^n = \phi_n - 2n\phi_{n-1} + \frac{2n(2n-1)}{2}\phi_{n-2}$$

$$-\dots(-1)^{n-1}\frac{2n(2n-1)\dots(n+2)}{1.2\dots(n-1)} \quad \text{......(21)},$$

giving

$$\Delta\delta = \phi_1, \quad \Delta^2\delta^2 = \phi_2 - 4\phi_1, \quad \Delta^3\delta^3 = \phi_3 - 6\phi_2 + 15\phi_1,$$

$$\Delta^4\delta^4 = \phi_4 - 8\phi_3 + 28\phi_2 - 56\phi_1,$$

whence, conversely,

$$\phi_1 = \Delta\delta,\ \phi_2 = 4\Delta\delta + \Delta^2\delta^2,\ \phi_3 = 9\Delta\delta + 6\Delta^2\delta^2 + \Delta^3\delta^3,$$
$$\phi_4 = 16\Delta\delta + 20\Delta^2\delta^2 + 8\Delta^3\delta^3 + \Delta^4\delta^4 \text{......(22)}.$$

Again, using the symbol F_m to denote $\left(\frac{\Delta}{\delta}\right)^m - \left(\frac{\delta}{\Delta}\right)^m$, we have

$$F_1 = \frac{\Delta}{\delta} - \frac{\delta}{\Delta} = \frac{\Delta^2 - \delta^2}{\Delta\delta} = \frac{\Delta^2 - \delta^2}{\Delta - \delta} = \Delta + \delta,$$

$$F_2 = \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta}\right)\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right) = (\Delta + \delta)(2 + \phi_1),$$

$$F_3 = \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta}\right)\left(\frac{\Delta^2}{\delta^2} + 1 + \frac{\delta^2}{\Delta^2}\right) = (\Delta + \delta)(3 + \phi_2),$$

$$F_4 = \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta}\right)\left(\frac{\Delta^3}{\delta^3} + \frac{\Delta}{\delta} + \frac{\delta}{\Delta} + \frac{\delta^3}{\Delta^3}\right) = (\Delta + \delta)(4 + \phi_1 + \phi_3),$$

$$\left.\begin{aligned} F_{2n} &= (\Delta + \delta)(2n + \phi_1 + \phi_3 + \ldots + \phi_{2n-1}) \\ F_{2n+1} &= (\Delta + \delta)(2n + 1 + \phi_2 + \phi_4 + \ldots + \phi_{2n}) \end{aligned}\right\} \text{...............(23)}.$$

Hence, by (22),

$$\left.\begin{aligned} &F_1 = \Delta + \delta,\ F_2 = (\Delta + \delta)(2 + \Delta\delta),\ F_3 = (\Delta + \delta)(3 + 4\Delta\delta + \Delta^2\delta^2) \\ &F_4 = (\Delta + \delta)(4 + 10\Delta\delta + 6\Delta^2\delta^2 + \Delta^3\delta^3) \\ &F_5 = (\Delta + \delta)(5 + 20\Delta\delta + 21\Delta^2\delta^2 + 8\Delta^3\delta^3 + \Delta^4\delta^4) \end{aligned}\right\} \text{...(24)}.$$

Another set of values are obtained from (6) and (7) by addition and subtraction. We have thus

$$\left.\begin{aligned} \phi_m &= n(\Delta - \delta) + \frac{n(n-1)}{2}(\Delta^2 + \delta^2) \\ &\qquad + \frac{n(n-1)(n-2)}{2.3}(\Delta^3 - \delta^3) + \&c. \\ F_m &= n(\Delta + \delta) + \frac{n(n-1)}{2}(\Delta^2 - \delta^2) \\ &\qquad + \frac{n(n-1)(n-2)}{2.3}(\Delta^3 + \delta^3) + \&c. \end{aligned}\right\} \text{....(25)},$$

F_m performed on $f(x)$ gives $f(x + mh) - f(x - mh)$, which, when developed by Taylor's theorem, contains only odd powers of mh.

ϕ_m gives $f(x+mh)+f(x-mh)-2f(x)$, which, when developed, contains only even powers.

It is easy to deduce from (8) and (9) that, when n is even,

$$\left.\begin{array}{l}\Delta^n+\delta^n=\phi_n-n\phi_{n-1}+\ldots-n\phi_1,\ \Delta^n-\delta^n=F_n-nF_{n-1}+\ldots-nF_1\\ \text{and, when } n \text{ is odd,}\\ \Delta^n-\delta^n=\phi_n-n\phi_{n-1}+\ldots+n\phi_1,\ \Delta^n+\delta^n=F_n-nF_{n-1}+\ldots+nF_1\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(26),$$

the coefficients being those of the expansion of $(a-b)^n$, with the last term omitted.

NO INTERPRETATION OF $\Delta^p\delta^q$ WITH $p+q$ FRACTIONAL.

Though we can interpolate any number of intermediate terms between two consecutive entries in a column, we cannot interpolate between two consecutive columns; for we can form no conception of fractional orders of differences. The general operator $\Delta^p\delta^q$ is interpretable when $p+q$ is zero, or a positive or negative integer, but not when it is fractional. It is zero for all powers of Δ/δ and δ/Δ, and is unity for $\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}}$. In the more general case in which p and q are any positive fractions whose sum is unity, we have

$$\Delta^p\delta^q=\left(\frac{\Delta}{\delta}\right)^p.\delta=\left(\frac{\delta}{\Delta}\right)^q.\Delta,$$

indicating a first difference interpolated between δ and Δ at distances from them as p to q.

Still more generally, when $p+q$ is any integer m, we have

$$\Delta^p\delta^q=\left(\frac{\Delta}{\delta}\right)^p.\delta^m=\left(\frac{\delta}{\Delta}\right)^q.\Delta^m,$$

indicating a difference of the m^{th} order at distances from δ^m and Δ^m as p to q.

DEDUCTIONS FROM THE VALUES OF $\left(\frac{\Delta}{\delta}\right)^n-1$ AND $1-\left(\frac{\delta}{\Delta}\right)^n$.

Equations (6), (7), (11), (12) give two expressions for $(\Delta/\delta)^n-1$, the operation which produces $f(x+nh)-f(x)$; and two expressions for $1-(\delta/\Delta)^n$, the operation which produces $f(x)-f(x-nh)$. All these expressions are divisible by n. If we put H/h for n, so that H is the increment nh, the two expressions for $(\Delta/\delta)^n - 1$ are

$$\left.\begin{aligned} &H\frac{\Delta}{h} + \frac{H(H-h)}{2}\frac{\Delta^2}{h^2} + \frac{H(H-h)(H-2h)}{2.3}\frac{\Delta^3}{h^3} + \&c. \\ &H\frac{\delta}{h} + \frac{H(H+h)}{2}\frac{\delta^2}{h^2} + \frac{H(H+h)(H+2h)}{2.3}\frac{\delta^3}{h^3} + \&c. \end{aligned}\right\} \ldots(27).$$

and corresponding expressions are obtained for $1 - (\delta/\Delta)^n$. All four, when h is diminished to zero, reduce to Taylor's theorem.

Again, dividing out by n, we have

$$\left.\begin{aligned} h\,\frac{\Delta^n\delta^{-n} - 1}{nh} &= \Delta + \frac{n-1}{2}\Delta^2 + \frac{(n-1)(n-2)}{2.3}\Delta^2 + \&c. \\ &= \delta + \frac{n+1}{2}\delta^2 + \frac{(n+1)(n+2)}{2.3}\delta^3 + \&c. \end{aligned}\right\} \ldots(28);$$

which, when n is diminished to zero, become

$$h\frac{d}{dx} = \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \&c. = \delta + \tfrac{1}{2}\delta^2 + \tfrac{1}{3}\delta^3 + \&c. \ldots(29);$$

whence, by squaring,

$$h^2\left(\frac{d}{dx}\right)^2 = \Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \&c. = \delta^2 + \delta^3 + \tfrac{11}{12}\delta^4 + \&c. \ldots(30).$$

As applied to values rigorously calculated from a formula, the two series in (29) will rigorously agree; but this will not be the case in dealing with experimental results, or with values correct only to a certain number of decimals. The accumulation of small errors increases as we go to higher orders of differences and interferes with regular convergence. In such cases it becomes a question how the two series in (29) can be best combined. The problem may be stated in this way: how to find the best value of $h\dfrac{d}{dx}$ when differences above a certain order are to be excluded from the reckoning. It can be solved by equating some of the first differences to their values as given by Taylor's theorem, and combining the equations in such a way as to eliminate one or more of the lower powers (after the first) of $h\dfrac{d}{dx}$.

Let a stand for $\frac{1}{3!}\left(\frac{d}{dx}\right)^3$; A for $\frac{1}{4!}\left(\frac{d}{dx}\right)^4$;

$$b \quad \text{for} \quad \frac{1}{5!}\left(\frac{d}{dx}\right)^5;\ B \ \text{,,}\ \frac{1}{6!}\left(\frac{d}{dx}\right)^6.$$

Then Taylor's theorem is

$$f(x+H)-f(x)$$
$$=\left\{H\frac{d}{dx}+\frac{H^2}{2}\left(\frac{d}{dx}\right)^2+aH^3+AH^4+bH^5+BH^6+\&c.\right\}f(x),$$

giving

$$f(x+H)-f(x-H)=2\left\{H\frac{d}{dx}+aH^3+bH^5+\&c.\right\}f(x),$$

that is, in the notation above employed,

$$F_m=2\left(mh\frac{d}{dx}+am^3h^3+bm^5h^5+\&c.\right).$$

Thus we have

$$\left.\begin{aligned}\tfrac{1}{2}F_1&=h\frac{d}{dx}+\ ah^3+\ \ bh^5+\&c.\\ \tfrac{1}{2}F_2&=2\ \text{,,}\ +8\ \text{,,}\ +\ 32\ \text{,,}+\&c.\\ \tfrac{1}{2}F_3&=3\ \text{,,}\ +27\ \text{,,}\ +243\ \text{,,}+\&c.\end{aligned}\right\}\text{.........(31).}$$

If third differences, and therefore $\left(\frac{d}{dx}\right)^3$, are to be ignored, a and b are zero, and we have

$$h\frac{d}{dx}=\tfrac{1}{2}F_1=\tfrac{1}{2}(\Delta+\delta)\ \text{...............(32)},$$

which is the best first approximation. It is better than $\frac{1}{4}F_2$, because the error ah^3 is less than the error $4ah^3$.

If the first four orders of differences are to be included and the fifth ignored, b is zero, and we are to eliminate ah^3 from the first two equations, giving

$$\left.\begin{aligned}&h\frac{d}{dx}=\tfrac{2}{3}F_1-\tfrac{1}{12}F_2=\tfrac{1}{2}(\Delta+\delta)\left(1-\tfrac{1}{6}\Delta\delta\right),\\ &\text{which may also be written } \tfrac{1}{2}(\Delta+\delta)-\tfrac{1}{12}(\Delta^2-\delta^2).\end{aligned}\right\}\text{...(33).}$$

If the first six orders of differences are to be included and the seventh ignored, we have to eliminate ah^3 and bh^5 from the three equations as they stand above, omitting the &c.'s. This gives

$$\left.\begin{array}{l} h\frac{d}{dx}=\frac{3}{4}F_1-\frac{3}{20}F_2+\frac{1}{60}F_3=\frac{1}{2}(\Delta+\delta)(1-\frac{1}{6}\Delta\delta+\frac{1}{30}\Delta^2\delta^2), \\ \text{which may also be written} \\ \quad \frac{1}{2}(\Delta+\delta)-\frac{1}{10}(\Delta^2-\delta^2)+\frac{1}{60}(\Delta^3+\delta^3) \end{array}\right\}\ldots(34).$$

In passing from the F_m notation to our standard notation, we have employed in each case, first, equations (24), and, secondly, equations (25). It will be noticed that the second and third approximations derived from (24) have the same coefficient of $\Delta\delta$; whereas, in the approximations derived from (25), the coefficient of $(\Delta^2-\delta^2)$ changes from $\frac{1}{12}$ to $\frac{1}{10}$.

In like manner we have

$$\left.\begin{array}{l} \phi_1 = h^2\left(\frac{d}{dx}\right)^2 + Ah^4 + Bh^6 + \&c. \\ \phi_2 = 4 \quad ,, \quad + 16 \,,, + 64 \,,, + \&c. \\ \phi_3 = 9 \quad ,, \quad + 81 \,,, + 729 \,,, + \&c. \end{array}\right\}\ldots\ldots\ldots\ldots(35)$$

giving as the first approximation, neglecting fourth differences,

$$h^2\left(\frac{d}{dx}\right)^2 = \phi_1 = \Delta\delta = \Delta - \delta \ldots\ldots\ldots\ldots(36);$$

as the second approximation, neglecting sixth differences,

$$h^2\left(\frac{d}{dx}\right)^2 = \frac{4}{3}\phi_1 - \frac{1}{12}\phi_2 = \Delta\delta - \frac{1}{12}\Delta^2\delta^2 = \frac{7}{6}(\Delta-\delta) - \frac{1}{12}(\Delta^2+\delta^2) \ldots\ldots\ldots\ldots(37);$$

and as the third approximation, neglecting eighth differences

$$\left.\begin{array}{l} h^2\left(\frac{d}{dx}\right)^2 = \frac{3}{2}\phi_1 - \frac{3}{20}\phi_2 + \frac{1}{90}\phi_3 = \Delta\delta - \frac{1}{12}\Delta^2\delta^2 + \frac{1}{90}\Delta^3\delta^3 \\ \qquad = \frac{37}{30}(\Delta-\delta) - \frac{7}{60}(\Delta^2+\delta^2) + \frac{1}{90}(\Delta^3-\delta^3) \end{array}\right\}\ldots(38).$$

Here again the coefficients change in passing from one approximation to another when the results are expressed in powers of Δ and δ; but not when they are expressed in powers of $\Delta\delta$.

CENTRAL DIFFERENCES.

The differences of even order $\Delta\delta$, $\Delta^2\delta^2$, $\Delta^3\delta^3$, &c., which stand at the same level as the operand, are called "central differences." The same name is also given to the arithmetical means, $\frac{1}{2}(\Delta+\delta)$, $\frac{1}{2}(\Delta^2\delta+\Delta\delta^2)$, $\frac{1}{2}(\Delta^3\delta^2+\Delta^2\delta^3)$, &c. of the two differences of any odd order which stand nearest to the level of the operand, one being just below and the other just above this level. The odd central differences can be written

$$\tfrac{1}{2}(\Delta+\delta),\ \tfrac{1}{2}(\Delta+\delta)\,\Delta\delta,\ \tfrac{1}{2}(\Delta+\delta)\,\Delta^2\delta^2,\ \tfrac{1}{2}(\Delta+\delta)\,\Delta^3\delta^3,\ \&c.$$

The values

$$\left.\begin{array}{c}\tfrac{1}{2}(\Delta+\delta)\,(1-\tfrac{1}{6}\Delta\delta+\tfrac{1}{30}\Delta^2\delta^2\\ \Delta\delta-\tfrac{1}{12}\Delta^2\delta^2+\tfrac{1}{90}\Delta^3\delta^3\end{array}\right\}\ldots\ldots\ldots\ldots(39)$$

above obtained for $h\dfrac{d}{dx}$, $h^2\left(\dfrac{d}{dx}\right)^2$, are expressed in "central differences."

In reducing a given expression to central differences, $\Delta-\delta$ should be replaced by $\Delta\delta$ wherever possible, and resolution into factors is often helpful.

Thus we have

$$\Delta^3+\delta^3=(\Delta+\delta)\,(\Delta^2-\Delta\delta+\delta^2)=(\Delta+\delta)\,(\Delta^2\delta^2+\Delta\delta),$$

$$\Delta^3-\delta^3=\Delta\delta\,(\Delta^2+\Delta\delta+\delta^2)=\Delta\delta\,(\Delta^2\delta^2+3\Delta\delta).$$

In reducing Δ^n the first step is to replace it by $\Delta^{n-1}(\delta+\Delta\delta)$ or $\Delta\delta\,(\Delta^{n-2}+1)$. In like manner we have

$$\delta^n=\delta^{n-1}(\Delta-\Delta\delta)=\Delta\delta\,(\delta^{n-2}-1).$$

These methods suffice for reducing expressions of the forms $\Delta^n\pm\delta^n$, $\dfrac{\Delta^n}{\delta^m}\pm\dfrac{\delta^n}{\Delta^m}$, n being any positive integer, and m a smaller one.

Expressions of the forms $\Delta^{n+1}\delta^n$, $\Delta^n\delta^{n+1}$, occurring not in conjunction, but separately, can be reduced as follows.

Let c denote the central difference $\frac{1}{2}(\Delta^{n+1}\delta^n+\Delta^n\delta^{n+1})$. Then we have

$$\Delta^{n+1}\delta^n+\Delta^n\delta^{n+1}=2c,$$

$$\Delta^{n+1}\delta^n-\Delta^n\delta^{n+1}=\Delta^n\delta^n\,(\Delta-\delta)=\Delta^{n+1}\delta^{n+1};$$

whence
$$\left.\begin{array}{l}\Delta^{n+1}\delta^n=c+\tfrac{1}{2}\Delta^{n+1}\delta^{n+1}\\ \Delta^n\delta^{n+1}=c-\tfrac{1}{2}\Delta^{n+1}\delta^{n+1}\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

But these last formulæ will not be required for any of the reductions in the present paper.

The following results of reduction to central differences are easily verified, and will be useful for reference:

$$\left.\begin{aligned}
&\Delta^2+\delta^2=2\Delta\delta+\Delta^2\delta^2;\quad \Delta^2-\delta^2=\Delta\delta\,(\Delta+\delta)\\
&\Delta^3+\delta^3=(\Delta+\delta)\,(\Delta\delta+\Delta^2\delta^2);\quad \Delta^3-\delta^3=3\Delta^2\delta^2+\Delta^3\delta^3\\
&\Delta^4+\delta^4=2\Delta^2\delta^2+4\Delta^3\delta^3+\Delta^4\delta^4\\
&\Delta^4-\delta^4=(\Delta+\delta)\,(2\Delta^2\delta^2+\Delta^3\delta^3)\\
&\Delta^3\delta+\Delta\delta^3=2\Delta^2\delta^2+\Delta^3\delta^3;\quad \Delta^3\delta-\Delta\delta^3=\Delta^2\delta^2\,(\Delta+\delta)\\
&\frac{\Delta^3}{\delta}+\frac{\delta^3}{\Delta}=2\Delta\delta+4\Delta^2\delta^2+\Delta^3\delta^3\\
&\frac{\Delta^3}{\delta}-\frac{\delta^3}{\Delta}=(\Delta+\delta)\,(2\Delta\delta+\Delta^2\delta^2)\\
&\frac{\Delta^3}{\delta^2}+\frac{\delta^3}{\Delta^2}=(\Delta+\delta)\,(1+3\Delta\delta+\Delta^2\delta^2)\\
&\frac{\Delta^3}{\delta^2}-\frac{\delta^3}{\Delta^2}=5\Delta\delta+5\Delta^2\delta^2+\Delta^3\delta^3\\
&\frac{\Delta^2}{\delta}+\frac{\delta^2}{\Delta}=(\Delta+\delta)\,(1+\Delta\delta)\\
&\frac{\Delta^2}{\delta}-\frac{\delta^2}{\Delta}=3\Delta\delta+\Delta^2\delta^2\\
&\frac{\Delta^2}{\delta^2}+\frac{\delta^2}{\Delta^2}=2+4\delta+\Delta^2\delta^2\\
&\frac{\Delta^2}{\delta^2}-\frac{\delta^2}{\Delta^2}=(\Delta+\delta)\,(2+\Delta\delta)\\
&\frac{\Delta}{\delta}+\frac{\delta}{\Delta}=2+\Delta\delta;\quad \frac{\Delta}{\delta}-\frac{\delta}{\Delta}=\Delta+\delta
\end{aligned}\right\}\ldots(41).$$

An interpolation formula in terms of central differences can be obtained as follows;

By (4) and (5) we have

$$\left.\begin{aligned}
&\left(\frac{\Delta}{\delta}\right)^n=(1+\Delta)^n=\left(\frac{\Delta}{\delta}\right)^{-1}(1+\Delta)^{n+1}=\frac{\delta}{\Delta}\,(1+\Delta)^{n+1}\\
&\left(\frac{\delta}{\Delta}\right)^{-n}=(1-\delta)^{-n}=\left(\frac{\delta}{\Delta}\right)^{-1}(1-\delta)^{-n+1}=\frac{\Delta}{\delta}\,(1-\delta)^{-n+1}
\end{aligned}\right\}\ldots.(42).$$

Each of these is an expression for the required interpoland. The two right-hand forms when expanded are

$$\frac{\delta}{\Delta}+(n+1)\,\delta+\frac{(n+1)\,n}{2}\,\Delta\delta+\frac{n\,(n^2-1)}{3\,!}\,\Delta^2\delta$$
$$+\frac{(n^2-2n)\,(n^2-1)}{4\,!}\,\Delta^3\delta+\&c.,$$

$$\frac{\Delta}{\delta}+(n-1)\,\Delta+\frac{(n-1)\,n}{2}\,\Delta\delta+\frac{n\,(n^2-1)}{3\,!}\,\Delta\delta^2$$
$$+\frac{(n^2+2n)\,(n^2-1)}{4\,!}\,\Delta\delta^3+\&c.$$

Their sum, or twice the interpoland, is

$$\frac{\Delta}{\delta}+\frac{\delta}{\Delta}+n\,(\Delta+\delta)-(\Delta-\delta)+n^2\Delta\delta+\frac{n\,(n^2-1)}{3\,!}\,(\Delta^2\delta+\Delta\delta^2)$$
$$+\frac{n^2\,(n^2-1)}{4\,!}\,(\Delta^3\delta+\Delta\delta^3)-\frac{2n\,(n^2-1)}{4\,!}\,(\Delta^3\delta-\Delta\delta^3)+\&c.$$

$$=2+\Delta\delta+n\,(\Delta+\delta)-\Delta\delta+n^2\Delta\delta+\frac{n\,(n^2-1)}{3\,!}\,(\Delta^2\delta+\Delta\delta^2)$$

$$+\frac{n^2\,(n^2-1)}{4\,!}\,(2\Delta^2\delta^2+\Delta^3\delta^3)-\frac{2n\,(n^2-1)}{4\,!}\,\Delta^2\delta^2\,(\Delta+\delta)+\&c.$$

Neglecting central differences of higher orders than the fourth, the expression for the interpoland is therefore

$$1+n\,\frac{\Delta+\delta}{2}+\frac{n^2}{2}\,\Delta\delta+\frac{n\,(n^2-1)}{3\,!}\,\frac{\Delta^2\delta+\Delta\delta^2}{2}+\frac{n^2\,(n^2-1)}{4\,!}\,\Delta^2\delta^2 \qquad \text{......(43).}$$

The next two terms are most easily found by regarding the interpoland as half the sum of

$$\frac{\delta^2}{\Delta^2}\,(1+\Delta)^{n+2} \text{ and } \frac{\Delta^2}{\delta^2}\,(1-\delta)^{-n+2}.$$

We shall thus obtain, in the expression for twice the interpoland, the sum of the four terms

$$\frac{n\,(n^2-1)\,(n^2-4)}{5\,!}\,\Delta^3\delta^2+\frac{(n^2-3n)\,(n^2-1)\,(n^2-4)}{6\,!}\,\Delta^4\delta^2,$$

$$\frac{n\,(n^2-1)\,(n^2-4)}{5\,!}\,\Delta^2\delta^3+\frac{(n^2+3n)\,(n^2-1)\,(n^2-4)}{6\,!}\,\Delta^2\delta^4.$$

The two left-hand terms give

$$\frac{n(n^2-1)(n^2-4)}{5!}$$

as the coefficient of $\frac{1}{2}(\Delta^3\delta^2+\Delta^2\delta^3)$ in their half sum.

Also, since $\Delta^4\delta^2+\Delta^2\delta^4$ is $\Delta^2\delta^2(\Delta^2\delta^2+2\Delta\delta)$, the coefficient of $\Delta^3\delta^3$ obtained from the half sum of the two right-hand terms is

$$\frac{n^2(n^2-1)(n^2-4)}{6!}.$$

Two more terms can in like manner be deduced from the expansions of

$$\frac{\delta^3}{\Delta^3}(1+\Delta)^{n+3} \text{ and } \frac{\Delta^3}{\delta^3}(1-\delta)^{-n+3},$$

and so on.

The general expression for any pair of terms thus found may be written

$$\frac{n^2(n^2-1^2)(n^2-2^2)\dots(n^2-m^2)}{(2m+2)!}\left\{\frac{m+1}{n}(\Delta^{m+1}\delta^m+\Delta^m\delta^{m+1})+(\Delta\delta)^{m+1}\right\} \quad \dots\dots(44);$$

which, by putting m successively equal to 0, 1, 2, 3, &c. and prefixing 1, gives the complete expression for the interpoland $(1+\Delta)^n$ in central differences.

Since dn is $\frac{dx}{h}$, we have $\frac{d}{dn}=h\frac{d}{dx}$.

The result of performing this operation on the series (43) will be identical with the coefficient of n. This is

$$\frac{\Delta+\delta}{2}\left\{1-\frac{1^2}{3!}\Delta\delta+\frac{1^2.2^2}{5!}\Delta^2\delta^2-\frac{1^2.2^2.3^2}{7!}\Delta^3\delta^3 + \dots (-1)^m\frac{1^2.2^2.3^2\dots m^2}{(2m+1)!}\Delta^m\delta^m\right\} \quad \dots\dots(45),$$

which includes the approximations to $h\frac{d}{dx}$ given in (32), (33), (34).

$h^2\left(\frac{d}{dx}\right)^2$ or $\left(\frac{d}{dn}\right)^2$ is twice the coefficient of n^2 in the series (43), (44), and is therefore

$$2\left\{\tfrac{1}{2}\Delta\delta - \frac{1^2}{4!}\Delta^2\delta^2 + \frac{1^2.2^2}{6!}\Delta^3\delta^3 - \frac{1^2.2^2.3^2}{8!}\Delta^4\delta^4 + \&c.\right\} \ldots(46),$$

which includes the approximations (36), (37), (38).

This last result can be checked in the following way.

Denoting $\frac{d}{dx}$ by D, we have the well-known symbolical equation $\Delta = e^{hD} - 1$. Similarly we have $\delta = 1 - e^{-hD}$. Hence

$$\Delta\delta = \Delta - \delta = e^{hD} + e^{-hD} - 2 = h^2D^2 + \frac{h^4D^4}{3.4} + \frac{h^6D^6}{3.4.5.6} + \&c. \ldots(47);$$

whence, by reversion of series, h^2D^2 can be expressed in a series of powers of $\Delta\delta$.

In like manner we have

$$\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}} = e^{\frac{1}{2}hD} - e^{-\frac{1}{2}hD} = hD + \frac{h^3D^3}{4.6} + \frac{h^5D^5}{4.6.8.10} + \&c. \ldots(48);$$

whence, by reversion, hD can be expressed in a series of powers of $\Delta^{\frac{1}{2}}\delta^{\frac{1}{2}}$.

When n is $\frac{1}{2}$, a more rapidly convergent series than (43) can be deduced as follows. Let u_0 and u_1 be the two consecutive values between which it is desired to interpolate a term midway. Then the relation between u_0 and u_1 is

$$\Delta u_0 = \delta u_1, \text{ or } \Delta^{m+1}\delta^m u_0 = \Delta^m\delta^{m+1}u_1 \ldots\ldots\ldots(49).$$

The differences represented by these formulæ are symmetrically placed with respect to u_0 and u_1, and are of odd order. They may be called central differences relative to the required midway interpoland; and the corresponding central differences of even order are expressed by the formula

$$\tfrac{1}{2}(\Delta^m\delta^m u_0 + \Delta^m\delta^m u_1);$$

which we shall sometimes write

$$\tfrac{1}{2}\Delta^m\delta^m(u_0 + u_1), \text{ or } \Delta^m\delta^m\frac{u_0 + u_1}{2}.$$

From the series (43) we can derive two expressions for the interpoland, one corresponding to $\left(\frac{\Delta}{\delta}\right)^{\frac{1}{2}}u_0$, the other to $\left(\frac{\Delta}{\delta}\right)^{-\frac{1}{2}}u_1$.

They are

$$\left.\begin{aligned}&\left(1+\frac{\Delta+\delta}{4}+\frac{\Delta\delta}{8}-\frac{\Delta^2\delta+\Delta\delta^2}{32}-\frac{\Delta^2\delta^2}{128}\right.\\&\qquad\left.+\frac{3\,(\Delta^3\delta^2+\Delta^2\delta^3)}{512}+\frac{\Delta^3\delta^3}{1024}\right)u_0\\&\left(1-\frac{\Delta+\delta}{4}+\frac{\Delta\delta}{8}+\frac{\Delta^2\delta+\Delta\delta^2}{32}-\frac{\Delta^2\delta^2}{128}\right.\\&\qquad\left.-\frac{3\,(\Delta^3\delta^2+\Delta^2\delta^3)}{512}+\frac{\Delta^3\delta^3}{1024}\right)u_1\end{aligned}\right\}\ldots(50).$$

Their sum, omitting terms which cancel by (49), is

$$\left.\begin{aligned}&u_0+u_1+\tfrac{1}{4}\,(\delta u_0-\Delta u_1)+\tfrac{1}{8}\Delta\delta\,(u_0+u_1)\\&\qquad-\tfrac{1}{32}\,(\Delta\delta^2u_0-\Delta^2\delta u_1)\\&-\tfrac{1}{128}\Delta^2\delta^2\,(u_0+u_1)+\tfrac{3}{512}\,(\Delta^2\delta^3u_0-\Delta^3\delta^2u_1)\\&\qquad+\tfrac{1}{1024}\Delta^3\delta^3\,(u_0+u_1)\end{aligned}\right\}\ldots(51).$$

But $\delta u_0-\Delta u_1$ is $(\Delta-\Delta\delta)\,u_0-(\delta+\Delta\delta)\,u_1=-\Delta\delta\,(u_0+u_1)$;

hence $\quad \Delta\delta^2u_0-\Delta^2\delta u_1=-\Delta^2\delta^2\,(u_0+u_1)$;

$$\Delta^2\delta^3u_0-\Delta^3\delta^2u_1=-\Delta^3\delta^3\,(u_0+u_1).$$

Substituting accordingly, and halving, we find for the midway interpoland the expression

$$(\tfrac{1}{2}-\tfrac{1}{16}\Delta\delta+\tfrac{3}{256}\Delta^2\delta^2-\tfrac{5}{2048}\Delta^3\delta^3+\&c.)\,(u_0+u_1)\ldots(52).$$

More generally, p and q being any positive fractions such that $p+q=1$, the interpoland

$$\left(\frac{\Delta}{\delta}\right)^p u_0,\ \text{or}\ \left(\frac{\Delta}{\delta}\right)^{-q} u_1,$$

has, by (43), the two expressions

$$\left.\begin{aligned}&\left(1+p\,\frac{\Delta+\delta}{2}+\frac{p^2}{2}\,\Delta\delta-\frac{pq\,(p+1)}{3\,!}\,\frac{\Delta^2\delta+\Delta\delta^2}{2}\right.\\&\qquad\left.-\frac{p^2q\,(p+1)}{4\,!}\,\Delta^2\delta^2+\&c.\right)u_0,\\&\left(1-q\,\frac{\Delta+\delta}{2}+\frac{q^2}{2}\,\Delta\delta+\frac{pq\,(q+1)}{3\,!}\,\frac{\Delta^2\delta+\Delta\delta^2}{2}\right.\\&\qquad\left.-\frac{pq^2\,(q+1)}{4\,!}\,\Delta^2\delta^2+\&c.\right)u_1.\end{aligned}\right\}\ldots(53).$$

Their sum, or twice the interpoland, is

$$u_0 + u_1 + \frac{\Delta + \delta}{2}(pu_0 - qu_1) + \frac{\Delta\delta}{2}(p^2u_0 + q^2u_1)$$
$$+ \frac{\Delta^2\delta + \Delta\delta^2}{2.3!} pq\{(1+q)u_1 - (1+p)u_0\}$$
$$- \frac{\Delta^2\delta^2}{4!} pq\{p(1+p)u_0 - q(1+q)u_1\} + \&c. \dots (54).$$

Hence, with the help of the same formulæ of reduction which were employed in deducing (52), we find

$$\left(\frac{\Delta}{\delta}\right)^p u_0 = \left(\frac{\Delta}{\delta}\right)^{-q} u_1 = \frac{u_0 + u_1}{2} + \frac{p-q}{2}\Delta u_0 - \frac{pq}{2}\Delta\delta\frac{u_0+u_1}{2}$$
$$- \frac{pq(p-q)}{2.3!}\Delta^2\delta u_0 + \frac{pq(1+p)(1+q)}{4!}\Delta^2\delta^2\frac{u_0+u_1}{2} + \&c. \dots (55),$$

the general term being

$$(-1)^m \frac{pq(1+p)(1+q)\dots(m-1+p)(m-1+q)}{2(2m)!}$$
$$\times\left\{\Delta^m\delta^m(u_0+u_1) + \frac{p-q}{2m+1}\Delta^{m+1}\delta^m u_0\right\}.$$

Interchanging p and q simply reverses the sign of the factor $p-q$ which occurs in the coefficients of the odd differences.

Putting $p = \frac{1}{2} + \theta$, $q = \frac{1}{2} - \theta$, expression (55) becomes

$$\frac{u_0+u_1}{2} + \theta\Delta u_0 - \frac{1-4\theta^2}{2!\,4}\Delta\delta\frac{u_0+u_1}{2} - \frac{\theta(1-4\theta^2)}{3!\,4}\Delta^2\delta u_0$$
$$+ \frac{(1-4\theta^2)(9-4\theta^2)}{4!\,4^2}\Delta^2\delta^2\frac{u_0+u_1}{2} + \frac{\theta(1-4\theta^2)(9-4\theta^2)}{5!\,4^2}\Delta^3\delta^2 u_0$$
$$- \frac{(1^2-4\theta^2)(3^2-4\theta^2)(5^2-4\theta^2)}{6!\,4^3}\Delta^3\delta^3\frac{u_0+u_1}{2} - \&c. \dots\dots (56).$$

The coefficient of θ in this expression is the value of $\frac{du}{d\theta}$ or $h\frac{du}{dx}$ for $\theta = 0$, that is for the value of u corresponding to the midway interpoland. It is

$$\left.\begin{aligned}&\left(\Delta - \frac{1}{3!\,4}\Delta^2\delta + \frac{1^2.3^2}{5!\,4^2}\Delta^3\delta^2 - \frac{1^2 3^2 5^2}{7!\,4^3}\Delta^4\delta^3 + \&c.\right)u_0\\ &\text{or } \left(\Delta - \tfrac{1}{24}\Delta^2\delta + \tfrac{3}{640}\Delta^3\delta^2 - \tfrac{5}{7168}\Delta^4\delta^3 + \&c.\right)u_0\end{aligned}\right\} \dots (57),$$

which contains only odd differences central with respect to the midway interpoland, as in (49).

The value of $h^2 \frac{d^2u}{dx^2}$ or $\frac{d^2u}{d\theta^2}$ for $\theta = 0$ is twice the coefficient of θ^2 in (56), and is

$$\left\{\frac{8}{2!\,4}\Delta\delta - \frac{8\,(1^2+3^2)}{4!\,4^2}\Delta^2\delta^2 + \frac{8\,(1^2.3^2+1^2.5^2+3^2.5^2)}{6!\,4^3}\Delta^3\delta^3\right.$$

$$\left. - \frac{8\,(1^2.3^2.5^2+1^2.3^2.7^2+1^2.5^2.7^2+3^2.5^2.7^2)}{8!\,4^4}\Delta^4\delta^4 + \&c.\right\}\frac{u_0+u_1}{2}$$

$$= (\Delta\delta - \tfrac{5}{24}\Delta^2\delta^2 + \tfrac{259}{5760}\Delta^3\delta^3 - \tfrac{3229}{322560}\Delta^4\delta^4 + \&c.)\,\frac{u_0+u_1}{2} \ldots(58).$$

More generally, the values of $h\frac{du}{dx}$ and $h^2\frac{d^2u}{dx^2}$, to the fifth order of differences, for *any value of* θ, are found, by differentiating (56), to be

$$\left.\begin{aligned} h\frac{du}{dx} = \frac{du}{d\theta} &= \left(\Delta - \frac{\Delta^2\delta}{24} + \frac{3\Delta^3\delta^2}{640}\right)u_0 \\ &+ \theta\left(\Delta\delta - \frac{5\Delta^2\delta^2}{24}\right)\frac{u_0+u_1}{2} + \theta^2\left(\frac{\Delta^2\delta}{2} - \frac{\Delta^3\delta^2}{16}\right)u_0\,; \\ h^2\frac{d^2u}{dx^2} = \frac{d^2u}{d\theta^2} &= \left(\Delta\delta - \frac{5\Delta^2\delta^2}{24}\right)\frac{u_0+u_1}{2} + \theta\left(\Delta^2\delta - \frac{\Delta^3\delta^2}{8}\right)u_0. \end{aligned}\right\}\ldots(59).$$

These formulæ enable us, from an ordinary table of values of u for equidifferent values of x, to find $\frac{du}{dx}$ and $\frac{d^2u}{dx^2}$ for the value of u corresponding to any intermediate value of x.

Results (56), (57), (58) were published (without proof) by Mr. W. F. Sheppard in *Nature*, August 24th, 1899 (Vol. LX., p. 391).

Comparison of Notations.

For passing from our notation to the ordinary notation, or, conversely, the most convenient formula is

$$\delta^n u_0 = \Delta^n u_{-n} \ldots\ldots\ldots\ldots(60),$$

which gives $$\Delta^n\delta^n u_0 = \Delta^{2n}u_{-n}.$$

Again, denoting, as usual, d/dx by D, we have

$$\left.\begin{aligned} &\Delta = e^{hD} - 1\,;\quad \delta = 1 - e^{-hD}\,;\quad \frac{\Delta}{\delta} = e^{hD}\,;\quad \frac{\delta}{\Delta} = e^{-hD} \\ &hD = \log\frac{\Delta}{\delta} = \log(1+\Delta) = -\log\frac{\delta}{\Delta} = -\log(1-\delta) \end{aligned}\right\}\ldots(61).$$

THE STRESS IN THE WEB OF A PLATE GIRDER.

By J. H. MICHELL.

1. IT is proposed to find the stress in a web of an I-shaped plate girder, supported at both ends and uniformly loaded, on the following assumptions:—

(*a*) The chords (top and bottom members) are perfectly flexible, so that the load is transmitted directly to the web, which takes all the vertical shearing stress.

(*b*) The longitudinal extensions of the web at top and bottom are, at each section, those of the two chords.

(*c*) The web can be treated as a uniform thin plate under forces in its plane.

(*d*) The girder is long in comparison with its depth, so that the principle of equipollent loads can be applied to the ends. The solution therefore does not apply near the ends.

Take the horizontal middle line of the web as the axis of x and the vertical through one end as the axis of y. Let w_1 be the load per unit length on the top chord at a section distant x from the origin, w_2 that on the bottom chord. The weight of the web is neglected. Let T_1, T_2 be the *tensions* of the two chords, $2h$ the depth of the girder, l its length.

If e_1, e_2 are the extensions of the chords at section x (and, therefore, by assumption (*b*), the longitudinal extensions of the top and bottom of the web)

$$T_1 = E_1 A_1 e_1, \qquad T_2 = E_2 A_2 e_2,$$

where E_1, E_2 are the Young's moduluses of the chords and A_1, A_2 their areas of cross-section.

Let P, Q, U denote, as usual, the elements of stress per unit length in the web; P_1, Q_1, U_1, P_2, Q_2, U_2 being their values at top and bottom respectively.

Then
$$E\tau e_1 = P_1 - \sigma Q_1 = P_1 + \sigma w_1,$$
$$E\tau e_2 = P_2 - \sigma Q_2 = P_2 - \sigma w_2,$$

where E is the Young's modulus of the web, σ its Poisson's ratio, and τ its thickness.

Resolving longitudinally for the equilibrium of the chords, we have

$$\frac{dT_1}{dx} = U_1, \qquad \frac{dT_2}{dx} = -U_2.$$

Substituting for T_1, T_2, these give

$$\left.\begin{aligned} \frac{d}{dx}\left\{\frac{E_1A_1}{E\tau}(P_1+\sigma w_1)\right\} &= \ U_1 \\ \frac{d}{dx}\left\{\frac{E_2A_2}{E\tau}(P_2-\sigma w_2)\right\} &= -\ U_2 \end{aligned}\right\} \text{............(1).}$$

These, with

$$\left.\begin{aligned} Q_1 &= -\ w_1 \\ Q_2 &= \ \ w_2 \end{aligned}\right\} \text{.........................(2),}$$

are the boundary conditions at the top and bottom of the web.

The areal equations for the web are

$$\frac{dP}{dx}+\frac{dU}{dy}=0, \quad \frac{dU}{dx}+\frac{dQ}{dy}=0,$$

which are satisfied quite generally by

$$P=\frac{d^2\psi}{dy^2}, \quad Q=\frac{d^2\psi}{dx^2}, \quad U=-\frac{d^2\psi}{dxdy},$$

where ψ satisfies $\nabla^2(P+Q)=0,$

or $\nabla^4\psi=0$ ....(3).

2. Suppose that the girder is uniform throughout its length and that the load is applied on top. Writing w for w_1, and putting $w_2=0$, the equations (1), (2) become

$$K_1\frac{dP_1}{dx}=U_1, \quad K_2\frac{dP_2}{dx}=-\ U_2,$$

$$\left.\begin{aligned} Q_1 &= -\ w \\ Q_2 &= \ \ 0 \end{aligned}\right\} \text{.......................(4),}$$

where $K_1=E_1A_1/E\tau, \quad K_2=E_2A_2/E\tau.$

Since $\dfrac{dP}{dx}+\dfrac{dU}{dy}=0$, the first two equations become

$$\left.\begin{aligned} K_1\frac{dU_1}{dy}+U_1=0 \\ K_2\frac{dU_2}{dy}-U_2=0 \end{aligned}\right\} \text{....................(5).}$$

From the general theory* of uniformly loaded beams, we

* Discussed in a paper on "Uniformly loaded beams," which will shortly appear in the *Quarterly Journal.*

know that if the principle of equipollent loads can be applied to the end conditions, a sufficiently general solution is obtained by making the stresses quadratic functions of x. We therefore take as the solution of (3)

$$\psi = c_0 y^2 + d_0 y^3 - \tfrac{1}{3} c_2 y^4 - \tfrac{1}{5} d_2 y^5$$
$$+ (b_1 y + c_1 y^2 + d_1 y^3)\, x$$
$$+ (a_2 + b_2 y + c_2 y^2 + d_2 y^3)\, x^2,$$

linear terms being omitted as of no significance.

This gives

$$U = -\psi_{xy} = -(b_1 + 2c_1 y + 3d_1 y^2) - 2(b_2 + 2c_2 y + 3d_2 y^2)\, x,$$

$$\frac{dU}{dy} = -(2c_1 + 6d_1 y) - 2(2c_2 + 6d_2 y)\, x.$$

The conditions (5) therefore give

$$K_1 (2c_1 + 6d_1 h) + b_1 + 2c_1 h + 3d_1 h^2 = 0,$$
$$K_1 (2c_2 + 6d_2 h) + b_2 + 2c_2 h + 3d_2 h^2 = 0,$$
$$K_2 (2c_1 - 6d_1 h) - (b_1 - 2c_1 h + 3d_1 h^2) = 0,$$
$$K_2 (2c_2 - 6d_2 h) - (b_2 - 2c_2 h + 3d_2 h^2) = 0.$$

To simplify the algebra suppose the two chords alike, so that $K_1 = K_2$. These conditions then give

$$\left.\begin{aligned} c_1 = c_2 = 0 \\ b_1 + 3d_1 h\,(h + 2K) = 0 \\ b_2 + 3d_2 h\,(h + 2K) = 0 \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(6).$$

Further $\quad Q = \dfrac{d^2\psi}{dx^2} = 2(a_2 + b_2 y + c_2 y^2 + d_2 y^3),$

and therefore, from (4),

$$a_2 + b_2 h + c_2 h^2 + d_2 h^3 = -\tfrac{1}{2} w,$$
$$a_2 - b_2 h + c_2 h^2 - d_2 h^3 = 0.$$

Hence, since $c_2 = 0$,

$$\left.\begin{aligned} a_2 = -\tfrac{1}{4} w \\ b_2 h + d_2 h^3 = -\tfrac{1}{4} w \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Therefore, from (6),

$$\left.\begin{aligned} b_2 &= -\tfrac{3}{8}\,\frac{h+2K}{h\,(h+3K)}\,w \\ d_2 &= \tfrac{1}{8}\,\frac{1}{h^2\,(h+3K)}\,w \end{aligned}\right\} \quad \text{.................(8).}$$

The other constants must be determined from the terminal conditions. We have

$$P=\frac{d^2\psi}{dy^2}=2c_0+6d_0y-4d_2y^3+6d_1yx+6d_2yx^2,$$

and hence

$$T_1=K(P_1+\sigma w)=K(\sigma w+2c_0+6d_0h-4d_2h^3+6d_1hx+6d_2hx^2),$$

$$T_2=KP_2=K(2c_0-6d_0h+4d_2h^3-6d_1hx-6d_2hx^2).$$

Since there is no resultant horizontal force across any section

$$4c_0\,(h+K)+K\sigma w=0,$$

or

$$c_0=-\tfrac{1}{4}\,\frac{K}{h+K}\,\sigma w \text{ ....(9).}$$

At the end $x=0$,

$$P=2c_0+6d_0y-4d_2y^3;$$

and, since the couple at the end vanishes,

$$\int_{-h}^{h} Py\,dy+(T_1-T_2)\,h=0,$$

that is, $4d_0h^3-\tfrac{8}{5}d_2h^5+K(12d_0h^2-8d_2h^4)+Kh\sigma w=0,$

or $\quad d_0h\,(h+3K)-2d_2h^3\,(\tfrac{1}{5}h+K)=-\tfrac{1}{4}K\sigma w;$

and, substituting the value of d_2 from (8),

$$d_0h\,(h+3K)^2=\tfrac{1}{4}h\,(\tfrac{1}{5}h+K)-\tfrac{1}{4}K\,(h+3K)\,\sigma w\text{...(10).}$$

Finally, the shearing force on the end is $wl/2$, so that

$$\int_{-h}^{h} U\,dy=-wl/2,$$

or

$$2b_1h+2d_1h^3=wl/2,$$

and hence, from (6),

$$\left.\begin{aligned} b_1 &= \tfrac{3}{8}\frac{h+2K}{h(h+3K)}wl \\ d_1 &= -\tfrac{1}{8}\frac{1}{h^2(h+3K)}wl \end{aligned}\right\} \quad \text{.............(11).}$$

From equations (6)—(10) the constants of the solution are

$$c_0 = -\tfrac{1}{4}\frac{K}{h+K}\sigma w,$$

$$d_0 = \tfrac{1}{20}\frac{h+5K}{(h+3K)^2}w - \tfrac{1}{4}\frac{K}{h(h+3K)}\sigma w,$$

$$b_1 = \tfrac{3}{8}\frac{l(h+2K)}{h(h+3K)}w,$$

$$c_1 = 0,$$

$$d_1 = -\tfrac{1}{8}\frac{l}{h^2(h+3K)}w,$$

$$a_2 = -\tfrac{1}{4}w,$$

$$b_2 = -\tfrac{3}{8}\frac{h+2K}{h(h+3K)}w,$$

$$c_2 = 0,$$

$$d_2 = \tfrac{1}{8}\frac{1}{h^2(h+3K)}w.$$

Substituting these values in the expressions for P, Q, U, they become

$$P = -\tfrac{1}{2}\frac{\sigma K w}{h+K} + \frac{3w}{h+3K}\left\{\frac{h+5K}{10(h+3K)} - \frac{\sigma K}{2h}\right\}y$$

$$- \frac{3w}{4h^2(h+3K)}x(l-x)y - \frac{w}{2h^2(h+3K)}y^3,$$

$$Q = -\frac{w}{4h^2(h+3K)}\{2h^3 + 6h^2K + 3h(h+2K)y - y^3\},$$

$$U = -\frac{3w}{8h^2(h+3K)}(h^2 + 2hK - y^2)(l-2x).$$

If K, h are of the same order and h^2/x^2 may be neglected, we may write these

$$P = -6\Gamma x(l-x)y,$$
$$Q = -2\Gamma\{2h^3 + 6h^2K + 3h(h+2K)y - y^3\},$$
$$U = -3\Gamma(h^2 + 2hK - y^2)(l-2x),$$

where $$\Gamma = \frac{w}{8h^2(h+3K)}.$$

The value of P is now that derived from the application of Bernoulli's formula for uniform flexure. The process commonly used by engineers to determine the shear U in the web is equivalent to the use of the equation

$$\frac{dP}{dx} + \frac{dU}{dy} = 0$$

and the given value of P. The stress Q can, for most purposes, be neglected.

The solution for a load on the bottom chord can plainly be derived from that obtained without further investigation.

ON THE HOLOMORPH OF THE CYCLICAL GROUP AND SOME OF ITS SUBGROUPS.

By G. A. Miller.

A FEW of the properties of the holomorph of a cyclical group were determined by Burnside.* In what follows several other important properties of this holomorph and its subgroups are determined. Most of the results were obtained by means of the commutator subgroups of the groups under consideration.

Since the group of isomorphisms of any cyclical group (G) must be Abelian, the commutator subgroup of the holomorph (K) of G must be contained in G. The number of different commutators of a group is not less than the number of the operators in its largest system of conjugate operators.† Hence it follows that every operator of G must be a commutator of K whenever the order of G is odd, and that at least one half of the operators of G are commutators of K when the order

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* Burnside, *Theory of Groups of a Finite Order*, 1897, p. 240.

† Miller, *Bulletin of the American Mathematical Society*, Vol. IV., 1898, p. 133.

of G is even. If we represent G as a regular substitution group, one half of its substitutions are negative when its order is even. Since the commutator of any two substitutions is positive, this proves that the number of the commutators of the holomorph of a cyclical group of an even order cannot exceed one-half the order of this group. These results, together with the known properties of the commutator subgroups, lead to the following theorems:

Theorem I. If the holomorph of a cyclical group of an odd order is isomorphic with an Abelian group, this Abelian group must have a 1, α *isomorphism with the group of isomorphisms of the given cyclical group.*

Theorem II. If the holomorph of a cyclical group of an even order is isomorphic with an Abelian group, this Abelian group must have a 1, α *isomorphism with the direct product of the group of isomorphisms of the given cyclical group and an operator of order two.*

Let K_1 be a subgroup of K composed of all its operators that correspond to a cyclical subgroup of its group of isomorphisms, and let g, k, k_1 represent the orders of G, K, K_1 respectively. If we represent by t any operator of K_1 such that G and this operator generate K_1, and if we represent by m the number of the operators of G that are commutative to t, then the number of the different commutators of K_1 cannot be less than $g \div m$. In order to prove that the number of these commutators is always equal to $g \div m$, it is only necessary to prove that the quotient group of K_1, with respect to the subgroup of G whose order is $g \div m$, is Abelian.

If s is any generator of G, then it follows from the preceding paragraph that $t^{-1}st = s^{\alpha}$ (where $\alpha = km + 1$, k being prime to g). All the operators of G which may be obtained by raising all of its operators to their kmth power, must correspond to identity in the given quotient group of K_1. Hence this quotient group is always Abelian. The following theorems depend directly upon this fact:

Theorem III. If K_2 is any subgroup of K that includes G, and if just m_2 operators of G are commutative to every operator of K_2, then the commutator subgroup of K_2 is of order $g \div m_2$.

Theorem IV. The group of cogredient isomorphisms of K_2 is Abelian whenever $m_2^2 \div g$ is an integer, and only then.

COROLLARY. *The group of cogredient isomorphisms of the holomorph of a cyclical group is Abelian whenever the order of this cyclical group is two or four, and only then.*

Every subgroup of K that contains G must correspond to some subgroup of the group of isomorphisms (I) of G. When the order of G is a power of an odd prime number, or twice such a power, its I is cyclical, and there is one and only one subgroup of K, containing G, for every factor of the order of I. When the order of G is 2^{α}, its group of isomorphisms contains just three subgroups of every order, greater than unity, that divides $2^{\alpha-2}$. Hence the corresponding K contains three subgroups of the same order that contain G. One-half of the operators of any one of these three subgroups are contained in each of the other two. We proceed to prove that no two of these three subgroups are simply isormorphic.

When their common order is $2^{\alpha+1}$, only one of them can contain two cyclical subgroups of order 2^{α}, since the operators in the tail of such a group have to transform the operators of G into their $2^{\alpha-1}+1$ power. Since the other two groups transform the operators of G into their $2^{\alpha-1}-1$ and $2^{\alpha}-1$ powers respectively, and contain only one cyclical subgroup of order 2^{α}, they must be distinct as abstract groups. When the order of the three groups under consideration exceeds $2^{\alpha+1}$, we may prove in the following manner that no two of them can be simply isomorphic. One of them contains operators of order 2 that are not common to the three, while the other two do not have this property. In one of these two groups each operator transforms each operator of G into its 5^{β} power. If t and G generate this group, we have $(ts)^2 = t^2 s^{\beta_1+1}$, s being any generator of G. Since $\beta_1 \equiv 1$, mod. 4, β_1+1 cannot be divisible by 4.

The tail of the other group will contain an operator t_1, which has the same square as t. The operators which this group has, in common with the preceding group together with t_1 must generate the entire group. As before we have $(t_1 s)^2 = t^2 s^{\beta_2+1}$, where $\beta_2+1 \equiv 0$, mod. 4, since $(2^{\alpha}-1)5^{\beta}+1$ must be divisible by 4. As the squares of the operators in the tail of this group give a smaller number of different operators than the squares of the operators in the tail of the preceding group, the groups cannot be simply isomorphic.

Cornell University,
October, 1899.

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Fig. 1.

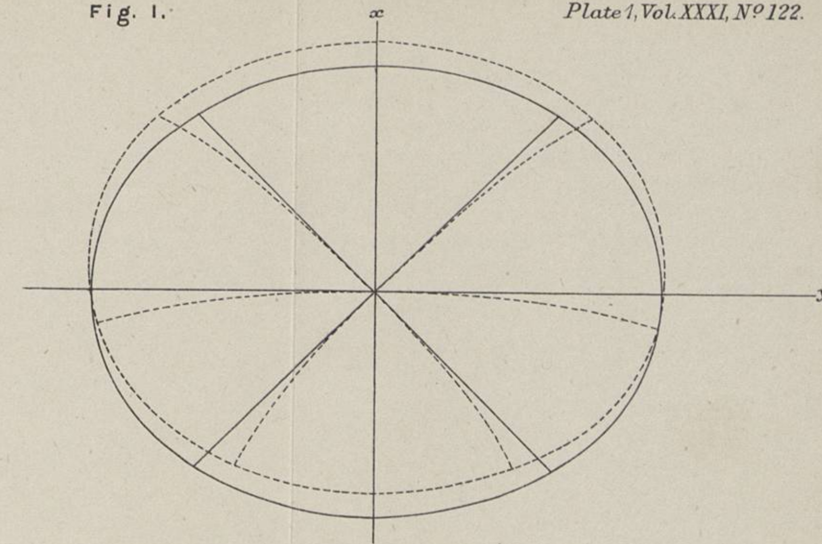
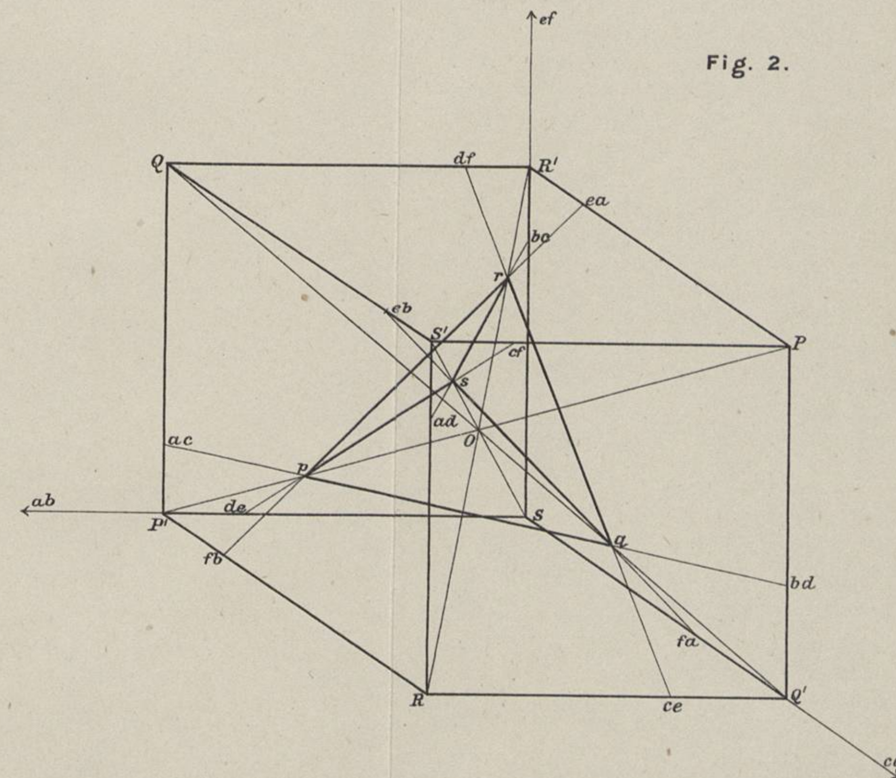


Fig. 2.



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