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PART I.—STATICS OF STRUCTURES.

CHAPTER I.

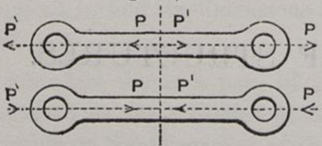
FRAMEWORK LOADED AT THE JOINTS.

1. *Preliminary Explanations and Definitions.*—A frame is a structure composed of bars, united at their extremities by joints, which offer no resistance to rotation. In the first instance we may suppose the centre lines of the bars all in one plane, and in that case the joints may consist simply of smooth pins passing through holes at the ends of the bars, which are to be imagined forked, if necessary, so as to allow the centre lines to meet in a point. A large and important class of structures, known to engineers as "trusses," approach so closely to frames that calculations respecting them may be conducted by treating them as if they were frames. The differences between a truss and a frame will appear as we proceed.

The frame may be acted on by forces applied at points in one or more of its bars, or at the joints which unite the bars together. An important simplification, however, is effected by supposing, in the first instance, that the joints only are loaded, an assumption which will be made throughout this chapter, except in a few simple examples. It will be shown hereafter that all other cases may be derived from this by means of a preliminary reduction (see Chapter IV.).

Assuming, then, that the frame is acted on by forces at the joints, due either to weights or other external causes, or to the reaction of supports on which the frame rests, the problem to be solved is to find the forces called into play on each of the bars of which it is constructed. These forces are caused by the pressure of the pins on the sides of the holes through which they pass, and it at once follows, since no other forces act on the bar, that for each bar these pressures must be equal and opposite, their common line of action being the line joining the centres of the holes. There are two possible cases shown in Figs. 1a, 1b; in the first the bar is acted on by a pair of equal and opposite forces tending to lengthen it, and in the second to shorten it. The pairs of forces are called a Pull and a Thrust respectively, while the bars subjected to their action are called Ties or Struts respectively. Between a pull and a thrust there is no statical difference but that of sign; the constructive difference, however, between a tie and a strut is great. The first may theoretically be a rope or chain, and the second may be made up of pieces simply butting against one another without fastening, while a rigid bar will serve either purpose, though its powers of resistance are generally entirely different in the two cases.

Figs. 1a, 1b.



It often happens that it is unknown whether a bar be a strut or a tie, and the pair of forces are then called a STRESS on the bar. This word "stress" was introduced by Rankine to denote the mutual action between any two bodies, or parts of a body, and here means, in the first instance, the mutual action between the parts of the frame united by the bar we are considering. If, however, we imagine the bar cut into two parts, A and B, by any transverse section, as shown in Figs. 1a, 1b, those parts are held together in the case of a pull, or thrust away from each other in the case of a thrust, by internal molecular forces called into play at each point of the transverse section, and acting one way on A and the other way on B. As A and B must both be in equilibrium, it is obvious that these internal forces must be exactly equal to the original forces, and thus it appears that the stress on the bar may also be regarded as the internal molecular action between any two parts into which it may be imagined to be divided. Stress, regarded in this way, will be fully considered in a subsequent division of this work; it will be here sufficient to say that its intensity is measured by dividing the total amount by the sectional area of the bar, and is limited to a certain amount, depending on the nature of the material of which the bar is constructed.

It is further manifest from what has been said, that the stress on a bar may likewise be regarded as a mutual action between the bar and either of the pins at its ends which are pulled towards the middle of the bar in the case of a pull, or thrust away from it in the case of a thrust; each pin is therefore acted on, in addition to any load which may be suspended from it, by forces, the directions of which are the lines joining the centres of the pins, from which it follows at once that *every joint may be regarded as a point kept in equilibrium by the load at that joint and by forces of which the bars of the frame are the lines of application.* This principle enables us to find the stress on each bar of a frame loaded at the joints, whenever such stress can be determined by statical considerations alone, without reference to the material or mode of construction, that is to say, in all cases which properly belong to the present division of our work.

Forces are measured in pounds-weight or, when large, in tons of 2240 lbs. They are often distributed over an area or along a line, and are then reckoned per square foot or per "running" foot, the last expression being commonly abbreviated to "foot-run."

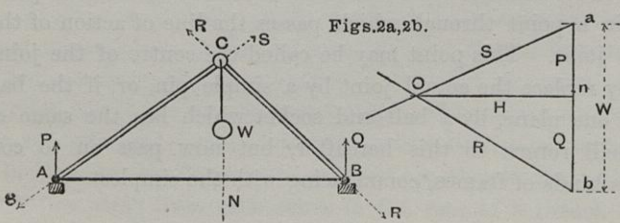
The bars need not be connected by simple pin joints as has been supposed for clearness, provided that their centre lines if prolonged meet in a point through which passes the line of action of the load on the joint. This point may be called the centre of the joint, and we may replace the actual joint by a simple pin, or, if the bars are not in one plane, by a ball and socket which has the same centre. We shall return to this hereafter, but now pass on to consider various kinds of frames, commencing with the simplest.

SECTION I.—TRIANGULAR FRAMES.

2. *Diagram of Forces for a Simple Triangular Frame.*—The simplest kind of frame is a triangle.

In Fig. 2a ACB is such a triangle; it is supported at AB so that AB is horizontal, and loaded at C with a weight W. Then evidently the effect of the weight is to compress AC, BC, and to stretch AB, which is conveniently indicated by drawing AC, BC in double lines, and AB in a single line. Also the weight produces certain vertical pressures on the supports A, B, which will be balanced by corresponding reactions P and Q.

To find the magnitude of the thrust on AC, BC, the pull on AB, and the reactions, the diagram of forces Fig. 2b is drawn: ab is a vertical line representing W on any convenient scale, while aO, bO are lines drawn through a, b respectively, parallel to AC, BC, to meet in O, and finally On is drawn parallel to AB, or, what is the same thing, perpendicular to ab. Now, applying the fundamental principle laid down above, we observe that C is a point kept in equilibrium by three forces, the load at C, namely W, the thrust of AC which we will call S, and the thrust of BC which we will call R. In the second figure the triangle Oab has its sides parallel to these forces, and hence it follows that Oa, Ob represent S, R on the same scale that ab represents W. Again A is a point kept in equilibrium by three forces, the thrust of AC, the pull of the tie AB, which we will call H, and the upward reaction P of the support A. But referring to the figure 2b, On, an, are respectively parallel to the two last forces, so that, by the triangle of forces, they represent H, P on the same scale that Oa represents S. The same reasoning applies to the point B, and therefore bn represents the other supporting force Q, as is also obvious from the consideration that $P + Q = W$. We thus see that all the forces acting upon and within the triangular frame ACB are represented by corresponding lines in Fig. 2b, which is thence called



Figs.2a,2b.

the "diagram of forces" for the triangular frame. Such a diagram can be drawn for any frame, however complicated, and its construction to scale is the best method of actually determining the stresses on the several parts of the frame.

The force H requires special notice: it is called the "*thrust*" of the frame. In the present case the thrust is taken by the tension of the third side of the triangle, but this may be omitted, and the supports A and B must then be solid and stable abutments capable of resisting a horizontal force H. In many structures such a horizontal thrust exists; and its amount and the mode of providing against it are among the first things to be considered in designing the structure. Besides the graphical representation just given, which enables us to obtain the thrust of a triangular frame by constructing a simple diagram, it may also be calculated by a formula which is often convenient. Let AC be denoted by b and BC by a, as is usual in works on trigonometry, and let AN, BN their projections on AB be called b', a', and let the height of the triangle be h and its span l, then by similar triangles,

$$\frac{P}{H}=\frac{an}{On}=\frac{CN}{AN}=\frac{h}{b'},$$

$$\frac{Q}{H}=\frac{bn}{On}=\frac{CN}{BN}=\frac{h}{a'}.$$

Therefore, by addition,

$$\frac{W}{H}=\frac{ab}{On}=h\left(\frac{1}{b'}+\frac{1}{a'}\right)$$

$$\text{or}\quad H=W\frac{a'b'}{lh}.$$

In practical questions it often happens that a', b', h are known by the nature of the question, whence H is readily determined. The case when the load bisects the span may be specially noticed; then $a'=b'=\frac{1}{2}l$ and

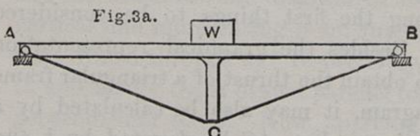
$$H=\frac{Wl}{4h}.$$

When the height of the frame is small compared with the span, this calculation is to be preferred to the diagram, which cannot then be constructed with sufficient accuracy.

The simple frame here considered may be inverted, in which case the diagram of forces and the numerical results are unaltered, the only change being that the two struts have become ties and the tie a strut.

3. *Triangular Trusses.*—Triangular frames are common in practice, and the rest of this section will be devoted to some of the commonest forms in which they appear.

Fig. 3a shows a simple triangular truss consisting of a beam, AB, supported by a strut at the centre, the lower extremity of which is carried by tie rods, AC, BC, attached to the ends of the beams. If now a weight, W, be placed at the centre, immediately over the strut, it does not bend the beam (sensibly) as it would do if there were no strut, but is transmitted by the strut to the joint C, so that the truss is equivalent to the simple triangular frame of the last article. This, however, supposes that the strut has exactly the proper length to prevent any bending of the beam; if it be too short or too long the load on the frame will be less or greater than W, a point which will be further considered presently. It should be noticed that D is not necessarily at the centre.



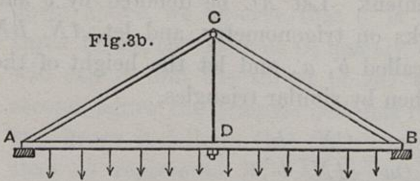


Fig. 3b shows the same construction inverted. CD is a tie by which D is suspended from C; we will suppose this rod to pass through AB and a nut applied below, by means of which D may be raised or lowered. Let AB now be uniformly loaded with a given weight, then the bending of AB is resisted by CD, which supports it and carries a part of the load, which may be made greater or less by turning the nut. If, however, we imagine AB, instead of being continuous through D, to be jointed at D, then the tie CD necessarily carries half the weight of AD and half the weight of BD, that is to say, half the whole load, whatever be its exact length. This simple example illustrates very well the most important difference between a truss and a mathematical frame; namely, that in the truss one or more of the bars is very often continuous through a joint. Such cases can only be dealt with on the principles of the present division of our work, by making the supposition that the bar in question, instead of being continuous, is jointed like the rest. The error of such a supposition will be considered hereafter; it is sufficient now to say that in order that it may be exact in the particular case we are considering, the nut must be somewhat slackened out so that D may be below the straight line AB, and that being dependent on accuracy of construction, temperature, and other varying circumstances, such errors cannot be precisely stated, but must be allowed for in designing the structure by the use of a factor of safety. The supposition is one which is usual in practical calculations, and will be made throughout this division of our work.

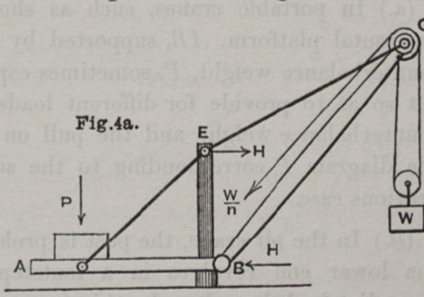
The foregoing is one of the simplest cases where, as is very common in practice, the bars of the frame are loaded and not the joints alone. When such bars are horizontal and uniformly loaded, the effect is evidently the same as if half the load on each division of the loaded bar were carried at each of the joints through which it passes. This is also true if the loaded bars be not horizontal, but the question then requires a much more full discussion, which is reserved for a later chapter (see Ch. IV.).

When one of the joints of the loaded bar is a point of support, like A in Fig. 3, the supporting force is due partly to the half weight of one or more divisions of the loaded bar, and partly to the downward pull or thrust of other bars meeting there: the first of these causes does not affect the stress on the different parts of the truss, and the calculations are therefore made without any regard to it. The explanations given in this article should be carefully considered, as they apply to many of the examples subsequently given.

The triangular truss in both the forms given in this article is frequently employed in roofs and bridges of small spans, as well as for other purposes.

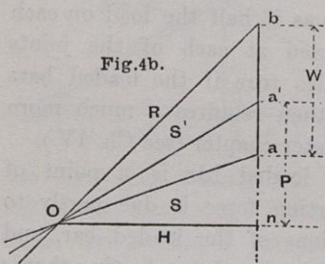
4. *Cranes.*—The arrangements adopted for raising and moving weights furnish many interesting examples of triangular frames. Fig. 4a shows one of the forms of the common crane, a machine the essential members of which are the jib, BC, supported by a stay, CE, attached to the crane-post, BE, which is vertical. In cranes proper this third member rotates, carrying BC and CE with it, but in the sailors' derrick a fixed mast plays the part of a crane-post,

Fig.4a.



and the stay, CE, is a lashing of rope frequently capable of being lengthened and shortened by suitable tackle, so as to raise and lower the jib, a motion very common in cranes and hence called a derrick motion. The weight is generally also capable of being raised and lowered directly by blocks and tackle, but for the present will be supposed directly suspended from C.

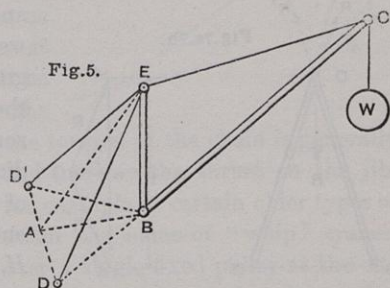
The diagram of forces now assumes the form shown in Fig. 4b, in which the lettering is the same as in Fig. 2b, page 4, the only difference in the diagrams being that in the present case AC, which is now a tie, is divided into two parts, AE and EC, inclined at an angle. The stress on AE is therefore not the same as on EC, but is got by drawing a third line, Oa', parallel to AE. The perpendicular On gives us in this instance not only the stress on AB and the horizontal thrust of CB at B, but also the horizontal pull of CE at E—we may call this H as before. There is an upsetting moment on the structure as a whole which is equal to the product of the weight W by its horizontal distance from B (often called the radius of the crane) and also to the force H, multiplied by the length of the crane-post, BE. One principal difference between different types of cranes lies in the way in which this upsetting moment is provided against.



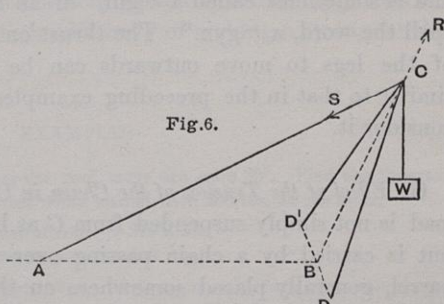
(α.) In portable cranes, such as shown in Fig. 4a, there is a horizontal platform, AB, supported by a stay, AE, and carrying a counterbalance weight, P, sometimes capable of being moved in and out so as to provide for different loads. The right magnitude of counterbalance weight and the pull on the stay AE are shown by the diagram P corresponding to the supporting force at A in the previous case.

(β.) In the pit crane, the post is prolonged below into a well and the lower end revolves in a footstep, the upper bearing being immediately below B. In this instance the post has to be made strong enough to resist a bending action at B, equal to the upsetting moment, and the bearings have to resist a horizontal force equal to H multiplied by the ratio of the length of the crane-post, BE, to that of its prolongation below the ground.

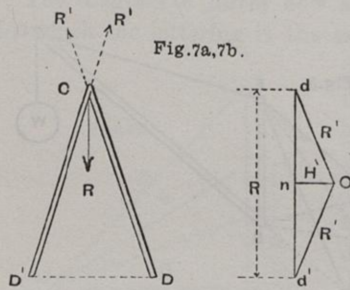
(γ.) The upper end of the crane-post may revolve in a headpiece, which is supported by a pair of stays anchored to fixed points in the ground. The upright mast of a derrick frequently requiring support in the same way, this arrangement is known as a derrick-crane. It is shown in Fig. 5, ED, ED' being the stays. To find the stress on the stays it is necessary to prolong the vertical plane through EC, to intersect the line DD', joining the feet of the stays in the point A, and imagine the two stays, ED, ED', replaced by a single stay EA: then a diagram of forces, drawn as in the previous case, determines S', the pull on this stay. But it is clear that S' must be the resultant pull on the two original stays, and may be considered as a force applied at E in the direction of AE to the simple triangular frame DED'. A second diagram of forces therefore will determine the pull on each stay, just as in the next following case.



5. *Sheer Legs and Tripods.*—Instead of employing an upright post to give the necessary lateral stability to the triangle, one of its members may be separated into two. Thus in moving very heavy weights sheer legs are used, the name being said to be derived from their resemblance to a gigantic pair of scissors (shears) partly opened and standing on their points. In Fig. 6, CD, CD' are spars, or tubular struts, often of great length, resting on the ground at DD' and united at C, so as to be capable of turning together about DD' as an axis. The load is carried at C and the legs are supported by a stay, CA,



which is sometimes replaced by a rope and tackle, capable of being lengthened or shortened so as to raise or lower the sheers. Drawing AB to the middle point of DD', the pair of legs are to be imagined replaced by a single one, CB, then the diagram of forces may be constructed just as in Fig. 4b, and we shall obtain the tension of the rope S and the resultant thrust on the pair of legs R. Now draw the triangle CDD', as in Fig. 7a, and imagine it loaded at C with a weight, R, then drawing the diagram of forces, Fig. 7b, we get R' the thrust on each leg. The horizontal force, H', in this second diagram represents the tendency of the feet of the legs to spread outwards laterally, while the force, H, of the original diagram represents their tendency to move inwards perpendicular to DD'. In some cases the guy rope and tackle CA are replaced by a third leg called the back leg, and the sheers are then raised and lowered by moving A by a large screw; the force H is then also the force to be overcome in turning the screw.

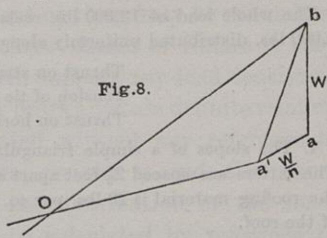


Instead of having only two legs, as in sheers, we may have three forming a tripod. This arrangement is frequently used to obtain a fixed point of attachment for the tackle required to raise a weight, and is sometimes called a "gin," or as military engineers prefer to spell the word, a "gyn." The thrust on each leg and the tendency of the legs to move outwards can be obtained by a process so similar to that in the preceding examples that we need not further consider it.

6. *Effect of the Tension of the Chain in Cranes.*—In most cases the load is not simply suspended from C as has been hitherto supposed, but is carried by a chain passing over pullies and led to a chain barrel, generally placed somewhere on the crane-post. The tension of the chain in this case is W/n, where n is a number depending on the nature of the tackle, and this tension is to be considered as an additional force applied at C to be compounded with the load W, the effect of which has been previously considered. Fig. 8 shows the

form the diagram of forces assumes in this case. Drawing ba as before to represent W, and aa' parallel to the direction in which the chain is led off from the pulley at C and equal to the tension W/n, the third side of the triangle, ba' must be the resultant force at C due to both forces, whence drawing $a'O$ parallel to the stay and bO parallel to the jib, and reasoning as before as to the equilibrium of the forces at C, we see that these lines must be the tension of the stay and the thrust on the jib. The effect of the tension of the chain is generally to diminish the pull on the stay and increase the thrust on the jib, sometimes very considerably, as for example in certain older types of crane still used for light loads under the name of "whip" cranes. In these cranes the chain passes over a single fixed pulley at the end of the jib, and is attached directly to the weight, so that the tension of the chain is equal to the weight. The other end of the chain is led off along a horizontal stay to a wheel and axle at the top of the crane post, a chain from the wheel of which passes to a windlass below. This arrangement, the double windlass of which facilitates changes in the lifting power corresponding to the load to be raised, is a development of the primitive machine in which the wheel was a tread wheel worked by men or animal power. In this case the pull on the stay is diminished by the whole weight lifted, and is thus reduced very much. Where a crane has to be constructed of timber only, this is a considerable advantage, from the difficulty of making a strong tension joint in this material.

Fig. 8.



EXAMPLES

1. The slopes of a simple triangular roof truss are each 30°. Find the thrust of the roof and the stress on each rafter when loaded with 250 lbs. at the apex.

Thrust of roof = 216·5 lbs.
Stress on rafters = 250 ,,

2. A beam 15 feet long is trussed with iron tension rods, forming a simple triangular truss 2 feet deep. Find the stress on each part of the frame when loaded with 2 tons in the middle.

Thrust on strut = 2 tons.
Pull of tension rods = 3·88 ,,
Thrust on beam = 3·75 ,,

3. The platform of a foot bridge is 20 feet span, and 6 feet broad, and carries a load of 100 lbs. per sq. ft. of platform. It is supported by a pair of triangular trusses each 3 feet deep, one on each side of the bridge. Find the stress on each part of one of the trusses.

The whole load of 12,000 lbs. rests equally on the two trusses, there is therefore 6,000 lbs. distributed uniformly along the horizontal beam of each truss.

Thrust on strut = 3,000 lbs.
Tension of tie rods = 5,220 ,,
Thrust on horizontal beam = 5,000 ,,

4. The slopes of a simple triangular roof truss are 30° and 45° and span 10 feet. The rafters are spaced $2\frac{1}{2}$ feet apart along the length of the wall, and the weight of the roofing material is 20 lbs. per sq. ft. Find by graphical construction the thrust of the roof.

Each rafter carries a strip of roof $2\frac{1}{2}$ feet wide, the load on rafter = 50 lbs. per foot length of rafter. Find the lengths by construction or otherwise. The virtual load at apex = $\frac{1}{2}$ weight on the two rafters = 311 lbs.

Thrust of roof = 198 lbs.

5. The jib AC of a ten-ton crane is inclined at 45° to the vertical, and the tension rod BC at an angle of 60°. Find the thrust of the jib, and the pull of the tie rod when fully loaded, the tension of the chain being neglected. If a back stay BD be added inclined at 45°, and attached to the end of a horizontal strut AD, find the counterbalance weight required at D to balance the load on the crane, and find also the tension of the back stay.

Thrust on jib AC = 33·5 tons.
Tension of tie rod = 27·5 ,,
Counterbalance weight = 23·5 ,,
Tension of back stay = 33·5 ,,

6. A pair of sheer legs are 40 feet high when standing upright, the lower extremities rest on the ground 20 feet apart, the legs stand 12 feet out of the perpendicular, and are supported by a guy rope attached to a point 60 feet distant from the middle point of the feet. Find the thrust on each leg, and the tension of the guy rope under a load of 30 tons.

Thrust on each leg = 19·5 tons.
Tension of guy rope = 12·8 ,,

7. In example 5 the tension of the chain is half the load, and the chain barrel is so placed that the chain bisects the crane post AB. Find the stress on the jib and tie rod.

Thrust of jib = 36 tons.
Pull of tie rod = 25 ,,

8. In a derrick crane the projections of the stays on the ground form a right-angled triangle, each of the equal sides of which is equal to the crane post. The jib is inclined at 45° and the stay at 60° to the vertical. Find the stress on all the parts (1) when the plane of the jib bisects the angle between the stays; (2) when it is moved through 90° from its first position. Load 3 tons.

9. A load of 7 tons is suspended from a tripod, the legs of which are of equal length and inclined at 60° to the horizontal. Find the thrust on each leg. If a horizontal force of 5 tons be applied at the summit of the tripod in such a way as to produce the greatest possible thrust on one leg, find that thrust and determine the stress on the other two legs.

Section II.—Incomplete Frames.

7. *Preliminary Remarks.*—A frame may have just enough bars and no more to enable it to preserve its shape under all circumstances, or the number of bars may be insufficient or there may be redundant bars. The distinction between these three classes of frames is very important: in the first the structure will support any load consistent with strength, and the stress on each bar bears a certain definite relation to the load, so that it can be calculated without any reference to the material or mode of construction; in the second, the frame assumes different forms according to the distribution of the load, but the stress on each bar can still be calculated by reference to statical considerations alone; in the third, where the frame has redundant bars, the stress on some or all of the bars depends on the relative yielding of the several bars of the frame. It is to the second class, which may be called incomplete frames, that the present section will be devoted.

In incomplete frames the structure changes its form for every distribution of the load, and, strictly speaking, therefore such constructions cannot be employed in practice, because the distribution of the load is always variable to a greater or less extent. But when the greater part of the load is distributed in some definite way the principal part of the structure may consist of an incomplete frame, designed for the particular distribution in question, and subsequent moderate variations of distribution may be provided for either by stiffening the joints or by subsidiary bracing. Such cases are common in practice, and investigations relating to incomplete frames are therefore of much importance.

8. *Simple Trapezoidal or Queen Truss.*—We will first consider a frame which is composed of four bars. The most common case

Fig.9a.

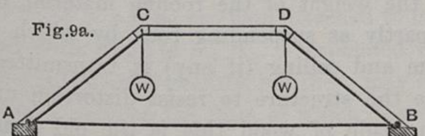
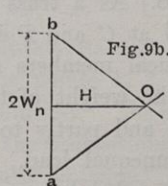


Fig.9b.



is that in which two of the bars are horizontal and the other two equal to one another, thus forming a trapezoid. The structure is called a *trapezoidal frame*.

It is suitable for carrying weights applied at the joints CD, either directly or by transmission through vertical suspending rods from the beam AB. From the symmetry of the figure it is evidently necessary for stability that the loads at C and D should be equal. This fact will also appear from the investigation. Consider first the joint C, and draw the triangle of forces, Oan, for that point; an being taken to represent W, aO will represent the thrust on AC and On that along CD. The triangle Obn will represent the forces at the joint D, Ob representing the thrust of BD; bn will represent the load at D, and from the symmetry of the figure must equal an, and hence weight at D must for equilibrium equal that at C. Now let us proceed to joint A, where there are also three forces acting, one along AC is now known and represented by aO, thus On will represent the tension of AB, and nb will be the necessary supporting force at A equal to W, as might be expected. The tension of AB is equal to the thrust on CD. We observe that the diagram of forces is the same as that of a triangular frame, carrying $2W$ at the vertex and of span equal to the difference between AB and CD.

Trapezoidal frames are employed in practice for various purposes.

(α.) A beam, AB (Fig. 10a), loaded throughout its length may be strengthened by suspending pieces, CN, OM, transmitting a part

Fig. 10a.

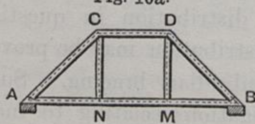
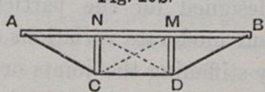


Fig. 10b.

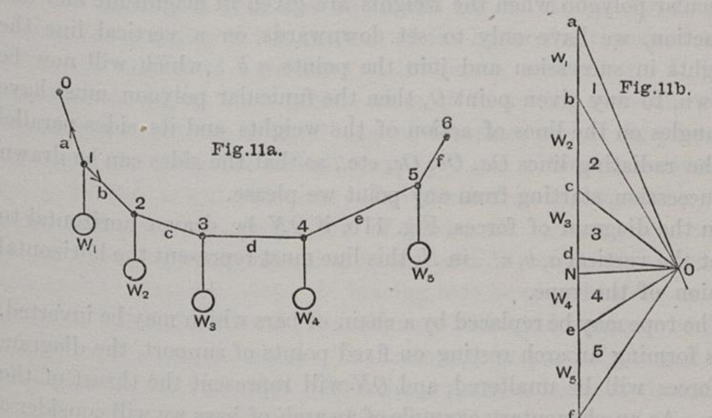


of the weight to the arch of bars AC, CD, BD, an arrangement common in small bridges.

(β.) As a truss for roofs, in which case there will be a direct load at C and D due to the weight of the roofing material, while vertical members serve partly as suspending rods by which part of the weight of tie beam and ceiling (if any) is transmitted to CD, and partly to enable the structure to resist distortion under an unequal load. When made of wood this is the old form of roof called by carpenters a "Queen Truss," CN, DM, being the "queen posts" (see Section III. of this chapter). This name is constantly used for all forms of trapezoidal truss erect or inverted which include the vertical "queens."

(γ.) Not less common is the inverted form, Fig. 10b, applied to the beams carrying a traversing crane, the cross girders which rest on the main girders of a railway bridge and carry the roadway, and many other purposes. The bars AC, CD, BD are now iron tie rods. In this case also if the two halves of the beam are unequally loaded there will be a tendency to distortion, to resist which completely, diagonal braces, CM, DN, must be provided, as shown in the figure by dotted lines. Such diagonal bars occur continually in framework, and their function will be fully considered in the next chapter. But in the present case they are quite as often omitted, the heavy half of the beam then bends downwards and the light half bends upwards (see Ex. 4, p. 97), but the resistance of the beam to bending is found to give sufficient stiffness.

9. *General case of a Funicular Polygon under a Vertical Load. Example of Mansard Roof.*—We next take a general case. In Fig. 11a



0 1 2 3 ... 6 is a rope or chain attached to fixed points at its ends and loaded with weights W_1 W_2... suspended from the points 1, 2, etc. The figure shows 5 weights, but there may be any number. The rope hangs in a polygon the form of which depends on the proportions between the weights. It is often called a "funicular polygon" and possesses very important properties. We shall find it convenient to distinguish the sides of this polygon by letters a, b, c, etc. We are about to determine the proportions between the weights when the

rope hangs in a given form, and, conversely, the form of the rope when the weights are given. In Fig. 11b draw ab vertical to represent W_1, the load suspended at the angle of the polygon where the sides a and b meet, then draw aO, bO parallel to a, b respectively to meet in O, thus forming a triangle Oab, which we distinguish by the number 1, which represents the forces acting on the point 1, so that the tensions of the sides a, b are thus determined. Now draw Oc parallel to the side c to meet the vertical in c; we thus obtain a triangle distinguished by the number 2, which represents the forces acting at that point, and as Ob is already known to be the tension of b it follows that bc must be the weight W_2, and Oc the tension of the side c. Proceeding in this way we get as many triangles as there are weights, and the sides of these triangles must represent the weights and the tensions of the parts of the rope to which they are respectively parallel. Thus, if the form of the rope is known and one of the weights, all the rest can be determined. Conversely, to find the form of the funicular polygon when the weights are given in magnitude and line of action, we have only to set downwards on a vertical line the weights in succession and join the points a b...,which will now be known, to any given point O, then the funicular polygon must have its angles on the lines of action of the weights and its sides parallel to the radiating lines Oa, Ob, Oc, etc., so that the sides can be drawn in succession, starting from any point we please.

In the diagram of forces, Fig. 11b, if ON be drawn horizontal to meet the vertical $a, b, c...$ in N, this line must represent the horizontal tension of the rope.

The rope may be replaced by a chain of bars which may be inverted, thus forming an arch resting on fixed points of support, the diagram of forces will be unaltered, and ON will represent the thrust of the arch. As an elementary example of an arch of bars we will consider a truss used for supporting a roof of double slope called a Mansard roof. We will take the usual case in which the truss is symmetrical about the centre. Suppose it is loaded at the joints. There is one proportion of load which the truss is able to carry without any bracing bars being added.

From symmetry the weights at 2 and 2′ (See Fig. 12a) must be equal. To find the proportion between the weights at 1, and at 2, 2′, together with the stresses on the bars of the frame, in Fig. 12b set down aa' to represent W at 1, and draw aO and $a'O$ parallel to a and

a', the thrusts along these bars will be determined. Then, considering the equilibrium of either 2 or 2′, say 2, one of the three forces acting at the joint, namely aO, along the bar a being known, the

Fig.12a.

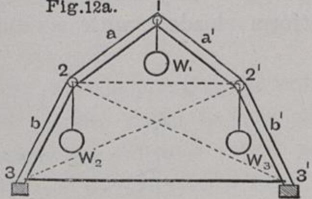
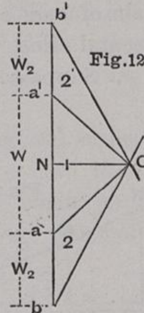


Fig.12b.



other two forces may be determined by drawing ab and Ob parallel to them, ba parallel to W_2, and Ob to the bar b. If ON be drawn horizontally it will give the amount of the horizontal thrust of the roof or the tension of a tie bar 3 3′, if there is such a bar. If the proportion of W_2 to W_1 is greater than ab to aa^1 the structure will give way by collapsing, 2 and 2′ coming together; and if the proportion is less, the structure will give way by 2 and 2′ moving outwards and 1 falling down between. In practice it is impossible to secure the necessary proportion of loads, on account of variation of wind pressure and other forces, and therefore stiffening of some kind is always needed. If bracing bars be placed as shown by the dotted lines 2 3′, 2′3, 2 2′, the structure will stand whatever be the proportion between the loads. The truss may be partially braced by the horizontal bar 2 2′ only. Then the proportion between the loads W_1 and W_2 may be anything we please, but the loads at 2 and 2′ must be equal, at least theoretically, but in practice the stiffness of the joints will generally be sufficient for stability, especially if vertical pieces be added connecting these points to the tie beam as in a queen truss.

10. *Suspension Chains. Arches. Bowstring Girders.*—We now go on to consider another important example, in which the number of bars composing the frame is very much increased, as found in the common suspension bridge.

Let AB (Fig. 13a) be the platform of a bridge of some considerable span, which has little strength to resist bending. Suppose it divided into a number of equal parts, an odd number for convenience, say nine, and each point suspended by a vertical rod from a chain of bars secured at the ends to fixed points, D and E, in a horizontal line. In the figure only half the structure is shown. Suppose the platform loaded with a uniformly

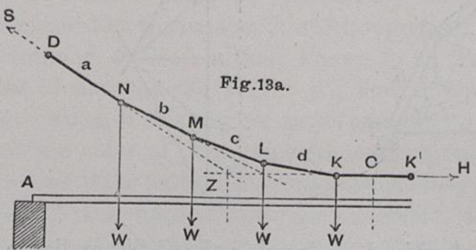


Fig.13a.

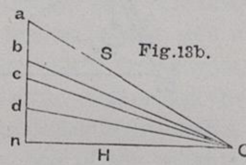


Fig.13b.

distributed weight; we require to know the stress on each bar and the form in which the chain will hang. Equal weights on each division of the platform will produce equal tensions in the vertical suspending rods, and if we neglect the differences of weight of the rods and bars themselves, the load at each joint of the chain of bars will be the same. (Compare Art. 11.) Let W = load at each joint. Now the centre link, KK', since there is an odd number and the chain is symmetrical, will be horizontal. Let us consider the equilibrium of the half chain between C and D. The four weights, W, hanging at K, L, M, N, are sustained in equilibrium by the tensions of the bars KK' and ND.

The resultant of the four W's will act at the middle of the third division from the left end, and since this resultant load together with the tensions of the middle and extreme links maintain the half chain in equilibrium, the three forces must meet in a point, the point Z shown in the figure. Thus the direction of the extreme link DN may be drawn. The direction and position of the other links may be found also. Considering the portion of the chain NC carrying three weights, the resultant of which is in the line through L, the link NM must be in such a direction as to pass through the point where this resultant cuts KK' produced. Having drawn NM, ML may be drawn in a similar way, and then LK. Returning to the consideration of the half chain, the three forces which keep it in equilibrium may be represented by the three sides of a triangle. Set down an (Fig. 13b) to represent $4W$, and draw aO and nO parallel to DZ and ZC; aO will be the tension of DN and nO of KK'. If an be divided into 4 equal parts, and the points b, c, d, joined to O, these lines will represent the tensions of links NM, ML, and LK. It may be easily shown that they will be parallel to those links. We see that the tension increases as we pass from link to link, from the centre to the ends.

In many cases in practice, the number of vertical suspending rods and links in the chain is very great. We may then, in what follows, without sensible error, regard the chain as forming a continuous curve. In such a case, C, the lowest point of the chain

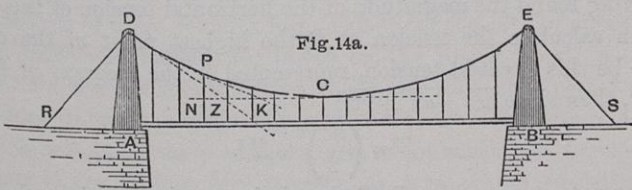


Fig. 14a.

(Fig. 14a), is over the middle of the platform. The tangent at C, which is horizontal, will meet the tangent to the chain at D, in a point Z, which will be over the middle of the half platform, for that will be a point in the line of action of the resultant load on the half chain. We can now draw a triangle of forces anO, for the half chain as before; On will represent the tension of the chain at the lowest point, or the horizontal component of the tension of the chain at any point. We can easily obtain a convenient expression for this horizontal tension thus:—Let l = span of the bridge, and w = load per foot run. Then $\frac{1}{2}wl$ = weight on the half chain represented by an. Let H = horizontal tension, then

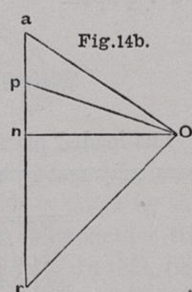


Fig. 14b.

$$\frac{H}{\frac{1}{2}wl} = \frac{On}{an}.$$

But if we drop a perpendicular from D to cut the horizontal tangent in a point V (not shown in the figure), DV will be the dip of the chain d, and comparing the triangles DVZ, aOn,

$$\frac{On}{an} = \frac{VZ}{DV} = \frac{\frac{1}{4}l}{d} = \frac{\frac{1}{2}wl}{H}$$

$$\therefore H = \frac{1}{8} wl \frac{l}{d},$$

which, since wl = total load on chain, may be written

$$H = \frac{1}{8} \text{load on chain} \frac{\text{span}}{\text{dip}},$$

This is the same as the horizontal thrust of a triangular frame of the same height which carries a uniformly distributed load of the same intensity.

Having found the magnitude of the horizontal tension of the chain we can calculate the tension at D, the highest point of the chain. Let S be this greatest tension, represented in the diagram of forces by aO, then since $\overline{aO} = \overline{an}^2 + \overline{nO}^2$

$$S^2 = \left(\frac{W}{2}\right)^2 + H^2.$$

The tension at any point P of the chain may be found by drawing from O a line op parallel to the tangent to the chain at P. It will cut an in a point p such that $np : na$ :: length of platform below $PC : \frac{1}{2}$ span.

$$\text{Since } \overline{Op}^2 = \overline{np}^2 + \overline{On}^2$$

$$\text{Tension at } P = \sqrt{\left(\frac{np}{na}\frac{W}{2}\right)^2 + H^2}.$$

The loaded platform, instead of being suspended from the chain of bars, may rest by means of struts on an arch of bars as in the figure.

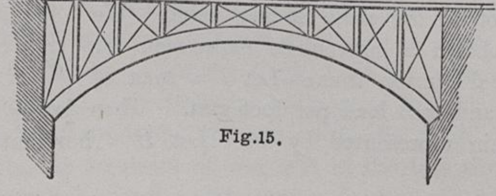


Fig.15.

In this case all the bars will be in compression instead of tension, as in the previous case. If the form of the arch is similar to that in which the chain hung, it will have no tendency to change its form under the load. There will be simple thrust of varying amount at different parts of the arch. The horizontal thrust at the top of the arch is given by the same expression as for the horizontal tension of the chain, and the thrust of any bar of the arch may be determined in a manner similar to that for finding the tension of any link of a chain. We shall show presently that the proper form of the arch and chain under a uniform load is a parabola. Hence, the structure just described is called a Parabolic Arch. In iron bridges the platform is not unfrequently carried by a number of ribs placed side by side. Each rib is approximately parabolic in form, usually of I. section, of depth from $\frac{1}{40}$th to $\frac{1}{60}$th the span at the crown, increasing somewhat towards the abutments. The roadway is supported sometimes by simple vertical struts, as in the ideal case just considered, sometimes by spandrils of more complex form, chiefly for the sake of appearance. When uniformly loaded, the stress on the ribs is nearly as found above: for resistance to variation in the load reliance is placed on the resistance to bending of the ribs and platform. The case of a stone or brick arch is far more complex, and is not considered here.

There is yet another very common structure whose construction is founded on the same principles as those just described. In this the

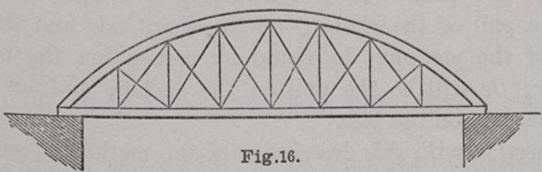


Fig.16.

platform, instead of resting on an arch below it, is suspended from an arch above it. In this case the thrust of the arch is taken by the platform, which serves as a tie, just as the string ties together the ends of a bow. Hence it is called a *Bowstring Girder*. In this, like the others, the loading proper to the parabolic form is a uniformly distributed one, and any variation of the loading will tend to distort the bow. The structure may, however, be enabled to sustain a varying load by the addition of bracing bars as shown by the diagonal lines. When the bridge is heavily loaded it will almost always happen that the greater part of the weight is uniformly distributed, and is sustained by simple thrust of the arch, so that the bracing is only a subsidiary part of the structure.

11. *Suspension Chains (continued). Bowstring Suspension Girder.*—In describing the suspension bridge we spoke of the chain as being secured at the ends to fixed points. In practice the securing of the ends is effected thus. The chain is led to the top of a pier of cast-iron or masonry, and instead of being simply attached to the top of the pier, and thus producing an enormous tendency to overturn the pier, the chain is secured to a saddle which rests on rollers on the top of the pier, and on the other side the chain is prolonged to the ground, passes through a tunnel for some little distance, and is finally secured by means of anchors to a heavy block of masonry. By this arrangement the only force acting on the pier is a purely vertical one, and a comparatively slender pier will be sufficient to sustain it. It is not necessary that the tension of the chain should be the same on each side of the pier, or that it should be inclined at the same angle. What is necessary is that the horizontal component of the tension on each side should be the same If an (Fig. 14b, page 19) = half weight on chain as before, and $On = H$, the horizontal tension (which may either be calculated from the formula just obtained, or found by construction), then aO will be the pull of the chain S at the top of the pier. Then considering the equilibrium of the saddle, the pull of the chain Q on the short side and the upward reaction of the pier may be found by completing the triangle of forces aOr; Or will be the pull on the anchor, and ar the total vertical pressure on the pier.

In connection with this description of the method of securing the ends of the suspension chain, we may mention a form of structure in which the arch and chain are combined, a good example of which occurs in the railway bridge at Saltash. The horizontal pull of the chain is here balanced by the thrust of an arch, so that the combined effect is to produce simply a vertical pressure on the piers. The suspending rods are secured to the chains and prolonged to the arch above, so that a portion of the load is carried by the arch, producing a thrust, and a portion by the chain, causing a pull. To prevent any tendency to overturn the piers, (this is insured by means of saddles resting on rollers) the horizontal component of the thrust of the arch must equal the horizontal component of the pull of the chain. The proportion between the loads on arch and chain will depend on the proportion between the rise of the arch and dip of the chain.

If W_1 = load on arch, and W_2 = load on chain,
d_1 = rise of arch, and d_2 = dip of chain,

then
$$H=\frac{W_1 l}{8d_1}=\frac{W_2 l}{8d_2};\ \therefore \frac{W_1}{W_2}=\frac{d_1}{d_2};$$

also
$$W_1+W_2=\text{total load on bridge}:$$

from which the stresses on the structure may be determined. It is known as a Bowstring Suspension Girder (pp. 47, 79).

We shall next show that the form of the curve of a chain carrying a uniformly loaded platform is a parabola. Referring to Fig. 14a, let P be any point in the chain, drop a perpendicular PN to meet the tangent at C, and bisect CN in K. Then KP must be the direction of the pull of the chain at P in order that the portion PC may be kept in equilibrium. The triangle PNK has its sides parallel to the three forces which act on PC, and the sides are therefore proportional to the forces. Let $CN=x$ so that the load hanging on $PC=Wx$, also let $PN=y$.

Then
$$\frac{H}{wx}=\frac{NK}{PN}=\frac{\frac{1}{2}x}{y}.$$

$$\therefore x^2=\frac{2H}{w}y;\ \text{or, since } H=\frac{wl^2}{8d},$$

$$x^2=\frac{l^2}{4d}y;$$

therefore x^2 is proportional to y.

Now the curve whose co-ordinates have this relation one to another is called a *parabola*.

If the load, instead of being uniformly distributed on a horizontal platform, were simply due to the weight of the chain itself, then the curve in which the chain would hang would deviate somewhat from the parabola; for in that case, since the slope increases as we approach the piers, the load also, per horizontal foot, would increase as we approach the piers, causing the chain near the piers to sink and become more rounded, and at the centre to rise and become more flattened. The curve in which the chain hangs by its own weight is called the *catenary*. In the catenary, as in the parabola, the tension increases as we approach the piers. This may be taken account of by proportioning the section of the chain to the tension at the various points; this would tend still more to make the weight of chain, per horizontal foot, increase as we approach the piers, and cause the chain to deviate still further from the parabolic form. Such a curve is called the catenary of uniform strength.

In an actual suspension bridge, where there is a uniformly loaded platform, as well as a heavy chain, the true curve in which it hangs will lie somewhere between the parabola and the catenary; but since in most cases the deviation from uniformity of the weight of chain is small compared with the load it carries, the deviation from the parabola is not great. The error involved in assuming the curve to be parabolic is generally greatest in bridges of large span; in such cases a preliminary calculation of approximate weights may be necessary so as to be able to apply the general process of article 9.

EXAMPLES.

1. A trapezoidal truss is 16 feet span and 4 feet deep, the length of the upper bar is 6 feet. Find the stress on each part when loaded with 2 tons at each joint.

Stress on sloping bars = 3·2 tons,
,, horizontal ,, = 2·5 ,,

2. The platform of a bridge, 8 feet broad and 27 feet span, is loaded with 150 pounds per square foot. It is supported on each side by an inverted queen truss placed below, the queen posts, each 3 feet deep, dividing the span into three equal portions. Find the stress on each part.

Load on each truss = half whole load on platform = 162,000.
$\frac{1}{3}$ 16,200 = 5,400 is the load at each of the two joints of one of the queen trusses.
Tension of sloping bars = 17,074 lbs.
Tension and thrust of horizontal bars = 16,200.

3. The height of a mansard roof without bracing is 10 feet and span 14 feet. The height of the triangular upper portion is 4 feet and span 8 feet. The load being 1 ton at the ridge, find the necessary load at each intermediate joint and the thrust of the roof.

By the construction described in the text, load at each intermediate joint = $\frac{1}{2}$ ton, and the thrust of the roof = $\frac{1}{2}$ ton.

4. If the roof in the last question be partly braced by a bar joining the intermediate joints, find the stress on the bar when the load at each intermediate joint is 1 ton.

Thrust on bar = $\frac{1}{4}$ ton.

5. The load on the platform of a suspension bridge, 600 feet span, is $\frac{1}{2}$ ton per foot run, inclusive of chains and suspending roads. The dip is $\frac{1}{13}$th the span. Find the greatest and least tensions of one of the chains.

Least tension = horizontal tension = $243\frac{3}{4}$ tons.
Greatest tension = 255 tons.

6. The load on a simple parabolic arch, 200 feet span and 20 feet rise, is 360 tons, determine the thrust and greatest stress on the arch.

Thrust = 450 tons; greatest stress = 484 tons.

7. The rise of a bowstring bridge is 15 feet and span 120 feet, find the thrust when loaded with 2,000 lbs. per foot run.

Thrust 240,000 lbs. = $107\frac{1}{7}$ tons.

8. In example 5 the ends of the chain are attached to saddles resting on rollers on the tops of piers 50 feet high, and prolonged to reach the ground at points 50 feet distant from the bottoms of the piers, where they are anchored. Find the load on the piers and the pull on the anchors.

Load on the pier = $637\frac{1}{2}$ tons;
Pull on each anchor = 344·6 tons.

9. A light suspension bridge is to be constructed to carry a path 8 feet broad over a channel 63 feet wide by means of 6 equidistant suspending rods, the dip to be 7 feet. Find the lengths of the successive links of the chain. Supposing a load of 1 cwt. per square foot of platform, find the sectional areas of the links of the chain, allowing a stress of 4 tons per square inch.

$\frac{6}{7}$ of the whole load is carried by the chains and the remaining portion by the piers directly. Tension of each suspending rod = 36 cwt.

Links.	Tensions.	Areas.	Lengths.
Centre	277·7	3·47	9·
2nd	280·	3·5	9·08
3rd	287·	3·6	9·3
4th	298·	3·72	9·66

10. Construct a parabolic arch, the thrust of which is half the total load.

Span = four times the rise.

11. If the weight of a uniformly loaded platform be suspended from a chain by vertical rods, show that the corners of the funicular polygon lie on a parabola.

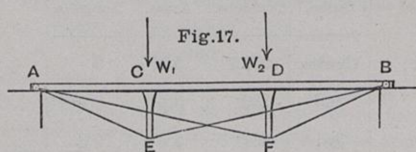
SECTION III.—COMPOUND FRAMES.

12. *Compound Triangular Frames for Bridge Trusses.* By a compound frame is meant a frame formed from two or more simple frames by uniting two or more bars. Many frames of common occurrence in practice may conveniently be considered as combinations of the simpler examples already described. They are generally dealt with by use of what we may call the principle of superposition, which may be thus stated:—*The stress on any bar due to any total load is the algebraical sum of the stresses due to the several parts of the load.*

We will now consider some examples of compound frames, which are used in bridge trusses. In these structures the object is to carry a distributed load by means of a comparatively slender beam. A prop in the centre may still leave the halves too weak to carry the weight on them, and the beam may be strengthened by supporting it in more than one point.

(1) Suppose the beam supported by a number of equidistant struts, the lower ends of which are carried by tension rods attached to the ends of the beam, we then have a structure called a *Bollman truss.* There may be any number of struts—2, 3, 4, or more; the structure has been used for bridges of comparatively large span. If the actual load is distributed in some manner over the beam, we must first reduce the case to that of a structure loaded at the joints only. The loads on the struts are due to the weights resting on the adjacent divisions of the beam, and may be determined by supposing the beam broken or jointed at the points where the struts are applied.

Fig.17.



Let us suppose the beam has three divisions, and that the load on the two struts are W_1 and W_2. These loads will be transmitted down the struts to the apices (Fig. 17) E and F, and will be independently supported, each by its own pair of tension rods. We may thus separately determine the stress on each part of either of the elementary triangular frames AEB or AFB. AB will be in compression on account both of the load at E and also at F. On account of W_1, using the formula previously obtained, the horizontal thrust $H_E = W_1 \dfrac{a'b'}{lh}$, and on account of W_2 at F, $H_F = W_2 \dfrac{a'b'}{lh}$.

$$\text{Tension of } AE,\ T_{AE} = H_E \sec EAB, \qquad T_{FB} = H_F \sec FBA;$$

$$\text{,,} \qquad EB,\ T_{EB} = H_E \sec EBC, \qquad T_{AF} = H_F \sec FAD.$$

The actual tensions of the sloping rods are simply as written, but since AB is a part of both triangular frames, the total thrust along it is found by summing the thrusts due to each; so

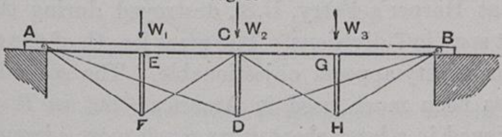
$$H = H_E + H_F.$$

This is an example of the *principle of superposition* stated above.

(2) Suppose the beam which carries the distributed load to be supported by a central strut forming a simple triangular truss, and further let the halves of the beam, not being strong enough to carry the load on them, each be subdivided and trussed by a simple triangular truss, the tension rods from the bottom of the subdividing struts proceeding only to the ends of each half beam. If the quarter spans are still too great, they may each of them be trussed in a similar way, and so on. Such a structure is called a *Finck truss.*

Suppose, for example, we have three struts. (Fig. 18.) We must first determine the load at the joints—that is, in this case the load on the struts due to the distributed load on the beam. Suppose that on account of the weights on the *adjacent* subdivisions those loads are W_1, W_2, W_3. If the load is uniformly distributed over the beam the W's are each of them equal to $\frac{1}{4}$ total weight on beam.

Fig. 18



We may now separately consider the triangular frame AFC carrying the load W_1. On account of it there will be a thrust on AC

$$H_F = W_1 \frac{AC}{4h} = W_1 \frac{l}{8h}.$$

The tensions of AF and FC are each $= H_F \sec FAE$. We get similar results from the triangle CHB. Just in the same way we may consider the principal triangular frame ADB, but in this case the thrust down the strut CD, which is the load at D, is not simply W_2, but greater by the amount of the downward pull of the two tension rods CF and CH. The vertical components of these tensions are $\frac{1}{2}W_1$ and $\frac{1}{2}W_3$, so that the total thrust down the strut $= W_2 + \frac{1}{2}(W_1 + W_3)$. This is the load which must be taken to act at D in determining the stresses on the members of ADB.

Thus $H_D = (W_2 + \frac{1}{2}W_1 + \frac{1}{2}W_3)\,\frac{l}{4h}$, and the tensions of AD and DB are each $= H_D \sec DAB$.

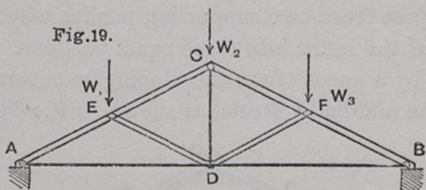
It will be seen that the thrust on the central strut and tensions of the longer rods are the same as if the secondary trusses had not been introduced. For example, if the W's each $= \frac{1}{4}$ whole load on beam, then the virtual load at $D = \frac{1}{2}$ weight on beam. The mere strengthening of each half the beam by trussing it can no more relieve the central strut of the load it has to carry, than the fact of strengthening a structure of any kind can relieve the two points of support from the duty each must have of bearing its own proper share of the weight. In stating the thrust on the beam we must divide it into two portions AC and CB. The portion AC is subjected to the thrust of the triangles AFC and ADB; $\therefore H_{AC} = H_F + H_D$, and CB being a portion of the triangles CHB and ADB, $H_{CB} = H_H + H_D$. When W_3 is not equal to W_1, the thrusts on the two portions will be different. This is quite possible although the beam AB may be a continuous one.

Both these simple forms of truss have been used for bridges of considerable span. As an example of the first may be mentioned the bridge at Harper's Ferry, U.S., destroyed during the war. It was 124 feet span in 7 divisions. The great length of the tension rods and their inequality appears objectionable. The second in 8 or 16 divisions has been much used in America; but in England other forms mentioned in a later chapter are much more common.

13. *Roof Trusses in Timber.*—In roofs of small span, 10 or 12 feet only, the roofing material, slates or tiles, rests on a number of laths set lengthways to the roof, and these laths rest on sloping rafters spaced 1 or 2 feet apart, with their feet resting on the walls of the building; the stability of the walls being depended on for taking the thrust.

When we come to larger and more important roofs we find additional members added for strength and security. The closely spaced rafters just mentioned are called common rafters. These being too long and slender to carry the weight of the roofing material and transmit it to the walls, are supported, not only at the ends by the walls and ridge piece, but also at the middle by a longitudinal beam of wood called a *purlin*, and the purlin is supported at intervals of its length by principal rafters. The principal rafters again are supported by struts at their central points, immediately below the purlins. To carry the lower ends of the struts, a vertical tension piece is introduced, by which they are suspended from the apex of the principals, while the thrust is taken by a tie beam connecting the feet of the rafters. In such a roof, a ceiling or floor may frequently be required to be supported by the tie beam, and to prevent it from sagging under the weight an additional tension will come on the vertical suspending rod. This rod is then a very important member of the structure, and is called the king post, and the whole structure, consisting of the principal rafters, king post, &c., is called a *king post truss.* This truss is often constructed entirely of wood. The sloping struts then for constructive reasons (Ch. xv.) butt on an enlarged part at the bottom of the king post above the point where the horizontal tie beam is attached, but for calculation purposes may be regarded as meeting at that point as shown in Fig. 19.

Fig.19.



By means of the purlins and the ridge piece the weight of the roofing material will produce loads at the joints $ECF = W_1 W_2 W_3$ suppose.

Now treat the structure as made up of three simple triangular frames AED, DFB, and ACB. First consider AED with the load W_1 at vertex E. The horizontal thrust of this frame $H_E = W_1 \dfrac{AD}{4h}$ where h is the height of point E above AD. Also the thrust along AE and ED due to the load at $E = H_E \sec EAD$. In an exactly similar manner we may consider the triangle DFB; the results for this will be to those for AED in the proportion of W_3 to W_1. Next as to the primary triangle ACB. There is at C a direct load of W_2 due to the weight between E and C, and F and C. But besides this, the king post pulls the point C downwards, so that the total load at $C = W_2 +$ tension of king post. In addition to a portion of the weight of the ceiling (if any) the post has to support D against the downward thrust of the two struts ED and FD. The vertical components of these thrusts are $\frac{1}{2}W_1$ and $\frac{1}{2}W_3$, therefore, neglecting the weight of ceiling, the virtual load at $C = W_2 + \frac{1}{2}(W_1 + W_3)$. Let us call the total load W, then H_C the horizontal thrust of $ACB = W \dfrac{AB}{4CD}$ and the thrusts along AC and CB due to load at $C = H_C \sec A$.

Now in the complete structure, since AD is a member both of the triangular frame AED and ACB, the total tension of $AD = H_E + H_C$. For the same reason tension of $DB = H_F + H_C$,

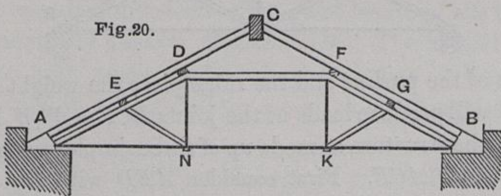
and thrust of $AE = (H_E + H_C) \sec A$,

" " $FB = (H_F + H_C) \sec A$.

The other members of the structure are portions of one elementary frame only, and the stress is due only to the load at the apex of that frame.

The king post truss serves for roofs of spans under 30 feet, but for spans greater than this trusses of more complicated construction are required. If the span is from 30 to 50 feet, then instead of supporting the common rafters by a purlin at the centre of its length only, as in the king post truss, two supporting purlins may be used, dividing the length of the rafter into three equal portions. These purlins may be carried by a queen truss, the sloping members of which are supported in the middle by struts, as shown in the figure (Fig. 20).

Fig. 20.



The vertical queen posts DN and FK serve to sustain the downward thrust of the struts EN and GK, and also to support the weight of a ceiling, if there is one. Supposing the weight of the ceiling omitted, let W be the weight of roofing material on one side for a length of roof equal to the spacing of the trusses, then $\frac{1}{3}W$ will, through the common rafters and purlins, act at E, and $\frac{1}{3}$ at D; and similarly for the other side. At the ridge C there will also be $\frac{1}{3}W$ acting; but this will be distributed equally amongst the common rafters which are carried by the truss, and will produce compression in those rafters without directly affecting the truss. The part of the thrust of the roof arising from this will, however, generally, like the rest, ultimately come on the principal tie beams.

To find the stresses on the different members of the truss. Consider first the small triangles AEN and BGK, each carrying $\frac{1}{3}W$ at the vertex. We then consider the trapezoidal truss $ADFB$. The loads at D and F will be $\frac{1}{3}W$ + tension of queen post. Since the tension of the queen post DN = the vertical component of the thrust along EN it will equal $\frac{1}{2} \cdot \frac{1}{3}W = \frac{1}{6}W$, and the total load at each joint of the trapezoidal truss will be $\frac{1}{3}W + \frac{1}{6}W = \frac{1}{2}W$, the same as would have acted if there had been no purlin at E and no strut EN. After having determined the respective stresses due to the triangles and trapezoid separately, we must add the results for any bar which is a part of both. Were it not for the friction at the joints and the power of resistance of the continuous rafters AC, CB to bending, this structure would be stable only under a symmetrical load. In practice, however, it is able to sustain an unsymmetrical load, such as roofs are frequently subjected to.

14. *Queen Truss for large Iron Roofs.*—As the span of the roof is still further increased we find other kinds of trusses employed to support them. A common form in iron roofs is constructed, as shown in Fig. 21. It is in reality a further development of

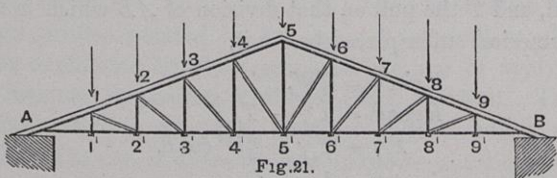


Fig. 21.

the wooden queen truss, and is known by the same name. AC and CB are divided into a number of equal parts, and sloping struts and vertical suspending rods are applied as shown. Suppose the load the same at each joint on one side of the roof, the load on the right, however, not being necessarily equal to that on the left. Let the upward supporting force at $A = P$. P will be $\frac{1}{2}$ total weight if the loading is symmetrical, but in any other case it may be found by taking moments of the loads about B. We might solve the problem of finding the stress on each member of the structure by treating separately each elementary triangle into which the structure may be divided, and summing the stresses for any bar which may form a part of two or more triangular frames. But we will describe another method.

First, to find the tension of the vertical suspending rods consider $A12'$ as an independent triangle, carrying a load W at its vertex. The slope of $12'$ being the same as that of $A1$, the tension rod $22'$

must supply a supporting force to the joint $2' = \frac{1}{2}W$. Considering next the triangle $A23'$ and its equilibrium about the point A. The forces along 23 and $3'4'$ have no moment about A, so that the moment of the two weights W at 1 and 2 about A must be balanced by the upward pull of the tension rod $33'$. $\therefore$ tension of $33' = W$.

In a similar way we can see that the tension of $44' = \frac{3}{2}W$. However many divisions of the roof there may be, the tensions of the vertical suspending rods will increase in arithmetical progression, with the same difference between each. The rod $11'$, except so far as may be due to the weight of the rod $A2'$, will have no tension on it. Calling this the 1st tension rod, the tension of the $n^{\text{th}} = \frac{n-1}{2}W$. We must notice that the rod $55'$ is common to both sides of the roof, and we must add the two tensions to get the total. Now consider any joint, say $4'$ in the tie bar AB, and resolve vertically and horizontally. If R = thrust of $34'$, θ its inclination to the horizontal, and T the pull on that division of AB which is indicated by the numerical suffix placed below it,

$$R \sin\theta = \tfrac{3}{2}W,$$
$$R\cos\theta = T_{3'4'} - T_{4'5'};$$
$$\therefore\ T_{3'4'} - T_{4'5'} = \tfrac{3}{2}W\cot\theta.$$

But from figure
$$\cot\theta = \tfrac{1}{3}\cot A;$$
$$\therefore\ T_{3'4'} - T_{4'5'} = \tfrac{1}{2}W\cot A.$$

Whichever joint we select we should find the same result—namely, that the difference between the tensions of two consecutive portions of the tie rod is a constant quantity $= \frac{1}{2}W\cot A$. So that these tensions are in arithmetical progression diminishing towards the centre.

If we call $A2'$ the 1st division of tie rod, then for the joint between the $n-1^{\text{th}}$ and n^{th} we have

$$R\sin\theta = \frac{n-1}{2}W.$$

$$R\cos\theta = T_{n-1} - T_n, \text{ and } \cot\theta = \frac{1}{n-1}\cot A;$$

$$\therefore\ T_{n-1} - T_n = \tfrac{1}{2}W\cot A.$$

If $A1$ is the 1st division of the rafter, then the thrust on the n^{th} division $= T_n \sec A$.

Now, the tension of the tie rod in the

$$\begin{aligned} 1^{\text{st}} \text{ division} &= P \cot A, \\ 2^{\text{nd}} \quad ,, \quad &= (P - \tfrac{1}{2}W) \cot A, \\ n^{\text{th}} \quad ,, \quad &= \left(P - \frac{n-1}{2}W\right) \cot A. \end{aligned}$$

The thrust on the n^{th} division of rafter $= \left(P - \dfrac{n-1}{2}W\right) \operatorname{cosec} A$.

The thrust on any strut may best be found by squaring and adding the two equations of equilibrium of the lower joint of it. We get

$$\text{Thrust of } n^{\text{th}} \text{ strut} = \frac{W}{2}\sqrt{n^2 + \cot^2 A}.$$

15. *Concluding Remarks—General Method of Constructing Diagrams of Forces.*—Cases of framework often occur which are much more complicated than those which we have hitherto considered, but if there are no redundant bars the stress on each part depends on statical principles only, without reference to the relative yielding of the several parts of the structure. Such cases may always be treated by use of the general principle stated in Art. 1, and we shall conclude this chapter by explaining briefly a graphical method of applying that principle invented by the late Professor Clerk Maxwell. The forces will be supposed all in one plane, and each of them will be supposed known, that is to say, if there be any unknown reactions at points of support they will be supposed previously found by a graphical or other process, from the consideration that the whole must form a set of forces in equilibrium. In Fig. 22a a frame is shown acted on by known forces PQR..., an ideal example is chosen which is better suited for the purpose of explaining the method than any case of common occurrence in practice. First seek out a joint where only two bars meet: there will usually be two such joints if there be no redundant bars in the frame, and in the present instance we will choose the joint where P acts. Distinguish all the triangles, making up the frame by letters A, B, C, &c., and place numbers or letters outside the frame, one for each bar. In Fig. 22b draw 18 parallel to the force P and representing it in magnitude, 8a parallel to 8, 1a parallel to 1, to intersect in the point a; then, as in previous examples, 8a, 1a represent the stress on the two bars to which they are parallel. Pass now to the joint where Q acts: this joint is chosen because only three bars meet there, on one of which we have just determined the stress; draw 12 parallel to Q and representing it, then ab parallel to the bar lying between the triangles A and B, and $2b$ parallel to the bar 2; we thus get a polygon $12ba$, the sides of which are parallel to the four forces acting at the joint where Q acts, while two of them represent two forces

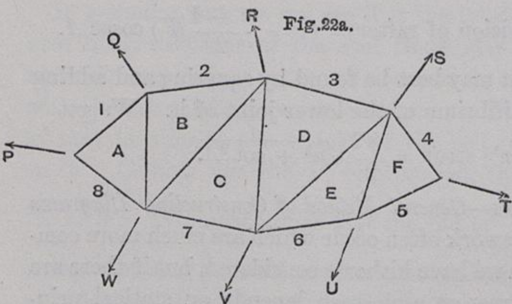


Fig. 22a.

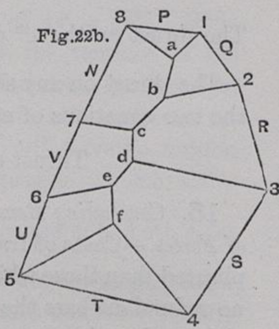


Fig. 22b.

already known, the other two, therefore, will represent the remaining two forces. Proceed now to the joint where W acts and complete in the same way the polygon $8abc7$, then to the joint where R acts, and so on. We at length arrive at the triangle $4f5$, the third side of which, if we have performed the construction accurately, and if the forces be really in equilibrium, must be parallel to the last force T. On examination of the diagram of forces (Fig. 22b) it will be seen that to every joint of the frame corresponds a polygon representing the forces at that joint, while each line, such as ab or $7c$, gives the stress on the bars separating those letters or numbers in the frame-diagram. The polygon 12...8 is the polygon of external forces, each side representing the force to which it is parallel.

The method here described is easy to understand in the general case we have considered, and with a little practice the transformations the diagram of forces undergoes will offer no difficulty. Some joints are usually unloaded, and the corresponding lines in the polygon of external forces vanish; the forces may be parallel, in which case the polygon becomes a straight line, while not unfrequently the sides of two of the polygons representing the forces at the joints coincide. The figure, however, always possesses the same properties.

In Mr. Bow's excellent work referred to at the end of this chapter over 200 examples will be found of the application of this method, including almost all known forms of bridge and roof trusses.

EXAMPLES.

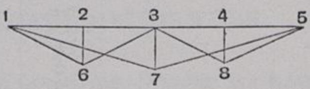
1. A Bollman truss of three divisions is 21 feet span, and is loaded uniformly with 1 ton per foot. The depth of the truss is $3\frac{1}{2}$ feet. Find the stress on each part.

Load on each strut = 7 tons,
Tension of short rods = 10·4 ,,
,, longer ,, = 9·6 ,,
Total thrust on beam = $18\frac{2}{3}$,,

being $9\frac{1}{3}$ due to each triangle.

2. A Finck truss of 4 divisions, 20 feet span and 3 feet deep, is loaded with 1 ton per foot, find the stress on each part.

Thrust on 26 and 48 = 5 tons.
,, 37 = 10 ,,
Tensions of 16, 63, 38, and 85 = 4·86 ,,
,, 17 and 75 = 17·4 ,,
Thrust on 13 and 35 = $4\frac{1}{6} + 16\frac{2}{3} = 20\frac{5}{6}$ tons.



3. In the last question suppose one half the truss loaded with an additional 1 ton per foot. Find the stress on each part.

Suppose the additional load on the right-hand side.

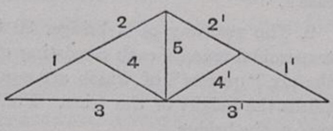
Thrusts.
On 26 = 5 tons.
,, 37 = 15 ,,
,, 48 = 10 ,,
,, 13 = $4\frac{1}{6} + 25 = 29\frac{1}{6}$.
,, 35 = $8\frac{1}{3} + 25 = 33\frac{1}{3}$.

Tensions.
On 16 and 63 = 4·86 tons.
,, 38 ,, 85 = 9·72 ,,
,, 17 ,, 75 = 26·1 ,,

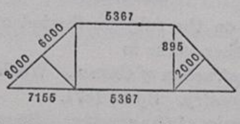
4. A roof 28 feet span, height 7 feet, rests on king-post trusses spaced 10 feet apart. The weight of roof is 20 lbs. per square foot. Find the stress on each part. Also obtain results when an additional load of 40 lbs. per square foot rests on one side.

Load at each joint. 1st case = 1566·6 lbs.

Bars.	Stress in lbs.		Bars.	Stress.	
	Equal Load.	Additional Load.		Equal Load.	Additional Load.
1	5254	8756	1′	5254	12261
2	3503	7006	2′	3503	7006
3	4700	7833	3′	4700	10966
4	1752	1752	4′	1752	5255
5	1566·6	3113			



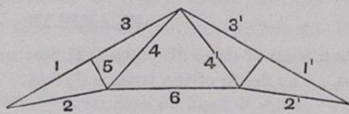
5. A roof 48 feet span, 12 feet high, rests on queen trusses 8 feet high, spaced 10 feet apart. Find the stresses for a load of 20 lbs. per square foot.



6. An A roof, braced as in the figure, is 40 feet span, and 10 feet high; the horizontal tie bar is 8 feet below the vertex. Find the stresses on each part

when loaded with 2 tons at each joint by constructing a diagram of forces or otherwise.

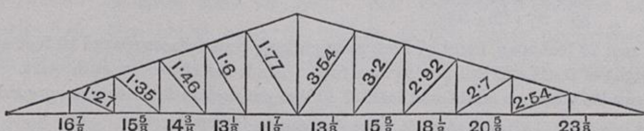
Bars.	Stress.
1	10·4
2	9·4
3	9·4
4	4·7
5	1·8
6	5·



7. In the last question suppose an accumulation of snow on one side equivalent to an additional load of 2 tons at the middle of the rafter, and 1 ton at the ridge. Find the stress on each part.

Bars.	Stress.	Bars.	Stress.
1	13·9	1′	17·3
2	12·5	2′	15·7
3	12·8	3′	15·4
4	5·5	4′	8·6
5	1·8	5′	3·6
6	7·5		

8. Suppose there are 11 suspending rods in iron roof shown in the figure, the height of which is $\frac{1}{10}$th the span. Find the stress on each part—1st, when loaded with $\frac{1}{2}$ ton at each joint on both sides, and, 2nd, when loaded with an additional $\frac{1}{2}$ ton at each joint on one side, not including the ridge.



Additional load is on right-hand side, and the figures on the diagram refer to case 2.

9. The roadway of a bridge, 80 feet span, is carried by a pair of compound trapezoidal trusses, each consisting of three simple trapezoids of the same height, the six "queens" of which are equidistant, forming six divisions of length four thirds the height of the truss. Find the stress on all the bars due to $\frac{1}{4}$ ton per foot run on the bridge.

10. Find the stress on each part of a "straight-link suspension" bridge formed by inverting the truss of the last question, assuming the pull at the centre of the platform zero.

REFERENCES.

For further information on the subjects treated of in the present chapter the reader may refer amongst other works to

GLYNN—*Construction of Cranes.* Weale's series.
HURST—*Carpentry.* Spon, 1871.
BOW—*Economics of Construction.* Spon, 1873.

CHAPTER II.

STRAINING ACTIONS ON A LOADED STRUCTURE.

16. *Preliminary Explanations.*—In the preceding chapter we have considered only those structures in which the parts are subject to compression and tension alone, except by way of anticipation in a few special cases. But the parts of a structure are generally subject to much more complex forces, and besides, although the forces acting on each bar have been determined, we should, if we stopped here, have a most imperfect idea of the way in which the load affects the structure as a whole.

If we imagine a structure to be made up of any two parts, A and B, united by joints, or distinguished by an ideal surface cutting through the structure in any direction, the whole of the forces acting on the structure may be separated into two sets, one of which acts on A, the other on B. Since the structure is in equilibrium as a whole, the two sets of forces must balance one another, and must therefore produce equal and opposite effects on A and B, effects which are counteracted by the union existing between the parts. The two sets of forces taken together constitute a STRAINING ACTION of which each set is an element, and the object of this and the next two chapters is to consider the straining actions to which loaded structures and parts of structures are subject.

Straining actions differ in kind, according to the nature of the effects which they tend to produce. Four simple cases may be distinguished :—

(1) The parts A and B may tend to move towards each other or away from each other perpendicular to a given plane. This effect is

called compression or extension, and the corresponding straining action is a thrust or a pull.

(2) A and B may tend to slide past each other parallel to a given plane. This effect is called shearing.

(3) A and B may tend to rotate relatively to each other about an axis lying in a given plane. This is called bending.

(4) A and B may tend to rotate relatively to each other about an axis perpendicular to a given plane. This is called twisting.

In the first two cases the straining action reduces to two equal and opposite forces, and in the second two to two equal and opposite couples. In general, straining actions are compound, consisting of two or more simple straining actions combined. The given plane with reference to which the straining actions are reckoned may always be considered as an ideal section separating A and B even when the actual dividing surface is different. We shall commence by considering the straining actions on a beam of small transverse section.

SECTION I.—BEAMS.

17. *Straining Actions on a Beam.*—The action of a simple thrust or pull on a bar has already been sufficiently considered in chapter I. They are usually considered as separate cases, and the simple straining actions on a bar are therefore reckoned as five in number. The other three are (1) shearing, (2) bending, and (3) twisting, of which the last rarely occurs, except in machines, and will, therefore, be considered in a later division of this work, under that head.

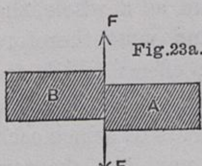


Fig.23a.

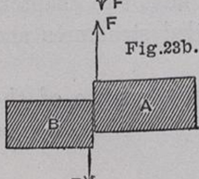


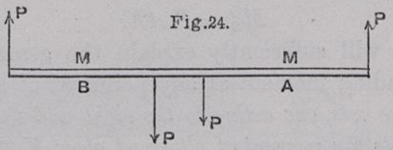
Fig.23b.

Shearing and bending are due to the action of forces, the directions of which are at right angles to the bar: in structures, the forces usually lie in one plane passing through the axis of the bar. A bar loaded in this way is called a beam.

Simple shearing is due to a pair of equal and opposite forces, F (Fig. 23), applied to points very near together, tending to cause the two parts A and B to slide past one another, as shown in the figure (Figs. 23a, 23b). Either element is called the shearing force, and is a measure of the magnitude of the shearing action, but in considering the sign we must consider both together. In this work, if the right-hand portion, A, tends to move upwards, and B downwards, as in Fig. 23b, the shearing action will usually be reckoned negative, while in the converse case (Fig. 23a) it will be reckoned positive.

Simple bending is due to a pair of equal and opposite couples applied to the bar, one acting on A, the other on B, as in Fig. 24,

Fig. 24.

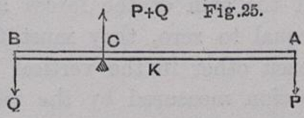


tending to make A and B rotate in opposite directions. The magnitude of the bending is measured by the moment of either couple which is called the bending moment. In this work bending moments will usually be reckoned positive when the left-hand half, B, rotates with the hands of a watch, and the right-hand half in the opposite direction. That is to say, when the beam tends to become convex downwards, as in the ordinary case of a loaded beam supported at the ends. In loaded beams shearing and bending generally exist together, and vary from point to point of the beam. We shall now consider various special cases.

18. *Example of a Balanced Lever. General Rules for calculating S.F. and B.M.*—First take the case of a beam, AB, supported at C (Fig. 25), and loaded with weights, PQ, at its ends.

If the weights are such that $P.AC = Q.BC$ the beam will be in equilibrium, but the two parts, AC, BC, tend to turn about C in opposite directions, there is therefore a bending action at C, of which the equal and opposite moments $P.AC$, $Q.BC$ are the elements. Either of these is the bending moment usually denoted by M, so that we write

Fig. 25.



$$M_C = P.AC = Q.BC.$$

Not only is there a bending action at C, but if we take any point, K, and consider the forces acting on AK, BK separately, we see that AK tends to turn about K under the action of the force P, while BK tends to turn about K under the action of the forces $P+Q$ at C and Q at B. The first tendency is immediately seen to be simply the moment $P.AK$, while the second is $Q.BK - (P+Q)CK$. The last quantity reduces to $Q.BC - P.CK$, or, remembering that $Q.BC = P.AC$ to $P.AK$ The two moments then, as before, are equal and opposite, and constitute a bending action at K, measured by the bending moment

$$M_K = P.AK.$$

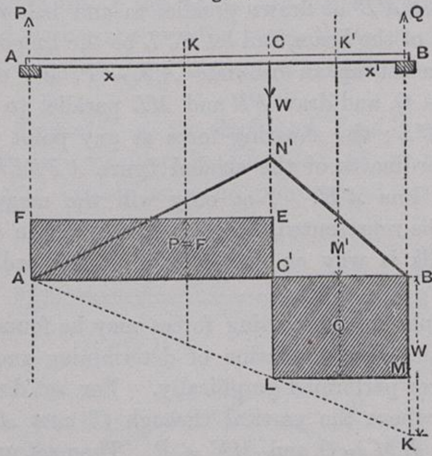
This example will sufficiently explain the general rule for calculating the bending moment at any point, K, of a beam. *Divide the forces into two sets, one acting to the right and the other to the left of K, and estimate the moment of either set about K, then the result will be the bending moment at K.* The example shows that the calculation of one of the two moments will generally be more simple than that of the other, and cases constantly occur, as where a beam is fixed at one end in a wall, where nothing is known about one set of forces except that they balance the other set. In each case the simplest calculation is of course to be preferred.

Moments are measured numerically by unit weight acting at unit leverage, as, for example, 1 ton acting at a leverage of 1 foot, for which the expression "foot-ton" is commonly employed. This phrase, however, is used also for a wholly different quantity, namely, the unit of mechanical work, and for this reason it would be preferable to call the unit of moment a ton-foot for the sake of distinction.

The peculiar action called shearing will be better understood when we come to consider the action of forces on a framework girder in the next section; it will here be sufficient to say that if the sum of the forces acting on AK, BK are not separately equal to zero, they must tend to cause AK, BK to move past each other in the vertical direction, thus constituting a shearing action measured by the magnitude of the shearing force, which may be thus calculated for any point K. *Divide the forces into two sets, one acting to the right of K and the other to the left of K, the algebraical sum of either set is the shearing force at K.* As before, either set may be chosen, whichever gives the result most simply. In the example just given the shearing force at any point of AC is P; and at any point of BC, Q.

19. *Beam Supported at the Ends and Loaded at an Intermediate Point.*—We will next consider the case of a beam supported at

Fig.26.



the ends and loaded at some intermediate point. Before we can apply the rules previously enunciated, to find the shearing force and bending moment at any point, we must first determine the supporting forces at the two ends. We find the force P acting at A, Fig. 26, by taking moments about B, thus,

$$P(a+b) = Wb; \therefore P = \frac{Wb}{a+b},$$

and similarly
$$Q = \frac{Wa}{a+b}.$$

First as to the shearing force. Taking any point K in AC, and considering the forces acting on AK, of which there is only one,

$$F_K = P = \frac{Wb}{a+b}.$$

At any point K' between C and B we have

$$F_{K'} = Q = \frac{Wa}{a+b}.$$

It will be noticed that at K the tendency is for the left-hand portion to slide upwards relatively to the right, whereas at K' the tendency is for the right-hand portion to slide upwards relatively to the left. It is advantageous to distinguish between these two tendencies, as previously stated, by calling the one positive and the other negative.

We may draw a diagram to represent the shearing force at any point thus. Let $A'B'$ be drawn parallel to and below AB to represent the length of the beam, and let $CC'L$ be the line of action of the weight. If we set up an ordinate $A'F = P$, and downwards an ordinate $B'M = Q$, and draw FE and ML parallel to $A'B'$ to meet the vertical $EC'L$; the shearing force at any point will be represented by the ordinates of the shaded figure $A'FELMB'$, measured from the base line $A'B'$. Not only will the magnitude of the shearing force be represented, but also the direction of the sliding tendency. This is why on one side of C' the ordinate was set downwards.

In this example the supporting forces may be found by construction, and thus the whole operation of determining and representing the shearing force performed graphically. For, set down $B'K = W$, join $A'K$, and where the vertical through C' cuts $A'K$, draw LM horizontal, then $B'M = Q$ and $MK = P$. Then set up $A'F = MK$, and draw FE horizontal.

Next as to the bending moment at any point. Take any point K in AC distant x from A, then

$$M_K Px = \;= \frac{Wb}{a+b}x,$$

and similarly at K' in CB distant x' from B,

$$M_{K'} = Qx = \frac{Wa}{a+b}x';$$

so for either side of C, the bending moment is greater the greater the distance of the point from the end of the beam. Thus the greatest bending moment is at C.

If in the value of M_K we put $x = a$,
or ,, $M_{K'}$,, $x' = b$,

we get the same result, viz., that

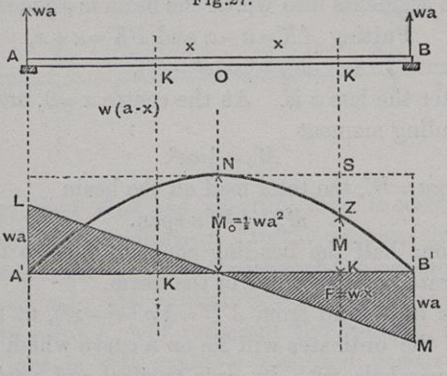
$$M_C = \frac{Wab}{a+b} = \text{greatest bending moment.}$$

The graphical representation of the bending moment at any point is very useful and instructive. We may construct the diagram thus:—$A'B'$ representing the length of the beam set up from C', $C'N$ the bending moment at $C' = \dfrac{Wab}{a+b}$ on some convenient scale, on such a scale for instance as 1 inch = 20 ft.-lbs. Then joining $A'N$ and $B'N$, the ordinate of the figure $A'NB'$, measured from the base line $A'B'$, will express on the scale chosen the bending moment at any point of the beam. If $a = b = \frac{1}{2}$ span, so that the load is applied at the centre of the beam, then

$$M_C = \tfrac{1}{4} W \times \text{span} = \text{greatest bending moment.}$$

20. *Beam Supported at the End and Loaded Uniformly.*—The next example for consideration is that of a beam supported at the ends and loaded uniformly throughout its length with w

Fig.27.



lbs. per foot. (Fig. 27.) Let the span $= 2a$. Take any point, K, distant x from the centre O. The load on AK is wAK, and therefore the shearing force at K, reckoning the forces on the left-hand side, must be

$$F_K = wa - wAK = wa - w(a - x) = wx.$$

That is, the shearing force is proportional to the distance of the point from the centre of the beam. At the end A where $x = a$,

$$F_A = wa,$$

and at B where $x = -a$,

$$F_B = -wa.$$

If from $A'B'$, below AB in the diagram, we set up and down ordinates at A' and $B' = wa$ on some scale, and join LM, the ordinates of the sloping line will represent the shearing force at any point. The shearing force at the centre of the beam is zero.

In finding the bending moment at K, reckoning still from the left-hand side, we must clearly take account not only of the supporting force at A, but also of the effect of the load which rests on the portion of the beam AK. The moment of this load about K is the same as if it were all collected at its centre of gravity, namely at the centre of AK. Thus

$$M_K = wa \,.\, AK - wAK \,.\, \frac{AK}{2}$$
$$= \frac{w}{2} AK\,(2a - AK) = \frac{w}{2} AK.KB.$$

That is to say, the bending moment at any point is proportional to the product of the segments into which the beam is divided by the point.

Putting $AK = a - x$ and $BK = a + x$,

$$M_K = \tfrac{1}{2}w(a^2 - x^2),$$

which is greater the less x is. At the centre $x = 0$, and we have the maximum bending moment

$$M_0 = \tfrac{1}{2}wa^2.$$

If we put $2wa = W$, the total load on the beam

$$M_0 = \tfrac{1}{8}W \times \text{span}.$$

This is only one half the bending moment due to the same load when concentrated at the centre of the beam.

If ordinates be set up from $A'B' = \frac{1}{2}w\ (a^2 - x^2)$, at all points, the extremities of the ordinates will lie on a curve which may easily be seen to be a parabola with its axis vertical and vertex above the middle point of the beam. For

$$SZ = SK - KZ = \tfrac{1}{2}wa^2 - \tfrac{1}{2}w(a^2 - x^2) = \tfrac{1}{2}wx^2.$$

So that SZ is proportional to SN^2, showing that the curve is a parabola.

21. *Beam Loaded at the Ends and Supported at Intermediate Points.*— Next, suppose a beam (Fig. 28) supported at A, B, and loaded with weights P, Q, at the ends C, D, which overhang the supports. If AC, AB, BD are denoted by a, l, b respectively, the supporting force S at A (by taking moments about B) is given by

$$Sl = P(a + l) - Qb.$$

Similarly R, the supporting force at B, is given by

$$Rl = Q(b + l) - Pa.$$

Take now a point K distant x from A; then

$$F_K = S - P = \frac{Pa - Qb}{l} = \frac{M_A - M_B}{l},$$

where M_A, M_B are the bending moments at A, B.

Also for the bending moment at K,

$$M_K = -Sx + P(a+x) = -\frac{M_A - M_B}{l} . x + M_A,$$

or, as we may write it,

$$M_K = M_A \frac{l-x}{l} + M_B \frac{x}{l}.$$

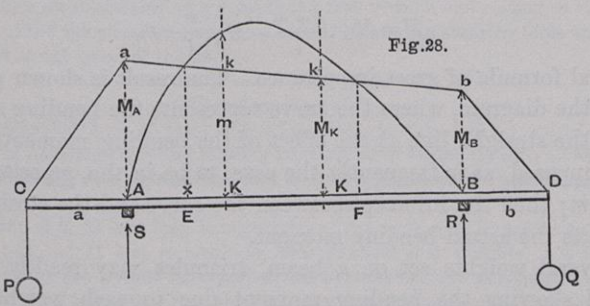


Fig.28.

These formulae show that the shearing force is constant while the bending moment varies uniformly. In the diagram this is indicated by setting up ordinates Aa, Bb, to represent the bending moments at A, B, and joining a, b; the ordinate Kk of this line corresponding to an intermediate point K, will represent the bending moment there. The moments are in this example reckoned positive for upward bending.

An important special case is when $M_A = M_B$; then the bending moment is constant, and the shearing force *zero*. We have then no shearing but only bending. Simple bending is unusual in practice, but an instance occurs in the axle of a carriage.

The ordinates of the straight lines Ca, Db, represent the bending moment at any point of the overhanging parts of the beam.

22. *Application of the Method of Superposition.*—When a beam is acted on by several loads, the principle of superposition already stated in Chap. I. is often very useful in drawing diagrams and writing down formulae for the straining action at any point. Thus, for example, in the preceding case, if there be many weights on the overhanging end of a beam, the bending moment and shearing force at each point must be the sum of that due to each taken separately; and hence it follows that, whatever be the forces acting on a beam, if there be a part AB under the action of no load, and the bending moments at the ends of that part be M_A, M_B, the straining actions at any intermediate point K will always be given by the formulae just written down. And, further, if there be a load of any kind on AB, and m be the bending moment, on the supposition that the beam simply rests on supports at A, B, then the actual bending moment must always be given by

$$M = M_A \cdot \frac{l-x}{l} + M_B \cdot \frac{x}{l} + m,$$

a general formula of great importance. The result is shown graphically in the diagram, where the curve represents the bending moment m, and the straight line ab the effect of the bending moments at the ends, supposed, as is frequently the case, to be in the opposite direction to m; then the intercept between the curve and the straight line represents the actual bending moment.

If several weights act on a beam, triangles may readily be constructed showing the bending moment due to each weight; then adding the ordinates of all the triangles at the points of application of the weights, and joining the extremities by straight lines, a polygon is obtained which is the polygon of bending moments for the whole load. This process may also be applied to shearing forces. It is simple, but somewhat tedious when there are many weights, and other methods of construction will be explained hereafter.

EXAMPLES.

1. A beam, AB, 10 feet long is fixed horizontally at A, and loaded with 10 tons distributed uniformly, and also with 1 ton at B. Find the bending moment in inch tons at A, and also at the middle of the beam.

$M = 720$ inch-tons at A.
$= 210$,, at the centre

2. In the last question find the shearing force at the two points mentioned.

$F = 11$ tons at A.
$= 6$,, at the centre.

3. A beam, AB, 10 feet long is supported at A and B, and loaded with 5 tons at a point distant 2 feet from A. Find the shearing force in tons, and the bending moment in inch-tons at the centre of the beam. Find also the greatest bending moment.

F at the centre $= 1$ ton.
M at the centre $= 60$ inch-tons.
Maximum bending moment $= 96$,,

4. In the last question suppose an additional load of 5 tons to be uniformly distributed. Find the shearing force and bending moment at the centre of the beam.

F at centre $= 1$ ton as before.
M at centre $= 11\frac{1}{4}$ foot-tons $= 135$ inch-tons.

5. A beam, AB, 20 feet long is supported at C and D, two points distant 5 feet from A and 6 feet from B respectively. A load of 5 tons is placed at each extremity. Find the bending moment at the middle of CD in inch-tons.

Moment = 330 inch-tons.

6. In the examples just given draw the diagrams of shearing force and bending moment at each point of the beam.

7. A foundry crane has a horizontal jib, AC, 21 feet long attached to the top of a crane post 14 feet high, which turns on pivots at A and B. The crane carries 15 tons, which may be considered as suspended at the extremity of the jib. The jib is supported by a strut attached to a point in it 7 feet from A, and resting on the crane post at B. Find the stress on crane post and strut, and the shearing force and bending moment at any point of the jib.

Tension of crane post = 30 tons.
Thrust on strut = 50 ,,

8. A rectangular block of wood 20 feet long floats in water; it is required to draw the curves of shearing force and bending moment when loaded (1) with 1 cwt. in the middle; (2) with $\frac{1}{2}$ cwt. at each end, and (3) $\frac{1}{2}$ cwt. placed at two points equidistant from the middle and each end.

9. A beam, AB, 20 feet long is supported at the ends, and loaded at two points distant 6 feet and 11 feet respectively from one end with weights of 8 tons and 12 tons: employ the method of superposition to construct the polygons of shearing force and bending moment. Find the maximum bending moment in inch-tons.

Maximum moment = 972 inch-tons.

10. A beam is supported at the ends and loaded uniformly throughout a part of its length: show that the diagram of moments for the part of the beam outside the load is the same as if the load had been concentrated at the centre of the loaded part, and for the remainder is a parabolic arc. Construct this arc.

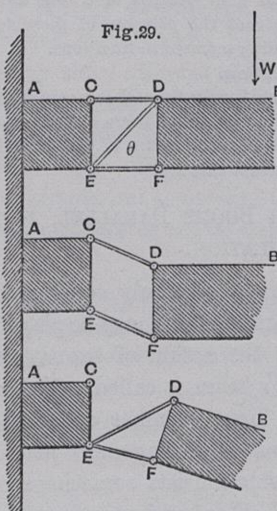
SECTION II.—FRAMEWORK GIRDERS WITH BOOMS PARALLEL, AND WEB A SINGLE TRIANGULATION.

23. *Preliminary Explanations.*—Hitherto we have only considered beams of small transverse section, but the part of a beam may be played by a framework or other structure under the action of transverse forces. Such a structure, when employed as a beam, is called a Girder, and consists essentially of an upper and a lower member called the Booms of the girder, connected together by a set of diagonally placed bars, called collectively the Web. The web consists sometimes of several triangulations of bars crossing each other, and may even be continuous. In the present section the booms will be supposed straight and parallel, and the web a single triangulation. The action of a load on such a girder furnishes the simplest and best illustration of the nature of the straining actions we have just been considering.

Suppose, in the first place, we have a rectangular beam of considerable transverse dimensions, which has one end fixed horizontally, and the other end loaded with a weight W. Now let a part of the length, CD (see Fig. 29), be cut away, and replaced by three bars, CD, EF, DE, jointed at their ends to the two parts of the beam—CD, EF forming a rectangle, of which DE is a diagonal. With this construction the load W will be sustained, as well as by the original beam, but the three bars will be subject to stresses which we shall now determine. To do this, suppose each of the three bars (in succession) removed, and examine the effect on the structure—an artifice which often enables us to see very clearly the nature of the stress on a given part of a structure.

In the first place, suppose CD removed; then the portion EB will turn about the joint E, as shown in the lower part of the diagram, so that the function of the bar CD must be to prevent this turning, which is exactly what we have previously described as bending. The tendency to turn round E—that is, the bending moment at E—is in this case simply $= W \times CB$. But if there is a system of loads, the bending moment at E may be found by methods previously described.

Fig.29.



Now let H = stress on CD. It may readily be seen to be a tensile stress, because, on the removal of the bar, the ends C and D separate from one another. Also, let $h = CE$ or DF, the depth of the beam. The power of CD to prevent EB from turning about E is measured by the moment about E of the force H which acts along it. Therefore

$$Hh = M_E.$$

And dividing the bending moment at E by the depth of the beam, we obtain the magnitude of the tension of CD.

Next, let the bar EF be removed. The structure will yield by turning round the joint D, the point F approaching E. Thus the bar EF is in compression, and by its thrust, $= H'$ say, towards F, it prevents FB from turning round D.

The tendency to turn round D, due to the action of the external forces $= M_D$, will be equal to the resisting moment $H'h$.

$$\therefore \quad H'h = M_D.$$

Therefore if we divide the bending moment for the joint D opposite to the bar, by the depth of the beam h, we obtain the magnitude of the compressive force H'.

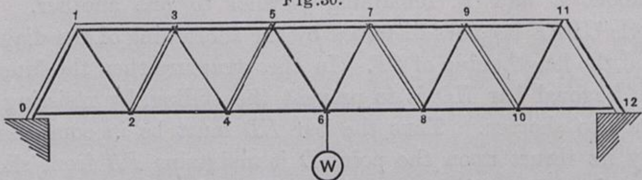
Lastly, let us suppose the diagonal bar ED to be removed, the effect is quite different from the two former cases; for instead of the overhanging portion of the beam turning about some point, it now gives way by sliding downwards (as shown in the centre of the diagram), remaining horizontal all the time. CD and EF turn about C and E, remaining parallel to one another. The rectangle $CDFE$ becomes distorted by the shortening of the diagonal ED and the lengthening of CF. In the structure then the function of the diagonal bar ED is to prevent the sliding, by resisting the tendency to shorten. Thus the bar ED must be in compression, and by its thrust upon the point D it maintains FB from sliding downwards. Let S = thrust along ED and θ = angle it makes with the vertical. The force S may be resolved into two components, a horizontal one, $S \sin \theta$, and a vertical one, $S \cos \theta$. It is the vertical component alone which resists the sliding action, and maintains D in its proper position. Now the tendency to slide is no other thing than the shearing force on the structure, which we have previously been investigating. In this example the shearing force is simply W for all sections between A and B. But in other cases of loading the shearing force may be estimated by previously given methods. Since the downward tendency of the shearing force is balanced by the upward thrust of the vertical component of S along ED we have

$$S \cos \theta = F.$$

Instead of the points E and D being joined there might have been a bar CF, which, by the resistance to lengthening which it would offer, would have sustained the portion FB from sliding downwards. Such a bar would be in tension just as the bar ED is in compression, and in finding the stress on it we should use exactly the same equation. Now instead of having 3 bars only, the whole structure may be built up of horizontal and diagonal bars. The same principles will apply. On removing any one of the horizontal bars, we see that the structure yields by turning round a joint opposite: so we say the function of the horizontal bars is to resist bending. This is expressed by the equation $Hh = M$. On the other hand, the function of the diagonal bars being to resist the shearing tendency, we have always $S \cos \theta = F$.

24. *Warren Girders under various Loads.*—Fig. 30 shows a Warren Girder, so called from the name of the inventor, Captain Warren, a type of girder much used for bridges since its first introduction about the year 1850. It consists of a pair of straight parallel booms connected

Fig.30.



together by a triangulation of bars inclined to each other, generally at 60°, so that the triangles formed are equilateral. The booms in the actual structure are generally continuous through the junctions with the diagonal bars, but, if well constructed, there is no sensible error in regarding the structure as a true frame, in which the several divisions are all united by perfectly smooth joints. Any three bars forming a parallelogram and its diagonal may be considered as playing the same part as regards the rest of the structure as in the case just considered.

When a Warren girder is used, it is generally supported at the ends, and the loads are applied at one or more joints in the lower boom. We will examine some examples.

(1) Suppose there is a single load applied at a joint in the centre of the span.

First as to the diagonal bars. It was shown above that the duty of these bars was to prevent the structure yielding under the action of the shearing force; the vertical component of the stress on either of the diagonal bars being equal to the shearing force for the interval of the length of the girder within which the diagonal bar lies. This is expressed by the equation

$$S \cos \theta = F.$$

Now in the example which we are considering with the load in the centre, the shearing force will be the same at all sections to the right and left, namely $= \frac{1}{2} W$. Therefore the stress on all the diagonal bars is of the same magnitude,

$$S = \frac{W}{2 \cos 30^\circ} = \frac{W}{\sqrt{3}}.$$

If we consider the effect of removing either of the bars, we shall find that commencing from one end they prevent alternately the shortening and lengthening of the diagonals which they join, so that, commencing with one end, the bars are alternately in compression and tension. The compression bars are shown in double lines.

Next as to the several portions of the length of the top and bottom booms. As was shown above, the stress on any division of the horizontal bars has the effect of preventing a bending round the joint opposite; so that the moment of the stress about the joint is equal to the bending moment at the joint, due to the external forces. This is expressed by the equation

$$Hh = M.$$

Let a = length of a division.

Then, since the supporting force at the joint 0 is $\frac{1}{2}W$, the bending moments at the joints numbered 1, 2, 3, &c., are

$$M_1 = \frac{W}{2}\frac{a}{2} = \frac{Wa}{4},$$

$$M_2 = \frac{W}{2} a = \frac{Wa}{2},$$

$$M_3 = \frac{W}{2}\frac{3}{2}a = \frac{3Wa}{4},$$

and so on, the bending moments increasing in arithmetical progression.

Since the depth of the girder h is the same at all parts of the length; if we divide the M's each of them by h, we obtain the magnitude of the stress on the bars opposite the respective joints. Thus

$$H_{02} = \frac{Wa}{4h}, H_{13} = \frac{Wa}{2h}, H_{24} = \frac{3Wa}{4h}, \text{ and so on.}$$

We see, then, that the stress on the several divisions increases in arithmetical progression as we proceed from the ends towards the centre. By observing the effect of removing either of the bars, we see that all the divisions of the upper boom are in compression. This is expressed by drawing them with double lines in the figure. All the divisions of the lower boom are in tension.

(2) Next suppose the load is applied at some other joint not in the centre—the joint 4 for example. We must first calculate the supporting forces. Suppose they are P at 0 and Q at 12. For the portion of the girder to the left of 4 the shearing force will be the same at all sections and be equal to P. So the stress on all the diagonals between 0 and 4 will be equal to $P \sec 30°$.

To the right of joint 4 the shearing force $= Q$, and the stress on all the diagonal bars from 4 to 12 will be $Q \sec 30°$.

Proceeding from either end towards the joint where the load is applied, we observe that the diagonal bars are alternately in compression and tension—so that the bar 56 is now in compression, whilst the bar 54 is in tension. On these bars the nature of the stresses is just opposite to that to which they were exposed when the load was at the centre joint. Thus by varying the position of the load we not only vary the magnitude of the stress, but we may in some cases change the character of the stress, requiring a diagonal bar to act sometimes as a strut and sometimes as a tie.

For the divisions of the horizontal booms on the left of W the stresses are

$$\frac{Pa}{2h}, \ \frac{2Pa}{2h}, \ \frac{3Pa}{2h}, \text{ \&c.,}$$

in arithmetical progression up to the bar opposite the joint to which the load is applied; and to the right of W,

$$\frac{Qa}{2h}, \ \frac{2Qa}{2h}, \ \frac{3Qa}{2h}, \text{ \&c.,}$$

in arithmetical progression also up to the bar opposite the load. The upper bars are all in compression and the lower in tension as before.

When there are a number of loads placed arbitrarily at the different joints, the simplest way of determining the stresses is often to find the stress on the bars due to each load taken separately, and then apply the principle of superposition. In applying the principle due regard must be paid to the nature of the stress. A compressive stress must be considered as being of opposite sign to a tensile stress, and, in compounding, the algebraical sum of the stresses for each load will be the total stress on the bars.

(3) There is one particular case, that in which the girder is uniformly loaded, which it is advisable to examine separately.

In general, the load on the platform of the bridge is by means of transverse beams or girders transferred to the joints of the lower boom. The transverse beams may be the same number as the joints in the lower boom. In that case the girder will be loaded with equal weights at all the bottom joints. If the transverse beams are more numerous their ends will rest on the bottom booms, and tend to produce a local bending action in each division, in addition to the tensile stress which, as the bottom member of the girder, it will have to bear. In some cases, to lessen or get rid of this bending action, vertical suspending rods are introduced, by which means the middle points of the lower divisions are supported, and the loads transmitted to the upper joints of the girder. In such a case we may take all the joints both in the upper and lower booms to be uniformly loaded.

We will, however, suppose equal weights applied to the joints of the lower boom only. First as to the shearing forces. Between the end and the 1st weight the shearing force = the supporting force, = half the total load = P say. In the next division the shearing force is less by the amount of the load at the 1st lower joint $= P - W$. In the third division of the lower boom from the end the shearing force $= P - 2W$, and so on. The stresses on the diagonals can now be found by multiplying the shearing force in the division within which any one diagonal lies by the secant of the angle which the diagonal makes with the vertical. The stresses will diminish in arithmetical progression as we pass inwards from the ends towards the centre. It will be observed that on the first and second diagonals from the end the stress is of the same magnitude. On the third and fourth it is alike also, and so on. The stresses are alternately compression and tension, commencing with compression on the first bar.

To find the stresses on the booms we must determine the bending moments at all the joints.

$$M_1 = \frac{P}{2}a. \qquad M_2 = \frac{P}{2}2a.$$

$$M_3 = \frac{P}{2}3a - W\tfrac{1}{2}a. \qquad M_4 = \frac{P}{2}4a - Wa$$

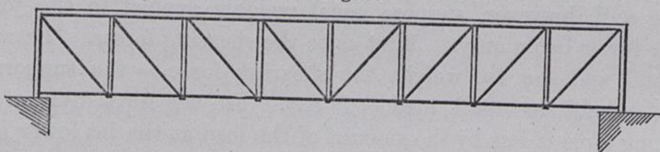
$$= \frac{a}{2}(3P - W). \qquad = \frac{a}{2}(4P - 2W).$$

$$M_5 = \frac{a}{2}(5P - 3W). \qquad M_6 = \frac{a}{2}(6P - 4W).$$

Division of the M's by h, the depth of the girder, will give the several horizontal stresses. They will be found to increase as we pass from the ends towards the centre.

25. *N Trusses.*—The web of the girder, instead of consisting of bars sloping both ways, forming a series of equilateral triangles, may be constructed of bars placed alternately vertical and sloping at an angle, so forming a series of right-angled triangles, looking like a succession of capital letters N. (See Fig. 31.) For this reason it is sometimes called an N girder. The ordinary practice is to divide the girder into a number of squares by means of the vertical bars, so that the diagonals slope at an angle of 45°. It is

Fig.31.



advantageous to place the diagonals so as to be in tension. For a load in the centre, or a uniformly distributed load, they should slope upwards from the centre towards the ends. The vertical bars will then be in compression. A short bar is better able to resist compression than a long one, whereas a tension bar is of the same strength whether short or long; so it is manifestly economical of material, and a saving of weight, to place the long bars, that is the sloping bars, so as to be in tension. The same methods will apply to find the stresses on the bars, since as before the web resists the shearing action, and the booms the bending.

The simple queen truss, considered in Chapter I., Section II., is another example of a web consisting of alternate vertical and diagonal bars, but the diagonal is not usually inclined at 45° to the vertical.

EXAMPLES.

1. A trapezoidal truss is 24 feet span and 3 feet deep. The central part is 8 feet long and is braced by a diagonal stay so placed as to be in tension. Find the stress on each part when loaded with 4 tons at one joint and 5 tons at the other.

Stress on diagonal stay = ·935 tons.

2. A bridge is constructed of a pair of Warren girders, with the platform resting on the lower booms, each of which is in 6 divisions. The bridge is loaded with 20 tons in the middle. Find the stress on each part.

3. In example 2 obtain the result when the load is supported at either of the other joints.

4. From the results of examples 2 and 3 deduce the stress on each part of the girder when the bridge is loaded with 60 tons, divided equally between the three pairs of joints from one end to the centre.

Results for questions 2, 3, 4, the bars being numbered as in Fig. 30.

Bars.	Stress on Booms.				Bars.	Stress on Diagonals.			
	Load at 6.	at 4.	at 2.	at 6, 4, and 2.		Load at 6.	at 4.	at 2.	at 6, 4, and 2.
02	2·88	3·85	4·8	11·53	01	− 5·76	− 7·7	− 9·6	−23·06
13	− 5·76	− 7·7	− 9·6	−23·06	12	5·76	7·7	9·6	23·06
24	8·64	11·55	8·64	28·83	23	− 5·76	− 7·7	1·92	−11·54
35	−11·52	−15·36	− 7·68	−34·56	34	5·76	7·7	− 1·92	11·54
46	14·4	13·44	6·72	34·56	45	− 5·76	3·85	1·92	0
57	−17·28	−11·52	− 5·76	−34·56	56	5·76	− 3·85	− 1·92	0
68	14·4	9·6	4·8	28·8	67	5·76	3·85	1·92	11·54
79	−11·52	− 7·68	− 3·84	−23·04	78	− 5·76	− 3·85	− 1·92	−11·54
8,10	8·64	5·76	2·88	17·28	89	5·76	3·85	1·92	11·54
9,11	− 5·76	− 3·84	1·92	−11·52	9,10	− 5·76	− 3·85	− 1·92	−11·54
10,12	2·88	1·92	·96	5·66	10,11	5·76	3·85	1·92	11·54
					11,12	− 5·76	− 3·85	− 1·92	−11·54

5. A bridge 80 feet span is constructed of a pair of N girders in 10 divisions, the platform resting on the lower booms, and the diagonals so arranged as to be all in tension. A load of 80 tons is uniformly distributed over the platform. Find the stress on each bar.

Section III.—Girders with Redundant Bars.

26. *Preliminary Explanations.*—Again, returning to the (p. 48) beam out of which a portion has been cut and replaced by bars, let us suppose that instead of one diagonal bar only, there are two. We require to find the stresses on the bars. First, on the diagonal bars. In this case also the stress on these bars will be due to the shearing force. Together they prevent the structure yielding under the shearing action, but the amount each one bears is indeterminate until we know how the diagonals are constructed and attached to the rest of the structure. Suppose, for example, the diagonals are simple struts placed across the corners of the rectangle, but not secured at the ends The struts will be incapable of taking tension; and the diagonal *ED*, which slopes in the direction, to be subject to compression will have to bear the whole shearing force. The other diagonal is ineffective. Secondly, suppose the diagonals to be simple ties, such as a chain or slender rod, and so incapable of withstanding compression. Then the bar *CF* will carry the whole shearing force. We may have any number of intermediate cases between these extreme ones according to the material of the diagonals and the method of attachment. In all cases one diagonal tends to lengthen, and the other to shorten, and according to their powers of resistance to these tendencies they offer resistance to the shearing. If S_1 and S_2 be stresses on the two bars, then in all cases

$$(S_1 + S_2)\cos\theta = F.$$

If the diagonals are exactly similar rigid pieces similarly secured at the ends, equal changes of length will produce the same stress whether in compression or tension, so that each will bear an equal share of the shearing force. We shall then have

$$S_1 = S_2 = \tfrac{1}{2}F\sec\theta.$$

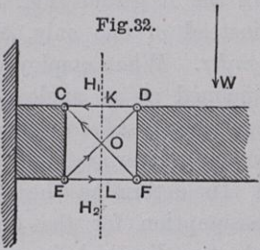
The foregoing is one of the simplest examples of a frame with redundant bars, and shows clearly why, in such cases, the stress on each bar cannot be determined by statical considerations alone, but depends upon the materials and mode of construction. In structures such as those considered in Chap. I., Sect. II., in which the principal part is an incomplete frame, stiffened by bracing or other means to provide against variations of the load, the bracing is usually redundant, and the stress on it cannot be calculated with certainty. Allowance has to be made for this in designing the structure by the use of a larger factor of safety. Redundant material is often no addition at all to the strength of the structure, and may even be a source of weakness, as will appear hereafter.

When framework girders were first introduced, it was objected by eminent engineers that failure of a single part would destroy the structure. Experience appears to have shown that risks of this kind are not serious, and the tendency of modern engineering design appears to be rather towards the employment of structures with as few parts as possible.

Next, as to the horizontal bars. These still sustain the bending moment, but not precisely in the same way as when there is only one diagonal. To find the magnitude of the forces, we employ a method similar to that used before, but instead of removing a bar we suppose the girder cut through one or more bars at any place convenient to our purpose; then the principle which we make use of is, that the action of each of the two halves on the other must be in equilibrium with the external forces which are applied to either half. In Figure 32 let us take a vertical section through the point of intersection of the diagonals, 4 bars are cut by the section, and through the medium of these 4 bars the structure to the left will act on the portion of the structure to the right of the section, and sustain it against the action of the external loads which rest on it.

First, there is the force H_1 pulling at K, and the force H_2 thrusting at L, and at O there are the two forces S_1 and S_2 on the two diagonals. Now, if we consider the tendency for the external forces to bend the right-hand portion round O, we see that the diagonal bars offer no resistance to this bending action, and must so far be left out of account. The whole resistance to bending is due to the bars CD and EF along which the forces H_1 and H_2 act, so that if M_O be the bending moment at O, due to the external forces,

Fig.32.



$$(H_1 + H_2)\frac{h}{2} = M_O.$$

This will be true whatever be the proportion between S_1 and S_2, and H_1 and H_2. Instead, therefore, of taking the bending moment about a joint, as we did previously, we have in this case to take the moment about the point where the two diagonals cross.

But besides the balancing of the bending moment, there are other conditions to which the forces are subject, in order that the right-hand portion may be in equilibrium. One is, that all the forces which act on this portion must balance horizontally. There are no external forces which have any horizontal action, so that it is only the four internal forces which act along the bars cut, of which we have to take any account, and these must, on the whole, have no resultant horizontal action. The two thrusts must equal the two pulls; that is,

$$H_2 + S_2 \sin\theta = H_1 + S_1 \sin\theta.$$

$$\therefore \quad H_2 - H_1 = (S_1 - S_2)\sin\theta.$$

This also is true whatever be the distribution of the shearing force between the two diagonals.

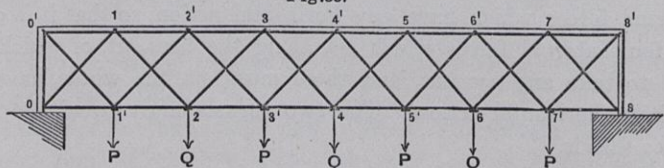
If, now, we suppose $S_2 = S_1$, then $H_2 = H_1 = H$, say. And the above formula becomes $Hh = M_O$, the same as we had before; but it must be applied a little differently, the moment now being taken about the point of intersection of the diagonals. If S_1 is not equal S_2, then H will be the mean of H_1 and H_2.

27. *Lattice Girders, Flanged Beams.* — Constructions with a double set of diagonals are common in practice. If, for example, in the N girder (Fig. 31) we place in each division two diagonals instead of one only, the construction is called a *lattice* or *trellis girder*. When employed for heavy loads, the diagonals are generally inclined at an angle of 45° to the vertical. In light structures, or when used for giving stiffness, they are often inclined at a much greater angle.

To determine the stresses, it will be necessary to make an assumption for the distribution of the shearing force between the two diagonals for each division of the girder, and it will generally be sufficiently correct to suppose each to carry half, and to write $S = \frac{1}{2}F \sec \theta$, and $Hh = M$ for the points where the diagonals intersect.

In lattice girders we more frequently find the double set of sloping bars introduced, but the vertical bars omitted. In this case it will not be true that the two diagonals in any one division are exposed to the same stress. We can determine the stresses otherwise. The structure may be divided into two elementary girders, each with its own system of diagonal bracing, and each with its own set of loads. Suppose, for simplicity, the number of divisions in the complete girder even, and each half girder loaded with equal weights applied

Fig. 33.



to all the lower joints. Then if we make the simple, and in most cases safe, assumption that the thrusts on the two end vertical bars are equal, the forces on all the bars of the structure will be determinate. In the example shown in Fig. 33 the thrusts on the vertical end bars will be $2P$.

After we have calculated the stresses on each bar in each elementary girder, then, for any bar which is a portion of both, we must compound to obtain the total stress.

We may further increase the number of diagonal bars and obtain a girder, the web of which is a network of bars. In this case it will not be exactly, but will be very nearly, true that the horizontal bars take the bending, and the sloping bars the shearing action, the shearing force being regarded as equally distributed between all the diagonals cut by any one vertical section.

We may go on adding diagonal bracing bars until the space between the booms is practically filled up, and even then assume that the bending is taken by the horizontal bars and the shearing by the web. The numerous bracing bars may then be replaced by a vertical plate, which will form a continuous web to the girder. Such a construction is a very common one in practice, the horizontal members are called the top and bottom flanges of what is still a girder, and often called so, but more often a flanged or I beam. In the smallest class of these beams, they are rolled or cast in one piece; but for large spans they are built up of plates and angle irons rivetted together. For figures showing the transverse sections of such beams see Part IV. In taking the depth of such a girder, to make use of in the equation $Hh = M$, we ought to measure the vertical distance between the centres of gravity of the parts which we consider to be the flanges of the beam or girder. In the simple rolled or cast beam this will be the distance from centre to centre of depth of flanges. In the built-up beam account must be taken of the effect of the angle irons.

It must be remembered that this method of determining the strength of an I beam is only approximate. Its strength will be determined in a more exact way hereafter, when it will be found that the web itself assists in resisting the bending moment, but, area for area, to the extent only of about one-sixth that borne by the flange. On the other hand, the effective depth is less than the distance from centre to centre of the flanges. In rough preliminary calculations we may often neglect this, and employ the same formula as for lattice girders.

Girders are often of variable depths, so that the booms are not parallel; when this is the case the booms assist in resisting the shearing action of the load, as will be seen hereafter.

EXAMPLES.

1. A beam of I section is 24 feet span, and 16 inches deep; the weight of the beam is 1,380 lbs. It is loaded in the centre with 5 tons. Assuming the resistance to bending to be wholly due to the flanges, find the maximum total stress on each flange and the sectional area of each—the resistance to compression being taken to be 3 tons and to tension 4 tons per square inch.

Maximum total stress = 53,505 lbs. = 23·88 tons.
Sectional area of upper flange = 8 square in.
" " bottom " = 6 "

2. A trellis girder, 24 feet span and 3 feet deep, in three divisions, separated by vertical bars, with two diagonals in each division, is supported at the ends and loaded (1) with 20 tons symmetrically distributed over the middle division of the top flange, (2) with 20 tons placed over one of the vertical bars. Find the stress on each part of the girder, assuming each diagonal to carry half the corresponding shearing force.

Stress on diagonals—Case 1.	14·2	0	14·2
Case 2.	$18\frac{8}{9}$	$9\frac{4}{9}$	$9\frac{4}{9}$

Remark.—These results show the unsuitability of this construction for carrying a heavy load on account of the great inclination of the diagonals to the vertical.

3. A water tank, 20 feet square and 6 feet deep, is wholly supported on four beams, each carrying an equal share of the load. The beams are ordinary flanged ones, 2 feet deep. Find approximately the maximum stress on each flange, assuming that the weight of the tank is one-fourth the weight of water it contains.

$$\text{Distributed load on one beam} = \frac{187{,}500}{4} = 46{,}875 \text{ lbs.}$$

$$H_{\text{max.}} = 58{,}593 \text{ lbs.} = 26 \cdot 1 \text{ tons.}$$

4. The Conway tubular bridge is 412 feet span. Each tube is 25 feet deep outside and 21 inside. The weight of tube is 1,150 tons, and the rolling load is estimated at $\frac{3}{4}$ ton per foot run. Find approximately the sectional areas of the upper and lower parts of the tube, the stress per square inch being limited to 4 tons.

$H_{\text{max.}} = 3{,}267$ tons.
Area = 817 square in.

REFERENCES.

For details of construction of girders the reader is referred to

Girder Making . . . in Wrought Iron. E. HUTCHINSON. Spon, 1879.

CHAPTER III.

STRAINING ACTIONS DUE TO ANY VERTICAL LOAD.

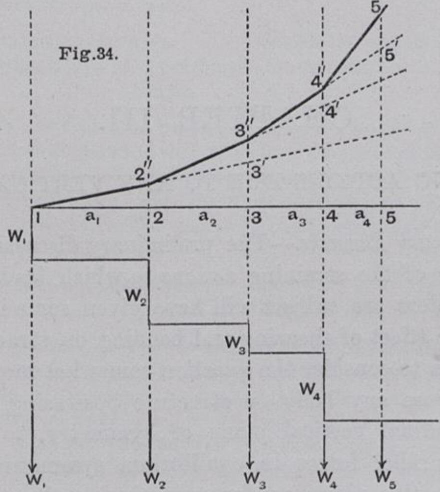
28. *Preliminary Remarks.*—The preliminary discussion in the preceding chapter of the straining actions to which loaded beams and framework girders are subject will have given some idea of the importance of the effect of shearing and bending on structures, and we shall now go on to consider the question somewhat more generally.

Let us suppose any body or structure possessing, as it usually will, a longitudinal vertical plane of symmetry, to be acted on by a set of parallel forces in equilibrium symmetrically disposed with respect to this plane, as, for example, gravity combined with suitable vertical supporting forces. Then these forces will be equivalent to a set of parallel forces in the plane of symmetry in question. Let the structure now be divided into two parts, A and B, by an ideal plane section, parallel to the forces and perpendicular to their plane. Then the forces acting on A may be reduced to a single force F lying very near the section considered and a couple M, while the forces acting on B may be reduced to an equal and opposite force F lying very near the section and an equal and opposite couple M. The pair of forces are the elements of the shearing action on the section, and the pair of couples are the elements of the bending action on the section. As the nature of the structure is immaterial, we may consider these straining actions for a given vertical section quite independently of any particular structure, and describe them as the Shearing Force and Bending Moment *due* to the given Vertical Load. We shall first consider the connection which exists between the two kinds of straining action and the method of determining them for any possible load.

CONNECTION BETWEEN SHEARING AND BENDING.

29. *Relation between the Shearing Force and the Bending Moment.*—Figure 34 shows the lines of action of weights W_1, W_2, &c., placed at the successive intervals a_1, a_2, &c.

Fig. 34.



In the first division the shearing force is

$$F_1 = W_1;$$

in the second $\quad F_2 = W_1 + W_2 = F_1 + W_2,$

$$\therefore \quad F_2 - F_1 = W_2;$$

in the third $\quad F_3 = W_1 + W_2 + W_3 = F_2 + W_3,$

$$\therefore \quad F_3 - F_2 = W_3;$$

and so on for all the divisions, so that in the n^{th} division

$$F_n - F_{n-1} = W_n.$$

We express this in words by saying that *the difference between the shearing forces on two consecutive intervals is equal to the load applied at the point between the two intervals;* or it may be written

$$\Delta F = W.$$

By setting down ordinates to a horizontal base line we obtain the stepped figure as the graphical representation of the shearing force at any point of the beam. It is drawn by first setting downwards at 1 an ordinate for the shearing force on the 1st interval, and then passing along the beam to the other end, on meeting the lines of action of the successive weights the length of the ordinates is increased by the amount of the weights. In so doing we make use of the proposition which has just been proved.

This is called the *Polygon of Shearing Force*, or more generally, when the loads are continuous, the *Curve of Shearing Force.*

Next as to the bending moment. At the first point where W_1 is applied $M_1 = 0$,

at the second point $M_2 = W_1 a_1 = F_1 a_1$;

,, third ,, $M_3 = W_1(a_1 + a_2) + W_2 a_2 = W_1 a_1 + (W_1 + W_2)a_2$
$= M_2 + F_2 a_2$,

$\therefore\ M_3 - M_2 = F_2 a_2$;

,, fourth point $M_4 = W_1(a_1 + a_2 + a_3) + W_2(a_2 + a_3) + W_3 a_3$,
$= W_1(a_1 + a_2) + W_2 a_2 + (W_1 + W_2 + W_3)a_3$,
$= M_3 + F_3 a_3$,

$M_4 - M_3 = F_3 a_3$;

and generally, $M_n - M_{n-1} = F_{n-1} a_{n-1}$.

We may express this in words by saying that the *difference between the bending moments at the two ends of an interval is equal to the shearing force, multiplied by the length of the interval.* Or the result may be written

$$\Delta M = Fa.$$

We will now take a numerical example and see how we may make use of this property to determine a series of bending moments.

Let AB be a beam fixed at one end, and loaded with weights of 2, 3, 5, 11, 13, 7 tons, placed at intervals of 3, 2, 3, 5, 4, 6 feet,

W.	F.	a.	Fa.	M.
2				0
	2	3	6	
3				6
	5	2	10	
5				16
	10	3	30	
11				46
	21	5	105	
13				151
	34	4	136	
7				287
	41	6	246	
				533

commencing from the free end. We adopt a tabular method of carrying out the work of calculation.

First set down a column of weights applied, as shown by the figures in the column headed W. In the next column write the shearing forces. Since the shearing forces are uniform over the intervals between the weights, it will be best to write the F's opposite the spaces between the weights. Any F is found by adding to the F above it the adjacent W. In the third column we set down the lengths of the intervals. Then multiplying the F's and corresponding a's together, set the results in column 4. Lastly, we can write down the column of bending moments by the repeated addition of the Fa's—the bending moment at any point being found by adding to the bending moment at the point above the value of Fa between the points.

If instead of all the forces acting one way some of them act upwards, a minus sign should be set opposite, and all the operations performed algebraically.

The method is equally applicable however the beam is supported.

For example, let a beam 23 feet long be supported at the ends and loaded with 3, 2, 7, 8, 9 tons, placed at intervals of 2, 2, 3, 4, 5, 7 feet, reckoning from one end.

First calculate one supporting force, say at the left-hand end

W.	F.	a.	Fa.	M.
16·17				0
	16·17	2	32·34	
−3				32·34
	13·17	2	26·34	
−2				58·68
	11·17	3	33·51	
−7				92·19
	4·17	4	16·68	
−8				108·87
	−3·83	5	−19·15	
−9				89·72
	−12·83	7	−89·81	
12·83				0

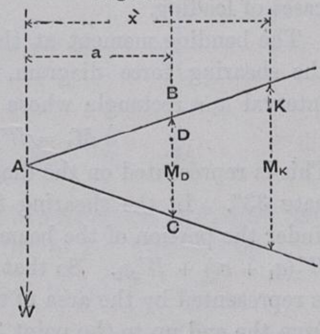
by taking moments about the other end. In the column of W's set this for the first force, and since all the loads act in the contrary direction, put negative signs opposite them, and in writing down the next column of F's add algebraically. We shall at the bottom of the column determine the supporting force at the right hand end. At the bottom of the column of M's, that is at the point where the right hand supporting force acts, we ought to get a zero moment. The obtaining of this will be a test of the accuracy of the work. In this example the small difference between 89·72 and 89·81 is due to our having taken the supporting force only to two places of decimals.

Observation of the process of calculation leads us to a very important proposition, viz., *where the shearing force changes sign, the bending moment is at that point a maximum.* This will be true for all important practical cases, but exceptional cases may be imagined in which, where the shearing force changes sign, the bending moment is a minimum. Since $\Delta M = Fa$, then, so long as F is positive, M will be an increasing quantity as we pass from point to point. But where F changes to negative there M commences to diminish.

We will now explain the construction of a diagram of bending moment for a system of loads: and first let us consider how the moment of a force about any point or succession of points may be graphically expressed.

Let W be a force and D any point, and suppose the numerical magnitude of the moment of W about D known. Draw a line through D parallel to the force at a distance a (Fig. 35), and anywhere in this line take a length BC to represent on some convenient scale the moment, $M_D = Wa$, of W about D. The scale must be so many inch-tons, foot-lbs., or similar units to the inch. Then choose any point A in the line of action of the force, join AB and AC, and produce these lines indefinitely. The moment of W about any point whatever is represented by the intercept by the radiating lines AB, AC of a line drawn through the point parallel to the force. For example, the moment about $K = M_K = Wx$, where x is the perpendicular distance of K from the line of action of W.

Fig.35.



$$\therefore \quad \frac{M_K}{M_D} = \frac{Wx}{Wa} = \frac{x}{a}.$$

By similar triangles the intercepts are to one another in the ratio $x : a$, so that they correctly represent the moments.

We will first draw the diagram of bending moments for a beam fixed at one end and loaded at intervals along its length.

Returning to Fig. 34, take a line representing the length of the beam as base line. Produce upwards the lines of action of the loads. Commence by setting up at the point where W_2 acts a line to represent the moment of W_1 about that point, that is, take $2'\,2$ to represent W_1a_1. If $1\,2'$ be joined and produced, then the intercept between this line and base line 1 5 will represent on the same scale the moment of W_1 about any point in the beam. Next at the point $3'$, where $12'$ cuts $W_3\,2'\,3'$, set up $3'\,3''$ to represent W_2a_2, join $2''\,3''$ and produce it. The intercept between $2'\,3'$ and $2''\,3''$ will represent the moment of W_2 about any point in the beam. Then at the point $4'$, where $2''\,3''$ cuts W_4, set up $4'\,4''$ to represent W_3a_3. Join $3''\,4''$, produce it, and so on with all the weights. The polygon 1, $2''$, $3''$, $4''$, $5''$ ... will be obtained, the ordinates of which measured from the base line AB will represent the bending moment at any point, due to all the weights on the beam. This is called the *Polygon of Bending Moment.* In the case of a continuous distribution of load it is called the curve of bending moment.

There is a very important relation between the polygons of shearing force and bending moment which have been drawn in all cases of loading.

The bending moment at the point $2 = W_1a_1$. Now referring to the shearing force diagram, we observe standing underneath the interval a_1 a rectangle whose area $= W_1a_1$. Next, for the point 3,

$$M_2 = W_1(a_1 + a_2) + W_2a_2.$$

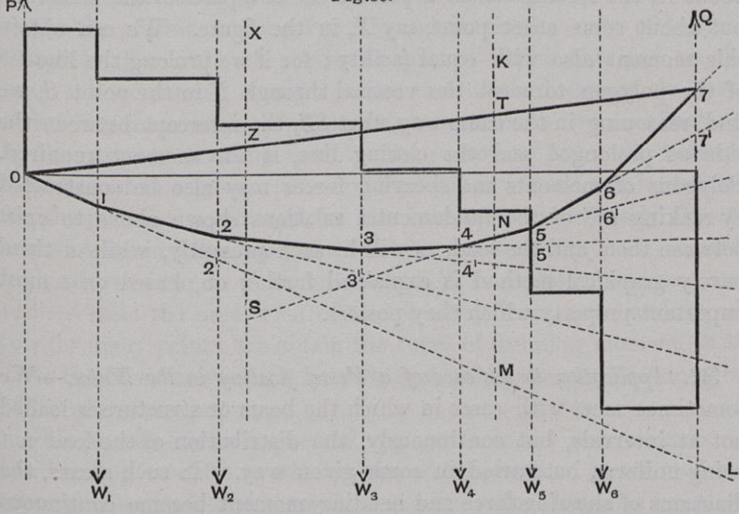
This is represented on the diagram of bending moment by the ordinate $33''$. In the shearing force diagram we notice that the area under the portion of the beam from 1 to 3 consists of two rectangles, $W_1(a_1 + a_2) + W_2a_2$. So that at this point also the bending moment is represented by the area of the polygon of shearing force, reckoned from the end up to the point 3. And so on for every point. This important deduction may be stated generally thus :—*The ordinate of the curve of bending moment at any point is proportional to the area of the curve of shearing force reckoned from one end of the beam up to that point.*

30. *Application to the case of a Loaded Beam.*—We will next take the case of a beam supported at the two ends.

First, calculate the supporting force P, set it up at the end of the base line as an ordinate, and draw the stepped polygon by continually subtracting the W's. At some point in the beam we shall cross the base line. At that point the shearing force changes sign, and there the bending moment is a maximum. The shearing force on the last interval will give the magnitude of the supporting force Q. The polygon thus drawn will be the polygon of shearing force.

The polygon of bending moment may be drawn without previously determining the supporting force at either end thus :—

Fig. 36.



Commencing at O (Fig. 36), the point of application of P, draw any sloping line 0 1 2′ cutting W_1 in 1, and W_2 in 2′. Then set up

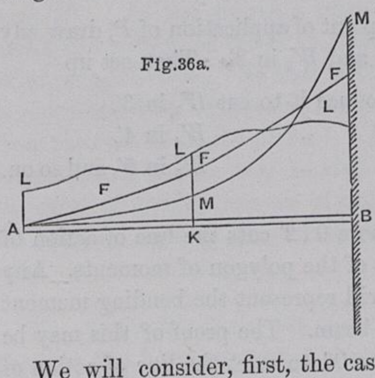
2′ 2 to represent W_1a_1, join 1 2, produce it to cut W_3 in 3′.
3′ 3 „ W_2a_2, „ 2 3, „ W_4 in 4′.
4′ 4 „ W_3a_3, „ 3 4, „ W_5 in 5′, and so on.
7′7 will represent W_6a_6.

Now join 7 with the point 0, where 0 1 2′ cuts the line of action of P. This is called the Closing Line of the polygon of moments. Any vertical intercept of this polygon will represent the bending moment at the corresponding point of the beam. The proof of this may be stated shortly thus :—If we produce 0 1 to meet the line of action of Q in L, then $L7$ will, from what has been said before, represent the sum of the moments of all the weights W about the end of the beam where Q acts. And from the conditions of equilibrium this must equal the moment of P about that end. Accordingly, if we take any point K, the vertical intercept MT below it will represent the moment of P about K. This is an upward moment. The four weights which lie to the left of K will together have a downward moment about K represented by MN. Therefore, the difference NT will represent the actual bending moment at the point K.

It sometimes happens that we want the moment of the forces not about K, the section which separates the two parts of the structure, but about some other point, say X, in the figure. We can obtain this moment also with equal facility; for if we prolong the line 4 5 of the polygon to meet the vertical through X in the point S, we find, reasoning in the same way, that SZ, the intercept between the side so prolonged and the closing line, is the moment required. Polygons of moments and shearing forces may also be constructed by making use of the fundamental relations shown above to exist between them and the load, as will be seen presently, while a third purely graphical method is explained farther on, based on a most important property which they possess.

31. *Application to the case of a Vessel floating in the Water.*—We sometimes meet with cases in which the beam or structure is loaded not at intervals, but continuously, the distribution of the load not being uniform, but varied in some given way. In such a case, the diagrams of shearing force and bending moment become continuous curves. The most convenient way of expressing how the load is distributed is by means of a curve, the ordinate of which at any point represents the intensity of the load at that point. Such a curve is called a *curve of loads.* It may be regarded as the profile of the upper surface of a mass of earth or other material resting on the beam.

Fig.36a.



We will consider, first, the case of a beam fixed at one end and loaded continuously throughout, in a manner expressed by a curve of loads LL. (Fig. 36a.) The total area inclosed by the curve of loads will represent the total load on the beam, and between the two ordinates of any two points will be the load on the beam between the two points. Now, the area of the curve of loads, reckoned from the end A up to any point, K say, since it represents the total load to the left of K, will be the shearing force at K. If at K we erect an ordinate, KF, to represent on some convenient scale the area ALK, and do this for many points of the beam, we shall obtain a second curve FF, the curve of shearing force. Having done this, we may repeat the process on the curve FF, and obtain the curve of bending moment. For we have previously proved that if the load on the beam is concentrated at given points, then the ordinate of the curve of bending moments is at any point proportional to the area enclosed by the curve of shearing force for the portion of the beam between the end and that point. The truth of this is not affected by supposing the points of application of the load to be indefinitely close to one another, in which case the load becomes continuous. Accordingly, if we set up at K an ordinate, KM, to represent on some convenient scale the area AFK of the shearing force curve, and repeat this for many points, we obtain the curve of bending moment, MM. Thus the three curves form a series, each being the graphical integral of the one preceding.

This process has an important application in the determination of the bending moment to which a ship is subjected on account of the unequal distribution of her weight and buoyancy along the length of the ship. On the whole, the upward pressure of the water, called the buoyancy, must be equal to the downward weight of the ship; and the lines of action of these two equal and opposite forces must be in the same vertical. But for any portion of the length, the upward pressure and the downward weight will not, in general, balance one another; so, on account of the difference, shearing and bending of the ship will be induced. In the case of a rectangular block of wood floating in water, the upward pressure of the water will, for every portion of its length, equal the downward weight, and there will be no shearing and bending action on it. But, in actual ships, the disposition of weight and buoyancy is not so simple. Taking any small portion of the length of the ship, the difference between the weight of that portion of the ship and the weight of the water displaced by that portion of the ship, will be a force which acts on the vessel sometimes upwards and sometimes downwards, according to which is the greater, just in the same way as forces act on a loaded beam producing shearing and bending. In the construction of the vessel, strength must be provided to resist these straining actions, and it is a matter of great practical importance to determine accurately the magnitude of them for all points of the length of the ship. We will select an example of very frequent occurrence, that in which at the ends of the ship the weight exceeds the buoyancy, whilst at the centre the buoyancy exceeds the weight. If the ship were very bluff ended, and carried a cargo of very heavy material in the centre hold, the distribution of weight and buoyancy would probably be the reverse of this.

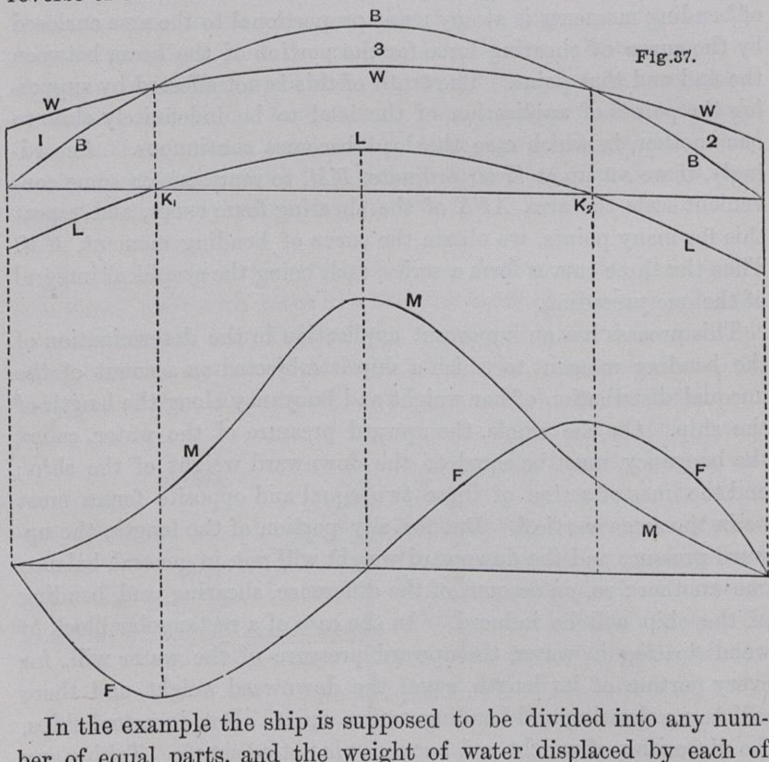


Fig. 37.

In the example the ship is supposed to be divided into any number of equal parts, and the weight of water displaced by each of those parts determined; ordinates are set up to represent those weights, and so, what is called a curve of buoyancy *BBB* (Fig. 37) is drawn. The whole area enclosed by the curve will represent the total buoyancy or displacement of the vessel, and is the same thing as the total weight of the vessel. Next we suppose that the weights of the different portions of the ship are estimated, and ordinates set up to represent these weights, then what is called a curve of weight, WWW, is obtained. In the figure it is set up from the same base line. The total area enclosed by this curve will also be the total weight of the ship, and must therefore equal the area enclosed by the curve of buoyancy. Thus the sum of the two areas marked 1 and 2 must equal the area marked 3. Not only must this be true, but also the centres of gravity must lie on the same ordinate. The difference at any point between the ordinates of the two curves will express by how much at the ends the weight exceeds the buoyancy, and in the middle portion by how much the buoyancy exceeds the weight, representing, in the first case, the intensity of the downward force, and, in the second, the intensity of the upward force. Where the curves cross one another and the ordinates are the same height, as at K_1 and K_2, the *sections are said to be water-borne.* If now we set off from the base line ordinates equal to the difference between the ordinates of the two curves BBB and WWW, we obtain the curve of loads LLL; some portions where the weight is in excess will lie below the base line, and the rest, where the buoyancy exceeds the weight, will lie above the base line. From what has been said before, the area above the base line must equal the area below. Having obtained the curve of loads, the curve of shearing force is to be obtained from it in the manner previously described, by setting up, at any point, an ordinate to represent the area of the curve LLL between the end of the ship and that point. In performing the operation, due regard must be paid to the fact that the loads on different parts of the ship act in different directions, and for one direction they must be treated as negative, and the corresponding area of the curve as a negative area.

Having thus determined the curve of shearing force FFF, the same operation must be repeated on that curve to determine the curve of bending moment. In drawing the curve of shearing force it will be found that at the further end of the ship we return again to the base line from which we started at first, for the shearing force at the end must be zero. Also the bending moment at the end must be zero. This gives us tests of the accuracy of our work.

In this example the bending is wholly in one direction, tending to make the ends of the ship droop or the ship to "hog" in the technical language of the naval architect, but in some examples the direction of bending changes once or more times. Curves of shearing force and bending moment were first explained in relation to a vessel floating in the water by the late Professor Rankine in his work on shipbuilding. It does not, however, appear that any such curves were ever constructed in any actual example until 1869, when some were drawn for vessels of war by direction of Mr. (now Sir E.) Reed, at that time chief constructor of the Navy. The results obtained by him are described in a paper read before the Royal Society (Phil. Trans. for 1871, part 2). They now form part of the ordinary calculations of a vessel.

Since the water exerts on the vessel not only vertical but also horizontal forces, the straining actions upon her do not consist solely of shearing and bending, but include also a thrust. The horizontal pressure also produces bending in a manner which we shall hereafter explain.

32. *Maximum Straining Actions.*—The set of forces we are considering are in equilibrium, and must therefore be partly upwards and partly downwards. The downward force is the total weight W, and is generally more or less distributed, the upward force is of equal magnitude, and is usually concentrated near two or more points. In the case of the vessel, however, the upward force is distributed like the weight, though not according to the same law. In any case the greatest shearing force must be some fraction of the weight, and the greatest bending moment must be some fraction of the weight multiplied by the length l over which the weight is distributed. We may therefore express the maximum straining actions by the formulae

$$F_0 = k \, . \, W \, ; \; M_0 = m \, . \, Wl,$$

where k, m are numerical quantities depending on the distribution of the load and the mode of support. Thus for a uniformly loaded beam supported at the ends $k = \frac{1}{2}$, $m = \frac{1}{8}$. The greatest value m can have in a beam resting on supports without attachment is $\frac{1}{4}$; this occurs when the beam is supported at the ends and the load concentrated in the middle or conversely. In vessels where the supporting force is distributed m is much less; its maximum value is estimated by Mr. White at $\frac{1}{35}$ in ordinary merchant steamers.

EXAMPLES.

1. A Warren girder with 12 divisions in the lower boom is supported at the ends and loaded with 250 tons, which may be supposed to be equally distributed among all the 25 joints. Find the stress on each bar by calculating the series of shearing force and bending moments.

RESULTS FOR LEFT-HAND HALF OF GIRDER.

F	115	105	95	85	75	65	55	45	35	25	15	5	5
S	132·2	120·7	109·2	97·7	86·2	74·7	63·2	51·7	40·2	28·7	17·2	5·7	
$\Delta H = F\frac{1}{\sqrt{3}}$	66·1	60·3	54·6	48·8	43·1	37·3	31·6	25·8	20·1	14·3	8·6	2·8	
H	66·1	126·4	181	229·8	272·9	310·2	341·8	367·6	387·7	402	410·6	413·4	

2. The buoyancy of a vessel is 0 at the ends and increases uniformly to the centre, while the weight is 0 at the centre and increases uniformly to the ends. Draw the curves of shearing force and bending moment, and find the maximum values of these quantities in terms of the displacement and length of the vessel.

Answer—$k=\frac{1}{4}$; $m=\frac{1}{12}$.

3. A beam, 48 feet span, is supported at the ends and loaded with weights of 6, 9, 10, 13, 5, and 7 tons, placed at intervals of 4, 5, 9, 7, 13, and 8 feet respectively, commencing at one end. Calculate the shearing force in each interval and the series of bending moments.

4. In the last question construct the polygons of shearing force and bending moment.

5. In the case of a uniformly loaded beam supported at the ends, verify the principle that the area of the curve of shearing force is proportional to the ordinate of the curve of bending moment.

6. When a beam is supported at the ends and loaded in any way, show that an ordinate at the point of maximum moment divides the area of the curve of loads into parts, which are equal to the supporting forces. Further, if a b are the distances of the centres of gravity of these parts from the ends of the beam, and l the span, show that the maximum moment is mWl where

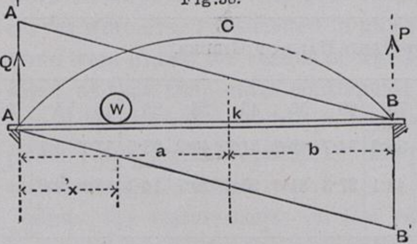
$$\frac{1}{m} = \frac{l}{a} + \frac{l}{b}.$$

TRAVELLING LOADS.

33. We have hitherto been investigating the effect of a permanent fixed load on a structure in producing straining actions on it. We next examine the effect of a load which is not permanent, but which at different times takes up different positions on the structure, and we require to know what position of the load will produce the greatest straining action at any particular part of the structure, and also the amount of that maximum straining action.

This question arises principally in the design of bridges across which a *travelling load*, such as a train, may proceed. We will take first the simple case of a beam of span l, supported at the ends and suppose a single concentrated load W to travel across it in the direction of the arrow. Let us consider any point K (Fig. 38) in the beam, distant a and b from the ends. As the load traverses the beam, each position of the load will produce a certain shearing force and bending moment at the point K. To find their greatest value let x = distance of W from A, then the supporting force at $B = P = W\frac{x}{l}$. So long as the weight lies between A and K the shearing force at K will be simply P.

Fig.38.



$$F_K = W\frac{x}{l},$$

consequently the shearing force will increase as x increases, until the load reaches the point K. So long as the weight lies to the left of K, the tendency will be for the portion KB to slide upwards relatively to the portion AK. This we will call a positive shearing force. Therefore, putting $x = a$,

$$\text{Max. positive shearing force at } K = W\frac{a}{l}.$$

Now, supposing the weight to move onward, it will in the next instant have passed to the other side of K, and the shearing force will have undergone a sudden change. It will now be equal to the supporting force at the end B,

$$Q = W\frac{b}{l}.$$

But not only is the magnitude of the shearing force suddenly changed, but the tendency to slide is now in the other direction, and the shearing force is negative. As the weight moves farther to the right of K the shearing force diminishes, thus

$$\text{Max. negative shearing force at } K = W\frac{b}{l}.$$

Wherever we take the point K it will always be true that the maximum positive shearing force will occur when the weight lies immediately to the left of K, and the maximum negative when the weight lies immediately to the right. The maximum positive shearing force for every point in the beam may be represented by the ordinates of a sloping line AB' below the beam, the length BB' being taken to represent W. And similarly the maximum negative shearing force at any point by the ordinates of the sloping line $A'B$ above, AA' also being taken to represent W.

Next as to the bending moment. When the weight lies to the left of K, and is at a distance from A equal to $-x$, the bending moment at K is given by

$$Pb = W\frac{b}{l}x.$$

This goes on increasing as x increases until the weight reaches the point K. After having passed K the bending moment at K must be differently expressed, being then

$$\frac{W(l-x)}{l}a,$$

which becomes smaller as x increases; so that the greatest bending moment at K occurs when the load is immediately over K, and then the

$$\text{Max. bending Moment at } K = \frac{Wab}{l}.$$

If the point K is taken in the centre of the beam,

$$\text{Max. Moment at centre} = \tfrac{1}{4}Wl \quad \text{as before.}$$

If ordinates be set up at all points to represent the maximum bending moments at these points, a parabola (ACB) will be obtained. For the expression for the maximum bending moment is just twice that previously obtained for the same weight distributed uniformly.

If there are more weights, W_1, W_2, &c., on the beam, and W_1 lie to the right of K, the shearing force at $K = P - W_1$, where P is the right-hand supporting force. Now, suppose we shift W_1 to the left of K, we shall diminish the supporting force to P' say, and this will be the new shearing force at K. The difference between P and P' will be less than W_1, and the shearing force will be increased by passing W_1 to the left of K. If we were to remove W_1 altogether the diminution of P will be less than the whole of W_1, and so the shearing force at K will be increased by so doing. We obtain the greatest positive shearing force at K when all the weights are to the left of K, but as near to K as possible. The greatest negative shearing force will occur when all the weights lie to the right of K, as near to K as possible.

The maximum bending moment at K will occur when the weights are as near K as possible, whether to the right or left. Any addition to the load, on whichever side of K it is placed, will cause an addition to the bending moment.

There is another important case, that in which we have a continuous load of uniform intensity passing over the beam, as when a long train passes on to a bridge. We observe that as the train approaches K, the supporting force at B, and therefore the shearing force at K, increases. When any portion of the weight lies to the right of K, the supporting force will be increased by a part of the weight lying to the right of K; but when we have subtracted the whole of that weight, the difference, which will be the shearing force at K, will be less than before; thus the maximum positive shearing force at K will occur when the portion AK is fully loaded, and no part of the load is on KB. To find its value we have only to determine the supporting force at B, by taking moments about it; then

$$F_K = wa\,\frac{\frac{1}{2}a}{l} = \frac{1}{2}\frac{wa^2}{l},$$

that is, the magnitude is proportional to the square of the distance of the point from the end A. It will be graphically represented by the ordinates of a parabola which has its vertex at A and axis vertical, cutting the vertical through B in a point B' such that $BB' = \frac{1}{2}wl$, that is, half the weight on the beam when fully loaded. As the load travels onward the shearing force diminishes at last to zero, and then changes sign, becoming negative, the numerical magnitude increasing as the rear of the load approaches K. The maximum negative shearing force will occur when the portion KB only is loaded. The ordinates of a parabola set below the line of the beam having its vertex at B and axis vertical, will represent the maximum negative shearing force.

The question of maximum bending moment is more simple. It will occur at any point when the beam is fully loaded; for at any point the bending moment is the sum of the bending moments due to all the small portions into which the load may be divided, and the removal of any one of them will cause a diminution of bending action throughout the whole length of the beam. A parabola, with its highest ordinate at the centre $= \frac{1}{8}wl^2$, will represent it at any point.

34. *Counter Bracing of Girders.*—In the design of a framework girder it is very important to take account of the maximum positive and negative shearing forces due to a travelling load.

In such a structure the shearing force is resisted by the diagonal bars, and in general these bars are so placed as to be in tension, for the bar may then be made lighter than if subject to a compressive force of the same amount. Suppose the diagonal bars so arranged as to be all in tension when the girder is fully loaded, or when there is only the dead weight of the girder itself to be taken account of. There may be ample provision made for withstanding the tensile forces, and yet it will be important to examine if there may not be some disposition of the travelling load which would cause a thrust on some of the diagonals. If so, the maximum amount of this must be calculated, and the structure made capable of withstanding it. If the shearing force at any section of the girder is what we have called a positive shearing force, that in which the right-hand portion tends to slide upwards relatively to the left, then, in order that it may be withstood by the tension of a diagonal bar, the bar must slope upwards to the right. If the bar so slopes, and by the movement of the travelling load the shearing force becomes negative, then the bar will be subjected to compression. Now, it will frequently happen that in the central divisions of a girder the positive or negative shearing forces due to the dead load are less than the negative or positive shearing forces due to the travelling load, so that if those bars are arranged to be in tension under the dead load, then, on the passage of the travelling load, the stress will be changed to compression. In some cases the bars are slender and not suited to sustain compression; the shearing force is then provided for by the addition of a second diagonal, sloping in the opposite direction, which, by its tension, will perform the duty the first bar would otherwise have to perform by compression. Such a bar is called a counter-brace. We frequently see such additional bars fitted to the middle divisions of framework girders.

Again, the powers of resistance of a piece of material to a given maximum load are greater the smaller the fluctuation in the stress to which it is exposed; and therefore, in determining its dimensions, it is important to know not only the maximum but also the minimum stress to which it is exposed. This can be done on the principles which have just been explained.

EXAMPLES.

1. A single load of 50 tons traverses a bridge of 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities for the quarter and half span.

2. A train weighing one ton per foot run, and more than 100 feet long, traverses a bridge 100 feet span. Draw the curves of maximum shearing force and bending moment, and give the values of these quantities at the quarter and half span.

3. In the last question, suppose the permanent load $\frac{3}{4}$ths ton per foot run. Find within what limits counterbracing will be required.

4. In Ex. 5, page 55, the maximum rolling load is estimated at 1 ton per foot run. Determine which of the diagonals will be in compression, and the amount of that compression, assuming a complete number of divisions to be loaded.

The two centre diagonals are the only ones which can be in compression, the maximum amount of which will be $= (3{\cdot}2 - 2)\sqrt{2} = 1{\cdot}7$. It will occur when the rolling load occupies 4 divisions only of the bridge.

5. In the last question, suppose a single load of 20 tons to traverse the bridge. Find the maximum stress, both tension and compression, on each part of the girder.

Divisions.	1	2	3	4	5
Max. tension, bottom boom, -	0	27	48	63	72
Max. compression, upper boom, -	27	48	63	72	75
Max. tension of diagonals, - -	38·1	31·1	24	17	9·8
Max. compression of diagonals, -	—	—	—	0	2·8

6. In the two preceding questions, find the fluctuation of stress on each part of the girder.

METHOD OF SECTIONS.

35. *Method of Sections applied to Incomplete Frames. Culmann's Theorem.*—The straining actions due to a vertical load may either be wholly resisted by internal forces called into play within the structure itself, or also in part by the horizontal reaction of fixed abutments: the supporting forces being in the first case vertical, and in the second having a horizontal component. The distinction is one of the greatest importance in the theory of structures, which are thus divided into two classes, Girders and Arches, including under the last head also chains. It is the first class alone which we consider in this chapter.

The general consideration of internal forces is outside the limits of this part of our work, and we shall here merely consider some cases of framework structures, commencing with that of an incomplete frame.

Incomplete frames are in general, as in Chapter I., structures of the arch and chain class, but by a slight modification we can readily convert such a frame into a girder and thus obtain very interesting results.

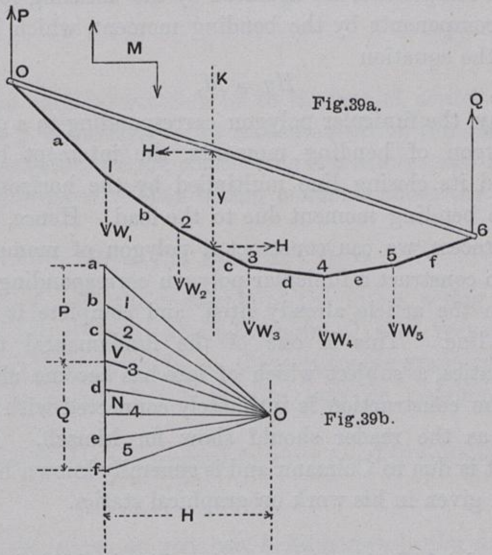


Fig.39a.

Fig.39b.

Fig. 39*a* shows a funicular polygon such as that in Fig. 11, page 15, except that the supports are removed and replaced by a strut $O6$. By this addition the polygon becomes a closed figure, and $O6$ is therefore called its "closing line." The structure is carried by suspending rods at the joints $O6$, and loaded as shown. The construction of the diagram of forces, Fig. 39*b*, has been sufficiently explained in the article referred to, and it only remains to observe that the supporting forces PQ are immediately derived from the diagram by drawing OV parallel to the closing line, which is not necessarily horizontal. The horizontal thrust of the strut and tension of the rope is found as before by drawing ON horizontal.

This structure may now be regarded as a girder, the load on which, together with the vertical supporting forces, produce definite straining actions M and F on any section. Let the section be KK' in the figure cutting one of the parts of the rope and the strut as shown in the figure: let the intercept be y. Consider the forces acting at the section on the left-hand half of the girder, the horizontal components of these forces are equal and opposite, acting as shown in the figure, each being H or ON in the diagram of forces. The vertical components are balanced by the shearing force, and the horizontal components by the bending moment, which last fact we express by the equation

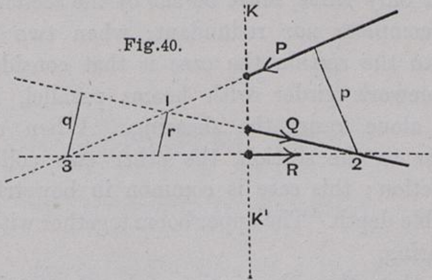
$$Hy = M,$$

that is to say, the funicular polygon corresponding to a given load is also a polygon of bending moments, the intercept between the polygon and its closing line multiplied by the horizontal force is equal to the bending moment due to the load. Hence, by a purely graphical process, we can construct a polygon of moments, for we have only to construct a funicular polygon corresponding to the load as shown in the article already cited, and complete it by drawing its closing line. This is one of the fundamental theorems of graphical statics, a subject which of late has become almost a new science. The construction is intimately connected with the process of Art. 29 as the reader should show for himself. In its complete form it is due to Culmann and is generally known by his name, having been given in his work on graphical statics.

36. *Method of Sections in general. Ritter's Method.*—In frames which are complete the number of bars cut by the section, instead of being two only, as in the preceding case, is in general three at least.

In Fig. 40 let KK' be the section cutting the three bars in three points which may be considered as the points of application of three forces PQR due to the reaction of the bars, which balance the shearing and bending actions to which the section is subject. Resolving horizontally and vertically, and taking moments, we should —remembering that the load being wholly vertical the sum of the horizontal components must be zero—obtain three equations which would determine P, Q, R. It is, however, simpler to employ a method introduced by Ritter which enables us to obtain the value of each force at once. Let the lines of action of P, Q intersect in the point 1, Q and R in 2, P and R in 3, and let the perpendicular

Fig. 40.



dropped from each intersection on to the line of action of the third force be r, p, q, respectively: by measurement on the drawing of the framework structure we are considering it is always easy to determine these perpendiculars. Then taking moments about the three points we get

$$Rr = L_1\,;\ Pp = L_2\,;\ Qq = L_3,$$

where L_1, L_2, L_3, are the moments of the forces acting on the left-hand half of the structure about the points 1, 2, 3, respectively. At page 68 it was shown how to get these moments graphically from the polygon of moments, but they also may be obtained by direct calculation.

We may write down a general formula for this method, thus—

$$Hh = L,$$

where H is the stress on any bar, h its perpendicular distance from the intersection of the two others cut by a section, and L is the moment of the forces about that intersection. The special case in which the intersection lies on the section considered so that the moment L becomes the bending moment (M) on the section, has already been considered in Chapter II. When the stress on a single bar is required as a verification of results obtained by graphical methods, or where the maximum stress due to a travelling load has to be determined, this method is often serviceable, but as a general method it is inconvenient from the amount of arithmetical labour involved, The shearing action on the section is resisted by the components parallel to the section of the stress on the several bars. In the case of the incomplete frame of Fig. 39, p. 79, these components are given at once by the diagram of forces. In general, however, three bars, and only three, must be cut by the section if the frame be neither incomplete nor redundant; when two of these are perpendicular to the section the case is that considered in Chap. III. of a framework girder with booms parallel, in which the diagonal bars alone resist the shearing. When one bar only is perpendicular to the section, the other two collectively resist the shearing action: this case is common in bowstring and other girders of variable depth. The upper boom together with the web here resists the shearing.

When more than three bars are cut by the section, the stress in each is generally indeterminate on account of the number of bars being redundant. On this question it will be sufficient for the present to refer to Chapter II., Section II.

EXAMPLES.

1. In example 3, page 73, construct the polygon of bending moments by Culmann's method.

2. In example 6, page 36, find the stress on each part of the roof by Ritter's method.

3. In example 7, page 36, find the stress on each bar by Ritter's method.

4. If a parabolic bowstring girder be subject to a uniform travelling load, represented by the application of equal weights to some or all of the verticals, show that the horizontal component of the maximum stress on each diagonal is the same for all.

REFERENCES.

For further information on subjects connected with the present chapter the reader may refer to

Naval Architecture. W. H. WHITE. Murray.

Graphical Statics. LIEUT. CLARKE, R.E. Spon.

Graphical Determination of Forces in Engineering Structures. J. B. CHALMERS. Macmillan.

Graphical Statics. H. T. EDDY. Van Nostrand.

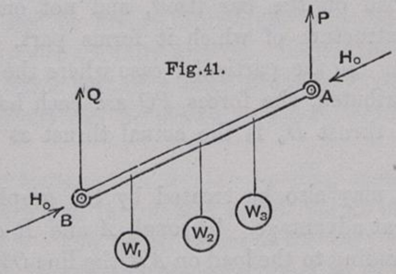
CHAPTER IV.

FRAMEWORK IN GENERAL.

37. *Straining Actions on the Bars of a Frame. General Method of Reduction.*—When the bars of a frame are not straight, or when they carry loads at some intermediate points, the straining action on them is not generally a simple thrust or pull, but includes a shearing and bending action. The present and two following articles will be devoted to some cases of this kind.

First suppose the bars straight, but let one or more be loaded in any way, and in the first instance consider any one bar AB (Fig. 41) apart from the rest of the frame, and suspended by

Fig. 41.



strings in an inclined position. Let any weights act on it as shown in the figure, then the tensions of the vertical strings will be just the same as in a beam, AB, supported horizontally at the ends and loaded at the same points with the same weights. Resolve the forces into two sets, one along the bar, the other transverse to the bar. The first set produce shearing and bending just as if applied to a beam in a horizontal position, while the second set produce a longitudinal stress, which will be different in each division of the bar. Let θ be the inclination of the bar to the vertical, then the pulls on the successive divisions are

$$P.\cos\theta : (P - W_3)\cos\theta : (P - W_3 - W_2)\cos\theta : \ldots.$$

the last being a thrust equal to $Q.\cos\theta$, so that the stress varies from $Q.\cos\theta$ to $-P.\cos\theta$. Now observe that we can apply to AB at its ends, in the direction of its length, a thrust, H_0, of any magnitude we please without altering P and Q, but that we cannot apply a force in any other direction, whence it follows that when AB forms one of the bars of a frame, its reaction on the joint A must be a downward force, P, and a force, H_0, which must have the direction BA, while the reaction on B in like manner consists of a downward force, Q, and an equal force, H_0, in the direction AB. The downward forces P, Q, are described as the part of the load on AB carried at the joints A, B, and it is now clear that if these quantities be estimated for each bar and added to the load directly suspended there, we must be able to determine the forces H_0 by exactly the same process as that by which we find the stress on each bar of a frame loaded at the joints. The actual thrust on AB evidently varies between $H_0 - P.\cos\theta$ at the top, to $H_0 + Q.\cos\theta$ at the bottom, so that H_0 may be described as the mean thrust on the bar, while the shearing and bending depend solely on the load on the bar itself, and not on the nature of the framework structure of which it forms part, or on the load on that structure. In the particular case where the load on the bar is uniformly distributed, the forces PQ are each half the weight of the bar, and the thrust H_0 is the actual thrust at the *middle point* of the bar.

This question may also be treated by the graphical method of Art. 35 with great advantage. Through A and B draw a funicular polygon corresponding to the load on AB, the line OV in the diagram of forces will be parallel to AB and may be taken to represent H_0. This funicular polygon will be the curve of bending moment for the bar, and the other straining actions at every point are immediately deducible. It will be seen presently that the bar need not be straight.

For simplicity it has been supposed that the forces acting on the bar are parallel: if they be not, the reduction is not quite so simple. It will then be necessary to resolve the forces into components along the bar and transverse to the bar, the second set can be treated as above, while the total amount of the first set must be considered as part of the force applied to the joints either at A or B. Such cases, however, do not often occur, and it is therefore unnecessary to dwell on them.

The joints have been supposed simple pin joints or their equivalents, but the method used for frames loaded at the joints wil apply even if the real or ideal centres of rotation of the bars are not coincident, provided only the centre lines prolonged pass through the point where the load is applied. The method of reduction just explained then requires modification. Such cases are of frequent occurrence, and the next article will be devoted to them.

38. *Hinged Girders. Virtual Joints.*—The case of a loaded beam, the ends of which overhang the supports on which it rests, has already been considered in Art. 21, where it was shown that the straining actions at any point might be expressed in terms of the bending moments at the points of support, which of course will be determined by the load on the overhanging part. If the overhanging parts be supported, as in the case of a beam continuous over several spans, or with the ends fixed in a wall, the same formula will serve to express the straining actions at any point in terms of the bending moments at the points of support, but those bending moments will not be known unless the material of the beam and the mode of support are fully known. Hence the full consideration of such cases forms part of a later division of our work. Certain general conclusions can be drawn, however, which are of practical interest.

The graphic construction for the bending moment at any point of a beam, CD, which is not free at the points of support, is given in Fig. 28, p. 45. The figure refers to the case where the bending action at C and D is in the opposite direction to the bending action near the centre, as it is easily seen must be the case in general. The points of intersection of the moment line with the curve of moments drawn, as explained in the article cited, on the supposition of the ends being free, show where the negative bending at the ends passes into the positive bending at the centre. Here there is no bending at all, and the central part of the beam (EF in figure) is exactly in the position of a beam supported but otherwise free at its ends. We may therefore treat the case as if E and F were joints, the position of which will be known if the bending moments at the ends are known, and conversely. In some cases there may be actual joints in given positions, while in others there will be "virtual joints," the position of which may be supposed known for the purposes of the investigation.

Fig.42.

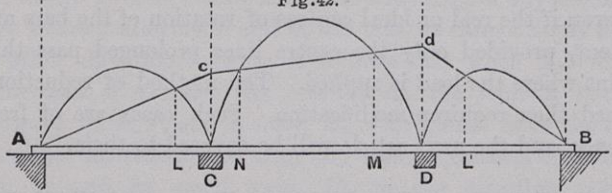


Fig. 42 shows a beam *AB* continuous over three spans, the moment curves for which will be known when the load resting on each span is known. It is evident from what has been said that the moment line must be the broken line *AcdB*, cutting the moment curve of the centre span in two points, and the moment curves of the end spans each in one point, the others being the ends of the beam. Thus there are four virtual joints, of which two must be supposed known in order to find the straining actions at any point. Their position will depend (1) on whether the supports are on the same level or not, (2) on the material and mode of construction of the beam, (3) on the load. Such a beam is in a condition analogous to that of a frame with redundant bars, considered in Chapter II. Section III.; the straining actions are indeterminate by purely statical considerations, for the same reason as before. We can, however, see that the bending action at each point is in general less than if the beam were not continuous.

In one particular case the position of the virtual joints can be foreseen. Suppose a perfectly straight beam, of uniform transverse section, to be continuous over an indefinite number of equal spans: let the weight of the beam be negligible, and let equal weights be placed at the centre of each span. Then since the pressure on each support must be equal to the weight, the beam is acted on by equal forces at equal distances alternately upwards and downwards, and there being perfect symmetry in the action of the upward and downward forces, the virtual joints must be midway between the centre and the points of support of each span.

In the special case where the beam is uniformly loaded we can further see that the load resting on the supports is not one half the weight of the parts of the beam resting there, as it would be if the beam were not continuous, but must in general be greater for the centre supports and less for the end supports. For if the virtual joints be $LNML'$, as in the figure, it is easily seen that A carries half the weight of AL, not of AC, while C carries half the weight of AL and NM, together with the whole weight of CL and CN. This observation shows that in trussed beams where, as is usually the case, the loaded beam is continuous through certain joints, the effect of the continuity is generally to transfer a part of the weight from the joints where the ends are free to the joints where the beam is continuous. We shall return to this point hereafter.

The principle of continuity is frequently taken advantage of in the construction of girders of uniform depth by making them continuous over several spans. The virtual joints, then, vary in position for each position of the travelling load, rendering it a complicated matter to determine the maximum straining actions, while there is always an element of uncertainty about the results, for reasons already referred to and afterwards to be stated more fully.

In some structures, however, the joints have a definite position.

Fig.43.

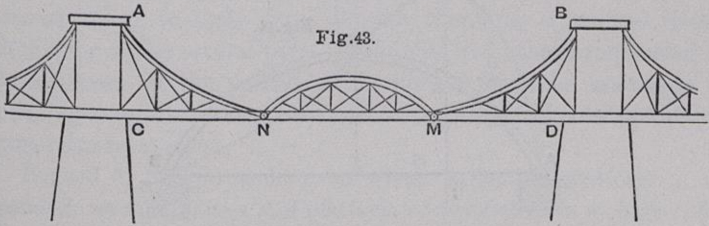


Figure 43 shows a cantilever bowstring girder, consisting of a central bowstring girder NM, the ends of which rest on parts ACN, BDM, projecting from the piers, technically described as cantilevers. The joints here are at N and M. In structures of great span, in which the weight of the structure is the principal element, so that the variations in distribution are small, this type of girder is economical in weight. In a bridge over the Forth now in process of execution (1883), the central portion for each of two principal openings consists of a bowstring suspension girder 350 feet span, while the cantilevers are each no less than 675 feet in length, making a total span of 1700 feet. These cantilevers are of great depth near the piers, and, to provide against wind pressure, they are there likewise greatly increased in breadth, and solidly united to them. For a description of this design, which, from its gigantic dimensions and other unusual features, deserves attentive study, the reader is referred to *Engineering* for September, 1882.

39. *Hinged Arches.*—In the second section of Chapter I. certain forms of arches were considered which are simply inverted chains, and require for equilibrium a load of a certain definite intensity at each point. We shall now take the case of an arched rib capable of sustaining a load distributed in any way. We shall suppose the load vertical, and, to take the thrust of the arch, we shall imagine a tie rod introduced so as to convert it into a bowstring girder. If the straining actions at each point of the rib are to be determinate without reference to the relative flexibility of the several parts of the rib, and other circumstances, we must have, as in the case of the continuous beam, joints in some given position. The necessary joints are in this instance three in number, and, we shall suppose, are at the crown C (Fig. 44), and one at each springing A and B.

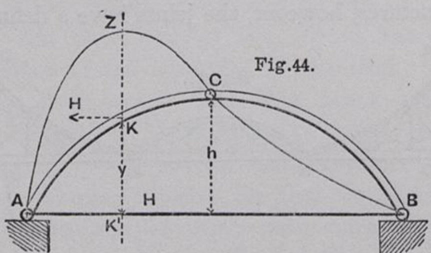


Fig.44.

Taking a vertical section KK' through the rib and tie, let the bending moment due to the vertical load and supporting forces be M. This bending moment is resisted, *first*, by the horizontal forces called into play; that is to say, the pull of the tie rod H at K', and the equal and opposite horizontal thrust of the rib at K; *secondly*, by the resistance to bending of the rib itself, the moment of which we will call μ. Hence if y be the ordinate of the point considered, we must have

$$M = Hy + \mu.$$

To determine H we have only to notice that at the crown where $y = h$ there is a joint, that is, $\mu = 0$,

$$\therefore \quad M_0 = Hh,$$

where M_0 is the bending moment due to the load for the central section. Thus, to determine μ we have the equation

$$\mu = M_0 - M \cdot \frac{y}{h}.$$

The graphic representation of μ is very simple. Let us imagine the curve of moments drawn for the given vertical load, and let it be so drawn as to pass through A, B, and C, which is evidently always possible. Then, if Y be the ordinate of the curve,

$$M = H \cdot Y.$$

Therefore, by substitution,

$$\mu = H(Y - y).$$

So that the bending moment at each point of the rib is represented graphically by the vertical intercept between the rib and the curve of moments. In the figure, the dotted curve $AZCB$ is the curve of moments, and KZ is the intercept in question.

Arched ribs in practice are rarely, if ever, hinged, and the straining actions on them occasioned by a distribution of the load not corresponding to their form depend, therefore, upon the relative flexibility of the several parts of the rib, and other complicated circumstances. If the position of the virtual joints be known, or the bending moments at any three points, the graphical construction just given can be applied.

Instead of a rigid arch, from which a flexible platform is suspended, we may have a stiff platform suspended from a chain. This is the case where a suspension bridge is adapted to a variable load by means of a stiffening girder. For this case it will be sufficient to refer to Ex. 3, page 97.

40. *Structures of Uniform Strength.*—In any framework structure without redundant bars, the stress on each bar may be determined as in Chapter I., by drawing a diagram of forces for any given load, W, and expressed by the formula

$$H = kW,$$

where k is a co-efficient depending on the distribution of the load. If A be the sectional area of the bar we find by division the stress per sq. inch, which must not exceed a certain limit, depending on the nature of the material as explained in Part IV. of this work. When the structure is completely adapted to the load which it has to carry, the stress per sq. inch is the same for all the bars, and it is then said to be of Uniform Strength. Uniformity of strength cannot be reached exactly in practice, but it is a theoretical condition which is carried out as far as possible in the design of the structure. Other things being equal, the weight of a structure of uniform strength is less than that of any other. Such a structure is therefore less costly, for weight is to a great extent a measure of cost.

Whenever the load is known, the weight of a structure of given type and of uniform strength can be calculated thus. Suppose A the sectional area of a piece, H, the stress on it, f, a co-efficient of strength, then

$$H = fA.$$

Next let w_0 be the weight of a unit of volume, usually a cubic inch, and assume

$$\lambda = \frac{f}{w_0},$$

then λ is a certain length, being in fact the length of a bar of the material which will just carry its own weight. Its value for various materials is given in Chapter XVIII. Then, assuming the piece prismatic and of length s, its weight is

$$w_0 A s = \frac{Hs}{\lambda},$$

and therefore the weight of the whole structure must be for the same value of λ,

$$W_0 = \frac{\Sigma Hs}{\lambda},$$

the summation extending to all the pieces in the structure, and being performed by integration in a continuous arch or chain. It will be observed that s is the length of any line in the frame-diagram, and H that of the corresponding line in the diagram of forces; we have only then to take the sum of the products of these lines and divide by λ, the result will be the weight of the structure. It is however generally necessary to find the weights, W_1, W_2, of the parts in compression and in tension separately, because the value of λ is generally different in the two cases.

A remarkable connection was shown by the late Prof. Clerk Maxwell to exist between W_1 and W_2. Let us take a structure of the girder class and suppose the total load upon it G, and the height of the centre of gravity of that load above the points of support h. Imagine this structure to become gradually smaller without altering either its proportions or the magnitude and distribution of the load G, then G descends and does work during the descent in overcoming the resistance (T) of the bars in compression to diminution of length, while at the same time the bars in tension (P) do work during contraction. The values of T and P do not alter, for the diagram of forces remains the same, and therefore if we conceive the process to continue till the structure has shrunk to a point,

$$Gh = \Sigma Ts - \Sigma Ps = \lambda_1 W_1 - \lambda_2 W_2.$$

In particular, if the centre of gravity of the load lies on the line of support, and if the co-efficients be the same, the weights of the parts in compression and tension will be equal. A corresponding formula may be obtained for structures of the arch-class by taking into account the thrust.

The weight of an actual structure is always greater than that found by this method. First, an addition must be made to allow for joints and fastenings. Thus, for example, in ordinary pin joints the eye of the bar weighs more than the corresponding fraction of the length of the bar, and in addition there is the weight of the pin. Secondly, in all structures there is more or less redundant material necessary to provide against accidental strains not comprehended in the useful load. Thirdly, there are local straining actions in the pieces occasioned by their own weight and other causes.

41. *Stress due to the Weight of a Structure.*—The total load on any structure consists partly of external forces applied to it at various points, and partly of its own weight: the total stress on any member is therefore the sum of that due to the external load and of that due to the weight of the structure itself. As that stress cannot exceed a certain limit, depending on the strength of the material, it necessarily follows that the stress due to the weight is so much deducted from the strength. Thus the consideration of the weight of a structure is an essential part of the subject, even if we disregard the question of cost.

The weight of each member is of course distributed over its whole length, and so also may be a part or the whole of the external load. Applying the general method of reduction explained in Art. 37, we suppose an equivalent load applied at each joint, and drawing a diagram of forces we determine the mean stress, H, on the member. If the unsupported length of the bars be not too great, a matter to be considered presently, this stress will be the principal part of the straining action on the bar, and the bending may be neglected as in the preceding article.

Now consider two structures similar in form and loaded with the same total weight, distributed in the same way, so that the only difference in the structures is in size: then the stress on corresponding bars must be the same, for the structures have the same diagram of forces. That is to say, in the formula

$$H = kW,$$

the coefficient k depends on the type of structure and the distribution of the load upon it, but not on its dimensions. Dividing by the sectional area the intensity of the stress is

$$p = k\frac{W}{A}.$$

Next let W_0 be the weight of the structure itself; and suppose the relative sectional areas of the several pieces the same, then

$$W_0 = w_0 \,.\, cAl,$$

where c is a coefficient depending on the type of structure, and l a length depending on the linear dimensions of the structure. For example, in roofs and bridges l may conveniently be taken as the span. Then if k_0 be the value of k, which corresponds to the distribution of the weight of the structure, which will be the same whether the structure be large or small,

$$p_0 = k_0 \,.\, \frac{W_0}{A} = w_0 k_0 cl,$$

will be the stress due to the weight of the structure. In other words, the stress due to the weight of similar structures varies as their linear dimensions.

Since p_0 cannot exceed f it follows at once that there must be a limit to the size of each particular type of structure, beyond which it will not carry its own weight. If L be that limit given by

$$L = \frac{\lambda}{k_0 c},$$

the stress due to the weight of any similar structure of smaller dimensions will be simply

$$p_0 = f \cdot \frac{l}{L},$$

and

$$f' = f - p_0 = f \cdot \frac{L - l}{L}$$

is the strength which may be allowed in calculations made irrespectively of weight. If the structure be of uniform strength throughout under its own weight, the value of p_0 will be the same for each member, but this is not necessarily the case, and there may be a different value of f' for each member. The actual limiting dimensions of the structure will of course be the least of the various values corresponding to the various members.

The conclusion here arrived at is obviously of the greatest importance, for it immediately follows that in designing a roof, bridge, or other structure of great size, the weight of the structure is the principal thing to be considered in estimating the straining actions upon it, while a certain limiting span can never be exceeded. On the other hand, in small structures the straining actions due to the weight are unimportant; it is the magnitude and variations of the external load which have the greatest influence. This remark also applies to the local straining actions which produce bending in the pieces, their relative importance increases with the size of the structure, and it is necessary to provide against them by additional trussing. A large structure is therefore generally of more complex construction than a small one, as is illustrated by the various types of roof-trusses considered in Chapter I.

The difference of type of large structures and small ones, as well as the circumstances mentioned at the close of the last article, render tentative processes generally necessary in calculations respecting weight. If the type of structure and the distribution of the total load, W, be supposed known, the value of the coefficients k and c will be known for some given member. By assuming the stress on that member equal to the co-efficient of strength f, we find

$$W_0 = W \cdot ck \cdot \frac{l}{\lambda},$$

a formula which gives the weight of the structure in terms of the load, but the co-efficients will generally vary according to the span. Among the circumstances on which they depend the ratio of the vertical to the horizontal dimensions of the structure is most important. For a given span k diminishes when the depth is increased, while on the other hand c generally increases, so that for a certain ratio of depth to span the weight of the structure is least. In ideal cases c may remain the same (Ex. 10, p. 97), but in actual structures the redundant weight of material necessary to give stiffness and lateral stability increases, so that the most economical ratio of depth to span is generally much less than would be found by neglecting such considerations. These points are illustrated by examples at the end of this Chapter and Chapter XII., where the question is again considered briefly; but for detailed applications to actual structures the reader is referred to works on bridges, in the design of which it is of the greatest importance.

42. *Straining Actions on a Loaded Structure in General.*—The results obtained in the last chapter for the case of parallel forces acting on a structure possessing a plane of symmetry in which the forces lie, may be readily extended to structures which have an axis of symmetry acted on by any forces passing through that axis and perpendicular to it. This is the case, for example, of a beam acted on by a vertical load, and also by some horizontal forces arising say from the thrust of a roof or from wind pressure. We have then only to consider the vertical and horizontal forces separately. Each will produce shearing and bending in its own plane, which may be represented by polygons as before. The total straining action will be simply shearing and bending, and will be as before independent of the particular structure on which the forces operate. The magnitude of the straining action, whether shearing or bending, will be the square root of the sum of the squares of its components, and may therefore be readily found by construction and exhibited graphically by curves. In shafts such cases are common, and some examples will be given hereafter.

Another entirely different kind of straining action sometimes occurs in structures proper (roofs, bridges, etc.), and in machines is one of the principal things to be considered. Imagine a structure of any kind to be divided by an ideal plane section into parts A and B, and to be acted on by forces parallel to that plane.

Let the forces acting on A reduce to a couple the axis of which is perpendicular to the section, the forces on B are equal and opposite, and the two equal and opposite couples tend to cause A and B to rotate relatively to each other. As already stated in Art. 16 this effect is called Twisting, and the magnitude of the twisting action is measured by the magnitude of either of the couples which form its elements.

Simple twisting sometimes occurs in practice, for example, when a capstan is rotated by equal forces applied to all the bars, but it is generally combined with shearing and bending. It is then necessary to know about what axis the twisting moment should be reckoned, which will depend on the nature of the structure. In shafts and other cases to be considered hereafter the geometrical axis is an axis of symmetry which at once determines this.

When twisting exists the shearing and bending are determined by the same method as before, for they are independent of the axis of reference. Should however the structure be subject to a thrust or a pull (Art 16), the axis about which the bending moment should be reckoned must be known, for it will depend on the nature of the structure.

These general observations will be illustrated hereafter, and are only introduced here to show how far straining actions can be regarded as depending solely on the external forces operating on the structure without reference to any other circumstances.

43. *Framework with Redundant Parts.*—In a complete frame, without redundant bars (pp. 13, 56), suppose a link applied to any two bars, one end attached to each. Let the link be provided with a right and left handed screw or other means of altering its length at pleasure, then by screwing up the link a pull may be produced in the link of any magnitude we please, while a corresponding stress will be produced in each bar of the frame which will bear a given ratio to the pull. Such a link may be called a straining link, and by its addition we obtain a frame with one redundant bar. The stress-ratio on the parts of a frame of this kind is completely definite, but the magnitude of the stress may be anything we please. Instead of one straining link we may have any number, and if the stress on each of these links be given, the same thing will be true. Thus it appears that a frame with redundant parts may be in a state of stress even though no external forces act upon it. This is of practical importance on account of the effect of changes of temperature. If all the bars of a frame with redundant parts are equally heated or cooled, the frame expands or contracts as a whole, but no other effect is produced; any inequality, however, causes a stress which may, under certain circumstances, be very great. This (at least theoretically) is one of the reasons why redundant parts are a source of weakness. The necessity of providing against expansion and contraction is well known in large structures resting on supports. The ground connecting the supports suffers little change of temperature, and the structure, therefore, cannot be attached to the supports, but must be enabled to move horizontally by the intervention of rollers. The magnitude of the stress produced when changes of length are forcibly prevented will be considered hereafter (Chapter XII.).

There is no essential difference between a frame the stress on the parts of which is due to the action of straining links, and a frame acted on by external forces; for every force arises from the mutual action between two bodies, and may therefore be represented by a straining link connecting the bodies. Even gravity may be regarded as a number of such links connecting each particle of the heavy body with the earth. Accordingly, if we include in the structure we are considering, the supports and solid ground on which it rests, we may regard it as a frame under no external forces, but including a number of straining links screwed up to a given stress. If the original frame be incomplete, its parts will be capable of motion, and it becomes a machine, as will be explained in Part III. of this work.

44. *Concluding Remarks.*—Various other questions relating to framework remain to be considered, especially with reference to the joints by which the parts are connected, but these, involving other than purely statical considerations, do not come within the present division of our work, but are referred to at a later period.

EXAMPLES.

1. In Ex. 4, page 12, if the weight be supposed uniformly distributed, find the thrust, shearing force, and bending moment at each point of each rafter, and exhibit the results graphically by drawing curves.

Diagrams of shearing force will be sloping lines crossing each rafter at the centre.
Max. shearing for short rafter = 91 lbs.
" " long " = 158·5 "
Diagrams of bending moment will be parabolas.

Max. moment at centre of short rafter = 117 ft.-lbs.

" " long " = 290 "

2. A triangular frame ABC, supported at A and C, with AC horizontal, is constructed of uniform bars weighing 10 lbs. per foot, the lengths being—$AB = 3$ feet, $BC = 4$ feet, and $AC = 5$ feet. Suppose, further, that AB and BC each carry 50 lbs. in the centre. Draw curves of thrust, shearing force, and bending moment for each bar.

3. The platform of a suspension bridge is stiffened by girders hinged at the centre and at the piers. The chains hang in a parabola, and the weight of the platform, chains and suspending rods may be regarded as uniformly distributed. Find the bending moment at any point of the stiffening girder, and exhibit it graphically by a curve when a single load W is placed (1) at the centre of the bridge, and (2) at quarter span.

First case. On account of W each half of the girder will tend to turn downwards about the ends, and will be supported by the uniform upward pull of the suspending rods. $\therefore$ total upward pull for each $\frac{1}{2}$ girder $= W$, because the centre of action is at $\frac{1}{4}$ span. Thus each $\frac{1}{2}$ girder will be in the state of a beam loaded uniformly with W, and supported at the ends. Max. moment at middle of each half

$$= \tfrac{1}{8} W \times \text{half span.}$$

Second case. The upward pull of the suspending rods will still be uniform, but for each half girder will now be only $\frac{1}{2}$ W, found by assuming an equal action and reaction at the centre joint, and taking moments of each half about the ends. For the half girder which carries the weight the bending moment will be the difference between that due to W concentrated in the centre and $\frac{1}{2} W$ distributed uniformly.

$$\therefore \quad \text{Max.} = \tfrac{3}{16} W \times \text{half span.}$$

On the other half it will be due simply to a distributed load of $\frac{1}{2} W$. Max. $= \frac{1}{16} W \times$ half span.

4. A timber beam 24 feet span is trussed by a pair of struts $8'$ apart, resting on iron tension rods forming a simple queen truss $3'$ deep without a diagonal brace. The beam is loaded with 5 tons placed immediately over one of the vertical struts. Find the shearing force and bending moment at any point of the beam, supposing it jointed at the centre and the centre only.

The thrust on each strut must be $2\frac{1}{2}$ tons; therefore, curves of shearing force and bending moment for each half of the beam are the same as those for a beam 12 feet long loaded at a point 4 feet from one end with $2\frac{1}{2}$ tons.

The problem should also be treated by the method of sections. Results should also be obtained for the case where one half the beam is uniformly loaded.

5. A beam uniformly loaded is fixed horizontally at the two ends, and jointed at two given points. Draw the diagrams of shearing force and bending moment. Show that the beam will be strongest when the distance of each point from centre is rather less than $\frac{3}{8}$ span.

6. The platform of a bowstring bridge of span $2a$ is suspended from parabolic arched ribs hinged at crown and springing. One half the platform only is loaded uniformly with w lbs. per foot run. Show that the greatest bending moment on the ribs is $\frac{1}{16} wa^2$.

7. In the last question, if a weight of W tons travel over the bridge, how great will be the maximum bending moment produced?

8. A girder is continuous over three equal spans, and is hinged at points in the centre span midway between centre and piers. Find the virtual joints in the end spans when uniformly loaded throughout.

9. The weight of the chains, platform and suspension rods of a suspension bridge may be treated as a uniform load per foot-run, which at the centre of the bridge is double the weight of the chain. The dip of the chain is $\frac{1}{13}$th the span. The weight of iron being 480 lbs. per cubic foot, and the safe load per square inch of sectional area of chain being 5 tons, find the limiting span, and deduce the sectional area of chains for a load of $\frac{1}{2}$ ton per foot-run on a similar bridge 300 feet span.

If A = sectional area of chains at centre in sq. ins., then $\frac{20}{3}A$ = weight of bridge per foot-run in lbs.

$$\text{Horizontal tension} = \tfrac{65}{6}AL = 5 \times 2240 \,.\, A.$$

$$\therefore L = 1034 \text{ feet.}$$

If A' = area of one chain of the bridge 300 feet span,

$$\text{Whole load on chain} = (\tfrac{20}{3}A' + \tfrac{2240}{4})\,300,$$

$$\text{Horizontal tension} = \tfrac{1}{8}(\tfrac{20}{3}A' + \tfrac{2240}{4})\,300 \times 13 = 5 \times 2240A',$$

$$\therefore A' = 34{\cdot}4 \text{ sq. in. each chain.}$$

Remark.—By the use of steel wire ropes and by lightening the platform and other parts of the structure as much as possible, the limiting span of suspension bridges is much increased, there being several examples of a span of 1250 feet and upwards.

10. In a girder with booms parallel and of uniform transverse section the weight of the web is equal to the weight of the booms. Assuming a co-efficient of strength of 9000 lbs per sq. inch, and the weight of a cubic inch $\frac{5}{18}$th of a lb., show that the limiting span in feet is

$$L = 5400N,$$

where N is the ratio of depth to span.

11. The weight of a rib of parabolic form, span l, rise nl, with transverse section varying for uniform strength under a uniformly distributed load W, is

$$W_0 = \left(\frac{1}{8n} + \tfrac{2}{3}n\right) W \frac{l}{\lambda}.$$

This is least when $n = \dfrac{\sqrt{3}}{4} = {\cdot}433$, then $W_0 = {\cdot}577\, W \dfrac{l}{\lambda}$.

The formula fails if W_0 be nearly equal to W, for the external load would then have to be partly acting upwards to secure uniform distribution of the total load.