CHAPTER II.

STRAINING ACTIONS ON A LOADED STRUCTURE.

16. *Preliminary Explanations.*—In the preceding chapter we have considered only those structures in which the parts are subject to compression and tension alone, except by way of anticipation in a few special cases. But the parts of a structure are generally subject to much more complex forces, and besides, although the forces acting on each bar have been determined, we should, if we stopped here, have a most imperfect idea of the way in which the load affects the structure as a whole.

If we imagine a structure to be made up of any two parts, *A* and *B*, united by joints, or distinguished by an ideal surface cutting through the structure in any direction, the whole of the forces acting on the structure may be separated into two sets, one of which acts on *A*, the other on *B*. Since the structure is in equilibrium as a whole, the two sets of forces must balance one another, and must therefore produce equal and opposite effects on *A* and *B*, effects which are counteracted by the union existing between the parts. The two sets of forces taken together constitute a STRAINING ACTION of which each set is an element, and the object of this and the next two chapters is to consider the straining actions to which loaded structures and parts of structures are subject.

Straining actions differ in kind, according to the nature of the effects which they tend to produce. Four simple cases may be distinguished :—

(1) The parts *A* and *B* may tend to move towards each other or away from each other perpendicular to a given plane. This effect is

called compression or extension, and the corresponding straining action is a thrust or a pull.

(2) A and B may tend to slide past each other parallel to a given plane. This effect is called shearing.

(3) A and B may tend to rotate relatively to each other about an axis lying in a given plane. This is called bending.

(4) A and B may tend to rotate relatively to each other about an axis perpendicular to a given plane. This is called twisting.

In the first two cases the straining action reduces to two equal and opposite forces, and in the second two to two equal and opposite couples. In general, straining actions are compound, consisting of two or more simple straining actions combined. The given plane with reference to which the straining actions are reckoned may always be considered as an ideal section separating A and B even when the actual dividing surface is different. We shall commence by considering the straining actions on a beam of small transverse section.

SECTION I.—BEAMS.

17. *Straining Actions on a Beam.*—The action of a simple thrust or pull on a bar has already been sufficiently considered in chapter I. They are usually considered as separate cases, and the simple straining actions on a bar are therefore reckoned as five in number. The other three are (1) shearing, (2) bending, and (3) twisting, of which the last rarely occurs, except in machines, and will, therefore, be considered in a later division of this work, under that head.

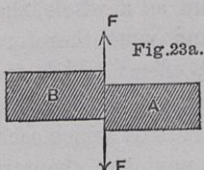


Fig.23a.

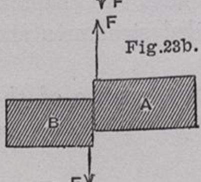


Fig.23b.

Shearing and bending are due to the action of forces, the directions of which are at right angles to the bar: in structures, the forces usually lie in one plane passing through the axis of the bar. A bar loaded in this way is called a beam.

Simple shearing is due to a pair of equal and opposite forces, F (Fig. 23), applied to points very near together, tending to cause the two parts A and B to slide past one another, as shown in the figure (Figs. 23a, 23b). Either element is called the shearing force, and is a measure of the magnitude of the shearing action, but in considering the sign we must consider both together. In this work, if the right-hand portion, A, tends to move upwards, and B downwards, as in Fig. 23b, the shearing action will usually be reckoned negative, while in the converse case (Fig. 23a) it will be reckoned positive.

Simple bending is due to a pair of equal and opposite couples applied to the bar, one acting on A, the other on B, as in Fig. 24,

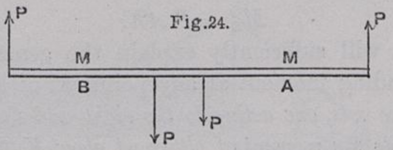


Fig. 24.

tending to make A and B rotate in opposite directions. The magnitude of the bending is measured by the moment of either couple which is called the bending moment. In this work bending moments will usually be reckoned positive when the left-hand half, B, rotates with the hands of a watch, and the right-hand half in the opposite direction. That is to say, when the beam tends to become convex downwards, as in the ordinary case of a loaded beam supported at the ends. In loaded beams shearing and bending generally exist together, and vary from point to point of the beam. We shall now consider various special cases.

18. *Example of a Balanced Lever. General Rules for calculating S.F. and B.M.*—First take the case of a beam, AB, supported at C (Fig. 25), and loaded with weights, PQ, at its ends.

If the weights are such that $P.AC = Q.BC$ the beam will be in equilibrium, but the two parts, AC, BC, tend to turn about C in opposite directions, there is therefore a bending action at C, of which the equal and opposite moments $P.AC$, $Q.BC$ are the elements. Either of these is the bending moment usually denoted by M, so that we write

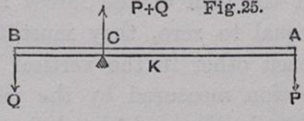


Fig. 25.

$$M_C = P.AC = Q.BC.$$

Not only is there a bending action at C, but if we take any point, K, and consider the forces acting on AK, BK separately, we see that AK tends to turn about K under the action of the force P, while BK tends to turn about K under the action of the forces $P + Q$ at C and Q at B. The first tendency is immediately seen to be simply the moment $P.AK$, while the second is $Q.BK - (P + Q)CK$. The last quantity reduces to $Q.BC - P.CK$, or, remembering that $Q.BC = P.AC$ to $P.AK$ The two moments then, as before, are equal and opposite, and constitute a bending action at K, measured by the bending moment

$$M_K = P.AK.$$

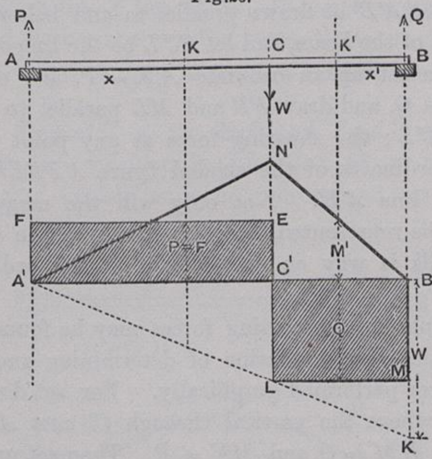
This example will sufficiently explain the general rule for calculating the bending moment at any point, K, of a beam. *Divide the forces into two sets, one acting to the right and the other to the left of K, and estimate the moment of either set about K, then the result will be the bending moment at K.* The example shows that the calculation of one of the two moments will generally be more simple than that of the other, and cases constantly occur, as where a beam is fixed at one end in a wall, where nothing is known about one set of forces except that they balance the other set. In each case the simplest calculation is of course to be preferred.

Moments are measured numerically by unit weight acting at unit leverage, as, for example, 1 ton acting at a leverage of 1 foot, for which the expression "foot-ton" is commonly employed. This phrase, however, is used also for a wholly different quantity, namely, the unit of mechanical work, and for this reason it would be preferable to call the unit of moment a ton-foot for the sake of distinction.

The peculiar action called shearing will be better understood when we come to consider the action of forces on a framework girder in the next section; it will here be sufficient to say that if the sum of the forces acting on AK, BK are not separately equal to zero, they must tend to cause AK, BK to move past each other in the vertical direction, thus constituting a shearing action measured by the magnitude of the shearing force, which may be thus calculated for any point K. *Divide the forces into two sets, one acting to the right of K and the other to the left of K, the algebraical sum of either set is the shearing force at K.* As before, either set may be chosen, whichever gives the result most simply. In the example just given the shearing force at any point of AC is P; and at any point of BC, Q.

19. *Beam Supported at the Ends and Loaded at an Intermediate Point.*—We will next consider the case of a beam supported at

Fig.26.



the ends and loaded at some intermediate point. Before we can apply the rules previously enunciated, to find the shearing force and bending moment at any point, we must first determine the supporting forces at the two ends. We find the force P acting at A, Fig. 26, by taking moments about B, thus,

$$P(a+b) = Wb; \therefore P = \frac{Wb}{a+b},$$

and similarly
$$Q = \frac{Wa}{a+b}.$$

First as to the shearing force. Taking any point K in AC, and considering the forces acting on AK, of which there is only one,

$$F_K = P = \frac{Wb}{a+b}.$$

At any point K' between C and B we have

$$F_{K'} = Q = \frac{Wa}{a+b}.$$

It will be noticed that at K the tendency is for the left-hand portion to slide upwards relatively to the right, whereas at K' the tendency is for the right-hand portion to slide upwards relatively to the left. It is advantageous to distinguish between these two tendencies, as previously stated, by calling the one positive and the other negative.

We may draw a diagram to represent the shearing force at any point thus. Let $A'B'$ be drawn parallel to and below AB to represent the length of the beam, and let $CC'L$ be the line of action of the weight. If we set up an ordinate $A'F = P$, and downwards an ordinate $B'M = Q$, and draw FE and ML parallel to $A'B'$ to meet the vertical $EC'L$; the shearing force at any point will be represented by the ordinates of the shaded figure $A'FELMB'$, measured from the base line $A'B'$. Not only will the magnitude of the shearing force be represented, but also the direction of the sliding tendency. This is why on one side of C' the ordinate was set downwards.

In this example the supporting forces may be found by construction, and thus the whole operation of determining and representing the shearing force performed graphically. For, set down $B'K = W$, join $A'K$, and where the vertical through C' cuts $A'K$, draw LM horizontal, then $B'M = Q$ and $MK = P$. Then set up $A'F = MK$, and draw FE horizontal.

Next as to the bending moment at any point. Take any point K in AC distant x from A, then

$$M_K Px = = \frac{Wb}{a+b}x,$$

and similarly at K' in CB distant x' from B,

$$M_{K'} = Qx = \frac{Wa}{a+b}x';$$

so for either side of C, the bending moment is greater the greater the distance of the point from the end of the beam. Thus the greatest bending moment is at C.

If in the value of M_K we put $x = a$,
or ,, $M_{K'}$,, $x' = b$,

we get the same result, viz., that

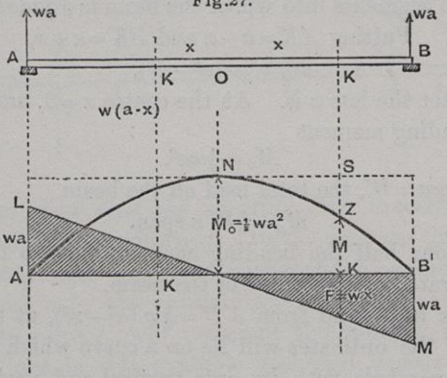
$$M_C = \frac{Wab}{a+b} = \text{greatest bending moment.}$$

The graphical representation of the bending moment at any point is very useful and instructive. We may construct the diagram thus:—$A'B'$ representing the length of the beam set up from C', $C'N$ the bending moment at $C' = \dfrac{Wab}{a+b}$ on some convenient scale, on such a scale for instance as 1 inch = 20 ft.-lbs. Then joining $A'N$ and $B'N$, the ordinate of the figure $A'NB'$, measured from the base line $A'B'$, will express on the scale chosen the bending moment at any point of the beam. If $a = b = \frac{1}{2}$ span, so that the load is applied at the centre of the beam, then

$$M_C = \tfrac{1}{4}W \times \text{span} = \text{greatest bending moment.}$$

20. *Beam Supported at the End and Loaded Uniformly.*—The next example for consideration is that of a beam supported at the ends and loaded uniformly throughout its length with w

Fig.27.



lbs. per foot. (Fig. 27.) Let the span = $2a$. Take any point, K, distant x from the centre O. The load on AK is wAK, and therefore the shearing force at K, reckoning the forces on the left-hand side, must be

$$F_K = wa - wAK = wa - w(a - x) = wx.$$

That is, the shearing force is proportional to the distance of the point from the centre of the beam. At the end A where $x = a$,

$$F_A = wa,$$

and at B where $x = -a$,

$$F_B = -wa.$$

If from $A'B'$, below AB in the diagram, we set up and down ordinates at A' and $B' = wa$ on some scale, and join LM, the ordinates of the sloping line will represent the shearing force at any point. The shearing force at the centre of the beam is zero.

In finding the bending moment at K, reckoning still from the left-hand side, we must clearly take account not only of the supporting force at A, but also of the effect of the load which rests on the portion of the beam AK. The moment of this load about K is the same as if it were all collected at its centre of gravity, namely at the centre of AK. Thus

$$M_K = wa \,.\, AK - wAK \,.\, \frac{AK}{2}$$
$$= \frac{w}{2} AK\,(2a - AK) = \frac{w}{2} AK.KB.$$

That is to say, the bending moment at any point is proportional to the product of the segments into which the beam is divided by the point.

Putting $AK = a - x$ and $BK = a + x$,

$$M_K = \tfrac{1}{2}w(a^2 - x^2),$$

which is greater the less x is. At the centre $x = 0$, and we have the maximum bending moment

$$M_0 = \tfrac{1}{2}wa^2.$$

If we put $2wa = W$, the total load on the beam

$$M_0 = \tfrac{1}{8}W \times \text{span}.$$

This is only one half the bending moment due to the same load when concentrated at the centre of the beam.

If ordinates be set up from $A'B' = \tfrac{1}{2}w\ (a^2 - x^2)$, at all points, the extremities of the ordinates will lie on a curve which may easily be seen to be a parabola with its axis vertical and vertex above the middle point of the beam. For

$$SZ = SK - KZ = \tfrac{1}{2}wa^2 - \tfrac{1}{2}w(a^2 - x^2) = \tfrac{1}{2}wx^2.$$

So that SZ is proportional to SN^2, showing that the curve is a parabola.

21. *Beam Loaded at the Ends and Supported at Intermediate Points.*— Next, suppose a beam (Fig. 28) supported at A, B, and loaded with weights P, Q, at the ends C, D, which overhang the supports. If AC, AB, BD are denoted by a, l, b respectively, the supporting force S at A (by taking moments about B) is given by

$$Sl = P(a + l) - Qb.$$

Similarly R, the supporting force at B, is given by

$$Rl = Q(b + l) - Pa.$$

Take now a point K distant x from A; then

$$F_K = S - P = \frac{Pa - Qb}{l} = \frac{M_A - M_B}{l},$$

where M_A, M_B are the bending moments at A, B.

Also for the bending moment at K,

$$M_K = -Sx + P(a+x) = -\frac{M_A - M_B}{l} \cdot x + M_A,$$

or, as we may write it,

$$M_K = M_A \frac{l-x}{l} + M_B \frac{x}{l}.$$

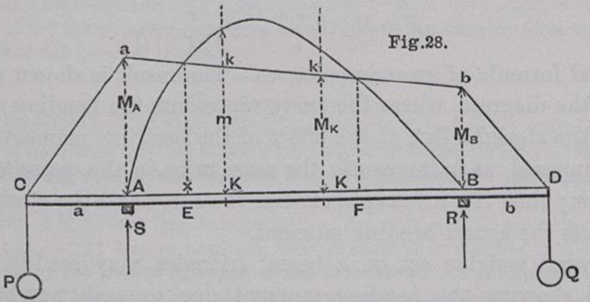


Fig.28.

These formulae show that the shearing force is constant while the bending moment varies uniformly. In the diagram this is indicated by setting up ordinates Aa, Bb, to represent the bending moments at A, B, and joining a, b; the ordinate Kk of this line corresponding to an intermediate point K, will represent the bending moment there. The moments are in this example reckoned positive for upward bending.

An important special case is when $M_A = M_B$; then the bending moment is constant, and the shearing force *zero*. We have then no shearing but only bending. Simple bending is unusual in practice, but an instance occurs in the axle of a carriage.

The ordinates of the straight lines Ca, Db, represent the bending moment at any point of the overhanging parts of the beam.

22. *Application of the Method of Superposition.*—When a beam is acted on by several loads, the principle of superposition already stated in Chap. I. is often very useful in drawing diagrams and writing down formulae for the straining action at any point. Thus, for example, in the preceding case, if there be many weights on the overhanging end of a beam, the bending moment and shearing force at each point must be the sum of that due to each taken separately; and hence it follows that, whatever be the forces acting on a beam, if there be a part AB under the action of no load, and the bending moments at the ends of that part be M_A, M_B, the straining actions at any intermediate point K will always be given by the formulae just written down. And, further, if there be a load of any kind on AB, and m be the bending moment, on the supposition that the beam simply rests on supports at A, B, then the actual bending moment must always be given by

$$M = M_A \cdot \frac{l-x}{l} + M_B \cdot \frac{x}{l} + m,$$

a general formula of great importance. The result is shown graphically in the diagram, where the curve represents the bending moment m, and the straight line ab the effect of the bending moments at the ends, supposed, as is frequently the case, to be in the opposite direction to m; then the intercept between the curve and the straight line represents the actual bending moment.

If several weights act on a beam, triangles may readily be constructed showing the bending moment due to each weight; then adding the ordinates of all the triangles at the points of application of the weights, and joining the extremities by straight lines, a polygon is obtained which is the polygon of bending moments for the whole load. This process may also be applied to shearing forces. It is simple, but somewhat tedious when there are many weights, and other methods of construction will be explained hereafter.

EXAMPLES.

1. A beam, AB, 10 feet long is fixed horizontally at A, and loaded with 10 tons distributed uniformly, and also with 1 ton at B. Find the bending moment in inch tons at A, and also at the middle of the beam.

$M = 720$ inch-tons at A.
$= 210$,, at the centre

2. In the last question find the shearing force at the two points mentioned.

$F = 11$ tons at A.
$= 6$,, at the centre.

3. A beam, AB, 10 feet long is supported at A and B, and loaded with 5 tons at a point distant 2 feet from A. Find the shearing force in tons, and the bending moment in inch-tons at the centre of the beam. Find also the greatest bending moment.

F at the centre $= 1$ ton.
M at the centre $= 60$ inch-tons.
Maximum bending moment $= 96$,,

4. In the last question suppose an additional load of 5 tons to be uniformly distributed. Find the shearing force and bending moment at the centre of the beam.

F at centre $= 1$ ton as before.
M at centre $= 11\frac{1}{4}$ foot-tons $= 135$ inch-tons.

5. A beam, AB, 20 feet long is supported at C and D, two points distant 5 feet from A and 6 feet from B respectively. A load of 5 tons is placed at each extremity. Find the bending moment at the middle of CD in inch-tons.

Moment = 330 inch-tons.

6. In the examples just given draw the diagrams of shearing force and bending moment at each point of the beam.

7. A foundry crane has a horizontal jib, AC, 21 feet long attached to the top of a crane post 14 feet high, which turns on pivots at A and B. The crane carries 15 tons, which may be considered as suspended at the extremity of the jib. The jib is supported by a strut attached to a point in it 7 feet from A, and resting on the crane post at B. Find the stress on crane post and strut, and the shearing force and bending moment at any point of the jib.

Tension of crane post = 30 tons.
Thrust on strut = 50 ,,

8. A rectangular block of wood 20 feet long floats in water; it is required to draw the curves of shearing force and bending moment when loaded (1) with 1 cwt. in the middle; (2) with $\frac{1}{2}$ cwt. at each end, and (3) $\frac{1}{2}$ cwt. placed at two points equidistant from the middle and each end.

9. A beam, AB, 20 feet long is supported at the ends, and loaded at two points distant 6 feet and 11 feet respectively from one end with weights of 8 tons and 12 tons: employ the method of superposition to construct the polygons of shearing force and bending moment. Find the maximum bending moment in inch-tons.

Maximum moment = 972 inch-tons.

10. A beam is supported at the ends and loaded uniformly throughout a part of its length: show that the diagram of moments for the part of the beam outside the load is the same as if the load had been concentrated at the centre of the loaded part, and for the remainder is a parabolic arc. Construct this arc.

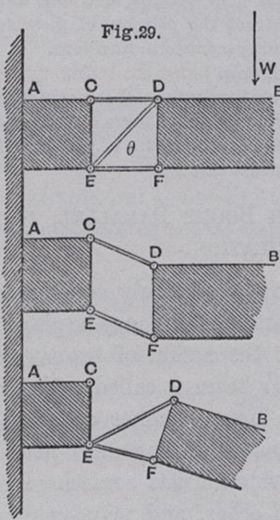
SECTION II.—FRAMEWORK GIRDERS WITH BOOMS PARALLEL, AND WEB A SINGLE TRIANGULATION.

23. *Preliminary Explanations.*—Hitherto we have only considered beams of small transverse section, but the part of a beam may be played by a framework or other structure under the action of transverse forces. Such a structure, when employed as a beam, is called a Girder, and consists essentially of an upper and a lower member called the Booms of the girder, connected together by a set of diagonally placed bars, called collectively the Web. The web consists sometimes of several triangulations of bars crossing each other, and may even be continuous. In the present section the booms will be supposed straight and parallel, and the web a single triangulation. The action of a load on such a girder furnishes the simplest and best illustration of the nature of the straining actions we have just been considering.

Suppose, in the first place, we have a rectangular beam of considerable transverse dimensions, which has one end fixed horizontally, and the other end loaded with a weight W. Now let a part of the length, CD (see Fig. 29), be cut away, and replaced by three bars, CD, EF, DE, jointed at their ends to the two parts of the beam—CD, EF forming a rectangle, of which DE is a diagonal. With this construction the load W will be sustained, as well as by the original beam, but the three bars will be subject to stresses which we shall now determine. To do this, suppose each of the three bars (in succession) removed, and examine the effect on the structure—an artifice which often enables us to see very clearly the nature of the stress on a given part of a structure.

In the first place, suppose CD removed; then the portion EB will turn about the joint E, as shown in the lower part of the diagram, so that the function of the bar CD must be to prevent this turning, which is exactly what we have previously described as bending. The tendency to turn round E—that is, the bending moment at E—is in this case simply $= W \times CB$. But if there is a system of loads, the bending moment at E may be found by methods previously described.

Fig. 29.



Now let H = stress on CD. It may readily be seen to be a tensile stress, because, on the removal of the bar, the ends C and D separate from one another. Also, let $h = CE$ or DF, the depth of the beam. The power of CD to prevent EB from turning about E is measured by the moment about E of the force H which acts along it. Therefore

$$Hh = M_E.$$

And dividing the bending moment at E by the depth of the beam, we obtain the magnitude of the tension of CD.

Next, let the bar EF be removed. The structure will yield by turning round the joint D, the point F approaching E. Thus the bar EF is in compression, and by its thrust, $= H'$ say, towards F, it prevents FB from turning round D.

The tendency to turn round D, due to the action of the external forces $= M_D$, will be equal to the resisting moment $H'h$.

$$\therefore \quad H'h = M_D.$$

Therefore if we divide the bending moment for the joint D opposite to the bar, by the depth of the beam h, we obtain the magnitude of the compressive force H'.

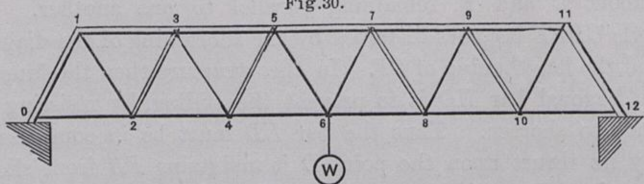
Lastly, let us suppose the diagonal bar ED to be removed, the effect is quite different from the two former cases; for instead of the overhanging portion of the beam turning about some point, it now gives way by sliding downwards (as shown in the centre of the diagram), remaining horizontal all the time. CD and EF turn about C and E, remaining parallel to one another. The rectangle $CDFE$ becomes distorted by the shortening of the diagonal ED and the lengthening of CF. In the structure then the function of the diagonal bar ED is to prevent the sliding, by resisting the tendency to shorten. Thus the bar ED must be in compression, and by its thrust upon the point D it maintains FB from sliding downwards. Let S = thrust along ED and θ = angle it makes with the vertical. The force S may be resolved into two components, a horizontal one, $S \sin \theta$, and a vertical one, $S \cos \theta$. It is the vertical component alone which resists the sliding action, and maintains D in its proper position. Now the tendency to slide is no other thing than the shearing force on the structure, which we have previously been investigating. In this example the shearing force is simply W for all sections between A and B. But in other cases of loading the shearing force may be estimated by previously given methods. Since the downward tendency of the shearing force is balanced by the upward thrust of the vertical component of S along ED we have

$$S \cos \theta = F.$$

Instead of the points E and D being joined there might have been a bar CF, which, by the resistance to lengthening which it would offer, would have sustained the portion FB from sliding downwards. Such a bar would be in tension just as the bar ED is in compression, and in finding the stress on it we should use exactly the same equation. Now instead of having 3 bars only, the whole structure may be built up of horizontal and diagonal bars. The same principles will apply. On removing any one of the horizontal bars, we see that the structure yields by turning round a joint opposite: so we say the function of the horizontal bars is to resist bending. This is expressed by the equation $Hh = M$. On the other hand, the function of the diagonal bars being to resist the shearing tendency, we have always $S \cos \theta = F$.

24. *Warren Girders under various Loads.*—Fig. 30 shows a Warren Girder, so called from the name of the inventor, Captain Warren, a type of girder much used for bridges since its first introduction about the year 1850. It consists of a pair of straight parallel booms connected

Fig. 30.



together by a triangulation of bars inclined to each other, generally at 60°, so that the triangles formed are equilateral. The booms in the actual structure are generally continuous through the junctions with the diagonal bars, but, if well constructed, there is no sensible error in regarding the structure as a true frame, in which the several divisions are all united by perfectly smooth joints. Any three bars forming a parallelogram and its diagonal may be considered as playing the same part as regards the rest of the structure as in the case just considered.

When a Warren girder is used, it is generally supported at the ends, and the loads are applied at one or more joints in the lower boom. We will examine some examples.

(1) Suppose there is a single load applied at a joint in the centre of the span.

First as to the diagonal bars. It was shown above that the duty of these bars was to prevent the structure yielding under the action of the shearing force; the vertical component of the stress on either of the diagonal bars being equal to the shearing force for the interval of the length of the girder within which the diagonal bar lies. This is expressed by the equation

$$S \cos \theta = F.$$

Now in the example which we are considering with the load in the centre, the shearing force will be the same at all sections to the right and left, namely $= \frac{1}{2} W$. Therefore the stress on all the diagonal bars is of the same magnitude,

$$S = \frac{W}{2 \cos 30^\circ} = \frac{W}{\sqrt{3}}.$$

If we consider the effect of removing either of the bars, we shall find that commencing from one end they prevent alternately the shortening and lengthening of the diagonals which they join, so that, commencing with one end, the bars are alternately in compression and tension. The compression bars are shown in double lines.

Next as to the several portions of the length of the top and bottom booms. As was shown above, the stress on any division of the horizontal bars has the effect of preventing a bending round the joint opposite; so that the moment of the stress about the joint is equal to the bending moment at the joint, due to the external forces. This is expressed by the equation

$$Hh = M.$$

Let a = length of a division.

Then, since the supporting force at the joint 0 is $\frac{1}{2}W$, the bending moments at the joints numbered 1, 2, 3, &c., are

$$M_1 = \frac{W}{2}\frac{a}{2} = \frac{Wa}{4},$$

$$M_2 = \frac{W}{2}a = \frac{Wa}{2},$$

$$M_3 = \frac{W}{2}\frac{3}{2}a = \frac{3Wa}{4},$$

and so on, the bending moments increasing in arithmetical progression.

Since the depth of the girder h is the same at all parts of the length; if we divide the M's each of them by h, we obtain the magnitude of the stress on the bars opposite the respective joints. Thus

$$H_{02} = \frac{Wa}{4h}, H_{13} = \frac{Wa}{2h}, H_{24} = \frac{3Wa}{4h}, \text{ and so on.}$$

We see, then, that the stress on the several divisions increases in arithmetical progression as we proceed from the ends towards the centre. By observing the effect of removing either of the bars, we see that all the divisions of the upper boom are in compression. This is expressed by drawing them with double lines in the figure. All the divisions of the lower boom are in tension.

(2) Next suppose the load is applied at some other joint not in the centre—the joint 4 for example. We must first calculate the supporting forces. Suppose they are P at 0 and Q at 12. For the portion of the girder to the left of 4 the shearing force will be the same at all sections and be equal to P. So the stress on all the diagonals between 0 and 4 will be equal to P sec 30°.

To the right of joint 4 the shearing force $= Q$, and the stress on all the diagonal bars from 4 to 12 will be Q sec 30°.

Proceeding from either end towards the joint where the load is applied, we observe that the diagonal bars are alternately in compression and tension—so that the bar 56 is now in compression, whilst the bar 54 is in tension. On these bars the nature of the stresses is just opposite to that to which they were exposed when the load was at the centre joint. Thus by varying the position of the load we not only vary the magnitude of the stress, but we may in some cases change the character of the stress, requiring a diagonal bar to act sometimes as a strut and sometimes as a tie.

For the divisions of the horizontal booms on the left of W the stresses are

$$\frac{Pa}{2h}, \ \frac{2Pa}{2h}, \ \frac{3Pa}{2h}, \ \&c.,$$

in arithmetical progression up to the bar opposite the joint to which the load is applied; and to the right of W,

$$\frac{Qa}{2h}, \ \frac{2Qa}{2h}, \ \frac{3Qa}{2h}, \ \&c.,$$

in arithmetical progression also up to the bar opposite the load. The upper bars are all in compression and the lower in tension as before.

When there are a number of loads placed arbitrarily at the different joints, the simplest way of determining the stresses is often to find the stress on the bars due to each load taken separately, and then apply the principle of superposition. In applying the principle due regard must be paid to the nature of the stress. A compressive stress must be considered as being of opposite sign to a tensile stress, and, in compounding, the algebraical sum of the stresses for each load will be the total stress on the bars.

(3) There is one particular case, that in which the girder is uniformly loaded, which it is advisable to examine separately.

In general, the load on the platform of the bridge is by means of transverse beams or girders transferred to the joints of the lower boom. The transverse beams may be the same number as the joints in the lower boom. In that case the girder will be loaded with equal weights at all the bottom joints. If the transverse beams are more numerous their ends will rest on the bottom booms, and tend to produce a local bending action in each division, in addition to the tensile stress which, as the bottom member of the girder, it will have to bear. In some cases, to lessen or get rid of this bending action, vertical suspending rods are introduced, by which means the middle points of the lower divisions are supported, and the loads transmitted to the upper joints of the girder. In such a case we may take all the joints both in the upper and lower booms to be uniformly loaded.

We will, however, suppose equal weights applied to the joints of the lower boom only. First as to the shearing forces. Between the end and the 1st weight the shearing force = the supporting force, = half the total load = P say. In the next division the shearing force is less by the amount of the load at the 1st lower joint $= P - W$. In the third division of the lower boom from the end the shearing force $= P - 2W$, and so on. The stresses on the diagonals can now be found by multiplying the shearing force in the division within which any one diagonal lies by the secant of the angle which the diagonal makes with the vertical. The stresses will diminish in arithmetical progression as we pass inwards from the ends towards the centre. It will be observed that on the first and second diagonals from the end the stress is of the same magnitude. On the third and fourth it is alike also, and so on. The stresses are alternately compression and tension, commencing with compression on the first bar.

To find the stresses on the booms we must determine the bending moments at all the joints.

$$M_1 = \frac{P}{2}a. \qquad\qquad M_2 = \frac{P}{2}2a.$$

$$M_3 = \frac{P}{2}3a - W\tfrac{1}{2}a. \qquad\qquad M_4 = \frac{P}{2}4a - Wa$$

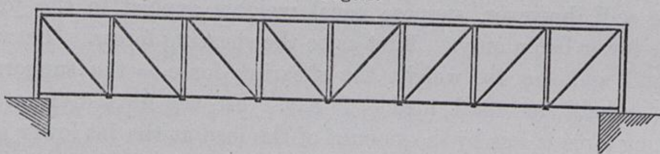
$$= \frac{a}{2}(3P - W). \qquad\qquad = \frac{a}{2}(4P - 2W).$$

$$M_5 = \frac{a}{2}(5P - 3W). \qquad\qquad M_6 = \frac{a}{2}(6P - 4W).$$

Division of the M's by h, the depth of the girder, will give the several horizontal stresses. They will be found to increase as we pass from the ends towards the centre.

25. *N Trusses.*—The web of the girder, instead of consisting of bars sloping both ways, forming a series of equilateral triangles, may be constructed of bars placed alternately vertical and sloping at an angle, so forming a series of right-angled triangles, looking like a succession of capital letters N. (See Fig. 31.) For this reason it is sometimes called an N girder. The ordinary practice is to divide the girder into a number of squares by means of the vertical bars, so that the diagonals slope at an angle of 45°. It is

Fig.31.



advantageous to place the diagonals so as to be in tension. For a load in the centre, or a uniformly distributed load, they should slope upwards from the centre towards the ends. The vertical bars will then be in compression. A short bar is better able to resist compression than a long one, whereas a tension bar is of the same strength whether short or long; so it is manifestly economical of material, and a saving of weight, to place the long bars, that is the sloping bars, so as to be in tension. The same methods will apply to find the stresses on the bars, since as before the web resists the shearing action, and the booms the bending.

The simple queen truss, considered in Chapter I., Section II., is another example of a web consisting of alternate vertical and diagonal bars, but the diagonal is not usually inclined at 45° to the vertical.

EXAMPLES.

1. A trapezoidal truss is 24 feet span and 3 feet deep. The central part is 8 feet long and is braced by a diagonal stay so placed as to be in tension. Find the stress on each part when loaded with 4 tons at one joint and 5 tons at the other.

Stress on diagonal stay = ·935 tons.

2. A bridge is constructed of a pair of Warren girders, with the platform resting on the lower booms, each of which is in 6 divisions. The bridge is loaded with 20 tons in the middle. Find the stress on each part.

3. In example 2 obtain the result when the load is supported at either of the other joints.

4. From the results of examples 2 and 3 deduce the stress on each part of the girder when the bridge is loaded with 60 tons, divided equally between the three pairs of joints from one end to the centre.

Results for questions 2, 3, 4, the bars being numbered as in Fig. 30.

Bars.	Stress on Booms.				Bars.	Stress on Diagonals.			
	Load at 6.	at 4.	at 2.	at 6, 4, and 2.		Load at 6.	at 4.	at 2.	at 6, 4, and 2.
02	2·88	3·85	4·8	11·53	01	− 5·76	− 7·7	− 9·6	−23·06
13	− 5·76	− 7·7	− 9·6	−23·06	12	5·76	7·7	9·6	23·06
24	8·64	11·55	8·64	28·83	23	− 5·76	− 7·7	1·92	−11·54
35	−11·52	−15·36	− 7·68	−34·56	34	5·76	7·7	− 1·92	11·54
46	14·4	13·44	6·72	34·56	45	− 5·76	3·85	1·92	0
57	−17·28	−11·52	− 5·76	−34·56	56	5·76	− 3·85	− 1·92	0
68	14·4	9·6	4·8	28·8	67	5·76	3·85	1·92	11·54
79	−11·52	− 7·68	− 3·84	−23·04	78	− 5·76	− 3·85	− 1·92	−11·54
8,10	8·64	5·76	2·88	17·28	89	5·76	3·85	1·92	11·54
9,11	− 5·76	− 3·84	1·92	−11·52	9,10	− 5·76	− 3·85	− 1·92	−11·54
10,12	2·88	1·92	·96	5·66	10,11	5·76	3·85	1·92	11·54
					11,12	− 5·76	− 3·85	− 1·92	−11·54

5. A bridge 80 feet span is constructed of a pair of N girders in 10 divisions, the platform resting on the lower booms, and the diagonals so arranged as to be all in tension. A load of 80 tons is uniformly distributed over the platform. Find the stress on each bar.

SECTION III.—GIRDERS WITH REDUNDANT BARS.

26. *Preliminary Explanations.*—Again, returning to the (p. 48) beam out of which a portion has been cut and replaced by bars, let us suppose that instead of one diagonal bar only, there are two. We require to find the stresses on the bars. First, on the diagonal bars. In this case also the stress on these bars will be due to the shearing force. Together they prevent the structure yielding under the shearing action, but the amount each one bears is indeterminate until we know how the diagonals are constructed and attached to the rest of the structure. Suppose, for example, the diagonals are simple struts placed across the corners of the rectangle, but not secured at the ends The struts will be incapable of taking tension; and the diagonal *ED*, which slopes in the direction, to be subject to compression will have to bear the whole shearing force. The other diagonal is ineffective. Secondly, suppose the diagonals to be simple ties, such as a chain or slender rod, and so incapable of withstanding compression. Then the bar *CF* will carry the whole shearing force. We may have any number of intermediate cases between these extreme ones according to the material of the diagonals and the method of attachment. In all cases one diagonal tends to lengthen, and the other to shorten, and according to their powers of resistance to these tendencies they offer resistance to the shearing. If S_1 and S_2 be stresses on the two bars, then in all cases

$$(S_1 + S_2)\cos\theta = F.$$

If the diagonals are exactly similar rigid pieces similarly secured at the ends, equal changes of length will produce the same stress whether in compression or tension, so that each will bear an equal share of the shearing force. We shall then have

$$S_1 = S_2 = \tfrac{1}{2}F\sec\theta.$$

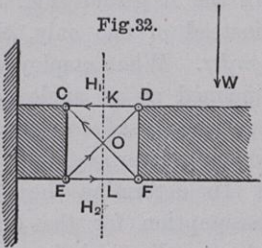
The foregoing is one of the simplest examples of a frame with redundant bars, and shows clearly why, in such cases, the stress on each bar cannot be determined by statical considerations alone, but depends upon the materials and mode of construction. In structures such as those considered in Chap. I., Sect. II., in which the principal part is an incomplete frame, stiffened by bracing or other means to provide against variations of the load, the bracing is usually redundant, and the stress on it cannot be calculated with certainty. Allowance has to be made for this in designing the structure by the use of a larger factor of safety. Redundant material is often no addition at all to the strength of the structure, and may even be a source of weakness, as will appear hereafter.

When framework girders were first introduced, it was objected by eminent engineers that failure of a single part would destroy the structure. Experience appears to have shown that risks of this kind are not serious, and the tendency of modern engineering design appears to be rather towards the employment of structures with as few parts as possible.

Next, as to the horizontal bars. These still sustain the bending moment, but not precisely in the same way as when there is only one diagonal. To find the magnitude of the forces, we employ a method similar to that used before, but instead of removing a bar we suppose the girder cut through one or more bars at any place convenient to our purpose; then the principle which we make use of is, that the action of each of the two halves on the other must be in equilibrium with the external forces which are applied to either half. In Figure 32 let us take a vertical section through the point of intersection of the diagonals, 4 bars are cut by the section, and through the medium of these 4 bars the structure to the left will act on the portion of the structure to the right of the section, and sustain it against the action of the external loads which rest on it.

First, there is the force H_1 pulling at K, and the force H_2 thrusting at L, and at O there are the two forces S_1 and S_2 on the two diagonals. Now, if we consider the tendency for the external forces to bend the right-hand portion round O, we see that the diagonal bars offer no resistance to this bending action, and must so far be left out of account. The whole resistance to bending is due to the bars CD and EF along which the forces H_1 and H_2 act, so that if M_O be the bending moment at O, due to the external forces,

Fig. 32.



$$(H_1 + H_2)\frac{h}{2} = M_O.$$

This will be true whatever be the proportion between S_1 and S_2, and H_1 and H_2. Instead, therefore, of taking the bending moment about a joint, as we did previously, we have in this case to take the moment about the point where the two diagonals cross.

But besides the balancing of the bending moment, there are other conditions to which the forces are subject, in order that the right-hand portion may be in equilibrium. One is, that all the forces which act on this portion must balance horizontally. There are no external forces which have any horizontal action, so that it is only the four internal forces which act along the bars cut, of which we have to take any account, and these must, on the whole, have no resultant horizontal action. The two thrusts must equal the two pulls; that is,

$$H_2 + S_2 \sin\theta = H_1 + S_1 \sin\theta.$$

$$\therefore \quad H_2 - H_1 = (S_1 - S_2)\sin\theta.$$

This also is true whatever be the distribution of the shearing force between the two diagonals.

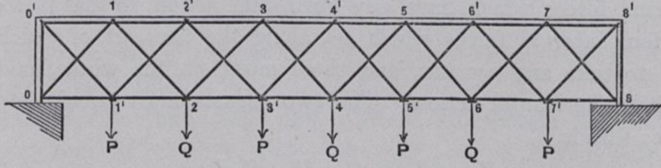
If, now, we suppose $S_2 = S_1$, then $H_2 = H_1 = H$, say. And the above formula becomes $Hh = M_O$, the same as we had before; but it must be applied a little differently, the moment now being taken about the point of intersection of the diagonals. If S_1 is not equal S_2, then H will be the mean of H_1 and H_2.

27. *Lattice Girders, Flanged Beams.* — Constructions with a double set of diagonals are common in practice. If, for example, in the N girder (Fig. 31) we place in each division two diagonals instead of one only, the construction is called a *lattice* or *trellis girder*. When employed for heavy loads, the diagonals are generally inclined at an angle of 45° to the vertical. In light structures, or when used for giving stiffness, they are often inclined at a much greater angle.

To determine the stresses, it will be necessary to make an assumption for the distribution of the shearing force between the two diagonals for each division of the girder, and it will generally be sufficiently correct to suppose each to carry half, and to write $S = \frac{1}{2}F \sec \theta$, and $Hh = M$ for the points where the diagonals intersect.

In lattice girders we more frequently find the double set of sloping bars introduced, but the vertical bars omitted. In this case it will not be true that the two diagonals in any one division are exposed to the same stress. We can determine the stresses otherwise. The structure may be divided into two elementary girders, each with its own system of diagonal bracing, and each with its own set of loads. Suppose, for simplicity, the number of divisions in the complete girder even, and each half girder loaded with equal weights applied

Fig. 33.



to all the lower joints. Then if we make the simple, and in most cases safe, assumption that the thrusts on the two end vertical bars are equal, the forces on all the bars of the structure will be determinate. In the example shown in Fig. 33 the thrusts on the vertical end bars will be $2P$.

After we have calculated the stresses on each bar in each elementary girder, then, for any bar which is a portion of both, we must compound to obtain the total stress.

We may further increase the number of diagonal bars and obtain a girder, the web of which is a network of bars. In this case it will not be exactly, but will be very nearly, true that the horizontal bars take the bending, and the sloping bars the shearing action, the shearing force being regarded as equally distributed between all the diagonals cut by any one vertical section.

We may go on adding diagonal bracing bars until the space between the booms is practically filled up, and even then assume that the bending is taken by the horizontal bars and the shearing by the web. The numerous bracing bars may then be replaced by a vertical plate, which will form a continuous web to the girder. Such a construction is a very common one in practice, the horizontal members are called the top and bottom flanges of what is still a girder, and often called so, but more often a flanged or I beam. In the smallest class of these beams, they are rolled or cast in one piece; but for large spans they are built up of plates and angle irons rivetted together. For figures showing the transverse sections of such beams see Part IV. In taking the depth of such a girder, to make use of in the equation $Hh = M$, we ought to measure the vertical distance between the centres of gravity of the parts which we consider to be the flanges of the beam or girder. In the simple rolled or cast beam this will be the distance from centre to centre of depth of flanges. In the built-up beam account must be taken of the effect of the angle irons.

It must be remembered that this method of determining the strength of an I beam is only approximate. Its strength will be determined in a more exact way hereafter, when it will be found that the web itself assists in resisting the bending moment, but, area for area, to the extent only of about one-sixth that borne by the flange. On the other hand, the effective depth is less than the distance from centre to centre of the flanges. In rough preliminary calculations we may often neglect this, and employ the same formula as for lattice girders.

Girders are often of variable depths, so that the booms are not parallel; when this is the case the booms assist in resisting the shearing action of the load, as will be seen hereafter.

EXAMPLES.

1. A beam of I section is 24 feet span, and 16 inches deep; the weight of the beam is 1,380 lbs. It is loaded in the centre with 5 tons. Assuming the resistance to bending to be wholly due to the flanges, find the maximum total stress on each flange and the sectional area of each—the resistance to compression being taken to be 3 tons and to tension 4 tons per square inch.

Maximum total stress = 53,505 lbs. = 23·88 tons.
Sectional area of upper flange = 8 square in.
" " bottom " = 6 "

2. A trellis girder, 24 feet span and 3 feet deep, in three divisions, separated by vertical bars, with two diagonals in each division, is supported at the ends and loaded (1) with 20 tons symmetrically distributed over the middle division of the top flange, (2) with 20 tons placed over one of the vertical bars. Find the stress on each part of the girder, assuming each diagonal to carry half the corresponding shearing force.

Stress on diagonals—Case 1.	14·2	0	14·2
Case 2.	$18\frac{8}{9}$	$9\frac{4}{9}$	$9\frac{4}{9}$

Remark.—These results show the unsuitability of this construction for carrying a heavy load on account of the great inclination of the diagonals to the vertical.

3. A water tank, 20 feet square and 6 feet deep, is wholly supported on four beams, each carrying an equal share of the load. The beams are ordinary flanged ones, 2 feet deep. Find approximately the maximum stress on each flange, assuming that the weight of the tank is one-fourth the weight of water it contains.

$$\text{Distributed load on one beam} = \frac{187{,}500}{4} = 46{,}875 \text{ lbs.}$$

$$H_{\text{max.}} = 58{,}593 \text{ lbs.} = 26{\cdot}1 \text{ tons.}$$

4. The Conway tubular bridge is 412 feet span. Each tube is 25 feet deep outside and 21 inside. The weight of tube is 1,150 tons, and the rolling load is estimated at $\frac{3}{4}$ ton per foot run. Find approximately the sectional areas of the upper and lower parts of the tube, the stress per square inch being limited to 4 tons.

$H_{\text{max.}}$ = 3,267 tons.
Area = 817 square in.

REFERENCES.

For details of construction of girders the reader is referred to

Girder Making . . . in Wrought Iron. E. HUTCHINSON. Spon, 1879.