PART III.—DYNAMICS OF MACHINES.

CHAPTER VIII.

PRINCIPLE OF WORK.

SECTION I.—BALANCED FORCES (STATICS).

88. *Preliminary Explanations. Definition of Work.* If the principal object of a piece of mechanism be to do some kind of work it becomes a machine. Many mechanisms—as for example clocks and watches—are not, properly speaking, machines; for though work is done during their action, yet the object of the mechanism is not the doing of the work but the measurement of time or some similar operation. Even in these cases, however, the forces in action cannot in general be excluded from consideration, and therefore in all mechanism a study of the manner in which forces are transmitted and modified is essential. This part of the subject is called the Dynamics of Machines.

A body can in general only be moved into a different position or be changed in form or size by overcoming resistances which oppose the change. This process is called doing WORK, and the amount of work is measured by the resistance multiplied by the space through which it is overcome. If there be many resistances, the total work done is the sum of that done in overcoming each resistance separately.

Consider the case of a weight raised vertically. Here the resistance is due to the action of gravity which is overcome by some external force, and the work done is simply the product of the weight and the height through which it is raised. The weight is measured by comparing it with that of a certain quantity of matter called a pound, the weight of which is taken as a unit for measuring forces. This mode of measurement has the disadvantage of giving a different unit for different points on the earth's surface, because the force of gravity varies according to the position of the point, and, for scientific purposes therefore, force is measured by the velocity which, when unbalanced, it produces in a given quantity of matter. In practical applications, however, gravitation measure is preferable, as the variation is very small, and the measure may be made precise when necessary by specifying the place on the earth's surface at which our operations are taking place. The unit of space is generally 1 ft., so that the unit of work is 1 lb. raised through 1 ft., or, as it is generally called, 1 foot-pound. Other units, however, such as, for example, "foot-tons," may also be employed for special purposes.

89. *Oblique Resistance.*—The resistance is here directly opposed to the movement which is taking place; if this be not the case it must be resolved into two components, one along and the other perpendicular to the direction of motion. The second of these is balanced by a constraint to which the motion is subject or by the opposition which the inertia of the body offers to a change in its direction at any finite rate; it is the first alone in overcoming which work is done. In Fig. 84 let R be a resistance applied at a point A which moves through a distance AB in a direction inclined at an angle θ to the direction of the resistance, then the work done is $R.\cos\theta.AB$, but if BN be drawn perpendicular to the direction of R to meet that direction in N,

$$AN = AB.\cos\theta,$$

and therefore the work done is $R.AN$.

Now AN is the distance through which A has moved in the direc-

Fig.84.

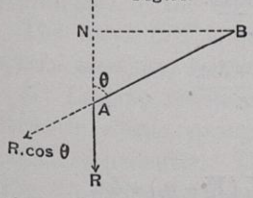
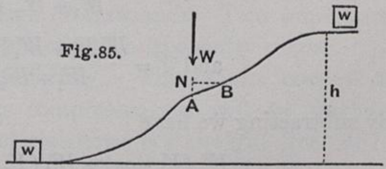


Fig.85.



tion of the resistance, so we obtain another rule for estimating the work done against an oblique resistance. It is equal to the product of the resistance into the distance moved in the direction of the resistance.

Suppose for example that a weight is raised, but that, instead of being lifted vertically, it is moved in any curved path—there being no friction or other resistance than that due to gravity.

Considering any small portion AB of the path (Fig. 85), the resistance being always vertical, the work done is $W.AN$. So the total work of raising the weight is $W.\Sigma AN$ or $W.h$, which is independent of the path described by the lifted weight, but depends simply on the height through which the weight is raised.

If there are a number of weights each of them raised through different heights, the total work done in raising all the weights is the sum of the works done in raising each weight separately; and the direct method of finding the total work is to add the separate results for each weight. But it may be determined by another method thus—

Let W_1, W_2, W_3 &c. be a number of weights which are at heights y_1, y_2, y_3 &c. above a given datum plane. Now suppose they are raised so that they are at heights Y_1, Y_2, Y_3 &c. above the same plane. The total work done in raising the weights will be the sum of the products,

$$W_1(Y_1-y_1)+W_2(Y_2-y_2)+W_3(Y_3-y_3)+\text{\&c.}$$

Now suppose the centres of gravity g and G for the initial and final positions of the weights to be at heights $\bar{y}$ and $\bar{Y}$ above the datum plane.

The centres of gravity g and G are such that if all the weights were collected at either centre, the moment of the collected weights about the plane is equal to the sum of the moments of each separate weight, before being collected, about the same plane. This is mathematically expressed thus

$$\bar{y}=\frac{W_1y_1+W_2y_2+W_3y_3+\text{\&c.}}{W_1+W_2+W_3+\text{\&c.}}$$

$$\text{and } \bar{Y}=\frac{W_1Y_1+W_2Y_2+W_3Y_3+\text{\&c.}}{W_1+W_2+W_3+\text{\&c.}}.$$

By subtracting we have

$$\bar{Y}-\bar{y}=\frac{W_1(Y_1-y_1)+W_2(Y_2-y_2)+W_3(Y_3-y_3)+\text{\&c.}}{W_1+W_2+W_3+\text{\&c.}};$$

hence the total work done in raising the weights may be expressed

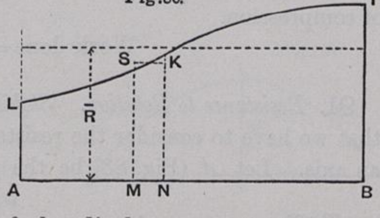
$$=(W_1+W_2+W_3+\&c.)\times(\bar{Y}-\bar{y})$$

$$\text{or } \Sigma W(\bar{Y}-\bar{y}).$$

That is to say, the total work of raising a number of weights is equal to the product of the sum of the weights by the vertical displacement of the centre of gravity of the weights.

90. *Variable Resistance.*—Let us next consider the work required to be done to overcome a variable resistance. The whole distance through which the resistance is overcome must then be divided into a number of parts, each being so small that, for that small space, the magnitude of the resistance may be treated as sensibly uniform. The work of overcoming the resistance through each of the small spaces being thus found, the total work will be the sum. The estimation can generally be most conveniently performed by a graphical construction. We will, for simplicity, take the case in which the direction of action of the resistance is that of the line of motion. Suppose a body moved from A to B against a resistance the magnitude of which varies from point to point in such a way that it is represented by the ordinates of the curve standing above AB. (Fig. 86.) For the small distance MN the resistance will vary slightly, but will have a mean value represented by SM or KN suppose, and the work of overcoming the resistance through the small space MN is $MN \times SM$ or is exactly represented by the area of the curve standing above MN; and so for any other small portion of the displacement of the body. Thus the total work of overcoming the resistance through AB is represented by the whole area $ALTB$ = mean resistance $\bar{R} \times AB$.

Fig.86.



The curve LST is called a curve of resistance. Two important special cases may be mentioned both of which frequently occur.

(1) Let the resistance vary uniformly. This is the case of a perfectly elastic spring which is compressed, as will be further explained hereafter. The curve of resistance is a straight line AST (Fig. 87a) where AB is the compression of the spring, BT the corresponding compressing force R_1. During the compression R is at first

zero and gradually increases to R_1, its value at any intermediate point being graphically represented by the ordinate SN corresponding to the compression AN. The work done is the area of the triangle, that is $\frac{1}{2}R_1.AB$, and the mean resistance $\frac{1}{2}R$.

(2) Let the resistance be inversely proportional to the distance of the point of application from a given point O. (Fig 87*b*.)

Fig.87a.

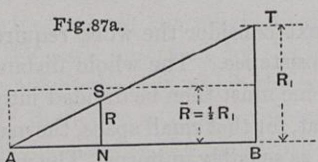
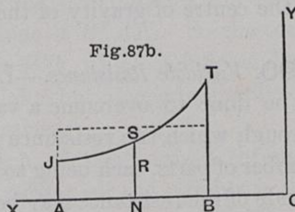


Fig.87b.

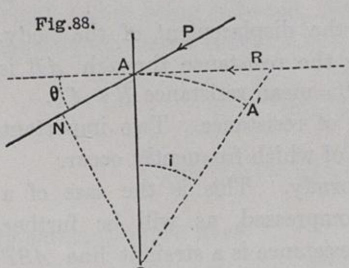


This applies to many cases of the compression of air and other elastic fluids. In the figure $NS = R$ is the resistance and $ON.NS$ is constant, so that the curve of resistance JST is an hyperbola. Let the ratio $OA : OB$ be called r, this is called the ratio of compression; then from the geometry of the hyperbola we know that the area of the curve is equal to the constant rectangle $ON.NS$ multiplied by $\log_e r$, the logarithm being Napierian, or as it is often called "hyperbolic" from this property of the hyperbola. If ON be denoted by V this gives a formula in frequent use for the work done in this kind of compression.

$$\text{Work done} = RV \log_e r.$$

91. *Resistance to Rotation. Stability of a Vessel.*—It often happens that we have to consider the resistance of a body to rotation about an axis. Let A (Fig. 88) be the point of application of a force P which resists the rotation of a body about an axis C perpendicular to the plane of the paper. If the resistance at A be not in the plane of rotation P must be supposed to be the component in that plane; the other component will be parallel to the axis of rotation and need not be considered. Let θ be the angle it makes with the direction of A's motion, then $R = P.\cos\theta$ is the effective resistance, the other com-

Fig.88.



ponent of P merely producing pressure on the axis. As the body turns through an angle i the resistance R will be overcome through the arc AA', and, assuming in the first instance R constant, the work done will be—

$$\text{Work done} = R.AA' = R.CA.i.$$

But, dropping a perpendicular CN on P's direction,

$$CN = CA.\cos\theta$$

$$\therefore \text{ Work done} = P.CN.i = Mi,$$

where M is the moment of the resistance about the axis of rotation. If there be many resistances then the same formula will hold if M be understood to mean the total moment of resistance.

We can readily extend this to the case of a variable moment by the graphical process already described for a linear resistance, the base of the diagram now representing the angles turned through and the ordinates the corresponding moments. As an example take the case of a heavy pendulum swinging about an axis O (Fig. 89a), let g be the centre of gravity, $Og = l$, and let it be swung through the angle i from the vertical, then the moment of resistance is

$$M = W.gN = W.l \sin i.$$

Fig. 89a.

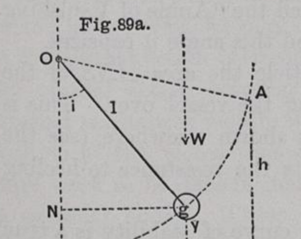
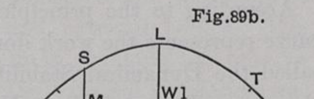


Fig. 89b.



In Fig. 89b draw a curve on the base AB such that the horizontal ordinate AN at every point represents the angle i on the same scale that AB represents two right angles, while the vertical ordinate represents M. This curve will be the curve of resistance, and in the present case is a curve of sines of which the maximum ordinate LE is Wl. The angles being supposed reckoned in circular measure so that $AB = \pi$, the area of the diagram from A up to any point S will represent the work done. We can, however, in this example find this work otherwise, for g rises through the height NZ, and therefore if U be the work

$$U = Wl\,(1 - \cos i).$$

By use of the integral calculus it can be verified that this is also the value of the area ASN.

It is not necessary that the axis of rotation should be fixed in estimating the work done during rotation, provided that the resistance be a couple, for then there is no pressure on the axis. An important example is that of a vessel floating in the water and steadily heeled over by the action of a couple M produced by external agency, or more frequently by shifting the weights on board in such a way that the displacement and trim remain constant. Then for each angle of heel this couple has a certain definite value which can be found either by calculation or by observation of the shift of the weights. The moment of resistance which is equal and opposite to M is called the Statical Stability of the vessel, and the curve of resistance drawn as above described is called the Curve of Stability. The construction of this curve is an important part of the design of the vessel. Such curves, though usually unsymmetrical, often bear a general resemblance to a curve of sines (Fig. 89*b*), the ordinate increases to a maximum which gives the maximum stability and then diminishes to zero at an angle of heel called the "Angle of Vanishing Stability." If the vessel be heeled beyond this angle it capsizes.

According to the principles of this article the area ANS of the curve represents the work done in heeling the vessel over. This is called the Dynamical Stability, and as is shown elsewhere (see the chapter on Impact in Part IV.) represents the resistance to heeling over to that angle by a sudden gust.

An important typical case is when the curve of stability is a true curve of sines. In this case suppose the angle of vanishing stability to be π/k, where k is some given number, then the ordinate S for any angle i is given by the equation

$$S = S_1 \,.\, \sin ki,$$

and the stability is the same as that of a heavy pendulum swinging through k times the angle. The dynamical stability is easily shown to be

$$U = \frac{S_1}{k}\,(1 - \cos ki).$$

92. *Internal and External Work.*—In all that precedes the position of a body has been changed by overcoming external resistances. All forces, however, arise from the mutual action between two bodies or between two parts of the same body, and every change of position must be with reference to some other body which is regarded as fixed. Work, then, consists in a change of relative position of two bodies notwithstanding a mutual action between the two which opposes the change. In raising weights the second body is the earth, but the pair of bodies may be such as occur in mechanism and the mutual action between the two may be due to springs or an elastic fluid, or to the resistance of some body to separation into parts. In scissors, nutcrackers, bellows, and other similar instruments, the elements of the pair are exactly alike and their existence is recognised in popular language.

In reckoning the work done either body may be regarded as fixed, the result must be the same and will be unaffected by any movement of the pieces common to both; thus when air is compressed in a cylinder the work done depends on the pressure of the air and the amount of compression, not on the movements of the cylinder within which the air is contained. In other words the motion to be considered is the motion of the pair as defined in Art. 46, p. 102.

In every case where we have to do with a number of pieces connected in any way, we may distinguish between the resistances due to the mutual action between the pieces themselves and those due to the mutual action between the pieces and external bodies. The internal resistances require work to be done in changing the relative position of the pieces themselves, while the external resistances require work to be done in changing the position of each piece relatively to external bodies. These two kinds of work are called Internal Work and External Work respectively. In two cases we can at once foresee that the internal work will be zero, first when the pieces are disconnected, secondly when they are rigidly connected. Thus for example if a heavy mass of matter be raised, we need only consider the rise of the centre of gravity (Art. 89) if the mass be rigid; but if not, any change of form which occurs ought to be taken into account. In raising ordinary solid bodies and masses of earth the internal work may usually be disregarded.

93. *Energy. Principle of Work.*—Hitherto we have been speaking

of the *resistance* which is being overcome during the process of doing work, let us now fix our attention on the *effort* which overcomes the resistance.

The forces arising from the mutual action between a pair of bodies, when not purely passive like the normal pressure between two surfaces in contact, are of two kinds. The first always oppose the motion of the pair, in other words they are always resistances. Friction between two surfaces is the simplest example of this, and hence such actions are called Frictional Resistances. The second on the other hand promote or oppose the motion of the pair according to the direction in which the motion is taking place, so that a resistance becomes an effort when the direction of motion is reversed. Such actions are conveniently described as Reversible; and systems of bodies, in which they occur, possess, when the parts are suitably disposed, the power of doing work. This power is called ENERGY. As examples of bodies possessing energy may be taken a raised weight, a compressed spring, or steam of high pressure. Change of velocity in a moving body likewise gives rise to efforts and resistances, but this is a matter for subsequent consideration. For the present we suppose all bodies with which we have to do to be in a state of uniform motion, or to move so slowly and steadily that no sensible action of this kind can arise.

Energy is measured by the quantity of work which it is capable of doing, and the process called doing work may also be described as the exertion or expenditure of energy, so that we write

$$\text{Energy exerted} = \text{Work done.}$$

If the effort which is being exerted and the resistance which is being overcome be applied to the elements of the same lower pair, as when a weight is lifted vertically or a spring wound up, the effort and the resistance are equal, and the equation shows that the energy exerted by an effort is the product of the effort and the space through which it is exerted. Thus all the examples given above of the doing of work will also serve as examples of the exertion of energy simply by supposing the direction of motion reversed. In short the exertion of energy and the doing of work are merely different aspects of the same process.

In this case the effort and the resistance may be regarded as applied at the same point, but the equation has a much wider application than this, for it is equally true if the points of application be different, provided only that they are rigidly connected. Thus, for example, if we dig in the ground, the energy we exert at the handle of the spade is—if the spade be perfectly rigid—exactly equal to the work done at the blade. This can be shown to be a necessary consequence of the forces we are considering being balanced, and the equation may be regarded as a concise statement of the conditions of equilibrium of forces applied to a rigid body. It is preferable, however, for our purposes to regard it as the simplest case of a fundamental mechanical principle continually verified by experience. This principle may be called the PRINCIPLE OF WORK.

We have now a means of transferring the power of doing work, that is to say energy, from one place to another: evidently we are not restricted to one piece as in the case of the spade. We may make use of a series of pieces through which energy may be transferred from piece to piece in succession; and if there were no frictional resistances to the relative motion of the pieces, there would be no loss of energy in the process. Thus the principle of work is true when the points of application of the effort and the resistance are mechanically connected in any way. Frictional resistances however absorb a portion of the energy whenever any relative motion occurs which they tend to prevent, and therefore a certain loss always accompanies the transmission of energy. Nevertheless the principle of work still holds good if overcoming friction be reckoned as part of the work done.

It may here be remarked that though frictional resistances are never a source of energy, yet friction may, like normal pressure between surfaces, transmit energy, and hence, in cases where one only of the bodies between which it is exerted belong to the set of bodies we are considering, may be an effort by means of which work is done on the set. Thus, for example, in the case of a shaft driven by a belt, the whole power of the engine is transmitted by friction closure between the belt and the pulleys; and if we consider the shaft alone apart from the rest of the mechanism, the friction may be regarded as the effort which drives the shaft. We cannot however in such cases properly speak of the friction as exerting energy; the source of energy is the steam, or other motive power, and the friction merely transmits it in the same way as the pressure between a connecting rod head and the crank pin transmits energy to the crank shaft.

Nevertheless in both of these cases the phrase "energy exerted" may be used conveniently, though "energy transmitted" would be more precise.

If a piece of material through which energy is transmitted yield under stress applied to it, as in fact it always does, the energy exerted will not be equal to the work done. Either the change of relative position of the several parts of the piece will require work to be done in order to overcome the mutual actions between the parts which resist the change, or, conversely, those mutual actions exert energy during the change. In the first case the work is done at the expense of the energy transmitted; in the second the piece of material is a source of energy which increases the energy transmitted. In perfectly elastic material the mutual actions are reversible, and any energy exerted in overcoming them is stored up in the piece and recovered when the piece resumes its original form, as in the case of a watch spring. (Compare Art. 98.)

94. *Machines.*—A mechanism becomes a machine if we connect together two of its elements by a link capable of changing its form or dimensions, and so moving the mechanism, notwithstanding a resistance applied by a similar link connecting two other elements.

The elements connected may be called the "driving pair" and the "working pair," and these pairs often, though by no means always, have one element common, namely the frame-link of the mechanism. The driving link is the source of energy. As examples, we may take steam which connects the piston and cylinder which form the driving pair in a steam engine, or gravity which, as in Art. 62, is to be conceived replaced by a link exerting the same effort. The working link is gravity in cranes and other hoisting machines, or a piece of material the deformation of which is the object of the machine, as in the case of machine tools.

In addition to the driving and the working links, the force of gravity acts on all the parts of the machine, and frictional resistances have to be overcome; but these are matters for subsequent consideration.

The driving and working pairs are very frequently kinematic pairs of the lower class. Let us suppose them in the first instance sliding pairs. Let the driving pair move through a space x, then the working pair will move through a space y, which is in a certain definite proportion to x depending on the nature of the mechanism. Let P be the driving effort, which, by taking x small enough, can be made as nearly uniform as we please; and let R be the resistance opposing the motion of the working pair, then

$$\text{Energy exerted} = Px;\ \text{Work done} = Ry,$$

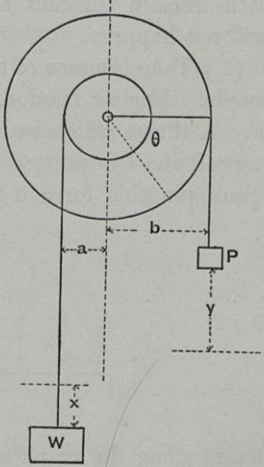
and these must be equal, therefore

$$\frac{P}{R}=\frac{y}{x}=\frac{\text{Velocity of Working Pair}}{\text{Velocity of Driving Pair}};$$

from which it appears that the ratio of the effort to the resistance, or as we may briefly call it, the "force ratio," is the reciprocal of the velocity ratio of the driving and working pairs. In works on mechanics this is also known as the Principle of Virtual Velocities.

If the pairs be turning instead of sliding pairs, then the effort and resistance are moments, and the velocities will be angular; and if one pair be sliding, the other turning, a suitable "radius of reference" must be selected (p. 103) to compare the motions and the forces, but the same principle holds good.

Fig.90.



In the simplest machines, known frequently as the "mechanical powers," we have a 2 or 3-linked chain, so that the driving pair and working pair are identical or very closely connected. But they may be separated by a long train of mechanism and have no common link. In all cases it must be carefully remembered that the effort and the resistance arise from the mutual action between the elements, each consisting of two equal and opposite forces, just as in the straining actions considered in Chapter II. and elsewhere. Either of these as before measures the magnitude of the action.

95. *Verification of the Principle of Work in Special Examples.*—We will now take some examples to illustrate and verify the principle of work, neglecting friction.

(1.) Take the common wheel and axle. Suppose P to be just sufficient to lift the weight W, so that the two forces exactly balance one another. Now let P descend through the distance y (Fig. 90), and W rise through the corresponding distance x.

As P falls it is said to exert energy. Energy exerted $= Py$. This is employed in overcoming the resistance to the rise of the weight W. Work done $= Wx$. The principle of work asserts that Energy exerted = Work done, that is $Py = Wx$.

Suppose the wheel and axle to turn through the angle θ, then $y = b\theta$ and $x = a\theta$. Then in order that the weights P and W may statically balance one another, $Pb = Wa$; from which it follows that $Py = Wx$, verifying the principle of work.

Also, we may write,

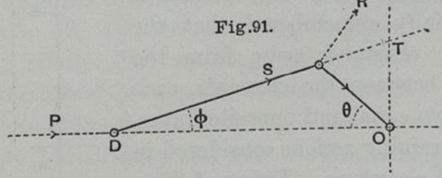
$$\frac{P}{W} = \frac{a}{b} = \frac{x}{y} = \frac{v}{V},$$

where v, V are the velocities of P, W respectively, thus showing that the force ratio is the reciprocal of the velocity ratio.

In this simple example both the force ratio and the velocity ratio remain constant throughout the movement. In general this will not happen.

(2.) Take the case of the mechanism of the steam engine for an example. Neglect friction and let the driving pressure on the piston be P. A thrust which we will call S will be produced along the connecting rod and transmitted to the crank pin as shown in Fig. 91. At the crank pin this force S may be resolved into two components, one

Fig. 91.



acting along the crank arm and the other, R, perpendicularly to it. The last alone will tend to turn the crank, the other component producing only a pressure on the shaft immediately balanced by the pressure of the bearings on the journals of the shaft.

This component R which tends to turn the shaft is called the *crank effort*. If the turning effort on the crank is perfectly balanced at all points of its revolution by some suitable resistance, then the resisting force which must be applied at the crank pin at right angles to the crank arm in order to balance perfectly the pressure of the steam on the piston must be equal and opposite to the component R previously referred to. The force ratio will be P/R. We have, with the notation employed in Chap. V., $S \cos \phi = P$ and $S \sin (\theta + \phi) = R$.

$$\text{Thus } \frac{R}{P} = \frac{\sin (\theta + \phi)}{\cos \phi} = \frac{\sin OBT}{\sin OTB} = \frac{OT}{OB}.$$

That is, the crank effort is to the steam pressure as the intercept OT is to the crank arm OB.

But we have previously shown (see p. 109) that this fraction expresses the velocity ratio of piston to crank pin; hence we have again found in this case that the force ratio is the reciprocal of velocity ratio, and the curve which we previously drew to represent the varying velocity of the piston, the crank pin moving uniformly, will represent also the varying crank effort, the pressure of the steam on the piston being uniform throughout the stroke. So we may call it the Curve of Crank Effort.

(3.) The same thing may be proved to be true for every mechanism, the forces acting on which balance one another. In some cases it may be easier to determine the force ratio than the velocity ratio or *vice versa*. In any case either may be inferred by taking the reciprocal of the other. As an additional example take the case of two pieces driving one another by simple contact (Fig. 92). We have already found the velocity ratio by a direct process (p. 165), but we may also determine it in the following way. When A presses on B there is a resistance R equal and opposite to the pressure,

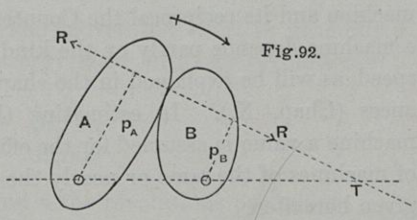


Fig. 92.

and normal to the portions of the surfaces in contact, if we suppose no friction to exist. Drop perpendiculars p_A and p_B on the common normal. Then the moment of the driving pressure R which A exerts on B or the turning moment due to $A = M_A = Rp_A$. Simi larly the moment of the resisting force which B exerts on A or the moment of resistance to turning which B opposes to $A = M_B = Rp_B$.

$$\text{Thus the ratio } \frac{\text{Driving moment}}{\text{Resisting moment}} = \frac{M_A}{M_B} = \frac{p_A}{p_B}.$$

But we have previously proved that this fraction is the angular velocity ratio of the piece B to the piece A, and thus we show that the moment ratio is the reciprocal of the angular velocity ratio.

96. *Periodic Motion of Machines.*—One of the most essential characteristics of a machine is the periodic character of its motion. Each part goes through a cycle of changes of position and velocity and returns periodically to its original place. When moving steadily the periods are equal and the velocity of each piece is the same at the beginning and end of each period. That this may be the case it is not necessary that the driving effort should balance the working resistance in every position; on the contrary, this seldom happens; it is sufficient if the mean effort be equivalent to the mean resistance, or as we may otherwise express it

Energy exerted during a period = Work done in the period;

a condition which always governs the action of a machine in steady motion. In reckoning the energy and work the action of gravity on any piece of the machine may be omitted, for, if the piece rise through any height during one part of the period, it will fall through an equal height during another part. The work done consists partly of the work which the machine is designed to do, and partly of frictional resistance to the relative motion of the parts of the machine, or in other words of Useful Work and Waste Work. The ratio of the useful work to the energy exerted is called the Efficiency of the machine and its reciprocal the Counter-Efficiency. The efficiency of a machine depends partly on the kind of machine and partly on the speed, as will be explained in the chapter devoted to frictional resistances (Chap. X.). In estimating the power required to drive a machine a value is assumed for the efficiency derived from experience of machines of the same or nearly the same type. Examples will be given hereafter.

97. *Power. Sources of Energy.*—The sources of energy are—

(1.) Living agents;

(2.) Gravity acting usually by means of falling water;

(3.) Springs and elastic fluids;

(4.) Gunpowder and other explosive agents.

The energy thus derived may be traced further back to the action of heat and chemical affinity, and we may add to the list electric and magnetic forces, but the foregoing is a sufficient statement for our present purpose. A machine which employs such agents directly is called a Prime Mover, or, more briefly, a Motor, but a number of machines may be driven from one prime mover which serves as their source of energy. In general, each source of energy has a motion and an effort peculiar to itself while the work is required to be done at a different place and under different circumstances. A machine, then, is a mechanism which transmits energy and converts it into a form suitable to the work to be done.

The rate at which energy is exerted is called Power; it is this which measures the value of a source of energy and the expense of the work which is being done. The ordinary unit of measurement is the conventional horse-power of 33,000 foot-pounds per minute or 550 per second, a quantity much greater than the working power of an ordinary draught horse on the average of a day's work. The unit of power employed universally on the Continent is somewhat less, being 75 kilogrammetres per second or 32,550 foot-pounds per minute.

In prime movers the effort may generally be regarded as applied at a point which moves with a known mean velocity; then the horse-power is given by the equation

$$\text{H.P.} = \frac{PV}{33{,}000},$$

where P is the mean value of the effort in lbs. and V the mean velocity in feet per minute.

In machines driven from a prime mover the effort is generally a moment M which exerts the energy $M.2\pi$ in every revolution of a driving shaft. We then have

$$\text{H.P.} = \frac{M.2\pi n}{33{,}000},$$

where M is the mean moment and n the revolutions per minute.

98. *Reversibility. Conservation and Storage of Energy.*—The resistance overcome at the working point may be either frictional as in machine tools or reversible as in machines for raising weights. In the second case, if the machine were stopped and set in motion in the reverse direction it would, if friction could be neglected, work equally well, the driving effort and working resistance would be interchanged, and constructive modifications might be required, but otherwise the action is unaltered. This may be described by saying that the machine is Reversible. Many machines actually occur in both their direct and their reversed forms; thus a pump is a reversed hydraulic motor. Hence it appears that in reversible machines the power of doing work, that is to say, energy, is not lost after being exerted, for by reversing the machine it may be employed a second time. Thus it is that we describe the action of reversible machines as a transfer of energy, and are led to conceive of energy as indestructible and independent of the bodies through which it is manifested. No machine, indeed, is completely reversible, for in all cases frictional resistances occur to a greater or less extent, while many machines are completely non-reversible; but we shall see as we proceed that even then energy is not lost but only converted into another form, so that we have in reversible machines the first and most simple example of the great natural law called the Conservation of Energy. The importance of reversibility as a test of maximum efficiency will be seen more fully hereafter.

Again, we can store up energy and use it as required when it is inconvenient to resort to any of the usual sources. For example, by a few turns of the watch key we store energy in the mainspring which is supplied at a regular rate to the watch throughout the day. So the hydraulic accumulator (Part V.) receives energy from the pumping engines and supplies it at irregular intervals to the hydraulic machines which lift weights and move gates in a dockyard or work the guns in a ship of war.

A large part of what follows in the present work is merely a development of what has been said here: in the succeeding chapters of the present division we consider machines comprising solid elements only, while in a future division we shall consider the transmission and conversion of energy by means of fluids.

EXAMPLES.

1. A waggon weighs 2 tons and its draught is $\frac{1}{50}$th of its weight. Find the work done in drawing it up a hill 1 in 20, half a mile long. Find also how long three horses will take to do it supposing each horse to work at the rate of 16,000 foot-pounds per minute.

Work done = 370 ft.-tons. Time occupied = 17′ 15″.

2. A force of 10 lbs. stretches a spiral spring 2″, find the work done in stretching it successively 1″, 2″, 3″, &c., up to 6″. *Ans.* $2\frac{1}{2}$, 10, $22\frac{1}{2}$, 40, $62\frac{1}{2}$ and 90 inch-lbs.

3. Find the H.-P. required to draw a train weighing 200 tons at the speed of 40 miles an hour on a level, the resistance being estimated at 20 lbs. per ton. Find also the speed of the train up a gradient of 1 in 100, the engine exerting the same power. H.-P. required = $426\frac{2}{3}$. *Ans.* Speed up the incline = 18·87 miles per hour.

4. The resistance of H.M.S. "Iris" at 17 knots is estimated at 40,000 lbs., what will be the H.P. required simply to propel the ship. Find also in inch-tons the moment, on each of the twin screw shafts, equivalent to this power, the revolutions being 80 per minute. *Ans.* H.-P. required = 2088. Moment on each shaft = 367 inch-tons.

5. The curve of stability of a vessel is a common parabola, the angle of vanishing stability 70″, and the maximum moment of stability 4,000 ft.-tons. Find the statical and dynamical stabilities at 30°. *Ans.* Statical stability = 3918 ft.-tons. Dynamical stability = 1283 ft.-tons.

6. Verify the principle of work, neglecting friction, in:—(*a*) The differential pulley (Art. 59). (*b*) A pair of 3-sheaved blocks. (*c*) The hydraulic press (Art. 62).

7. From the results in question 3, p. 112, deduce the crank efforts for the given positions of the piston and the mean crank effort, supposing the effective steam pressure on the piston 20 tons and neglecting friction.

Crank effort at quarter stroke in the { forward stroke = 18·4 tons. backward ,, = 16·6 tons. Mean = 12·74 tons.

8. Show that the efficiency of a machine is equal to the velocity ratio divided by the force ratio.

SECTION II.—UNBALANCED FORCES (KINETICS).

99. *Kinetic Energy of a Particle.*—We now proceed to consider the cases in which efforts or resistances arise from the changes of velocity of the parts of a system, which changes thus become a source of energy or require energy in order to produce them. The commonest observation is sufficient to show the importance of such cases: a cannon ball possesses a great power of doing work, and a railway train requires energy to be exerted by the steam to obtain the requisite speed, quite irrespectively of that necessary to maintain the speed when once produced.

First, suppose a weight under the action of gravity only. Unless it be supported by a vertical force exactly equal to the weight it will fall with a gradually increasing velocity. Let it be wholly unresisted, let it start from rest and fall through a height h, then we know that it will acquire a velocity v given by the formula

$$v^2 = 2gh,$$

where g is a number which for velocities in feet per second ranges from 32·117 at the equator to 32·227 at the pole, and having intermediate values at other points on the earth's surface according to the intensity of gravity at the point. The average value 32·2 is usually adopted for this important constant, and the height h is called the "height due to the velocity."

During the whole fall, the weight W of the body has been exerting an effort upon it which overcomes an equal resistance occasioned by the change of velocity which is taking place; thus an amount of energy has been exerted, and an amount of work done equal to Wh. Resistance of this kind is of the reversible kind, for if we imagine the weight, after reaching the ground, projected up again with the same velocity, it will, if wholly unresisted, attain the height from which it originally fell. Hence we describe the weight as possessing energy, and the amount it possesses when moving with velocity v is

$$Wh = \frac{Wv^2}{2g}.$$

Energy due to motion is called Kinetic Energy, to distinguish it from that kind of energy considered previously, which is a consequence of the relative position of the parts of a system, and which is called Potential Energy. The kinetic energy of a body depends on its velocity only, not on the direction of its motion nor on the way in which its motion has been produced; and the energy exerted in changing the motion of a body is always represented by an exactly equivalent increase of kinetic energy, whether this effort be uniform or variable, or whether its direction coincide with the direction of motion or not. To illustrate this, consider the following cases.

(1) Let the body move in a straight line under the action of a force P, in that line let it start with velocity V, and after moving through a space x let its velocity be v, then, it is shown in works on elementary dynamics, that v is given by a formula which may be written

$$Px = \frac{Wv^2}{2g} - \frac{WV^2}{2g}.$$

Now, the left-hand side of the equation is the energy exerted by P, and the right-hand side is the increase of kinetic energy of the body.

If P be a resistance instead of an effort, then work is done at the expense of the kinetic energy which is now diminished. If P be variable we must represent it graphically by a curve as in Art. 90, and it should be especially remarked that the ordinate of the curve of areas deduced as in Art. 31 will, on affixing a suitable scale and measuring the ordinates from a suitable base line, represent the height due to the velocity of the body.

(2) Let the body be constrained by means of a smooth guiding curve to move along a given path by a force P in any direction, then the energy exerted by P is the same as that exerted by the resolved part of P in the direction of motion. But this resolved part accelerates the motion just as if the body moved in a straight line, so that this case is reduced to the last.

(3) The pressure on the guiding curve will be the difference between the normal component of P and the force necessary to change the direction of P's motion. If the two are equal the guiding curve may be removed, and we obtain the case where the body moves freely, as in the case of a projectile in vacuo.

100. *Partially Unbalanced Forces. Principle of Work.*—Again, the effect which is changing the motion of the body may be partly balanced by an external resistance to which the body is subject. If this be the case we can imagine it separated into two parts, a part which is, and a part which is not, balanced. The energy exerted by the first is employed in overcoming the external resistance, while that exerted by the second is employed in increasing the kinetic energy of the body. Or the resistance may be greater than the effort, then the excess is overcome at the expense of the kinetic energy of the body, the velocity of which now diminishes.

In the present treatise we shall use the phrases "energy exerted" and "work done" only in reference to efforts and resistances other than those due to inertia, subject to which convention, we may state the principle of work as applied to cases where the forces are partially unbalanced, as follows—

$$\text{Energy exerted} = \text{Work done} + \text{Change of Kinetic Energy.}$$

In this statement the work done may be greater or less than the energy exerted. In the first case the change of kinetic energy is a decrease, in the second an increase.

Not only does this principle apply to a single body, but—subject to the observations of the preceding section—to a set of bodies mechanically connected in any way, provided that one of them be fixed to the earth; or, in other words, that a body of great mass like the earth be one of the set. When no one of the set predominates over the rest it is necessary to consider further how the kinetic energy should be reckoned: for the present, however, we shall suppose this condition satisfied.

A simple case is that of Atwood's machine. Let the descending weight P be greater than the rising one Q. Neglecting friction, the excess sets the two weights in motion. Let P descend through a distance y, then Q rises through the same distance, and therefore

$$\text{Energy exerted} = Py.$$
$$\text{Work done} \quad = Qy.$$

Let v be the velocity of the two weights; then supposing them to start from rest,

$$\text{Kinetic energy acquired} = (P+Q)\,\frac{v^2}{2g}$$

From principle of work

$$Py = Qy + \frac{(P+Q)v^2}{2g}\,; \quad \therefore\ v^2 = \frac{P-Q}{P+Q}\,2gy.$$

The law of increase of velocity is, therefore, the same as that of a body falling freely, but the rate of increase is less. This formula is the same as that obtained by other methods, and we have therefore here a verification of the principle of work.

101. *Kinetic Energy of the Moving Parts of a Machine.*—Instead of a single body, suppose we have a system of bodies, and we require to know the total kinetic energy of the system. The direct method is to find the energy of each separate particle of the system and add the results. In the particular case of a rotating rigid body we are able to express the result of the summation in a convenient and simple form. First consider a ring of small section rotating about an axis in the centre perpendicular to its plane. Every portion of the ring will move with the same velocity, v say, and the kinetic energy of the ring may, as before, be written $Wv^2/2g$.

We may express this another way, as follows:—If n be the revolutions per second, and a the radius, $v = 2\pi an$,

$$\therefore \frac{Wv^2}{2g} = W\,.\,\frac{4\pi^2n^2}{2g}\,a^2.$$

If the ring is not complete, but W is the weight of a portion which has the same centre of rotation, the expression will still hold.

Now, suppose we have a body consisting of a number of particles rigidly connected together, rotating about a centre O, at n revolutions per second.

Let the weights of the particles be w_1, w_2, w_3, w_4, etc.,
rotating about O at distances y_1, y_2, y_3, y_4, etc.

By adding together the results for each particle, we obtain for the kinetic energy of the system,

$$\frac{4\pi^2 n^2}{2g}(w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.})$$

Now suppose a is such a radius that

$$a^2 = \frac{w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.}}{w_1 + w_2 + w_3 + \text{etc.}},$$

then substituting, we may write

$$\text{Kinetic energy} = \frac{4\pi^2 n^2}{2g}(w_1 + w_2 + w_3 + \text{etc.})\, a^2 = \frac{4\pi^2 n^2}{2g} W a^2.$$

By this method we are always able to reduce any system of bodies to a ring, which ring is often called the *Equivalent Fly Wheel*, and the radius a is called the *Radius of Gyration*. The quantity Wa^2/g is usually called the *Moment of Inertia*, and denoted by the symbol I.

However numerous the particles are, the expression obtained above will hold, and so will be true if they are sufficient in number to make up a solid body. In a continuous body, the separate weights w_1, w_2, w_3, etc., must be taken indefinitely small and close together to get accurate results, and the results of the summation may be most conveniently arrived at by the use of the calculus. The quantity W/g is called the mass of the body, and but for the introduction of this factor the symbol I would have the same meaning as in Chap. XII. Hence all the results there given may be used here for thin plates simply by multiplication by the mass of a unit of area. In addition, the following simple cases will be sufficient. The fourth is a particular case of the second.

1. Solid cylinder rotating about its axis. $\qquad a^2 = \dfrac{r^2}{2}$
   Radius $= r$.

2. Rectangular parallelopiped rotating about an axis. Diagonal of either end = $2d$. $a^2 = \frac{d^2}{3}$

3. Sphere rotating about a diameter. Radius $= r$. $a^2 = \frac{2r^2}{5}$

4. Rod rotating about an axis perpendicular to it through one end. Length $= l$. $a^2 = \frac{l^2}{3}$

In other cases such as occur in practice, the body is generally too irregular and complex in form to render mathematical formulæ useful; we then apply the rule given in Ch. XII. for plane areas, which by a similar process can readily be extended to solids. That is to say, if I be the moment of inertia of a body about any axis, I_0 that about a parallel axis through the centre of gravity at a distance h,

$$I = I_0 + mh^2,$$

where m is the mass of the body. In applying this rule the body is cut up into portions to which the values just given apply exactly or with sufficient approximation, just as in the chapter cited.

In estimating the kinetic energy of a fly-wheel, which consists of rim, arms, and boss, since the rim is by far the most important part for storing energy, it is generally sufficient to consider it alone. If it be desired to take the remaining parts into account, an addition of about one-third the weight of the arms may be made to the weight of the rim. The combined effect of arms and boss is said to amount to an addition of, on the average, about 8 per cent. to the weight of the rim.

If the body have a motion of translation, combined with a motion of rotation about its centre of gravity, it will be shown hereafter that its total kinetic energy is the sum of that due to the translation and the rotation taken separately, so that the whole can be found by preceding rules. As an example of the use of this principle, consider the case of a ball rolling down an inclined plane, the ball and plane being sufficiently rough that slipping does not take place between them; and suppose the resistance to rolling, called the rolling friction, is insensible. In this case the whole energy due to the descent of the ball is employed in generating kinetic energy in the ball, which will be stored in it by virtue of its two motions of translation and rotation. Let V be the velocity of translation, A the angular velocity, r the radius of sphere; then since no slipping occurs $V = Ar$.

Let the ball descend through a vertical height h, then the energy exerted is Wh, equating which to the kinetic energy stored we obtain

$$Wh = \frac{WV^2}{2g} + \frac{WA^2a^2}{2g},$$

where a = radius of gyration is given by $a^2 = \frac{2}{5}r^2$.

$$\therefore Wh = \frac{WV^2}{2g} + \frac{W}{2g} \cdot \frac{2}{5}r^2 = \frac{7}{5} W \frac{V^2}{2g}.$$

$$\therefore V^2 = \frac{5}{7} 2gh.$$

Thus the velocity of the ball will be less than if it simply slid down the plane without rotating in the proportion $\sqrt{5} : \sqrt{7}$.

The total kinetic energy of the moving parts of a machine in any position may be found by drawing a diagram of velocity for that position in the manner explained in Chaps. V. and VI. Each part may be divided into a number of small portions, and the centre of each portion may be laid down on the diagram, as explained on page 125. If now the diagram be imagined to represent a set of particles rigidly connected, of masses equal to those of the particles in question, the moment of inertia of those particles about the pole of the diagram must be the total kinetic energy required; the radius vector of each particle representing the velocity of the corresponding portion.

102. *Conservation of Energy.*—The principle of work may also be stated in another form, which, though not so convenient in practical applications, is much employed by scientific writers. It has already been explained that, when there are no frictional resistances, the power of doing work (energy) exerted in doing a given amount of work is not lost but merely transferred from one place to another (Art. 98), while it appears from the present section that any energy exerted in changing the motion of a body is represented by an exactly equivalent amount of kinetic energy stored up in the moving body; hence it follows that in any dynamical system, which receives no energy from without and supplies none to external bodies, the total amount of energy is always the same if there be no frictional resistances. We express this by the equation

Kinetic Energy + Potential Energy = Total Energy = Constant,

and call it the principle of the Conservation of Energy. In all actual motions frictional resistances occur which gradually absorb the energy, but we shall find hereafter that this process is accompanied by the generation of heat which is equivalent to the energy absorbed a fact which leads us to conclude that heat is a form of energy, so that the principle still holds good.

103. *Examples.*—Let us now illustrate and verify the principle by some examples.

(1) Suppose a weight suspended by a string and oscillating under the action of gravity, forming the simple pendulum Og (Fig. 89*a*, p. 197), of length l.

Let the pendulum start from the position OA, and when it reaches the position Og let its velocity be v. Let the height of g above the tangent at the lowest point be y, and that of A, h, then we know that

$$v^2 = 2g(h - z),$$

which may be written, if W be the weight,

$$W\frac{v^2}{2g} + Wy = Wh.$$

Here the first term on the left-hand side is the kinetic energy of the weight and the second term Wy the potential energy, that is to say, the power of doing work which the weight possesses, in virtue of its height y above the lowest position it is capable of occupying. The sum of the two is the total energy Wh, and the motion consists in a continual interchange between the kinetic and potential energies. It is, of course, supposed that the resistance of the air is neglected; this is a resistance of the frictional kind, and continually absorbs energy from the weight which is thus at last reduced to rest.

The time of an unresisted double oscillation is shown in works on dynamics to be

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

when the oscillations are small enough to be sensibly isochronous.

Larger oscillations are sensibly slower, as shown by the approximate formula,

$$T = T_0 \left\{ 1 + \frac{\theta^2}{16} \right\} = T_0 \left\{ 1 + \frac{n^2}{52521} \right\},$$

where θ is the angle of swing in circular measure, and n is the same angle in degrees.

(2) The pendulum has been here supposed to be merely a heavy particle attached to the end of a string without weight. Let us next suppose a rigid body, the centre of gravity of which is g, oscillating about a centre O. Let v be the velocity of g, then

$$\text{Kinetic Energy} = W\frac{v^2}{2g} + W\frac{k^2A^2}{2g} \text{ (p. 214)},$$

where k is the radius of gyration about the centre of gravity, and A the angular velocity. If L be the length Og of the compound pendulum, this may be written

$$\text{Kinetic Energy} = \frac{Wv^2}{2g}\left\{ 1 + \frac{k^2}{L^2} \right\}.$$

The potential energy is the same as if the whole weight were concentrated at g; therefore, assuming the pendulum to start from the position OA, as before,

$$\frac{Wv^2}{2g}\left\{ 1 + \frac{k^2}{L^2} \right\} + Wy = Wh.$$

Comparing this with the result previously obtained for the simple pendulum, it is not difficult to see that the motion is identical if

$$l = \sqrt{L^2 + k^2}$$

which is the length of the simple equivalent pendulum.

(3) Take the case of a projectile unresisted by the air. Let A be the point from which the projectile starts with velocity V. If we draw through A a horizontal line AL, from this set up an ordinate $AH = h = V^2/2g$, and then draw a horizontal line HK, this line will be the directrix of the parabola in which the projectile moves. When the projectile has reached any point in its path, which is at a height y from the ground and at which it has the velocity v, the total energy possessed by the projectile $= W\left(y + \frac{v^2}{2g}\right)$. This being

equal to that which it had at starting $= W\dfrac{V^2}{2g} = Wh$, $\dfrac{v^2}{2g} = h - y$, and so the projectile will, at every point of its path, have a velocity due to its having fallen from the directrix.

EXAMPLES.

1. The energy of 1 lb. of pebble powder is 70 foot-tons. Find the weight of charge necessary to produce an initial velocity of 1300 feet per second in a projectile weighing 700 lbs., neglecting the recoil of the gun and the rotation of the shot.

Wt. of powder required = 117 lbs.

2. In Example 1 suppose the gun fired at an elevation of 30°, and resistance of the atmosphere neglected, find the kinetic and potential energies of the shot at its greatest elevation. Also deduce the greatest elevation.

Horizontal velocity = velocity at highest point $= 1300\ \dfrac{\sqrt{3}}{2}$,
Kinetic energy at highest point = 6150 ft.-tons,
Potential ,, ,, = 2050 ,,
$\dfrac{\text{Potential energy}}{\text{Wt. of shot}} = 6560{\cdot}6$ feet = maximum elevation.

3 A train is running at 40 miles an hour, find the resistance in pounds per ton necessary to stop the train in 1000 yards on a level. Also find the distance in which the train would be brought up by the same brake power on a gradient of 1 in 100, both when going up and when going down.

Resistance = 39·9 lbs. per ton.
Distance required to bring up the train when ascending the gradient ... ... = 640 yards.
When descending ... ... = 2280 ,,

4. The reciprocating parts of an engine running at 75 revolutions per minute weigh 25 tons, of which parts weighing 20 tons have a stroke of 4 feet and parts weighing 5 tons a stroke of 2 feet. Find the energy stored in the parts, assuming a pair of cranks OP, OQ at right angles and neglecting obliquity of connecting rod.

$$\text{Velocity of parts attached to crank } P = PN\frac{V}{OP}.$$
$$\text{,, ,, ,, } Q = QM\frac{V}{OP}.$$

Where V is the velocity of the crank pin and PN, PM are perpendiculars on the line of centres.

Assuming weights attached to these cranks each equal W. Then energy stored in these weights together $= \dfrac{WV^2}{2g}(PN^2 + QM^2)\dfrac{1}{OP^2} = \dfrac{WV^2}{2g}$.

In example, total kinetic energy = 40·7 ft.-tons.

5. One weight draws up another by means of a common wheel and axle. The force ratio is 1 to 8 and the velocity ratio is 9 to 1. Find the revolutions per minute after 10 complete revolutions have been performed, neglecting frictional resistances and the inertia of the wheel and axle. Diameter of axle 6 inches.

Revolutions per second = 2·14.

6. In Ex. 1 suppose the gun rifled so that the projectile makes 1 turn in 40 diameters, find the additional powder charge required to provide for the rotation of the shot, the diameter of shot being 12 inches and the radius of gyration $4\frac{1}{2}$ inches.

Additional powder required = ·407 lb.

7. A disc of iron rolls along a horizontal plane with velocity 15 feet per second, and comes to an incline of 1 in 40 on to which it passes without shock. Find how far it will ascend the incline, neglecting friction.

Distance along incline it will run = 209·6 feet.

8. In Ex. 5 suppose the weight of wheel = weight of axle, and the two together = sum of weights, obtain the result, taking account of the inertia of the wheel and axle.

After 10 revs. it will rotate at 1·22 revs. per second.

9. Assuming that when a vessel rolls her dynamical stability is the same as when steadily heeled over (Art. 91), and neglecting that part of her kinetic energy which is due to the motion of her centre of gravity (Art. 101), write down her equation of energy (Art. 103). If the curve of stability be a true curve of sines, show that the vessel will keep time with a pendulum of length l swinging through k times her angle of heel, where

$$k\theta_0 = \pi\ ;\ l = \frac{r}{\sqrt{k}},$$

θ_0 being her angle of vanishing stability and r her radius of gyration.

Note.—The rolling is here supposed unresisted. Observe that the deviation from isochronism is much greater than in a simple pendulum swinging through the same angle, k being always greater than unity.

REFERENCES.

Numerous elementary examples on the application of the Principle of Work will be found in Twisden's *Practical Mechanics.*

CHAPTER IX.

DYNAMICS OF THE STEAM ENGINE.

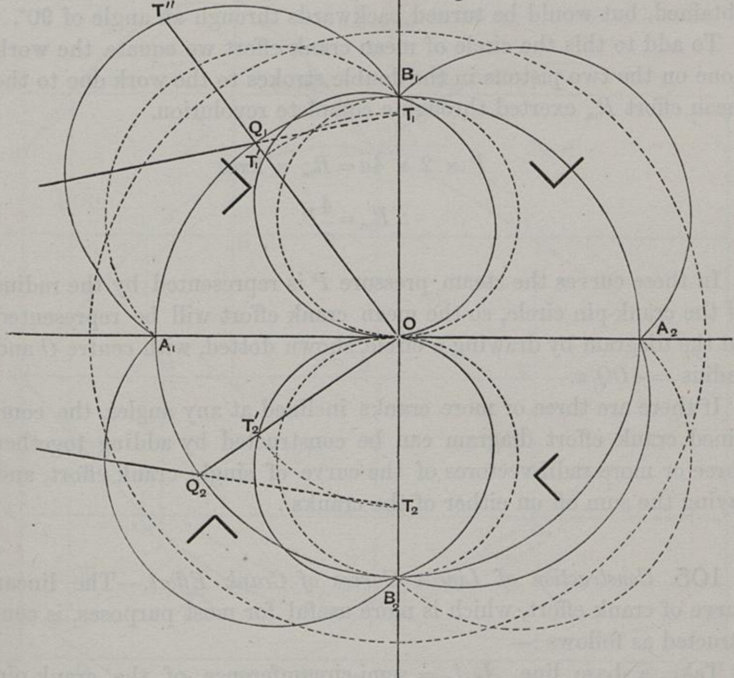
104. *Construction of Polar Curves of Crank Effort.*—One of the most common and important applications of the principles of the preceding chapter is to the working of steam engines, and we shall investigate this question, chiefly with reference to fluctuations of stress, energy and speed. Throughout, frictional resistances are neglected.

In Ch. V. a curve was constructed which shows the velocity ratio of piston and crank pin, and it has been proved (p. 204) that this curve must also give the ratio of the effort tending to turn the crank to the pressure of the steam on the piston, so that it may also be called a Curve of Crank Effort. If there are two or more cranks, the crank effort can be obtained by suitably combining the results for each taken separately, and a curve may then be drawn representing the combination. There are two kinds of such curves, the Polar and the Linear. First suppose two cranks at right angles, steam pressure uniform, and the same on both pistons. Let us commence with the polar curve.

Suppose $OT_1'B_1$, $OT_2'B_2$ (Fig. 93) to represent the polar curve of crank effort for an engine constructed as in Art. 49, and let the two cranks be in the positions OQ_1, OQ_2, each pointing towards the cylinder. Add together the corresponding crank efforts OT_1', OT_2', which are given by the curve, and set off their sum along OQ, we thus obtain a radius OT'', which represents the total crank effort for the two engines taken together. It may also be considered as the leverage at which the pressure on one piston must act to produce the same turning moment. Performing this construction for a number of positions of the cranks, we obtain a polar curve showing the crank effort in every position.

If the connecting rod is indefinitely long the single curve of crank effort consists of the pair of circles on OB_1, OB_2, shown dotted in the diagram. If we add together radii of these circles, the combined curve of crank effort will consist of four portions of circles passing the points $A_1B_1A_2B_2$; each of the circular arcs if produced would pass

Fig. 93.



through the point O. These arcs are also dotted in the diagram. When the crank is in a quadrant lying towards the engine, the actual crank effort is in excess of that due to a long connecting rod. So for the positions OQ_1, OQ_2, shown, for each the crank effort is in excess, and thus the curve of combined effort will for the quadrant A_1B_1 lie outside the circular arc. When the cranks are in the two upper quadrants the effort for the leading crank is less than when the connecting rod is long, whereas for the following crank it is greater; and the diminution of one is very approximately equal to the excess of the other; and the sum is the same as that, neglecting the shortness of the rod. The true combined effort is then for the quadrant B_1A_2 represented by the circle. In the next quadrant both are in diminution; and the true curve will lie inside the circle A_2B_2, while for the fourth quadrant it will again coincide with the circular arc.

We may, if we please, lay off the sum of the radii on the following crank instead of the leading; the same series of curves would be obtained, but would be turned backwards through an angle of 90°.

To add to this the circle of mean crank effort we equate the work done on the two pistons in the double strokes to the work due to the mean effort R_m exerted through a complete revolution.

$$P \times 2 \times 4a = R_m \times 2\pi a.$$

$$\therefore R_m = \frac{4}{\pi}P.$$

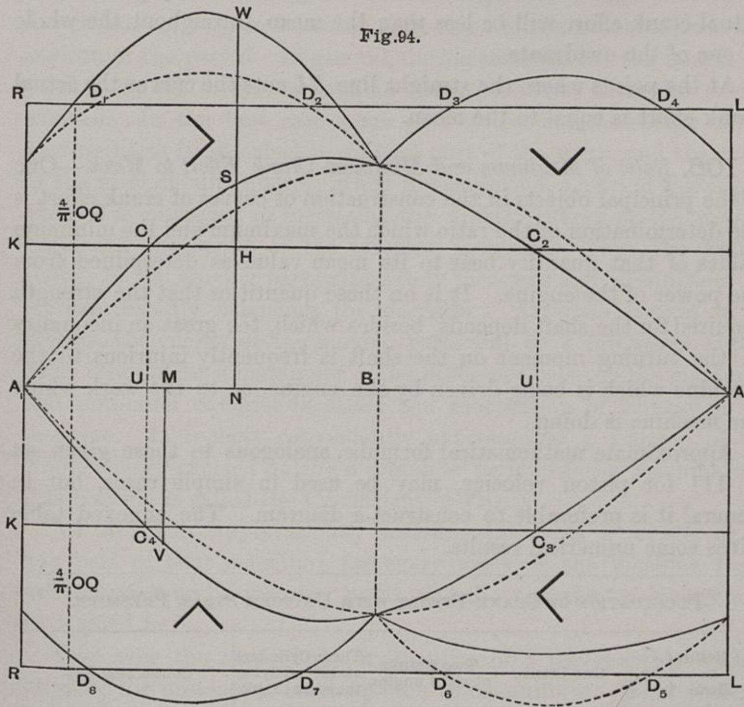
In these curves the steam pressure P is represented by the radius of the crank-pin circle, so the mean crank effort will be represented on the diagram by drawing a circle, shown dotted, with centre O and radius $= 4OQ/\pi$.

If there are three or more cranks inclined at any angles, the combined crank effort diagram can be constructed by adding together three or more radii vectores of the curve of single crank effort, and laying the sum off on either of the cranks.

105. *Construction of Linear Curves of Crank Effort.*—The linear curve of crank effort, which is more useful for most purposes, is constructed as follows:—

Take a base line A_1A_2 = semi-circumference of the crank-pin circle, and let the circle and this base line be divided into the same number of equal parts, and at the points of division of the base line set off ordinates such as SN, VM both above and below the base equal to lengths of the common ordinates of the single crank effort diagram such as OT'_1, OT'_2, and so we construct the linear crank effort diagram for a single crank. Neglecting the obliquity of the connecting rod, the diagram will consist of two curves of sines shown dotted, one above, the other below (Fig. 94). To get the combined crank effort diagram we have only to add together proper ordinates according to the angle between the cranks, just as we did in drawing the polar diagram. When the cranks are at right angles it will be seen that when the leading crank is, for example, at Q_1 or N the following crank is at Q_2 or M; and if the ordinate MV is laid off on the top of ordinate NS we obtain a point W on the curve of combined crank effort. If the same process be followed throughout we obtain the diagram shown in Fig. 94, consisting of four curves. If

Fig. 94.



the connecting rod be taken as indefinitely long, and ordinates of the dotted curve be added together the combined diagram will consist of four curves, also curves of sines shown dotted in the diagram, all alike and all of the same height. But taking proper account of the shortness of the rod, we observe that for one quadrant of the revolution when both cranks lie towards the cylinder, each ordinate added is in excess of that, neglecting obliquity, and then we obtain the highest curve. In the next quadrant the height of the curve is less and is the same as if we neglected the shortness of the rod. In the next quadrant when both cranks are away from the cylinder the shortness of the rod makes the crank effort for each engine less, and we get a very low curve for the combination. This is followed in the last quadrant by a curve like the second.

The mean crank effort will be represented by a horizontal line at a height $4OQ/\pi$, as before. Setting off this line we observe that unless the connecting rod is longer than is usual in ordinary practice, the actual crank effort will be less than the mean throughout the whole of one of the quadrants.

At the points where the straight line RL cuts the curves the actual crank effort is equal to the mean.

106. *Ratio of Maximum and Minimum Crank Effort to Mean.*—One of the principal objects in the construction of curves of crank effort is the determination of the ratio which the maximum and the minimum values of that quantity bear to its mean value as determined from the power of the engine. It is on these quantities that the strength required for the shaft depends, besides which, too great an inequality in the turning moment on the shaft is frequently injurious to the machine which is being driven by the engine, or to the work which the machine is doing.

Approximate mathematical formulæ, analogous to those given on p. 111 for piston velocity, may be used in simple cases, but in general it is preferable to construct a diagram. The annexed table gives some numerical results.

FLUCTUATION OF CRANK EFFORT WITH UNIFORM STEAM PRESSURE.

Ratio to Mean for	One Crank.	Two Cranks at right angles.	Three Cylinders at 120°, driving the same Crank.	Connecting Rod.
Maximum.	1·57	1·112	1·047	Indefinitely long.
Minimum.	0	·785	·907	
Maximum.	1·62	1·31	1·077	Four Cranks.
Minimum.	0	·785	·794	

The great influence which the length of the connecting rod has on the results should be especially noticed; we shall return to this hereafter, but now go on to consider the motion of the engine under the action of the varying crank effort.

107. *Fluctuation of Energy.*—We have already referred to the periodic character of the motion of a machine, and explained that when the mean motion is uniform we have for a complete period

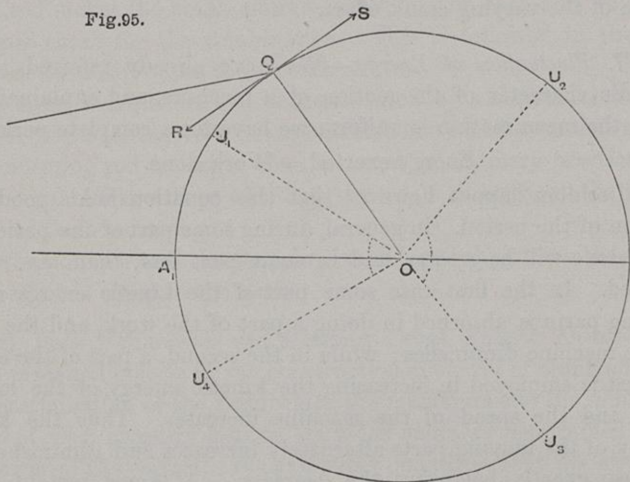
Energy exerted = Work done.

It will seldom happen however that this equation holds good for a portion of the period. In general, during some part of the period the work done will be greater, and in some part less, than the energy exerted. In the first case some part of the kinetic energy of the moving parts is absorbed in doing a part of the work, and the speed of the machine diminishes; while in the second, a part of the energy exerted is employed in increasing the kinetic energy of the moving parts and the speed of the machine increases. Thus the kinetic energy of the moving parts alternately increases and diminishes, the increase exactly balancing the decrease. At some instant in its motion, the energy of the moving parts will be a minimum, and at some other point a maximum. The difference between the maximum and minimum energies is called the Fluctuation of Energy of the machine. It is most conveniently expressed as a fraction of the whole energy exerted during a complete period of the machine, and this fraction is called the Co-efficient of Energy.

All this will apply to any machine taken as a whole, or to any part of that machine; for every piece of the machine has a driving point and a working point, and the equation of energy may be applied to it.

Take now the case of the mechanism of a direct-acting engine. Suppose the pressure P on the piston to be uniform. This through the connecting rod will produce a crank effort S, the magnitude of which for each position of the crank may be found as just now shown. To the crank and shaft S is the driving force and furnishes the energy exerted. At every point of the revolution of the shaft a certain resistance will be overcome, which resistance will tend to prevent the shaft from turning; it will not depend on the steam pressure, but on the sort of work that is being done. As the most simple ordinary case we will suppose the resistance overcome to be uniform, and we will neglect the inertia of the reciprocating parts (Art. 110). We may represent this constant resistance by a constant force R applied to the crank pin Q (Fig. 95), at right angles to the

Fig.95.



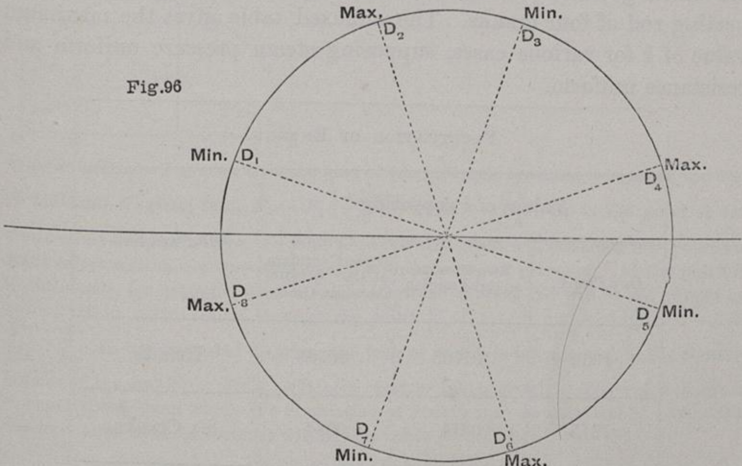
crank arm, resisting its motion. The magnitude of R is immediately determined by the application of the principle of work to a complete period, say one revolution. We have

$$P4a = R \times 2\pi a. \quad \therefore R = \frac{2}{\pi}P.$$

This constant resisting force is the same as the mean crank effort. Then, so long as $S > R$ the speed of the crank shaft will increase, and when $S < R$ it will diminish.

Referring to the linear curve of crank effort (Fig. 94, p. 223) let $A_1N =$ the arc A_1Q (Fig. 95), then $NS =$ crank effort S for this position of the crank. If an ordinate A_1K be set up to represent the constant resistance or mean crank effort, and a horizontal line parallel to base line be drawn, then NH being the representation of R the resistance overcome, the effort S will be greater for this position of the crank, and the difference HS will be employed in accelerating the motion of the machine. From the commencement of the revolution up to this position, the energy exerted is represented by the area A_1NS, whereas the work done is represented by the area A_1KHN. As the crank revolves from the position A_1 the crank effort increases until when at U_1 it is equal to the resistance. Up to this point the speed of rotation will have been diminishing. After passing the point U_1 the effort will be greater than the resistance and the speed of the engine will increase. Thus U_1 is a point of minimum speed at which the kinetic energy is a minimum. When the crank reaches the position U_2 the effort will again be equal to the resistance; and, since from U_1 to U_2 the effort has been greater than the resistance, during the whole of which time the engine has been increasing its speed, it follows that at the point U_2 the speed and the kinetic energy will have reached a maximum. The energy stored during this interval will be equal to the area C_1SC_2, and this will be the fluctuation of energy. During all the movement from U_2 to U_3 the speed of the engine will diminish, so that U_3 is another point of minimum kinetic energy. The kinetic energy stored from U_2 to U_3 is negative and represented by $C_2A_2C_3$, which quantity also is the fluctuation of energy. Again at U_4 the kinetic energy is a maximum. If the resistance had not been uniform, but its varying magnitude represented by the ordinates of some curve of resistance, then where the curve of resistance intersected the curve of crank effort would be the points where the kinetic energies would be maximum and minimum, as just explained. By the graphical construction of such a curve of resistance the fluctuation of energy may be estimated by measuring

Fig.96



the area of the crank-effort curve cut off above or below the curve of resistance, which area will lie between consecutive points of maximum and minimum energies. If the energy be E, the fluctuation of energy may properly be denoted by ΔE. It is convenient to express this as a fraction of the total energy $4Pa$ exerted in a revolution. We have then for the co-efficient of fluctuation of energy $\frac{\Delta E}{4Pa} = k$.

The value of k does not depend on the size of the engine, but only on the length of the connecting rod and the way in which the steam pressure and resistance vary. If the connecting rod is indefinitely long, steam pressure and resistance uniform, $k = \cdot 1052$. The shorter the connecting rod the greater will be the value of k.

An equally important case is that of two cranks at right angles also shown in Fig. 94. Neglecting the shortness of the connecting rod, then the line of resistance cuts each of the four curves in two points, the first of which is a point of minimum energy as shown in Fig. 96, on the preceding page. For this case $k = \cdot 01055$ or one-tenth of its value for a single crank: eight fluctuations of equal magnitude occur in each revolution. When the connecting rod is short the curves of crank effort are not the same in each quadrant (see Fig. 94), and one of them lies wholly below the line of resistance. There are then six fluctuations in each revolution: four of these are nearly the same as before, but the other two are much greater, the values of k being ·037 and ·042, with a connecting rod of four cranks. The annexed table gives the maximum value of k for various cases, supposing steam pressure uniform and resistance uniform.

FLUCTUATION OF ENERGY.

Values of k supposing			Length of Rod.
One Crank.	Two Cranks, at right angles.	Three Cylinders, at 120°, driving the same Crank.	
·1052	·01055	·00325	Infinite.
·1245	·0314	·0084	Six Cranks.
·1358	·0418	·0115	Four Cranks.

As before, the great influence of length of connecting rod on the results should be noticed. Frictional resistances, which are here neglected, generally increase the value of k.

In general the pressure of the steam in the cylinder of an engine varies throughout the stroke, and the construction of the curve of crank effort previously described must be modified on account of this. Suppose, instead of the steam being admitted throughout the stroke, it is cut off at a certain point and expanded so that the expansion curve is hyperbolic. For simplicity neglect the back pressure. At the point N in the stroke (Fig. 97) the pressure will have fallen to P, such that $\frac{P}{P_1}=\frac{OK}{ON}$ If we

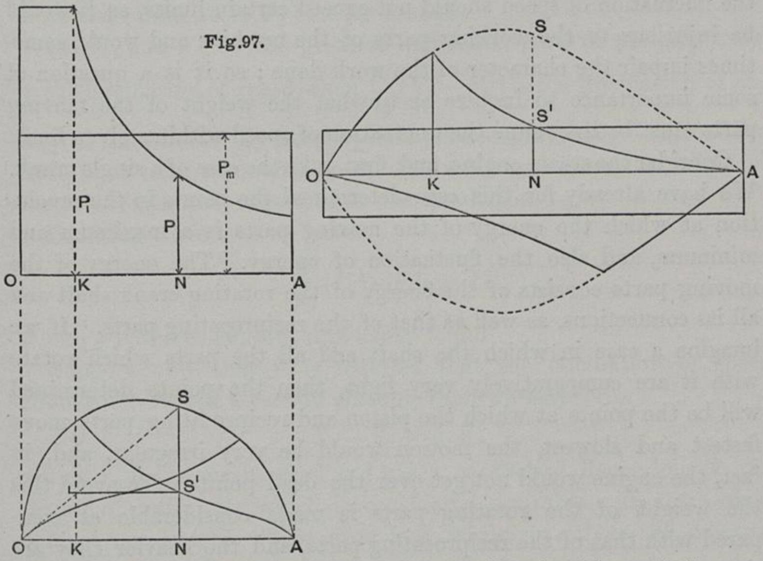


Fig. 97.

draw an ordinate P_m such that the area of the rectangle enclosed is equal to the area of indicator diagram, then $P_m = P_1 \frac{1+\log_e r}{r}$ where $r=\frac{OA}{OK}$. Up to the point K the crank-effort diagram will be the same as previously described, but after that point the crank effort will be less than that due to a uniform steam pressure. At the point N in the stroke, for example, the crank effort instead of being NS will be NS', found by joining OS to cut a vertical through the point K of cut-off and making $NS' = KL$, $\frac{NS'}{NS}=\frac{P}{P_1}$. In the expanded diagram, the base of which is taken equal to the circumference of the crank-pin circle, ordinates must be taken equal to NS', and a diagram so constructed, from which the fluctuation of energy may be calculated. Assuming the resistance to be uniform, it will have a value R such that

$$R\pi a = P_m 2a = 2aP_1\frac{1+\log_e r}{r},$$

$$R=\frac{2}{\pi}P_1\frac{1+\log_e r}{r};$$

and drawing a horizontal line above the base at a height to represent R, it will cut off an area above it which will be the fluctuation of energy. The diagram for the return stroke is shown below. It is not exactly the same as that for the forward stroke, because the effect of obliquity is different. A general method of procedure applicable with any given indicator diagram is explained at the end of this chapter.

108. *Fluctuation of Speed. Fly-Wheels.*—Fluctuation of energy in an engine or any other machine is necessarily always accompanied by a fluctuation of speed; but the heavier the moving parts the less will be the fluctuation of speed. In most cases it is necessary that the fluctuation of speed should not exceed certain limits, as it would be injurious to the working parts of the machine and would sometimes impair the character of the work done; so it is a question of some importance to inquire as to what the weight of the moving parts must be to confine the fluctuation of speed within a given limit.

Consider the steam engine, and, first, take the case of a single crank. We have already for this case determined the points in the revolution at which the energy of the moving parts is a maximum and minimum, and also the fluctuation of energy. The energy of the moving parts consists of the energy of the rotating crank shaft and all its connections, as well as that of the reciprocating parts. If we imagine a case in which the shaft and all the parts which rotate with it are comparatively very light, then the points determined will be the points at which the piston and reciprocating parts move fastest and slowest, the motion would be very irregular, and, in fact, the engine would not get over the dead points. To avoid this the weight of the rotating parts is made considerable as compared with that of the reciprocating parts, and the heavier they are the more uniform the motion of the engine will be. To increase the uniformity, the weight must generally be artificially increased by the addition of a heavy fly-wheel to the shaft, and the inertia of this is predominant over that of the other moving parts of the engine. For the present we may neglect the inertia of the reciprocating parts and consider the fly-wheel alone.

On this supposition the energy and speed of the fly-wheel will be greatest and least at the points previously described, viz., where the curve of crank effort is cut by the line of uniform resistance. Let W be the weight, V the velocity of rim of fly-wheel; then

$$\frac{WV^2}{2g} = \text{Energy of Rim.}$$

The energy of the arms and boss may be estimated by the addition of a percentage to the weight of the rim, or be considered as furnishing a margin in favour of uniformity. On account of the danger of fracture the speed of periphery V should not exceed 80 feet per second. This is the limit of speed commonly stated, but the liability to fracture depends very much on the straining action on the arms of the wheel due to inequality between the crank effort and the resistance, and not merely on centrifugal forces. (See Ch. XI.). In large wheels the rim is in segments, and the speed is not more than from 40 to 50 feet per second.

Let V_1 and V_2 be the greatest and least speed of periphery due to the fluctuation of speed, then $\frac{W}{2g}(V_1^2 - V_2^2)$ is the fluctuation of energy of the wheel. By the graphical process previously described we have been able to determine the fluctuation of energy in terms of the total energy E_0 expended in one revolution.

Equating these two we have

$$\frac{W}{2g}(V_1^2 - V_2^2) = kE_0,$$

where k is the co-efficient previously found.

Suppose now that it is required that the fluctuation of speed should not exceed a certain amount, then we may write

$$V_1 - V_2 = q \, . \, V_0,$$

where V_0 is the mean speed and q is a co-efficient depending on the degree of uniformity which is considered desirable. In some cases q must not exceed ·02 or even less, whilst in others ·05 or even more may be sufficient.

We may generally assume with sufficient accuracy that

$$V_0 = \frac{V_1 + V_2}{2}$$

(see next Article), then we find by substitution that, at the mean speed,

$$\text{Energy of Wheel} = \frac{k}{2q} \, . \, E_0.$$

In a single crank non-expansive engine the value of k ranges, as we have seen, from ·1 to ·14 when the resistance is uniform. In expansive engines k may be ·25 even with a uniform resistance, and when an engine is doing very irregular work k may be unity.

If we have a pair of cranks at right angles, the kinetic energy of the reciprocating parts is the same, at the same speed, for all positions of the cranks. (Ex. 4, p. 218.) Consequently these parts may be considered as so much added to the weight of the fly-wheel. Besides this the value of k is much less, seldom reaching ·1 if the resistance is approximately uniform. Hence a lighter fly-wheel may be used. The difference however is not so great as it might appear, for in estimating the weight of wheel required, it is important to consider not merely the change of speed, but also the time in which the change takes place. A small change taking place rapidly may be as injurious as a much greater change taking place slowly. The values of the acceleration and retardation at any instant are proportional to the difference between the crank effort and resistance at that instant, which can be found from tables such as that on page 224, and some regard should be paid to these numbers in considering what value of q should be employed.

In any case then we may write

$$\text{Energy of Wheel} = K \, . \, E_0,$$

where K is a co-efficient, which will vary within much narrower limits than the two co-efficients of speed and energy on which it depends. In general, in the very cases in which the resistance is most irregular a greater variation in speed is admissible.

The old rule for fly-wheels, dating from the time of Watt, was that the energy of the wheel should be 3·75 times the energy exerted per stroke. This corresponds to $K = 1{\cdot}875$, and would be satisfied by $k = 1$, $q = {\cdot}267$, or by $k = {\cdot}125$, $q = \frac{1}{30}$th. The first of these cases would be a very irregular resistance with a great variation in speed, and the second a moderately uniform resistance with a uniformity of speed which would be sufficient for most purposes. Heavier wheels are not unusual in modern practice, and it may be here remarked that the minimum weight necessary may depend partly on the rigidity of the shafting.

There is another method of obtaining the fluctuation of energy which, though not practically so convenient, is for some purposes advantageous. A curve representing the energy exerted may be constructed in this way: Suppose the steam pressure P constant, then in the movement of the crank pin from A to Q the piston moves from A to N and the energy exerted $= P \times AN$, which will be proportional to AN. Now in Fig. 98 take a base line AA' equal to the semi-circumference, and at the various points, such as Q, set up ordinates $QK = AN$,

$A'A'' = AA'$, and so on; a curve $AKLA''$ will be obtained, which will represent by its ordinates the energy which has been exerted from the commencement up to the various points in the stroke. At the same time, the resistance being uniform,

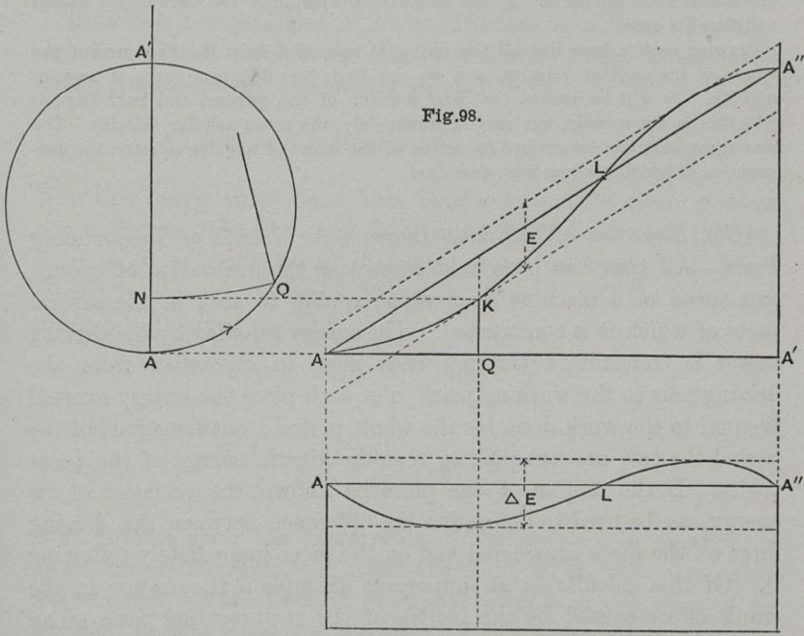


Fig. 98.

the work done will be proportional to the length of the arc AQ, since work done $= R \times AQ$. If from the base line AA' we set up ordinates to represent the work done, a straight sloping line will be obtained. If the work done = energy exerted in the complete stroke, they will both be represented by the same ordinate $A'A''$, and so the sloping line will meet the curve at the point A''. The intercept between the curve and line AA'' measured on the vertical ordinate will at any point be the difference between the energy exerted and the work done reckoned from the commencement of the stroke up to that point, and what we have called the fluctuation of energy will be the vertical intercept between two tangents to the curve $AKLA''$ drawn parallel to AA''.

From this we can derive a curve which will represent the varying angular velocity of the crank; but, in order to simplify the measurement and description, let the vertical intercepts of the curve just described be laid off from a horizontal base line, as shown below.

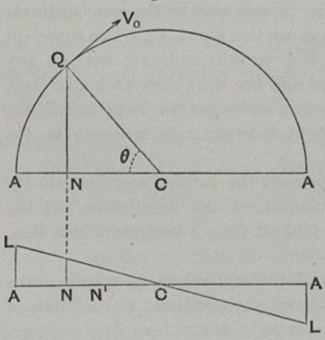
For suppose we know the moment of inertia of the equivalent fly-wheel of the engine and the angular velocity of the crank in some one position: the ordinate of the curve ALA'' at this point measured from a properly taken base line must represent the energy of the moving parts. Thus, if the base line be drawn in proper position, all ordinates measured from it will represent the square of the velocity of revolution of the crank shaft. If the speed of the machine is great, the base line will be some distance below the curve. On the other hand, if the speed is small, the base line will be close to the curve. There is manifestly a minimum speed at which the machine can be kept revolving ; it is that which corresponds to the case in which the base line touches the curve. At one instant of the period of the machine the energy will then be zero.

Drawing such a base line all the ordinates measured from it will represent the square of the angular velocity, and we can from this deduce a curve of angular velocity. It will be noticed that half the sum of the greatest and least angular velocities is not exactly, but only approximately, the mean angular velocity. The true mean may be determined by means of the curve of angular velocity, the construction of which has just been described.

109. *Correction of Indicator Diagram for Inertia of Reciprocating Parts.*—All that has been said respecting the fluctuation of energy and speed of a machine as a whole, applies to each of the several parts of which it is constructed. The energy supplied by the driving power is transmitted through each piece in succession from the driving pair to the working pair. For each piece the energy exerted is equal to the work done for the whole period ; but for a part of the period the two are unequal, so that the kinetic energy of the piece varies. If the motion of the piece be known, the variation of its energy can be used to determine the difference between the driving force on the piece considered and on the piece immediately following it. Of this calculation an important example is the change in the crank effort caused by the inertia of the reciprocating parts of an engine.

In this calculation we neglect, in the first instance, the obliquity of the connnecting rod, and suppose the crank to rotate uniformly. Let Q (Fig. 99) be the centre of the crank pin describing a circle AQA with velocity V_0, then the position of the piston is represented by N, and its velocity is

Fig.99.



$$V = V_0 . \sin \theta,$$

from which it follows that the kinetic energy of the reciprocating parts must be given by

$$\text{Kinetic Energy} = \frac{WV_0^2 \sin^2 \theta}{2g} = \frac{WV_0^2}{2g}\left(1 - \frac{x^2}{a^2}\right),$$

where W is the weight of the piston, piston rod, and other reciprocating parts, and x is the distance of the piston from the centre of its stroke.

Take now two positions N, N', at distances x_1, x_2 from the centre, and find by this formula the change of kinetic energy as the piston moves from N to N'. Evidently we shall have

$$\text{Change of Kinetic Energy} = \frac{WV_0^2}{2g} \cdot \frac{x_1^2 - x_2^2}{a^2}.$$

Now this energy must have been obtained from the steam pressure which drives the piston and accelerates its motion. Let P be the mean value of that part of the whole steam pressure which is employed in this way between N and N', then $P \, . \, NN'$ is the energy exerted in this way, so that

$$P(x_1 - x_2) = \frac{WV_0^2}{2g} \cdot \frac{x_1^2 - x_2^2}{a^2},$$

or dividing by $x_1 - x_2$,

$$P = \frac{WV_0^2}{2g} \cdot \frac{x_1 + x_2}{a^2}.$$

This formula gives the mean value of the pressure in question between any two points N, N', and therefore, if we take the points near enough, we shall obtain the actual pressure at any point of the stroke. Putting $x_1 = x_2 = x$ we get

$$P = \frac{WV_0^2}{ga} \cdot \frac{x}{a}.$$

It is convenient to express our result as a pressure in lbs. per square inch by dividing by the area of the piston in square inches, then

$$p = p_0 \cdot \frac{V_0^2}{ga} \cdot \frac{x}{a},$$

where p_0 is the weight of the reciprocating parts divided by the area of the piston, or, as we may call it, the "pressure equivalent to the weight of the reciprocating parts."

When $x = a$ we get the pressure at the commencement of the stroke required to start the piston: here the pressure is greatest, and elsewhere varies as the distance from the centre. At the centre the pressure is zero: the piston then for the moment moves with uniform velocity and requires no force to change its motion. When past the centre the pressure is so much addition to the steam pressure because the piston is at every instant being stopped: this is shown by the formula, since x is then negative. All this is shown graphically by drawing a straight line LCL through C such that

$$AL = p_0 \cdot \frac{V_0^2}{ga}.$$

The ordinate of that straight line represents the pressure due to inertia for each position of the piston. After subtracting this from the actual steam pressure the effective pressure is found, which is transmitted to the crank pin, and furnishes the crank effort.

The value of p_0, the pressure equivalent to the weight of the reciprocating parts, varies considerably according to the size and type of engine, but in ordinary cases ranges from $1\frac{1}{2}$ to 3 lbs. per square inch. In return connecting rod engines, and in some other types where the reciprocating parts are exceptionally heavy, p_0 may reach $4\frac{1}{2}$ or 5 lbs. per square inch. This being given, the pressure due to inertia will vary inversely as the stroke and directly as the square of the speed; in the short-stroke high-speed marine engines common in the present day, the correction for inertia is sometimes very considerable. It is hardly necessary to say that it is only the value of the crank effort at particular points of the stroke which is affected. The mean value must remain unaltered, for any energy employed in overcoming inertia at one part of the stroke must be given out again at another part, so that the total energy exerted by the steam remains the same. Further, when there are a pair of cranks at right angles the total crank effort is little altered.

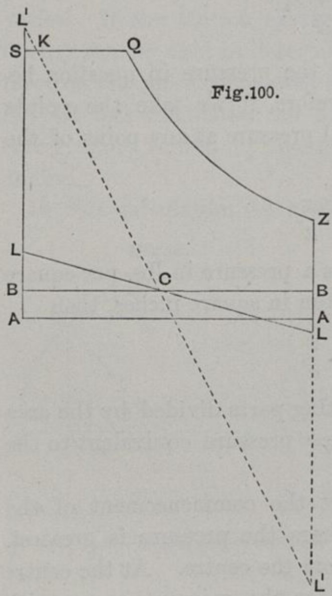


Fig.100.

The effect is best seen by correcting an indicator diagram for the inertia of reciprocating parts in the following way. Consider, for simplicity, a theoretical indicator diagram (Fig. 100) $SQZA$, in which BB is the back-pressure line, QZ the expansion curve, then, but for inertia, the ordinates reckoned from BB of SQZ give the effective pressure of the steam. Set up BL equal to the pressure necessary to start the piston found above and draw the straight line LCL, then the actual effective pressure will be obtained by measuring the ordinates to the sloping base LCL instead of the original base BB. It will be seen that the general effect is to equalize the steam pressure throughout the stroke.

In engines running at a very high speed the pressure necessary to start the piston at the commencement of the stroke may be greater than the steam pressure (see Ex. 11, p. 243), which will be shown on the diagram by the point L rising above S, as shown by the dotted line $L'CL'$ of the figure. The direction of stress on piston rod and connecting rod is then reversed, which will produce a shock if the brasses are at all loose. This gives a limit to the speed with which the engine can safely be driven (see p. 244).

In obtaining the preceding results it has been supposed, first, that the crank rotates uniformly and, secondly, that the connecting rod is indefinitely long. To take account of the variation in the velocity of the crank, it would be necessary to draw a curve representing that velocity, and deduce from it a curve showing the kinetic energy of the piston in every position. In general, however, the inertia of the rotating parts will be sufficient to reduce the variation in speed within narrow limits, and the error caused by neglecting it may be disregarded. The effect of obliquity is of more importance: to obtain it we may either use the formula for piston velocity given on p. 111 instead of the simpler formula employed above (Ex. 13, p. 243), or we may derive a curve of kinetic energy from the known curve of piston velocity and take the differences of equidistant ordinates. For the sake of variety, however, we will employ a method depending on a different principle, which is perhaps more simple in practical application.

Divide the crank-pin circle into a number of equal parts, and supposing the connecting rods drawn, let them cut the vertical through 0 in the points $1'$, $2'$, $3'$ in Fig. 101. Also find and mark off the corresponding positions of the piston $1''$, $2''$, $3''$, &c. Now, since the lengths $01'$, $02'$, $03'$, &c., represent the velocities of the piston and reciprocating parts when in positions $1''$, $2''$, $3''$, &c., the difference between any two consecutive lengths, for example $1'$, $2'$, will represent the change of velocity that has taken place in the corresponding movement of the piston $1''$, $2''$. If we suppose the crank pin to revolve uniformly and divide the circle into equal parts, equal times will be occupied in the motions from point to point, and therefore equal times in the motions between consecutive positions $1''$, $2''$, $3''$, $4''$, &c., of the piston. Accordingly the differences $01'$, $1'2'$, $2'3'$, &c., will represent the force required to change the velocity of the reciprocating parts; and if we set them up as ordinates between the corresponding positions of the piston, we shall obtain the curve expressing the effect of inertia. The ordinate should be erected from the position of the piston when the crank-pin is at the middle of the intervals 1, 2, 3, &c.

It will be seen that the greater the number of parts into which we divide the crank-pin circle the less will be the ordinates representing the effect of inertia, though in all the curves the same character will be preserved. Accordingly it is possible to determine the number of parts into which the crank circle should be divided, or to determine the angle between consecutive radii, 01, 02, &c., such that the ordinates of the inertia curve be of such a length that they represent the pressure

Fig. 101.

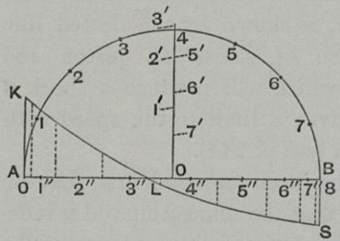
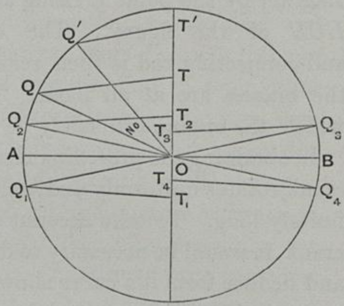


Fig. 102.



per square inch of piston area required for inertia on the same scale that the indicator diagram is drawn. The ordinates of the resulting inertia curve may then be directly employed to correct the indicator diagram.

Let N be the number of revolutions per minute; Q, Q' consecutive points on the crank-pin circle; and let $QOQ' = n°$. Further suppose that the crank-pin circle is drawn on a scale of x inches to the foot. Then

$$\frac{\text{change of velocity of piston}}{\text{velocity of crank pin}} = \frac{TT'}{OQ},$$

$$\therefore \text{ change of velocity of piston } \Delta v, \text{ in feet per second} = \frac{2\pi N}{60} TT',$$

where TT' is to be measured in feet on the scale x inches $= 1$ foot.

$$\therefore \Delta v = \frac{2\pi N}{60} \frac{TT'}{x}, \text{ where } TT' \text{ is to be measured in inches.}$$

Now this change of velocity takes place in the time occupied by the movement

$$QQ' = \Delta t \text{ seconds} = \frac{60}{N} \cdot \frac{n°}{360} = \frac{n°}{6N}.$$

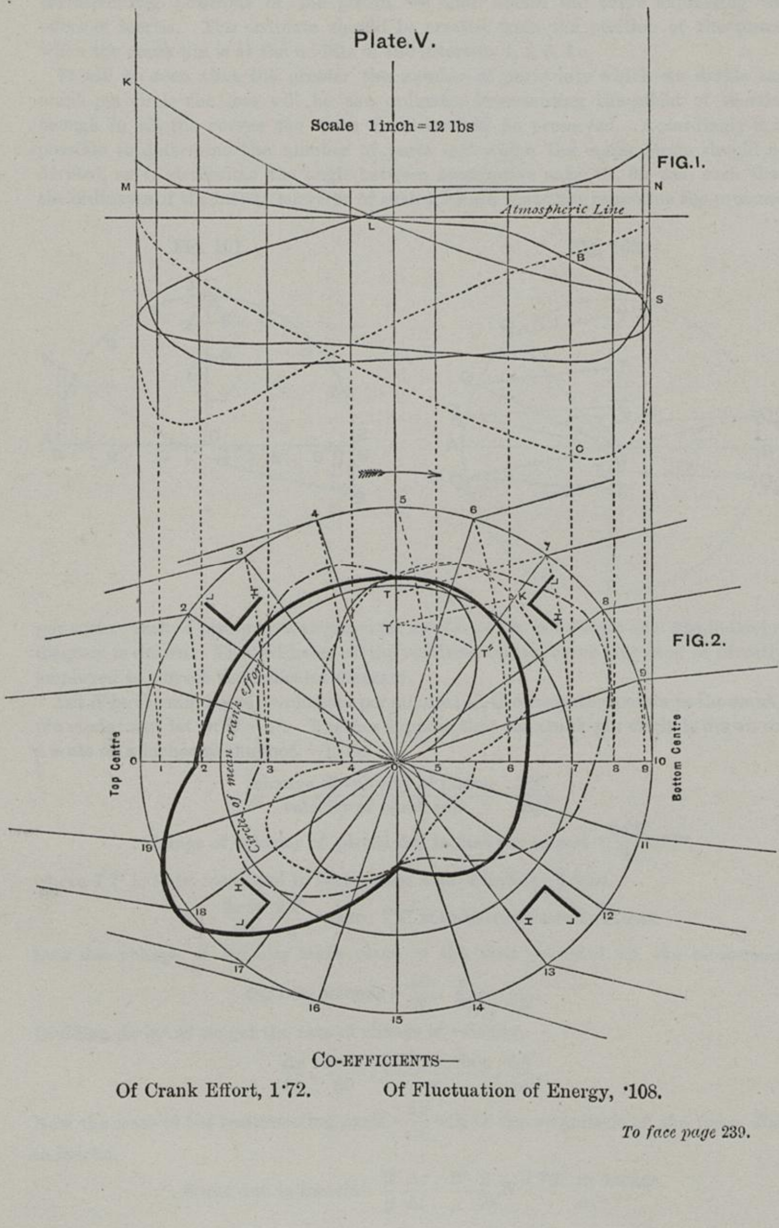
Dividing Δv by Δt we get the rate of change of velocity,

$$\frac{\Delta v}{\Delta t} = \frac{2\pi N}{60} \cdot \frac{(TT') \text{ inches}}{x} \cdot \frac{6N}{n°}.$$

Now the mass of the reciprocating parts $\times \dfrac{\Delta v}{\Delta t}$ will be the magnitude of the force due to inertia.

$$\therefore \text{ Force due to inertia} = \frac{W}{g}\frac{\Delta v}{\Delta t} = \frac{W}{g}\frac{2}{10} N^2 \frac{(TT' \text{ in inches}}{x n°}.$$

Plate. V.



CO-EFFICIENTS—

Of Crank Effort, 1·72. Of Fluctuation of Energy, ·108.

To face page 239.

Divide both sides by the area of the piston, and let p = pressure per square inch due to inertia, which will be in lbs. if W is taken in lbs. Also let p_0 = pressure equivalent to weight of reciprocating parts in lbs. per square inch.

$$\therefore p = p_0 \frac{2\pi}{10g} N^2 \frac{(TT' \text{ inches})}{xn^\circ}.$$

If now the indicator diagram be drawn on a scale of y lbs. to the inch, the pressure p equivalent to inertia will be represented on the same scale by taking a length in inches—

$$\frac{p}{y} = p_0 \frac{2\pi}{10g} \frac{N^2}{xyn^\circ} (TT' \text{ inches}).$$

Now it is required that n° be so taken that p/y in inches shall be the same thing as TT' in inches. Consequently

$$p_0 \frac{2\pi}{10g} \frac{N^2}{xyn^\circ} = 1.$$

$$\therefore n^\circ = \frac{2\pi}{10g} p_0 \frac{N^2}{xy} = \cdot 0195\, p_0 \frac{N^2}{xy}.$$

If now we draw a number of cranks inclined to each other at an angle of n°, we may obtain as many points on the curve of inertia as we please.

In practice it will in general be sufficient if we determine the end ordinates AK, BS, and the point L (Fig. 101) and draw a fair curve through these points. The ordinates AK and BS will be determined if we take (Fig. 102) AOQ_1, AOQ_2, BOQ_3, BOQ_4 each equal $\frac{1}{2}n^\circ$, then Q_1OQ_2 and Q_3OQ_4 being each equal n°, T_1T_2 will equal AK and $T_3T_4 = BS$.

Further, the curve will cross the base line at the point L, at which the piston will have its maximum velocity, which will occur approximately when the crank is at right angles to the connecting rod.

$$\therefore OL = \sqrt{(\text{con. rod})^2 + (\text{crank})^2} - \text{connecting rod}.$$

110. *Construction of Curves of Crank Effort for any given Indicator Diagram.*—If the varying magnitude of the steam pressure is given by the actual indicator diagram of the engine we may deduce the true crank effort as follows:—Let Fig. 1, Plate V., be a pair of indicator diagrams. The examples chosen are from the low-pressure cylinder of H.M.S. "Nelson." * Before proceeding to make use of them they should be corrected for inertia, and, where the engines are vertical, for the weight of the reciprocating parts. The curve of pressure due to inertia is KLS in Fig. 1, which has been drawn, as just described, to the same scale as the indicator diagram. If we draw a line MN parallel to the base line of the inertia curve to represent p_0, the pressure due to the weight of the reciprocating

* I am indebted to Mr. T. Hearson for the example here given, and for the method of drawing the curve of inertia which has just been described.

parts, then the intercept between MN and KLS will be the necessary correction for inertia and weight combined. In applying the correction, the forward pressure in one of the pair of diagrams should be taken in conjunction with the back pressure of the other, for it is the difference between these which gives the true effective pressure on the piston. Let the dotted lower curves be the result of the correction, so that the virtual pressure which is transmitted to the crank pin is to be measured by the vertical intercept between the upper steam curve and the dotted curve, such as BC for example. Immediately below the diagram draw a crank-pin circle with diameter equal to the length of the indicator diagrams. Divide the crank-pin circle into, say 20, equal parts, and suppose the crank pin to be successively at these points of division; determine the corresponding positions of the piston in its stroke. Whilst doing this, mark the directions in which the connecting rod lies when the crank pin is in these several positions. Let the positions of the piston in the line of stroke be set off along the diameter 0, 10. Through these points draw verticals to intersect the indicator diagrams. The intercepts of these verticals will give us the virtual steam pressure at each of the points of the stroke and corresponding to each position of the crank in its revolution. Next, having in Fig. 2 drawn a number of radii through the points 1, 2, 3, &c., lay off from the centre O along each, the respective intercepts of the indicator diagram which represent the vertical pressures of the steam when the cranks are in those positions. We thus draw what we may call a polar curve of virtual steam pressure. We have for example taken OK equal to BC in the figure, and similarly for all other radii.

Now, referring to page 204, we observe that if the connecting rod in any position be drawn to cut the vertical through O, in a point T, as for example in Fig. 2 when the crank is at 7, then the length OT will represent the crank effort on the same scale that the length of the crank arm $O7$ represents the magnitude of the steam pressure. If now through K we draw KT' parallel to $7T$, then by similar triangles $\dfrac{OT'}{OK} = \dfrac{OT}{O7}$, and thus on the same scale that OK represents the steam pressure OT' will represent the crank effort. Now along the crank $O7$ set off a length $OT'' = OT'$, and perform a similar operation for each of the positions of the crank. If through the points so obtained we draw a continuous curve it will be the polar curve of crank effort which we require, for it will represent by its radii in any position the actual crank effort when the crank is in that position; and we see that, in the construction, account is taken not only of the angular position of the crank, but also of the steam pressure which is available for turning the crank. Taking both indicator diagrams we thus draw the curve for the complete revolution of the engine. By transfer of the radii of the polar curve to the crank circle unrolled we can construct a linear curve (Art. 105), and thus determine the fluctuation of energy.

In Fig. 2 the thick curve has been drawn to show the crank effort due to the high and low pressure cylinders combined, by adding to the radii of the original curve the corresponding radii of the high pressure curve (not shown in the figure). In this engine the high pressure crank is 90° in advance of the low: if it had been 90° behind the low the fluctuation of crank effort would have been less. This is shown by the large dotted curve in the figure. The circle of mean crank effort is added to facilitate comparison. The values of the co-efficients of crank effort and energy are given in the diagram.

111. *Periodic Motion of Machines in General.*—The motion of a steam engine, which we have been describing in detail in this chapter, may be taken as a typical example of the transmission of energy by any machine whatever. Neglecting frictional resistances the energy is transmitted without alteration from a driving pair to a working pair—when the complete period of the machine is considered; but the rate of transmission varies from instant to instant during the period. The alternate excess and deficiency of energy is provided for by the moving parts of the machine, which serve as a store of energy or "kinetic accumulator," which can be drawn upon at pleasure. For equable motion it is necessary that they should be sufficiently heavy, and that the rotating pieces should greatly predominate over the reciprocating pieces. If the speed be very great reciprocating pieces are to be avoided altogether, especially in cases of higher pairing with force closure (Ex. 17, p. 244).

It has been supposed that the mean resistance at the working pair is exactly equal to the mean effort at the driving pair. If this be not the case the machine will rapidly alter its mean speed, till the equality is restored by alteration of the effort or the resistance or both. The equality seldom exists for long, and some means of controlling the machine is therefore generally indispensable, but this is a matter for subsequent consideration.

EXAMPLES.

1. In the case of a pair of cranks at right angles, draw the polar diagram of crank effort when the connecting rod is indefinitely long, and find the ratio of maximum crank effort to mean. Find also the position of the cranks when the actual crank effort is equal to the mean.

Maximum crank effort = 1·11 mean.

2. Draw the diagram and obtain the results as in the last question, when the length of connecting rod is equal to 4 cranks.

Maximum crank effort = 1·307 mean.

3. Draw the linear diagram of crank effort, assuming two cranks at right angles and connecting rod = 4 cranks.

4. What is the maximum length of connecting rod for which the crank effort is less than the mean throughout one quadrant?

Connecting rod = 7·1 cranks.

5. From the diagram of crank effort constructed in question 3, determine the coefficient of fluctuation of energy, 1st. When the connecting rods are indefinitely long; 2nd. When the length equals 4 cranks.

Connecting rod indefinitely long. Co-efficient of fluctuation of energy = ·011.
Connecting rod = 4 cranks. Co-efficients are ·011, ·042, ·011, ·009, ·038, ·009.

6. A pair of engines of 500 h.p., working on cranks at right angles with connecting rods = 4 cranks, are running at 70 revolutions per minute. Find the maximum and minimum moments of crank effort, and the fluctuation of energy in ft.-lbs.; assuming the steam pressure and resistance uniform.

Maximum moment of crank effort = 49,125 ft.-lbs.
Minimum moment of crank effort = 29,465 ft.-lbs.
Mean moment of crank effort = 37,500 ft.-lbs.

Fluctuation of energy = 9,900 ft.-lbs. Co-efficient = ·042.

7. In the case of a single crank the steam is cut off at one-fourth of the stroke. Neglecting back pressure and inertia, find the ratio of maximum to mean crank effort, and also the ratio of the fluctuation of energy to the energy of one revolution.

Maximum = 2·9 mean crank effort.
Fluctuation of energy = $\frac{1}{4}$ energy of one revolution.

8. Construct a diagram of crank effort for three cranks at angles of 120°. The lines of stroke of the three pistons are parallel, the steam pressure constant, and the resistance uniform. Find the ratio of maximum to mean crank effort, and the coefficient of fluctuation of energy for a connecting rod of 4 cranks.

Maximum = 1·077 mean crank effort.
k = ·0115.

9. In a pair of cranks at right angles, connecting rod 4 cranks long, the reciprocating parts have a stroke of 4 feet and weigh 20 tons. The steam pressure is uniform, and equal to 50 tons on each piston, and the resistance moment is uniform. Find the least number of revolutions the engines can make without the aid of a fly-wheel, and draw a curve of angular velocity ratio for this case.

Ans. At the point of maximum speed the least number of revolutions will be 50 per 1′. To obtain the curve and the least number of *complete* revolutions, see p. 233.

10. The pressure equivalent to the weight of the reciprocating parts of an engine is 4 lbs. per square inch, the stroke is 4 feet. Find the pressure necessary to start the piston, when the engines are making 75 revolutions per minute. If the steam pressure be initially at 30 lbs. above the atmosphere, and the cut-off at $\frac{1}{4}$th the stroke, find the effective pressure at each eighth of the stroke, taking account of the inertia of the piston, and assuming a constant back pressure of 3 lbs.

Pressure equivalent to inertia at commencement of stroke = 15·3 lbs. per sq. in.

Effective pressure at commencement			= 26·4
,,	,,	1st eighth	= 30·3
,,	,,	2nd ,,	= 34·0
,,	,,	3rd ,,	= 23·
,,	,,	4th ,,	= 19·4
,,	,,	5th ,,	= 18·7
,,	,,	6th ,,	= 19·5
,,	,,	7th ,,	= 21·2
,,	,,	8th ,,	= 23·5

11. In the last question find the number of revolutions per minute necessary to produce a shock near the commencement of the stroke. If the steam be cut off at $\frac{1}{8}$th, or earlier, show that a shock occurs also at other points of the stroke. *Ans.* 124.

12. In question 10 construct a curve showing the kinetic energy of the piston at each point of the stroke, and deduce a curve showing the pressure due to inertia of the piston.

Take the curve of piston velocity previously constructed, and PN being any ordinate of it, the kinetic energy of the piston will be proportional to the square of PN, so we have only to draw a curve whose ordinates vary as $(PN)^2$.

Having drawn the curve of kinetic energy, take the difference between consecutive equi-distant ordinates of that curve and set them as an ordinate from a new base line AB as Cd, and so construct a curve whose ordinates will be proportional to the pressure equivalent to inertia.

13. By use of the formula

$$V = V_0\left(\sin\theta + \frac{1}{n}\cdot\sin\theta\cdot\cos\theta\right)$$

(page 111) for the velocity of the piston, prove that the pressure necessary to start and stop the piston at the ends of the stroke is given by

$$p' = p_0\frac{V_0^2}{ga}\left(1 + \frac{1}{n}\right).$$

14. Draw a curve of kinetic energy of an oscillating cylinder, assuming a mean radius of gyration for the cylinder and piston, and deduce the bending moment on the piston rod.

NOTE.—The force of inertia in this case is so great that the speed of oscillating engines is limited.

15. If n be the revolutions per minute of a fly-wheel and d its diameter : show that the weight of wheel necessary for a given regularity in an engine of given indicated power is

$$W = C.\frac{IHP}{n^3 d^2},$$

where C is constant.

NOTE.—The diameter is generally about $3\frac{1}{2}$ times the stroke (S), and according to a well-known empirical rule for piston speed (V) employed in calculating nominal horse-power $V^3 \propto S$. If this be assumed $n^3 d^2$ is constant, and the weight of wheel is then proportional to the indicated horse-power, a rule sometimes employed, 100 lbs. being allowed for each horse-power.

16. The fluctuation of energy of an engine of 150 $I.H.P.$ is 13 per cent. of the energy exerted in one revolution. The revolutions are 35 per minute, find the weight of a fly-wheel 20 feet in diameter, that the fluctuation in speed may not exceed one-fortieth. *Ans.* 8 tons.

17. In the cam movement shown in Fig. 1, Plate IV., page 173, suppose the cam a circular disk of radius equal to the stroke of the sliding piece. Supposing the force of the spring twice the weight of sliding piece : find the greatest number of revolutions per 1′ the mechanism can make when the cam rotates uniformly.

Ans. If S be the stroke in inches, n the revolutions,

$$n = \frac{216}{\sqrt{S}}.$$

18. In a 3-cylinder Brotherhood engine, the stroke is S inches, the revolutions n per 1′, the total pressure on one piston P; show that, to avoid reversal of the stress on the piston rod, the weight of a piston and rod must not exceed

$$W = 70{,}500\frac{P}{n^2 S}.$$

NOTE.—In a double-acting engine there is necessarily reversal at the ends of the stroke : in the Brotherhood this is avoided by the use of 3 cylinders at 120°, the inner ends of which communicate constantly with a central chamber containing steam at full pressure. These engines therefore may run at a very high speed if the cut off at the outer end be sufficient.

CHAPTER X.

FRICTIONAL RESISTANCES.

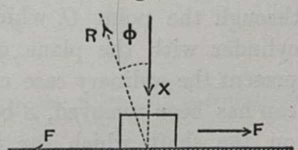
112. *Preliminary Remarks.*—The action of a machine consists, as we have seen, in a transmission of energy from a driving pair to a working pair, through a number of intermediate pairs, which change in a given way the motions proper to the source of energy. In the absence of friction, the energy transmitted from piece to piece in a complete period would be the same for all the pairs, but, in consequence of frictional resistances, a certain part of the energy is lost at each transmission. These frictional resistances are of two kinds, one due to the relative motion of the elements of the pairs one upon another, the other to the changes of form which the flexible parts of the machine undergo, for example to the bending of ropes and belts. It is to the first kind that the word "friction" is specially appropriated, although it is not essentially different from the second kind which in some cases is also called "stiffness."

We commence with the case of linkwork mechanisms in which the friction is due simply to the sliding of one surface upon another. The pairing is in this case of the lower class.

SECTION I.—EFFICIENCY OF LOWER PAIRING.

113. *Ordinary Laws of Sliding Friction.*—If one body rests on another (Fig. 102) and is pressed against it with a force X, a mutual action takes place between the two which resists sliding. The magnitude of this mutual action or tangential stress (Ch. XII.) is measured by the force F which is necessary to produce sliding, and the ratio F/X is called the co-efficient

Fig. 102.



of friction and will be denoted by f. The value of f depends on the nature and condition of the surfaces in contact, whether rough or smooth, dry or lubricated. Under certain circumstances and within certain limits it is independent of the area of the surfaces in contact and of the velocity of sliding. These statements may be called the "ordinary" laws of friction. The evidence on which they rest and the limitations to their truth will be considered hereafter; for the present we assume them as applicable to all the cases we consider.

The work done in overcoming friction may be estimated just as in the case of any other resistance. If the body move through a space x the work done is Fx or $f.Xx$ if X be uniform, and if it be not, a curve is constructed giving X at every point, then the area under that curve multiplied by the co-efficient f is the work done (see Ex. 2). If R be the re-action of the surface upon which the body we are considering rests, ϕ the angle its direction makes with the normal to the plane,

$$R \, . \cos \phi = X : R \, . \sin \phi = F ;$$
$$\therefore \tan \phi = f,$$

an equation which shows, that the total mutual action between two plane surfaces, which slide over one another, makes an angle with the normal to the plane, the tangent of which is the co-efficient of friction. The magnitude of this angle then is fixed, but its direction varies according to the direction of the sliding. It may therefore be called the "friction angle," but it is also often called the "angle of repose," because it is the greatest inclination of a plane on which the body can rest under the action of gravity without slipping. In the solution of questions respecting friction, graphically or otherwise, it is often convenient to suppose it known.

114. *Friction of Bearings.*—Next suppose the surfaces in contact cylindrical. In Fig. 103 ABA represents a cylinder pressed down into a semicircular bearing by a force S, the direction of which passes through the point O, which is the intersection of the axis of the cylinder with the plane of the paper. We may take this to represent the ordinary case of a shaft and its bearing from which the cap has been removed, S being the resultant of all the forces acting on the shaft which for the moment are supposed to have no tendency to turn the shaft. The force S is balanced by the reaction of the bearing which, when the bearing is in good condition, consists of a pressure distributed over the whole semi-cylindrical surface. Let DE be a small element of the surface,

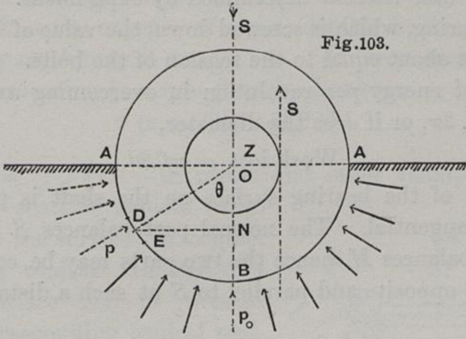


Fig.103.

p the pressure, θ the angle the radius of DE makes with the direction of S, then we must have

$$\Sigma p\, DE \cos\theta = S.$$

If now we knew the law according to which p varies from point to point, we could by use of this equation find the actual value of p and also find the total amount of the distributed pressure, that is to say, $\Sigma p . DE$ which we will call X. Evidently then we shall have

$$X = k . S,$$

where k is a co-efficient depending on the law of distribution and therefore to some extent uncertain. When a bearing is well worn it is probable that (see Art. 115) if p_0 be the pressure at B

$$p = p_0 . \cos\theta,$$

that is, that the intensity of the pressure at any point varies as ON the distance of the point below the centre. This is the same law as that which the pressure of a heavy fluid follows, supposed occupying the semicylinder ABA, and it is shown in books on hydrostatics that

$$\frac{\text{Total pressure}}{\text{Resultant pressure}} = \frac{4}{\pi} = k.$$

Next suppose the shaft to be turned by the action of a couple M applied to it, then if a be the radius

$$M = \Sigma f . p . DE . a = f . Xa = fk . Sa.$$

In this formula we have some doubt as to the value of k, and we are not sure that the co-efficient f would be the same for a curved as for a plane surface; we therefore replace fk by f', where f' is a special co-efficient of axle friction determined by experiment. If there is a cap on the bearing, which is screwed down, the value of S is increased by an amount about equal to the tension of the bolts.

The loss of energy per revolution in overcoming axle friction is evidently $M \cdot 2\pi$, or if d be the diameter,

$$\text{Work lost} = \pi f' S d.$$

The reaction of the bearing surface on the shaft is partly normal and partly tangential. The normal part balances S and the tangential part balances M, hence the two parts may be combined into a single force opposite and parallel to S at such a distance z from O that

$$Sz = M, \text{ or } 2z = f'd,$$

that is to say, the line of action of the mutual action between the shaft and its bearing always touches a circle, the diameter of which is f' times the diameter of the shaft. This circle is called the Friction Circle of the shaft or pin considered. When the bearing has a cap on, the force S must be increased by the tension of the bolts in calculating M, but not for any other purpose, and the diameter of the friction circle is consequently increased, it may be very considerably. The utility of this rule will be seen presently.

The real pressure between a shaft and its bearing varies from point to point, as we have seen. What is conventionally called the "pressure on the bearing" is something different. Let l be the length of the bearing, then ld is the area of the diametral section, and

$$p = \frac{S}{ld}$$

is the quantity in question. It is a sort of mean value of the actual pressure, and will bear some definite relation to it depending on the law of pressure. For the particular law of pressure given above

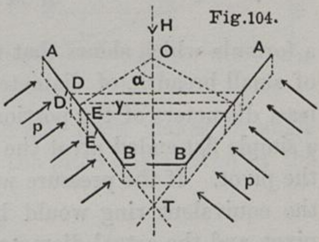
$$p = p_0 \cdot \frac{\pi}{4}.$$

The work lost by friction per square inch of bearing surface per 1′ is evidently proportional to pv, where v is the rubbing velocity in feet per minute. An equivalent amount of heat is generated as we shall see hereafter, and it is upon the rate at which this heat can be abstracted by the cooling influences to which the bearing is exposed that the amount of bearing surface required depends. In marine engine bearings the value of pv is sometimes as much as 60,000, though at the expense of a considerable liability to heating, and in railway machinery it is not less. At lower speeds the value is smaller. According to a rule given by Rankine,

$$p\,(v + 20) = 44{,}800.$$

115. *Friction of Pivots.*—In pivots and other examples in which the revolving shaft is subject to an endways force the surfaces in contact are frequently conical. In Fig. 104 a conical surface AB is pressed against a corresponding conical seating by a force H, and revolves at a given rate. If the surface be divided into rings, one of which is seen in section at DE, the pressure on those rings may be resolved vertically upwards, and must then balance H. Hence if p be the pressure on DE a ring the radius of which is y,

Fig.104.



$$\Sigma p \,.\, DE \,.\, 2\pi y \cos \alpha = H,$$

where α is the angle a normal to the conical surface makes with the axis.

When the bearing is somewhat worn the conical surface will have descended through a certain space, and it may be assumed that all points such as DE will descend through an equal space, so that the wear of the surface measured normal to itself is proportional to $\cos \alpha$. But if v be the velocity of rubbing of the ring DE, the wear will be proportional to pv, that is to py: hence

$$py \propto \cos \alpha.$$

This principle determines the most probable distribution of the pressure on worn surfaces in any case, and has already been used above for the case of a journal. In the present case α is constant, and we have

$$py = \text{constant} = p_1 y_1 = p_2 y_2,$$

where the suffixes 1 and 2 refer to the upper and lower edge; hence, by substitution, if l be the length AB of the conical surface,

$$py \,.\, 2\pi l \,.\, \cos \alpha = H,$$

a formula which determines the pressure at every point. The moment of friction is evidently

$$M = f\Sigma pDE2\pi y^2$$
$$= f.py.2\pi.\Sigma y.DE = \frac{fpy2\pi\Sigma y\Delta y}{\cos\alpha},$$

where Δy is written for the projection of DE on the transverse plane. By use of the integral calculus this is readily seen to be

$$M = fpy2\pi\frac{y_1^2 - y_2^2}{2\cos\alpha} = fpy2\pi l.\frac{y_1 + y_2}{2},$$

$$\text{or } M = f.H.\frac{y_1 + y_2}{2\cos\alpha},$$

a formula which shows that the friction is the same as that of a ring of small breadth, of diameter equal to the mean of the greatest and least diameters of the portion of a cone considered. In the case of a simple flat-ended pivot the equivalent ring is half the diameter of the pivot. If the pressure were uniform throughout, the diameter of the equivalent ring would be $\frac{2}{3}$ instead of $\frac{1}{2}$ the diameter of the pivot, and the actual diameter in practice will probably vary between these limits.

Pivots are sometimes used in which the surfaces in contact are not cones, but are curved, so that in wearing the pressure and wear are the same throughout (Schiele's pivots). That this may be the case we must have, since p is constant,

$$y \propto \cos\alpha,$$

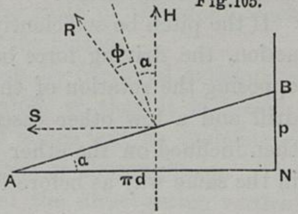
that is to say, if we draw a tangent DET to meet the axis in T, ET must be constant. The curve which possesses this geometric property is called the "tractrix." It is traced readily by stepping from point to point, keeping the tangent always of the same length. Pivots of this kind are very suitable for high speeds, as the wear is very smooth.

116. *Friction and Efficiency of Screws.*—In any case of a machine in steady motion the principle of work takes the form (Art. 96)

$$\left.\begin{matrix}\text{Energy exerted}\\ \text{in a period}\end{matrix}\right\} = \left\{\begin{matrix}\text{Useful work done + Work wasted}\\ \text{in overcoming frictional resistance.}\end{matrix}\right.$$

The simplest case is that of a screw which we will suppose to be square threaded and applied to a press, or to some similar purpose. The pressure between the nut and the thread is distributed uniformly along the thread, if the screw be accurately constructed and slightly worn. As shown in the last article in the similar case of a pivot, the friction may be regarded as concentrated on a spiral traced on a cylinder the diameter of which may be expected to be about the mean of the external and internal diameter of the screw. Fig. 105 shows one convolution of this spiral unrolled. AB is the thread, BN, parallel to the axis of the screw, is the pitch p, and AN is the circumference πd. H is the thrust of the screw, being the force which the screw is overcoming by means of a couple applied to turn it about its axis. R is the action of the screw thread which (Art. 113) makes an angle ϕ with the normal, where ϕ is the angle of repose. The normal itself makes an angle α with the axis of the screw, where α is the pitch angle given by the formula

Fig.105.



$$\tan \alpha = \frac{p}{\pi d}.$$

This force R arises from the turning forces applied to the screw, and must have the same moment M about the axis of the screw; its vertical component therefore must be H and its transverse component a force S such that

$$S \cdot \frac{d}{2} = M.$$

Hence the equations

$$M = \frac{Rd}{2} \cdot \sin(\alpha + \phi),$$

$$H = R \cdot \cos(\alpha + \phi).$$

Also considering a complete revolution of the screw,

$$\text{Energy exerted} = M \cdot 2\pi = R\pi d \cdot \sin(\alpha + \phi),$$

$$\text{Useful work done} = H \cdot p = Rp \cdot \cos(\alpha + \phi),$$

from which it follows that the efficiency of the screw is

$$\text{Efficiency} = \frac{\tan \alpha}{\tan(\alpha + \phi)}$$

It is not difficult to show that this fraction is greatest when $\alpha = 45° - \frac{1}{2}\phi$, and its value is then

$$\text{Maximum efficiency} = \left(\frac{1 - \frac{1}{2}f}{1 + \frac{1}{2}f}\right)^2 \text{ approximately.}$$

For ordinary values of f then, the best pitch angle is approximately 45° and the efficiency is considerable.

In practice, however, the pitch angle is much smaller, its value in bolts and the screws used in presses ranging from ·035 in large screws to ·07 in smaller ones; the efficiency is then less, often much less, than one third, the object aimed at being not efficiency but a great mechanical advantage.

If the pitch be sufficiently coarse, it will be possible to reverse the action, the driving force being then H and the resistance a moment opposing the rotation of the screw. In a well known kind of hand drill and a few other cases this occurs in practice; the force R is then inclined on the other side of the normal, and the efficiency is in the same way as before found to be

$$\text{Efficiency} = \frac{\tan(\alpha - \phi)}{\tan \alpha}.$$

In most cases, however, α is less than ϕ, and the screw is then incapable of being reversed. Non-reversibility is often a most valuable property in practical applications, the friction then serving to hold together parts which require to be united or to lock a machine in any given position.

In estimating the efficiency of screw mechanisms the friction of the end of the screw acting like a pivot or of the nut upon its seat must be included; in screw bolts this item is generally as great as the friction of the threads. The friction due to lateral pressure of the screw on its nut may usually be neglected, but when necessary it may be estimated by the same formula as is used for shafts. The above investigation, strictly speaking, applies only to square-threaded screws; it has, however, been shown that the efficiency is only slightly diminished by the triangular or other form of thread usually adopted for the sake of strength.* The formulæ here given for screws may be applied to any case of a sliding pair in which the driving effort is at right angles to the useful resistance. A simpler case is that in which the driving effort is parallel to the direction of sliding. This is given in Example 1, page 271. In all cases observe that the efficiency diminishes rapidly when the velocity-ratio is increased. This, which is common to most mechanisms, limits the

* *Cours de Mécanique Appliquée aux Machines*, par J. V. Poncelet, p. 386. Paris, 1874.

mechanical advantage practically attainable. The hydraulic press is an exception, as will be seen hereafter.

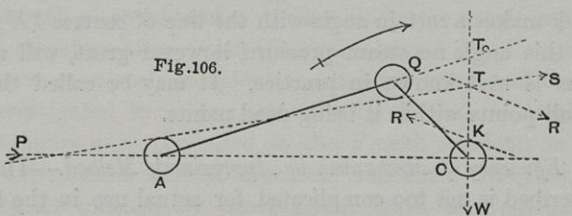
117. *Efficiency of Mechanism by Exact Method.*—In the preceding cases the efficiency is the same for any motion of the mechanism whether large or small. Generally, however, it will be different in each position of the mechanism, and by the "efficiency of the mechanism" is then to be understood the ratio of the useful work done in a period to the energy exerted in the period.

The exact calculation of the loss of work by frictional resistances in mechanism is generally very complicated, so that it is best to proceed by approximations the nature of which will be understood on considering an example with some degree of thoroughness. The case we select is that of the mechanism of the direct-acting vertical steam engine such as is represented in Plate I., p. 119.

The losses by friction are (1) the loss by piston friction, (2) friction of guide bars, (3) friction of crosshead pin, (4) friction of crank pin, (5) friction of crank-shaft bearings. Of these, the first two are considered separately (Ex. 2, p. 271), and for the present neglected, while the last three are treated by a graphical method as follows.

In Fig. 106 CQA are the friction circles of the three parts in

Fig.106.



question, which for the sake of clearness are drawn on a very exaggerated scale while the bearings themselves are omitted. We will neglect the weight of the connecting rod and its inertia; of these the first is generally relatively inconsiderable, but in high-speed engines the last is often very large and makes the friction very different at high speeds and low speeds (see Ch. XI.) The weight of the crank shaft and all the parts connected with it is supposed to act through the centre of the shaft; for simplicity we will call it W. The pressure on the piston after correction for piston and guide-bar friction is denoted by P. Then, in the absence of friction, the line of action of the thrust on the connecting rod is the line joining the centres of the friction circles, and the moment of crank effort is $P.CT_0$, where T_0 is the intersection of that line with the vertical through C. But the line of action in question must now touch the friction circles (Art. 114), and the true moment of crank effort on the same principle must be $P.CT$, where T is the intersection of this common tangent with the vertical CT. Thus $P.TT_0$ is the correction for friction of the crosshead and crank pins. Next observe that the forces acting on the crank shaft are W the weight and S the thrust of the connecting rod; these may be compounded into one force R passing through T as shown in the diagram. The reaction of the crank-shaft bearing is an equal and opposite force R which must touch the friction circle and cut CT in a certain point K. Now the horizontal component of R is the same as that of S, namely P; therefore the true moment of crank effort after allowing for friction is $P.TK$.

By performing this construction for a number of positions, as in the last chapter, we obtain a diagram of crank effort corrected for friction. The area of this curve will give us the useful work done in a revolution, the ratio of which to the energy exerted is the efficiency of the mechanism: and its intersections with the line of mean resistance will give the points of maximum and minimum energy and the fluctuation of energy as corrected for friction. When the crank makes a certain angle with the line of centres TK vanishes. Within this angle no steam pressure, however great, will move the crank, as is well known in practice. It may be called the "dead angle," all points within it being dead points.

118. *Efficiency of Mechanism by Approximate Method.*—The process just described is not too complicated for actual use in the foregoing example, but in many cases it would be otherwise, and it may therefore be frequently replaced with advantage by a calculation of the efficiency of each of the several pairs of which the mechanism is made up taken by itself.

Each pair consists of two elements, one of which transmits energy to the other, with a certain deduction caused by the friction between the elements. The ratio of the energy transmitted to the energy received may be called the efficiency of the pair. If e_1, e_2, e_3 ... be the efficiencies of all the pairs in the mechanism it is evident from the definition that the efficiency of the whole mechanism must be

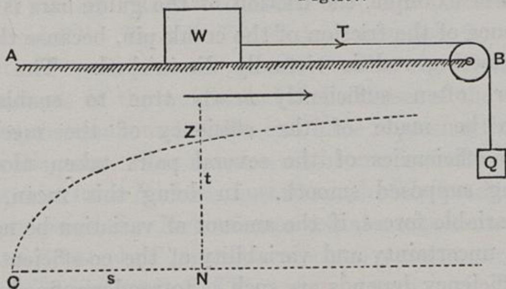
$$e = e_1 . e_2 . e_3 \dots$$

In some cases the efficiency of each pair will be independent of the frictional resistances of all the other pairs, and may be found separately. In general this is approximately, but not exactly, true, a point which will be best understood by a consideration of the foregoing diagram. For example, the friction of the guide bars is diminished in consequence of the friction of the crank pin, because the obliquity of the connecting rod is virtually diminished. The supposition is, however, often sufficiently nearly true to enable a rough estimate to be made of the efficiency of the mechanism by finding the efficiencies of the several pairs taken alone, all the others being supposed smooth. In doing this mean values are taken for variable forces, if the amount of variation be not considerable. The uncertainty and variability of the co-efficients on which frictional efficiency depends are such as to render refined calculations of little practical value.

119. *Experiments on Sliding Friction* (*Morin*).—The ordinary laws of friction, which may be comprised in the single statement that the co-efficient of friction depends on the nature of the surfaces alone, and not on the intensity of the pressure or on the velocity of rubbing, were originally given by Coulomb in a memoir, published in 1785, although some facts of a similar kind were previously known. They are therefore often called Coulomb's laws. Yet Coulomb's experiments were scarcely sufficient to establish them, and the subject was reinvestigated by others, especially by the late General Morin, whose memoirs were presented to the French Academy in 1831-4. Morin's experiments were so elaborate and exact that they may be considered as conclusively proving the truth of Coulomb's laws within certain limits of pressure and velocity, and under the circumstances in which they were made: it will therefore be advisable to explain them briefly.

A sledge loaded with a given weight was caused to slide along a horizontal bed AB more than 12 feet long (Fig. 107), the rubbing surfaces being formed of the materials to be experimented on. The necessary force was supplied by a cord passing over a pulley at B to a descending weight Q. The tension of the cord T was measured by a spring dynamometer, and could likewise be inferred from the magnitude of the weight after correction for the stiffness of the cord and the friction of the pulley. In one form of experiment the weights were so arranged that the sledge moved nearly uniformly: the corresponding friction was measured and found to be constant. In a second form, the times occupied by the sledge in reaching given

Fig.107.



points were automatically measured and compared with the spaces traversed, by setting them up as ordinates of the curve CZ shown below. The curve proved to be a parabola, showing that the space varied as the square of the time, from which it was inferred that the acceleration of the sledge was constant.

From both methods it appeared that the co-efficient of friction was exactly the same, whatever the pressure and whatever the velocity, provided the nature and condition of the surfaces were the same. A few important results are given in the annexed table; they are taken from Morin's latest memoir,* containing, besides many new experiments, tables of the results of the whole series. The limits to their application will be considered presently.

Nature of Surfaces.	Condition of Surfaces.	Co-efficient of Friction.
Wood on Wood,	Perfectly dry and clean,	·25 to ·5
Metal or Wood on Metal or Wood,	Slightly oily,	·15
Do. do.,	Well lubricated,	·07 to ·08
Do. do.,	Lubricant constantly renewed,	·05

* Nouvelles Experiences . . . . faites à Metz en 1834. Page 99.

Full tables of Morin's results will be found in Moseley's work cited on page 267. The friction between surfaces at rest is often greater than when they are in motion, especially when the surfaces have been some time in contact: the excess, however, cannot be relied on, as it is liable to be overcome by any slight vibration.

120. *Exceptions to the Ordinary Laws in Plane Surfaces.*—From the exactitude with which Coulomb's laws were verified by Morin's experiments the inference was naturally drawn that they were universally true, but this is probably erroneous. Although no complete and thorough investigation has been made, it can hardly now be a matter of doubt that there are cases in which the laws of friction are widely different. The known cases of exception for plane surfaces may be grouped as follows:—

(1) At low pressures the co-efficient of friction increases when the pressure diminishes. This has been shown by various experimentalists, as, for example, by Dr. Ball.* The lowest pressure employed by Morin was about three fourths of a lb. per square inch, and this is about the pressure at which the deviation noticed by Ball becomes insensible. This effect may be due to a slight adhesion between the surfaces independent of friction proper.

(2) At high pressures, according to certain experiments by Rennie,† the co-efficient increases greatly with the pressure. The upper limit of pressure in Morin's experiments was from 114 to 128 lbs. per square inch. At 32·5 lbs. per square inch Rennie found for metallic surfaces at rest ·14 to ·17, nearly agreeing with Morin; but on increasing the pressure the co-efficient became gradually greater, ranging from ·35 to ·4 at pressures exceeding 500 lbs. per square inch. The metals tried were wrought iron on wrought and cast iron, and steel on cast iron. Tin on cast iron showed only a slight increase in the co-efficient. In fully lubricated surfaces in motion we shall see presently the results are exactly opposite. This increased friction at high pressures may be due to abrasion of the surfaces.

(3) At high velocities the co-efficient of friction, instead of being independent of the velocity, diminishes greatly as the velocity increases. This was shown by M. Bochet in 1858. Similar results

* *Experimental Mechanics*, by R. S. Ball, page 78. Macmillan, 1871.
† *Phil. Trans.* for 1829.

have been obtained by others, especially by Capt. Galton in some important experiments on railway brakes.* The limit of velocity in Morin's experiments was 10 feet per 1″, and at somewhat greater velocities than this the diminution becomes perceptible. Morin's results have been shown to be applicable at the very lowest velocities by Professor F. Jenkin and Mr. Ewing.†

It appears difficult to explain the diminution at high speeds merely by a change in the condition of the surfaces; it should, probably, be regarded as part of the law of friction. Professor Franke in the *Civil Ingenieur* for May, 1882, has proposed the formula

$$f = f_0 \cdot \epsilon^{-av},$$

where f_0 is about ·29, and a (for velocities in metres per 1″) ranges from ·02 to ·04, according to the nature and state of the surfaces.

121. *Axle Friction.*—It has already been pointed out that the co-efficient of axle friction is not necessarily the same as that for plane surfaces sliding on one another, and, besides, the continuous contact of a shaft and its bearing is very different from the brief contact occurring in sledge experiments. Morin however made special experiments on the friction of axles and showed that the co-efficients were constant and nearly the same in the two cases. The diameters employed however were 4 inches and under, while the revolutions did not exceed 30 per minute, so that the rubbing velocity was not more than 30 feet per minute. The pressures were not great, the value of pv not exceeding 5,000.

Much greater values of pv than this occur in modern machinery, and then it is tolerably certain that the value of the co-efficient is much less and diminishes with the pressure. Already in 1855 M. Hirn had made a long series of experiments on friction, especially of lubricated surfaces. The following summary of his results is given by M. Kretz, editor of the third edition of the *Mécanique Industrielle.*‡

(*a*) That a lubricant may give a regular and minimum value to the friction it must be "triturated" for some time between the rubbing surfaces.

* See *Engineering*, vol. 25, pages 469–472.

† *Phil. Transactions*, vol 167, part II.

‡ *Introduction à la Mécanique Industrielle*, par J. V. Poncelet. Troisième édition. Paris, 1870. Page 516.

(*b*) The friction of lubricated surfaces diminishes when the temperature is raised, other things being equal.

(*c*) With abundant lubrication and uniform temperature friction varies directly as the velocity. When the temperature is not maintained uniform, the relation between friction and velocity depends on the law of cooling of the special machine considered. In ordinary machinery friction varies as the square root of the velocity.

(*d*) The friction of lubricated surfaces is nearly proportional to the square root of the area and the pressure.

The last result is equivalent to saying that the co-efficient of friction varies inversely as the square root of the pressure per unit of area. It is remarkable that this law has also been deduced by Professor Thurston from experiments made apparently without any knowledge of what Hirn had done* with pressures from 100 to 750 lbs. per square inch and a velocity of 150 per 1′.

It may be open to question whether Hirn's experiments are sufficient to establish all the above statements, but it cannot be doubted that for values of pv exceeding 5000 the co-efficient of friction of well lubricated bearings of good construction diminishes with the pressure, and may be much less than the value at low speeds as determined by Morin. How far the diminution can be regarded as due to a change of condition consequent on continuous wear is uncertain.

We now proceed to consider higher pairing, commencing with the case of rolling contact. The friction is then described as "rolling friction."

SECTION II.—EFFICIENCY OF HIGHER PAIRING.

122. *Rolling Friction.*—When a wheel rolls on soft ground the resistance to rolling is due to the fact that the wheel makes a rut and depresses the ground as it advances over it. Thus the resistance to motion is proportioned to the product of the weight moved into the depth of the depression. The depth of the rut depends on the radius as well as the breadth of the wheel. It is found that the resistance may be expressed by

$$R = \frac{bW}{r},$$

where W = weight, r = radius of wheel, and b is approximately a constant length. This might have been anticipated, since the depth

* *Friction and Lubrication*, by R. H. Thurston. New York, 1879.

of the rut is the versed sine of the arc of contact, and therefore for a given small arc is inversely as the radius. If the wheel roll on hard ground over a succession of obstacles of small height the law of resistance will be expressed by the same formula.

When the surface rolled over is elastic and the pressure on it is not sufficient to produce a permanent rut, the resistance to rolling is not so easily explained. If we consider an extreme case, as for instance a heavy roller rolling on india-rubber, we shall be able to see to what action the resistance is due. The wheel will sink into the rubber, which will close up around it both in advance and behind as shown in Fig. 108. At C the rubber will be most compressed.

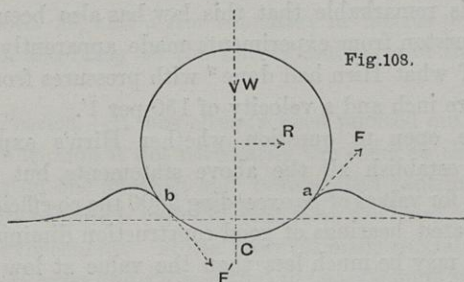


Fig.108.

As the wheel advances and commences to crush the rubber in advance of it the rubber moves away to avoid the compression, heaping itself up continually in advance of the wheel. In this movement it rubs itself over the surface Ca of the wheel, exerting on it a frictional force in the direction shown by the arrow F, which opposes the onward motion of the wheel. Again, the rubber in the rear is continually tending to recover its normal position and form of flatness, and in doing so rubs itself over the surface bC of the wheel in the direction shown by the arrow F', which also tends to oppose the onward motion of the wheel. The effect of this creeping action of the rubber over the surface of the wheel is to cause the onward advance of the centre of the wheel to be different from that due to the circumference rolled out.* Moreover the vertical component of the reaction of the surface no longer passes through the centre of the wheel as it must do in the absence of friction, but is in advance by a small quantity b such that Wb is the moment of resistance to rolling.

* See a paper by Prof. Osborne Reynolds, *Phil. Trans.*, vol. 166, to whom the true explanation of resistance to rolling in perfectly elastic bodies is due.

Experiments on rolling resistance present considerable discrepancies, but within the limits of dimension of rollers which have been tried it appears that b is independent of the radius; this leads to a formula of the same form as before for the force necessary to draw the roller, namely

$$R = \frac{Wb}{r},$$

where b is a constant which for dimensions in inches is from ·02 to ·09 according to the nature of the surfaces. With very hard and smooth surfaces of wood or metal, the lower value ·02 may be employed. Rolling friction is not sensibly diminished by lubricants, but depends mainly on smoothness and hardness of the surfaces. It is probably influenced by the speed of rolling, but this does not appear to have been proved by experiment unless in cases where the resistance of the atmosphere and other causes make the question more complicated.

In many cases of rolling the surfaces are partly elastic and partly soft, so that the resistance to rolling is partly due to surface friction and partly to permanent deformation. The value of the constant b is then much increased. For wagon wheels on macadamized roads in good condition the value of b is about ·5″, and on soft ground four to six times greater. The draught of carts is said to be increased by the absence of springs.

123. *Friction of Ropes and Belts.*—Frictional resistances are also produced by the changes of form and dimension of the parts of a machine occasioned either by the stresses necessarily accompanying transmission of energy or by shocks. In the present chapter we consider tension elements only, that is to say, chiefly ropes and belts.

In Fig. 109 AB is a pulley, the centre of which is O, over which a rope passes embracing the arc AKB and acted on by forces T_1T_2 at its ends. If there be sufficient difference between T_1 and T_2 the rope will slip over the pulley notwithstanding the friction which tends to prevent it. Let the rope be just on the point of slipping, then its tension will gradually diminish from T_1 at A to T_2 at B. Let T, T' be the tensions at the intermediate points K, L, then the portion KL of the rope is kept in equilibrium by the forces T, T' at its ends, and a third force S due to the reaction of the pulley, the three forces meeting in a point E.

On OL set off to Ol to represent T, and draw lk perpendicular to S to meet OK in k, then the sides and the triangle Okl will be proportioned by the three forces, so that Ok represents T' and ak S. The angle S makes with the radius will be the same for all arcs of the same length, and if KL be taken small enough will be the angle of friction (Art. 113).

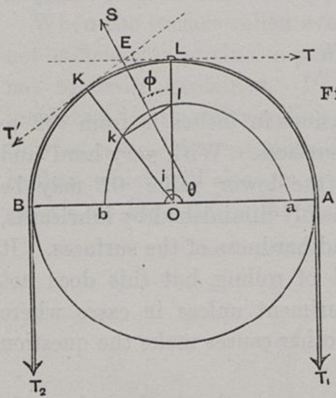


Fig.109.

This construction can, if we please, be commenced at A and repeated for a number of small portions of the rope till we arrive at B; we shall obtain a spiral curve $alkb$, the last radius Ob of which represents T_2 on the same scale as the first Oa represents T_1. It is convenient however to have an algebraical formula to calculate T_2. Let the angle KOL be i and the angle S makes with the radius ϕ, then

$$\frac{T}{T'}=\frac{Ol}{Ok}=\frac{\sin\ Okl}{\sin\ Olk}=\frac{\cos\ (i+\phi)}{\cos\ \phi}=\cos\ i+\sin i \tan\ \phi.$$

If now the angle i be diminished indefinitely we may write $\cos\ i=1$ and $\sin\ i=i$, so that

$$\frac{T-T'}{T'}=i\ .\tan\ \phi.$$

Replacing i by $\Delta\theta$, $T-T'$ by ΔT, and proceeding to the limit

$$\frac{1}{T}\frac{dT}{d\theta}=\tan\ \phi=f,$$

which being integrated gives

$$\frac{T_1}{T_2}=\epsilon^{f\theta},$$

where f is the co-efficient of friction, θ the angle subtended by the part of the pulley embraced by the rope, and ϵ the number 2·7288 being the base of the Napierian system of logarithms. The formula is applicable even if the pulley be not circular. For a circular pulley the spiral curve, representing graphically the tension at every point, is the equiangular or logarithmic spiral of which the formula may be regarded as the equation. In constructing it graphically, the value of ϕ, for a small yet finite angle i, is found by replacing T/T' by ϵ^{fi} and expanding the exponential: we thus get approximately

$$1 + fi = \cos i + \sin i \,.\, \tan \phi = 1 - \tfrac{1}{2} i^2 + i \,.\, \tan \phi,$$
$$\therefore \tan \phi = f + \tfrac{1}{2} i.$$

With small values of the co-efficient $2f$ may be a sufficiently small angular interval, but in general it will be advisable to take the angular interval equal to the angle of friction, then the value of ϕ is $1\frac{1}{2}$ times that angle. The construction being one in which errors accumulate, the formula is preferable when great accuracy is desired.

124. *Driving Belts.*—When a belt is stretched over a pulley by equal weights, the tension of the belt is not necessarily the same everywhere in the first instance; but if the pulley move steadily and the stiffness of the belt be disregarded, it must be so. Assuming this, let one of the weights be increased by a certain quantity Q and the pulley be held fast, then the tension of that side of the belt will be increased by an amount equal to Q at A, but diminishing to zero at L, a point determined by the intersection of the friction spiral $a_1\, l_1$ (Fig. 110) with the circle alb, the radius of which represents the weight W.

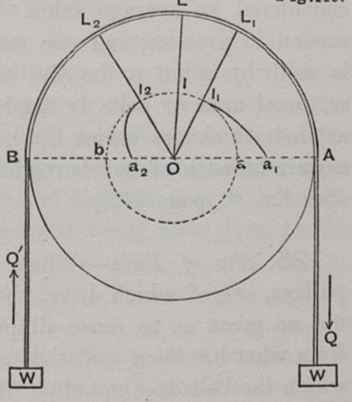


Fig.110.

Similarly, if the other weight be diminished by Q', the tension will be diminished by an amount equal to Q' at B, but diminishing to zero at L_2. The portion L_1L_2 will remain at the original tension W. If QQ' be increased sufficiently, L_1, L_2 will coincide in one point L, the position of which will depend on the proportion between Q and Q'. While these changes take place in tension, corresponding changes of length must occur in the parts of the belt exposed to them, AL increases and BL_2 diminishes in length. Hence both these parts slip over the pulley and work is lost by friction, while L_1L_2 remains fixed. If now, instead of altering the weights W, we imagine these weights held fast and the pulley forcibly rotated so as to increase A's tension by Q, and diminish B's tension by Q', L_1L_2 will rotate with the pulley, and the total increase of length of the one side must be equal to the total diminution on the other, from which consideration it is possible to calculate the ratio Q bears to Q'. In practical cases, however, the difference between Q and Q' is so small that it may be neglected without sensible error, and therefore, in all questions relating to the working of belts, it may be assumed that the mean tension of the two sides of the belt is independent of the power which is being transmitted. The difference of tensions, however, is directly proportional to the power, and may at once be calculated if the speed be known, while the ratio of tensions may be determined, so that the belt shall just not slip, by means of the formula above obtained. The value of the co-efficient of friction of leather on iron ranges from ·15 to ·46 according to the degree of lubrication: under ordinary circumstances ·25 may be considered an average value. This, however, is often greatly exceeded in practice, and one reason why large values are admissible is said by some to be the effect of atmospheric pressure. The sectional area of belts is fixed by considerations of strength, and as their thickness varies little, this is equivalent to saying that a certain breadth of belt is required for each horse-power transmitted. (See Ex. 10, page 272).

125. *Slip of Belts.*—When a belt is stretched over a pair of pulleys, one of which drives the other, notwithstanding a resistance not so great as to cause slipping of the belt as a whole, it appears from what has been said that a certain arc exists on each pulley on which the belt does not slip. The length of these arcs has already been found, but in the present cases the movement of the pulleys causes them to place themselves where the belt winds on to the pulleys, so that the driving pulley has the speed of the tight side of the belt and the driven pulley that of the slack side. The two sides have different speeds, because the same weight of belt must pass a given point in a unit of time, wherever that point be situated, and therefore the speed must be greater the greater the elongation, that is to say the greater the tension. Hence the driving pulley moves quicker than the driven pulley by an amount which can be calculated when the tensions and the elasticity of the leather are known, and this "slip" measures the loss of work due to the creeping of the belt over the pulleys described above. In ordinary belting

this loss is small, not exceeding 2 per cent. The length of belts, however, must not be too great, or its extensibility will be inconvenient, especially if the motion of the machine be not sufficiently uniform.* Within moderate limits extensibility is favourable to smooth working.

126. *Stiffness of Ropes.*—When a rope is bent it is found that a certain moment is required to do it depending on the dimensions of the rope and, besides, on its tension. The reason of this is best understood by referring to the corresponding case in a chain with flat links united by pin joints. If d be the diameter of the pin, T the tension of the chain, there will be a certain moment of friction resisting bending which, if the pin be an easy fit, will be simply $\frac{1}{2} fTd$, but if it be tight will be

$$M = \tfrac{1}{2}fTd + \tfrac{1}{2}fT_0d,$$

where T_0 is a constant depending on the tightness. If the chain pass over a rotating pulley without slipping, this frictional moment has to be overcome both when bending on and when bending off the pulley. The effect shows itself by a shift outwards on the advancing and inwards on the retiring side of the chain, so as to increase the leverage of the resistance and diminish that of the effort. In the present case the two shifts are equal, being each given by the formula

$$x = \tfrac{1}{2}fd\left\{1 + \frac{T_0}{T}\right\}.$$

The case of a rope differs from this only in being more complex: in the act of bending, the fibres move over each other, and the relative motion is resisted by friction due to pressures which are partly constant and partly proportional to the tension. The shift of the centre line of the rope is visible on the side of the resistance, but hardly perceptible on the side of the hauling force, showing that most of the loss of work is due to the bending on the pulley. The magnitude of the shift varies so much according to the mode of manufacture and the condition of the rope that it is useless to attempt more than a very rough estimate. According to a formula given by Eytelwein, if d be the diameter of the rope,

$$x = c \,.\, d^2,$$

* See a footnote by M. Kretz, *Cours de Mécanique Appliquée aux Machines par Poncelet*, page 264.

where c is a constant, which for dimensions in inches is taken as ·47 for hemp ropes; but this value is too large, except for light loads, and small diameters of pulley. The loss of work per revolution is $T \cdot 2\pi x$, and if D be the effective diameter of the pulley,

$$\text{Efficiency} = \frac{D}{D + 2x}.$$

There is a loss of work by the stiffness of belts of a similar kind, but of uncertain amount. By most authorities it is considered so small as to be negligible.

The shift of the line of action of the tension of a rope due to its stiffness has the effect of diminishing its strength.

127. *Friction of Toothed Wheels and Cams.*—The friction of toothed wheels is partly rolling and partly sliding, but the first is relatively small and may be neglected. To determine the sliding friction, let $PT = z$ (see Fig. 71, page 161), then (page 166) the velocity of rubbing is given by the formula

$$v = (A + A')z,$$

which may be written, if V be the speed of periphery of the pitch circles, R, R' the radii,

$$v = z\left\{\frac{1}{R} + \frac{1}{R'}\right\}V.$$

If, therefore, the wheels be supposed to turn through a small space δx measured on the pitch circles, the pair of teeth will slide on one another through the small space δy, given by the formula

$$\delta y = \left(\frac{1}{R} + \frac{1}{R'}\right)z\delta x.$$

This enables us to find the work done in overcoming friction, for if P be the pressure between the pairs of teeth,

$$\text{Work done} = f\int Pdy = f \cdot \left(\frac{1}{R} + \frac{1}{R'}\right)\int Pzdx.$$

The pressure between the teeth will vary as the wheels turn according to some unknown law, depending on the way the teeth wear, but the variation is probably not great. Assuming it constant, and further, supposing that the chord PT (Fig. 71) is equal to the arc PT, and therefore to x the arc turned through by the wheels after the teeth pass the line of centres,

$$\text{Work done} = f \cdot P \cdot \left(\frac{1}{R} + \frac{1}{R'}\right)\frac{x^2}{2}.$$

The same formula applies before the line of centres, and if we assume the arcs of approach and recess each equal to the pitch p, we shall have for the whole work lost by the friction of a pair of teeth,

$$\text{Whole Work lost} = fP\Big(\frac{1}{R} + \frac{1}{R'}\Big)p^2.$$

The energy transmitted during the action of a pair of teeth is $2Pp$, therefore the counter efficiency is

$$1 + e = 1 + f\Big(\frac{1}{R} + \frac{1}{R'}\Big)\frac{p}{2} = 1 + f\pi\Big(\frac{1}{n} + \frac{1}{n'}\Big),$$

where n, n' are the numbers of teeth in the wheels. A smaller arc of action is sometimes employed in practice, and the friction will then be less. This is also the case in bevel gear. The formula shows that the friction is diminished by increasing the number of teeth.

A more exact solution of this question * can be obtained on the assumption that P varies as it would do if there were only one pair of teeth; but as this is uncertain it is not practically useful.

In all cam and wheel mechanisms the efficiency for a small movement in any position can be determined exactly by a graphical or other process. For the velocity ratio can be found, as shown in Part II., and the force-ratio is determinate by the principles of statics, therefore the quotient which gives the efficiency can also be found. In the case of toothed wheels this method shows at once † that the friction of the teeth before the line of centres is greater than the friction after the line of centres. The difference appears insufficient to account for the injurious effects generally ascribed to friction before the line of centres, which however may be due to other causes. In cam mechanisms the efficiency in one position is little guide to the efficiency in a complete period, which can only be found by a process too intricate to be useful, or by making some supposition as the mean value of the pressure between the rubbing surfaces.

The counter efficiency of a train of m equal pairs of wheels is

$$1 + e = 1 + mf\pi\Big(\frac{1}{n} + \frac{1}{n'}\Big).$$

Assume now that a given velocity-ratio is to be provided by the train, and that the number of teeth in one wheel is given, then it is

* See Moseley's *Mechanical Principles of Engineering.*

† Ibid., page 286.

possible to find the value of m that the friction may be least. The solution of this problem is the same as that of finding the least possible number of teeth, and it was shown by Young that, for this, we ought to take m, so that the velocity-ratio for each pair of wheels is, as nearly as possible, 3·59. For example, if the train is to give a total velocity-ratio of 46, there should be three pair of wheels. The gain over a single pair in this case is one third, but will be much greater for higher velocity-ratios. The solution (first given by Mr. Gilbert) takes no account of axle friction, a circumstance which would greatly modify the result.

SECTION III.—FRICTIONAL RESISTANCES IN GENERAL.

128. *Efficiency of Mechanism in general.*—It appears from what has been said that an exact calculation of the frictional resistances is impracticable, partly because the process is too complex to be useful, but chiefly because the co-efficients to be employed are variable according to circumstances, and within limits, which are not precisely known. Hence when possible the efficiency of a machine is estimated, not by considering each particular element, but by direct experiment on the machine as a whole, and we conclude this chapter with some general principles which bear on this question.

The effort employed to drive a machine may be greater or less, according to the resistance which is being overcome, and therefore the stress between each element will also vary according to this effort. As, however, these stresses depend also on other forces, such as weight and elasticity, which have no connection with the effort, but are always the same, they will not increase so fast, and the frictional resistances will accordingly be proportionally less the greater the effort. Some resistances are absolutely constant, for example, the friction of bearings, the load on which is simply the weight of a fly-wheel or other moving part: or the friction of a piston rod in its stuffing box. Others are sensibly proportional to the driving effort or the useful resistance, in which case, when the ordinary laws of friction apply, the loss of work increases in direct proportion to these quantities. The greater number depend on both variable and constant forces, but these may be in great measure separated into two parts, one of which is approximately constant and the other approximately proportional either to the driving effort or to the useful resistance. Hence, if U be the useful work done and E the energy exerted in a period of the machine,

$$E = U + kU + k' . E + B,$$

where k, k' are numerical co-efficients and B the work done in overcoming the constant resistances. In hydraulic and other machines, where fluid resistances occur, terms depending on the speed of the machine must be added, indeed this is so in all machines when driven at a high speed; because forces due to inertia increase the friction, and besides shocks and the resistance of the atmosphere have to be considered. Such cases, however, are not considered here.

If we transfer the term $k'E$ to the other side of the equation and divide by $1 - k'$, we get

$$E = (1 + e)U + E_0,$$

where e, E_0 are two new constants derived from the former ones, of which E_0 is the work done in driving the machine when unloaded, and $1 + e$ the counter-efficiency when the load is very great.

The same formula may also be written in a way which is sometimes more convenient. Let P be the mean value of the driving effort and R that of the useful resistance during a complete period, r the mean value of the velocity-ratio of the working and driving pairs, then

$$P = (1 + e)Rr + P_0,$$

where P_0 is now the effort required to drive the machine when unloaded. In hoisting machines R is the weight lifted and P the hauling force usually called the power, R/P is the mechanical advantage or purchase.

In the steam engine, if p_m be the actual mean effective pressure, p'_m the part of that pressure employed in overcoming the useful resistance, p_0 the pressure necessary to drive the engine when unloaded,

$$p_m = (1 + e)\, p'_m + p_0.$$

The value of e may be taken as ·15 or in large engines somewhat less. The constant p_0, often called the "friction pressure," is from 1 to $1\frac{1}{2}$ lbs. or in marine engines 2 lbs. or more per square inch. At high speeds and pressures the ordinary laws of friction fail and e is diminished, the constant friction is then relatively of more importance.

If the direction of motion of the machine be reversed so that the original resistance becomes the driving effort and the effort the resistance, the same general formula is approximately true, but the constants k, k' are interchanged. Unless under special conditions the efficiency is not the same in the two cases, and in fact is generally very different. Let us suppose that in a machine working against a known reversible resistance, the driving effort is gradually diminished until the machine reverses, and let E' be the work done when reversing, we have the equations

$$E = U + kU + k'E + B,$$
$$U = E' + k'E' + kU + B,$$

from which by subtraction and dividing by U we find

$$\frac{E'}{U} = \frac{2}{1+k'} - \frac{1-k'}{1+k'} \cdot \frac{E}{U},$$

a formula which gives the efficiency when reversing. If the original efficiency be less than $\frac{1}{2}(1-k')$, the machine will not reverse even when the driving force is entirely removed. In most forms of hoisting machines k' is small enough to be neglected, and we have the important principle that a machine will not reverse if its efficiency is less than ·5. It will not reverse under any circumstances if $k > 1$. As previously explained in the case of a screw, non-reversibility is a property so valuable in practical applications as to be worth obtaining at the sacrifice of efficiency. The differential pulley block is a common example.

129. *Friction Brakes.*—Frictional resistances are not only a source of loss, they are also usefully employed in machines for various purposes. In screws and driving belts we have already found them employed for the purpose of locking a pair or closing a kinematic chain, and many instances of the same kind might be referred to. Another application of equal importance is for the purpose of absorbing surplus energy, which might otherwise produce dangerous effects, or which requires to be disposed of in order to stop a machine. An apparatus for this purpose is called a "brake."

The most powerful brakes are those in which fluid resistances are used, but when the amount of energy is small as compared with the surfaces available, the friction of solids may be employed. The energy thus absorbed is converted into heat, and is dissipated by radiation and conduction. Sufficient surface must be provided to prevent the temperature rising too high.

A brake is generally applied to a rotating wheel or drum, and consists either of a solid block of wood or metal pressed against the wheel by some suitable mechanism; or else of a strap of metal, often lined with small blocks of wood, embracing the drum and tightened by a lever or otherwise. Three common forms are shown in Plate VII.; two of these (Figs. 1 and 2) are used as dynamometers, and will be referred to as such in the next chapter.

EXAMPLES.

1. A weight is moved up a plane inclined at 1 vertical to n horizontal by an effort parallel to the plane: show that the counter-efficiency is $1+nf$, where f is the co-efficient of friction. Find the value of n for a mechanical advantage of 10 : 1 and a co-efficient ·05. *Ans.* $n=20$.

2. Show that the pressure on the guide bars of a direct-acting engine is approximately proportional to the ordinates of an ellipse, and deduce the work lost per stroke.

Referring to Fig. 91 let X be that pressure, then

$$X=S.\sin\phi=P.\tan\phi=\frac{P}{n}\sin\theta \text{ approximately.}$$

If the radius of the crank circle represent P, and an ellipse be drawn with the same major axis, and minor axis $=P/n$, X will be the ordinate of the ellipse at a point representing position of piston.

$$\text{Loss of work per stroke}=f\times\text{Area of semi-ellipse}$$

$$=\tfrac{1}{2}f.\pi.a\frac{P}{n}=\frac{\pi fsP}{4n},$$

where s is the stroke and f the co-efficient of friction.

3. A bearing 16″ diameter is acted on by a horizontal force of 50 tons and a vertical force of 10 tons. Find the work lost by friction per revolution, using a co-efficient of one-eighteenth. Find also the horse power lost by friction at 70 revolutions per minute. *Ans.* Loss of work = 11·87 foot-tons. H.P. = 56·4.

4. The thrust of a screw propeller is 20 tons, the pitch 20 feet. The thrust block is 18″ diameter at the centre of the rings. Find the efficiency with a co-efficient of friction of ·06. *Ans.* Efficiency = ·986.

5. Find the efficiency of a common screw and nut with pitch angle 45° and co-efficient ·16. *Ans.* Efficiency = ·72.

6. A screw bolt is ½″ diameter outside and ·393″ at the base of the thread. The effective diameter of the nut is ¾″, and the co-efficient of friction ·16; supposing it screwed up by a spanner two feet long, find the mechanical advantage.

Tension of bolt = 234 × pull on spanner.

7. Find the efficiency of a pair of wheels, the numbers of teeth being 10 and 75, and the co-efficient of friction ·15. *Ans.* ·954.

8. The stroke of a direct-acting engine is 4 feet, piston load 50 tons, load on crank-shaft bearings 10 tons, connecting rod 4 cranks: trace the curve of crank effort when friction is taken into account, assuming all bearings 16″ diameter and co-efficient one eighteenth. Find the "dead angle."

9. In the last question, if the engine drive the screw propeller of question 4, find the efficiency of the mechanism, including thrust block, by the approximate method. The connecting rod may be supposed indefinitely long except for the purpose of estimating the efficiency of the guide bars.

$$\text{Efficiency} = \cdot989 \times (\cdot97)^2 \times \cdot986 = \cdot92.$$

10. A rope is wound thrice round a post, and one end is held tight by a force not exceeding 10 lbs. What pull at the other end would be necessary to make the rope slip, the co-efficient of friction being supposed ·366? *Ans.* 1,000 lbs.

11. Find the necessary width of belt three sixteenths inch thick to transmit 1 h.p., the belt embracing 40 per cent. of the circumference of the smaller pulley and running at 300 feet per 1′. Co-efficient = ·25. *Ans.* Breadth = $4\frac{1}{2}''$.

12. In question 10 construct the friction spiral showing the tension of the rope at every point.

13. The axles of a tramway car are $2\frac{1}{2}''$ diameter, and the wheels 2′ 6″: find, the resistance being given, that the co-efficient of axle friction is ·08 and that for rolling ·09. *Ans.* Resistance = $28\frac{1}{2}$ lbs. per ton.

14. Find the efficiency of a pulley 6″ diameter, over which a rope $\frac{1}{2}''$ diameter passes, the axis of the pulley being $\frac{1}{2}''$ diameter, and the load on it twice the tension of the rope. Co-efficient of axle friction ·08. Co-efficient for stiffness of rope ·47. *Ans.* Efficiency = 94 per cent.

15. From the result of the preceding question deduce the efficiency of a pair of three-sheaved blocks. *Ans.* Efficiency = 71 per cent.

16. A wheel weighing 20 lbs., radius of gyration 1′, is revolving at 1 revolution per second on axles 1″ diameter. It is observed to make 40 revolutions before stopping: find the co-efficient of axle friction. *Ans.* Co-efficient = ·059.

17. In a pair of three-sheaved blocks it is found by experiment that a weight of 40 lbs. can be raised by a force of 10 lbs., and a weight of 200 lbs. by a force of 40 lbs. Find the general relation between P and W, and the efficiency when raising 100 lbs.

$$P = \tfrac{3}{16}W + \tfrac{5}{2}. \quad \text{Efficiency} = \cdot784 \text{ when raising 100 lbs.} \quad e = \tfrac{1}{8}.$$

18. Find the distance to which power can be transmitted by shafting of uniform diameter, with a loss by friction due to its weight of n per cent, assuming that the angle of torsion is immaterial, and co-efficient for strength 9,000 lbs. per square inch.

If f be co-efficient of friction, then the length of shafting is $13\frac{1}{2} \cdot \dfrac{n}{f}$.

References.

On the graphical determination of the efficiency of mechanism the reader is referred to two papers by Prof. F. Jenkin in the *Transactions* of the Royal Society of Edinburgh. On the stiffness of ropes, see Weisbach, *Ingenieur-Mechanik*, vol. I., 3rd German edition, p. 300.

CHAPTER XI.

INCOMPLETE CONSTRAINT. STRAINING ACTIONS ON MACHINES.

130. *Preliminary Remarks.*—In the motion of a machine the relative movements of the several parts are completely defined by the nature of the machine, and the principal action consists in a transmission and conversion of energy. Hence it is that the principle of work is of such importance in all mechanical operations that it is desirable to consider it as an independent fundamental law verified by daily experience. Even in applied mechanics, however, we have sometimes to do with sets of bodies, the relative movements of which are not completely defined by the constraint to which they are subject, but partly depend on given mutual actions between them. When this is the case, the principle of work, though still of great importance, is not by itself sufficient to determine the motions.

Again, if we wish to study the forces which arise when the direction of a body's motion is changed, the principle of work does not help us, for no work is done by such forces. For example, the position of the arms of a governor, revolving at a given speed, cannot be found, except, perhaps, indirectly, by the methods hitherto employed. We then resort to the ordinary laws connecting matter and motion, which form the base of the science of mechanics, and of which the principle of work itself is often considered as a consequence.

The present chapter will be devoted in the first place to a brief summary of elementary dynamical principles, and afterwards to various questions relating to machines and the forces to which they are subject.

SECTION I.—ELEMENTARY PRINCIPLES OF DYNAMICS.*

131. *Impulse and Momentum.*—The effect of an unbalanced force P, acting during a certain time t, on a piece of matter, is to generate a velocity v, which is proportional to P and t directly and the quantity of matter inversely. When the force P is equal to the weight W, as in the case of a body falling freely, the velocity generated in $1''$ is known to be g, where g is a number which varies slightly for different positions on the earth's surface (Art. 99), but is precisely the same for all sorts of matter. We may express this by the equation

$$Pt = \frac{W}{g}v.$$

In this formula we may take W to mean the weight of the piece of matter as compared with that of a unit piece at a given point on the earth's surface. As formerly stated (Art. 88) this is called "gravitation measure," and has the defect of giving a varying unit of force, so that considerations of convenience alone induce us to employ it. If, instead of measuring W in units of weight, we compare it with the force P, which produces unit velocity in unit time, we have

$$W = Pg,$$

that is, the weight of the unit piece of matter is g units of force. Such units depend on nothing but the size of the unit piece of matter, and are hence called "absolute" units. For scientific purposes, and especially in electrical measurements, they are much employed.

Quantity of matter is called MASS, and, when absolute measure is used, is simply measured by comparing it with that of a standard piece, for example, in Britain, with a certain piece of platinum called a pound. The unit of force is then that which is necessary to produce a velocity of 1 foot per $1''$ in this piece, a quantity for which the name "Poundal" was suggested by the late Professor Clerk Maxwell; the weight of a piece is then g poundals, so that what is called a pound-weight in the common gravitation measure is about 32·2 of these units. When absolute measure is used, however, the Continental system of units depending on the mètre and gramme is likely to be universally employed. No more need be

* The brief statement here made of principles assumed in subsequent articles of this treatise is not intended as a substitute for a treatise on elementary dynamics.

said on this point, as gravitation measure is exclusively used in this treatise.

When gravitation measure is used the unit of mass employed is that piece of matter in which a pound weight generates a velocity of 1 foot per second, that is the above mentioned piece of platinum divided by the numerical value of g, so that the unit of mass as well as the unit of force varies according to the place. If m be the mass, W the weight,

$$W = mg,$$

where g is taken equal to 32·2.

This explanation being premised we have

$$Pt = mv.$$

The products Pt, mv are called IMPULSE and MOMENTUM respectively, and the equation may be written

Impulse exerted = Momentum generated.

A unit of impulse is unit force exerted for unit time, usually 1 lb. for 1″, a quantity for which the expression "second-pound" may conveniently be used. If P be variable, then impulse is calculated in the same way as the energy exerted by a variable force (Art. 90), the abscissæ of the diagram now representing time instead of space.

The body we are considering may have a velocity at the commencement of the time t, and the force may be partially balanced; if so, v must be understood to be the *change* of velocity, and P the unbalanced part of the force.

132. *Centrifugal Force.*—So far the equation of momentum is analogous to the equation of work, impulse representing the time-effect of force as energy represents its space-effect. There are, however, two important differences.

Change of kinetic energy arises from a change in the magnitude of the velocity irrespectively of direction, whereas change of momentum must be estimated in the direction of the force producing it, and includes change of direction. Hence the equation is applicable when the direction of the force is perpendicular to the direction of motion, so that the only effect produced is change of direction. The rate of change of velocity, taken in the most general sense, is called Acceleration, and the equation of momentum may also be written

$$P = mf,$$

where f is the acceleration estimated in the direction of the force. By taking the force perpendicular to the direction of motion we get the equation which connects the curvature of the path of a moving body with the force R, which compels it to deviate from the straight line, namely,

$$R = \frac{mv^2}{r},$$

where v is the velocity and r the radius of the circle in which it is moving at the instant considered. Like other forces this arises from the mutual action between two bodies: one of these is the moving body; the other, the fixed body which furnishes the necessary constraint. If we are thinking of the fixed body instead of the moving body, we call the force R the Centrifugal Force, being the equal and opposite force with which the moving body acts on the body which constrains it. The two forces together constitute what we have already called a Stress (Art. 1). To determine a stress of this kind it is necessary to refer the direction of motion to some body which we know may be regarded as fixed, and we are not at liberty to choose any body we please for this purpose, as in kinematical questions. What constitutes a fixed body is a question of abstract dynamics, into which we need not enter. For practical purposes the Earth is taken as fixed.

If a body rotate about a fixed axis the centrifugal forces, arising from the motion of each particle, will not balance one another unless the axis be one of three lines, passing through the centre of gravity, which are called the "principal axes of inertia" at that point. In most cases occurring in practical applications the position of these lines can be at once foreseen as being axes of symmetry. This is the case, for example, in homogeneous ellipsoids and parallelopipeds. In the common case of a homogeneous solid of revolution, the axis of revolution, and any line at right angles to it through the centre of gravity, are principal axes. If the axis of rotation be parallel to one of these axes, but do not pass through the centre of gravity, the centrifugal forces reduce to a single force, which is the same as if the whole mass were concentrated at the centre of gravity. In all other cases there is a couple depending on the direction of the axis of rotation, as well as the force just mentioned. (Ex. 16, p. 297.)

133. *Principle of Momentum.*—Again, every force arises from the mutual action between two bodies, consisting in an action on one accompanied by an equal and opposite reaction on the other. Hence, if we understand by the total momentum of two bodies in any direction, the sum or the difference of the momenta of each, according as the bodies move in the same or in the opposite direction, it appears that the total momentum will not be affected by the mutual action between the two. And more generally, if there be any number of bodies we shall have

Total impulse exerted = Change of total momentum,

where, in reckoning the impulse, we are to take into account external forces alone, and not the internal forces arising from the mutual action of the parts of the set of bodies we are considering. This equation expresses one form of what we may call the Principle of Momentum; other forms will be explained hereafter in connection with questions relating to fluid motion (Part V.).

The total momentum of a number of bodies may be reckoned by direct summation, with due regard to sign, but it may also be expressed in terms of the velocity of the centre of gravity; for, let m be the mass of any particle of the system, the ordinate of which, reckoned from a given origin parallel to a given line, is x; also, let Σmx denote the sum of all the separate products mx, for all the particles of the system, and let M be the total mass, then we know that the ordinate of the centre of gravity * is given by the formula

$$\bar{x} = \frac{\Sigma mx}{M}.$$

Let the velocity of a particle parallel to the given line be u, then if x_1, x_2 be the ordinates at the beginning and end of 1″ we shall have

$$u = x_2 - x_1.$$

Hence, if $\bar{u}$ be the velocity of the centre of gravity parallel to the same line,

$$\bar{u} = \bar{x}_2 - \bar{x}_1 = \frac{\Sigma m (x_2 - x_1)}{M} = \frac{\Sigma mu}{M},$$

which equation may be written

$$M\bar{u} = \Sigma mu,$$

showing that the total momentum of the system is the same as if its total mass were concentrated in its centre of gravity. We conclude from this that the motion of the centre of gravity can only be

* Called more correctly by modern writers on mechanics the "centre of mass."

influenced by external forces and not by any action between the parts of the system.

134. *Internal and External Kinetic Energy.*—If we multiply the equation just obtained by $2\bar{u}$ and remember that $\bar{u}$ being constant may be placed within the sign of summation, we obtain

$$2M\bar{u}^2 = \Sigma m \,.\, 2u\bar{u},$$

which, adding Σmu^2 to each side and re-arranging the terms, may be written

$$M\bar{u}^2 + \Sigma m(\bar{u} - u)^2 = \Sigma mu^2.$$

This is true in whatever direction the velocities are estimated, and we can therefore write down two similar equations for the velocities in two directions at right angles to the first. Now the resultant of three velocities at right angles is the square root of the sum of the squares of the components, also $\bar{u} - u$ is the velocity parallel to x relatively to the centre of gravity; hence if U be the resultant velocity of the centre of gravity, v, $\bar{v}$ the velocities of any particle relatively to the body regarded as fixed and relatively to the centre of gravity respectively, we have, adding the three equations together, and dividing by 2,

$$\tfrac{1}{2}MU^2 + \Sigma\tfrac{1}{2}m\bar{v}^2 = \Sigma\tfrac{1}{2}mv^2.$$

The first term on the left-hand side of this equation is what the energy would be, if the whole mass were concentrated at its centre of gravity, a quantity which may be described as the External Energy, or otherwise as the Energy of Translation of the system. The second term is the energy relatively to the centre of gravity considered as fixed, which may be called the Internal Energy. The right-hand side is the total energy of motion, and we see therefore that this is the sum of the internal and external energies. In the case of a single rigid body the motion relatively to the centre of gravity is always a rotation about some axis, and therefore

Energy of Motion = Energy of Translation + Energy of Rotation,

a principle already employed in a preceding chapter (p. 214).

In the case of a set of rigid bodies the internal energy is the sum of the energies of rotation of each together with the internal energy of a set of particles of the same mass occupying the centres of gravity of the bodies and moving in the same way.

135. *Examples of Incomplete Constraint.*—In the cases which

occur in applications to machines and structures we usually have to consider two bodies moving in straight lines without rotation.

CASE I. *Recoil of a Gun.*—When a cannon is fired the shot is projected and the cannon recoils with velocities dependent on the relative weights of the shot, the cannon, and the charge of powder.

Here, the motion is due to the pressure of the gases generated by the combustion of the powder one way on the shot, the other way on the cannon. If the inertia of these gases could be neglected these pressures would be exactly equal at each instant and would cease as soon as the shot left the bore. The impulse exerted on shot and cannon would then be equal. In fact, the inertia of the powder gases causes the pressure to be greater and to last longer on the cannon than on the shot, so that the impulses on the two are not nearly equal. For the present we shall neglect this, and shall further suppose that the material of both shot and gun is sensibly rigid.

In general, recoil is checked by an apparatus called a "compressor," which supplies a gradually increasing resistance to the backward movement of the gun, while friction and the resistance to rotation of the shot resist the forward movement of the shot. In the first instance suppose there are no such resistances, let V be the velocity of recoil and M the mass of the gun, v the velocity and m the mass of the shot; then, since the impulse exerted is the same for both,

$$MV = mv.$$

Further, if the weight of the charge and the amount of work 1 lb. of it is capable of doing be known, the explosion will develop a definite amount of energy (E) which will be all spent in giving motion to the shot and the cannon.

$$\text{Energy of Explosion} = \tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2.$$

Here E is the sum of two parts—

$$\text{Energy of Shot} = \frac{M}{M+m}E,$$

$$\text{Energy of Recoil} = \frac{m}{M+m}E.$$

The energy of recoil has to be absorbed by the compressor, usually an hydraulic brake, which will be considered hereafter (see Part V.).

CASE II. *Collision of Vessels.*—When two vessels come into collision an amount of damage is done depending on the size and velocities of the vessels.

Here we may suppose the vessels moving in given directions with given velocities; let the velocities parallel to a given line be u_1, u_2, and the masses m_1, m_2, then, as in Art. 133, the velocity of the centre of gravity parallel to the same line is

$$\bar{u} = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2},$$

and therefore the velocities of the vessels relatively to their common centre of gravity must be

$$u_1 - \bar{u} = \frac{m_2(u_1 - u_2)}{m_1 + m_2}\,;\ u_2 - \bar{u} = \frac{m_1(u_2 - u_1)}{m_1 + m_2}.$$

Two similar equations may be written down for the velocities in a direction at right angles to the first. Square and add corresponding equations, multiply by $\frac{1}{2}m_1$, $\frac{1}{2}m_2$, and add the pair of products, then (Art. 134)

$$\text{Internal Energy} = \tfrac{1}{2} \cdot \frac{m_1 m_2}{m_1 + m_2} V^2,$$

where V is the velocity of either vessel relatively to the other, a quantity found immediately from the given velocities of the vessels by means of a triangle of velocities.

The total kinetic energy of the vessels is found by adding the energy of translation. As, however, this quantity cannot be altered by the collision, it is clear that the amount of work done must depend on the internal energy alone: we may properly call it therefore the "energy of collision." If the displacements in tons of the vessels be W_1, W_2, we shall have, in foot-tons,

$$\text{Energy of Collision} = \frac{W_1 W_2}{W_1 + W_2} \cdot \frac{V^2}{2g}.$$

It is not, however, to be supposed that the whole of this is necessarily expended in damage to the vessels; if the circumstances of the collision be such that the vessels, even though completely devoid of elasticity, would have a motion of rotation or a velocity of separation of their centres of gravity, then the corresponding internal energy must be deducted. Also the influence of the water surrounding the vessels has been left out of account; this somewhat augments the effect by increasing the virtual mass of the vessels.

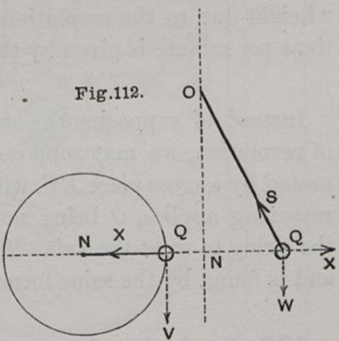
The same formula may be used for other cases of impact, but the effects of impact depend so much on the strength and stiffness of the colliding bodies that the subject must be postponed (Ch. XVI.).

Section II.—Centrifugal Regulators.

136. *Preliminary Remarks.*—Centrifugal forces may be employed in machines to do the work which is the object of the machine, as in certain drying machines where the substance to be dried is caused to rotate with great rapidity so that the fluid is expelled at the outer circumference: or, partially, in centrifugal pumps. More frequently they serve to move a kinematic chain connected with a shifting piece which regulates the speed of the machine. Such mechanisms are called Centrifugal Regulators or, more briefly, Governors.

137. *Simple Revolving Pendulum.*—In Fig. 112 Q is a heavy particle attached by a string to a fixed point O and revolving in a horizontal circle the centre of which is N vertically below O. This will be possible if the centrifugal force due to the motion of the particle just balances the horizontal component of the tension of the string. Let S be that tension, W the weight of the particle,

Fig.112.



and let the string make an angle θ with the vertical, then the horizontal and vertical components of S are

$$X = S \,.\, \sin\theta\,;\;\; W = S \,.\, \cos\theta.$$

Let A be the angular velocity of the revolving particle, then it is shown in works on elementary dynamics that the centrifugal force is

$$X = \frac{W}{g} \,.\, A^2 \,.\, QN.$$

Equating these values of X and eliminating S,

$$W \,.\, \tan\theta = \frac{W}{g} \,.\, A^2 \,.\, QN.$$

Since $QN = ON \,.\, \tan\theta$, this reduces to the simple formula

$$ON = \frac{g}{A^2},$$

which shows that the vertical distance of Q below the point of suspension depends on the speed, not on the length of the string or the magnitude of the weight.

This distance is called the height of the revolving pendulum, and will be denoted by h. If t be the period, that is the time of a complete revolution, we find, since $At = 2\pi$,

$$t = 2\pi \sqrt{\frac{h}{g}},$$

showing that the period is the same as that of a double oscillation of a simple pendulum of length h (see Art. 103). The height of a simple revolving pendulum may often be conveniently adopted as a measure of a speed of revolution, and will then be spoken of as the "height due to the revolutions." Its value in inches for n revolutions per minute is given by the formula

$$n^2 h = 35{,}232.$$

Instead of supposing the string attached to a point O in the axis of revolution, we may suppose it attached to a point K, rigidly connected by a cross-piece KE, with a revolving spindle ON. The same reasoning applies, O being now an ideal point, found by prolonging the string to meet the axis. The height of the pendulum is still ON, and is found by the same formula.

138. *Speed of a Governor to overcome given Frictional Resistances. Loaded Governors.*—In the simplest centrifugal governors two heavy balls are attached to arms, which are jointed either directly to a revolving spindle, or to the ends of a cross-piece attached to a spindle. Motion is communicated by links from the arms to a piece sliding on the spindle, the movement of which is communicated by a train of linkwork, either to a throttle valve directly controlling the supply of steam, or to an expansion valve which regulates the cut-off. In either case an upward movement of the arms has the effect of diminishing the mean effective pressure, and a downward movement of increasing it. Two forms of this mechanism are shown in the figures of Plate VI.: in one of these (Fig. 1) the weight of the sliding piece is increased by a large additional weight, the governor is then said to be loaded; while in the other (Fig. 2) the arms cross each other, the spindle being slotted, or the arms bent to permit this. The object of these arrangements we shall see presently.

Plate. VI.

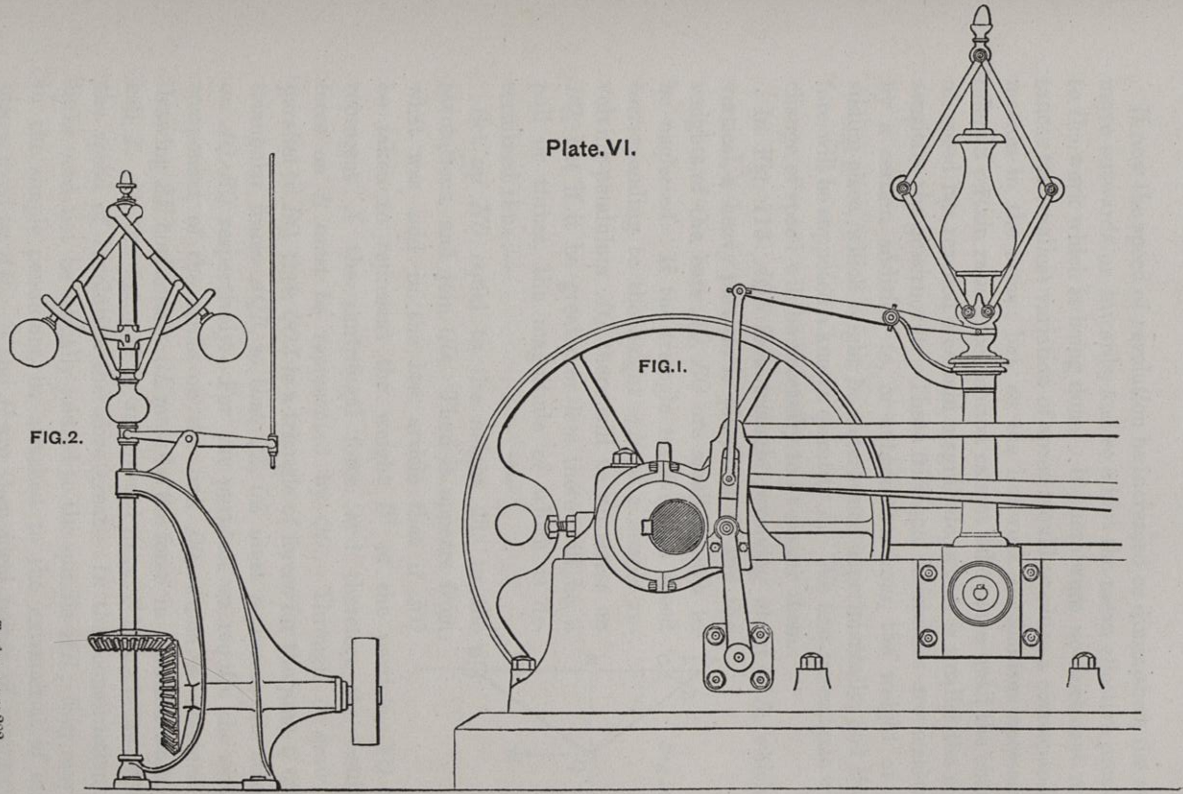


FIG. 2.

FIG. 1.

To face page 282.

If now the speed of revolution be increased or diminished, the arms move outwards or inwards, and so adapt the mean effective pressure to the work which is being done. If there were no frictional resistances the smallest variation of speed would produce a corresponding motion in the arms; but, as the linkwork mechanism necessarily offers a certain resistance, motion cannot take place until the change of speed has reached a certain magnitude, which is smaller the more sensitive the governor is. These frictional resistances are measured by a certain addition to, or subtraction from, the weight of the sliding-piece, which might be determined experimentally, and therefore will be supposed a known quantity F. We first investigate what change of speed will be necessary to overcome them.

In Fig. 113 AQB is a triangle revolving about AB, which is vertical, a heavy particle is placed at Q, and the weights of the bars AQ, BQ are small enough to be neglected. If the triangle revolve at a speed corresponding to the height AN of a simple revolving pendulum AQ, there will be no stress on BQ, but if it be greater or less there will be a pull or thrust, the magnitude of which is determined thus:—

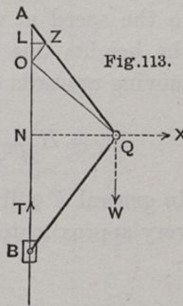


Fig. 113.

Set up NO equal to the height due to the revolutions, and join QO. Then it appears from what was said in the last article that if NO be taken to represent the weight W of the particle, NQ will represent X the centrifugal force, and therefore the resultant force on B must be represented by QO. Through O draw OZ parallel to BQ, then QOZ is a triangle of forces for the joint Q of the triangular frame AQB, so that QZ, OZ must represent the stresses on AQ, BQ respectively. For our purposes we require the vertical component of the stress on the link BQ, which is obtained by drawing ZL horizontal: OL must be the force in question which we call T. In the figure T is an upward force, O being below A, and the speed of revolution therefore great. In this construction the links need not be actually jointed to the spindle AB; they may, as in the simple pendulum, be attached to the extremities of cross-pieces fixed to AB. A and B are then ideal points of intersection of the links with the axis of rotation.

In general AQ and BQ are equal; we may then obtain a simple formula for T. Let $NO = h$, a quantity given by the same formula as before for a given speed, and let AN, the actual height of the governor, be denoted by H, then $OA = H - h$; but in the case supposed, $OA = 2OL$, therefore

$$2T = W.\frac{H-h}{h};\ h = H.\frac{W}{W+2T}:$$

formulæ which give the pull for any speed, and conversely the speed for which the pull will have a given value. In practical applications there are always two balls, so that if W be the weight of one, $2T$ will be the pull due to both.

We can now find within what limits of speed the mechanism can be in equilibrium. Let w be the weight of the sliding-piece B, inclusive of any load which may be added to it, h the height due to the speed at which there is no tendency to move the arms, h_1, h_2 the heights due to the speeds at which they are on the point of moving upwards or downwards respectively, then

$$h_1 = H\frac{W}{W+w+F};\ h = H\frac{W}{W+w};\ h_2 = H\frac{W}{W+w-F}.$$

In general F will be small compared with $W + w$, and then we have very approximately,

$$h_2 - h = h - h_1 = h\frac{F}{W+w}.$$

These results show that loading a governor gives it a power of overcoming frictional resistances, which would otherwise require a weight of ball equal to the sum of the load and the actual weight. Light balls may therefore be used as in the figure (Plate VI.) without sacrificing power, as the load may be made great without inconvenience. The speed of a loaded governor is greater than that of a simple governor of the same actual height, as appears from the formula for h. It may be altered at pleasure by altering the load. This arrangement is known as Porter's governor, from the name of the inventor.

139. *Variation of Height of a Pendulum Governor by a Change of Position of the Arms.*—Next suppose the speed to alter so much that the arms actually change their position, then if H remained the same, the tendency to move would also be the same, and the movement must therefore continue until the speed is brought back within the limits for which rest is possible. In the ordinary pendulum governor, however, H alters in a way which depends on the mode of attachment and arrangement of the arms, as will appear from the annexed diagram (Fig. 114) which shows three cases.

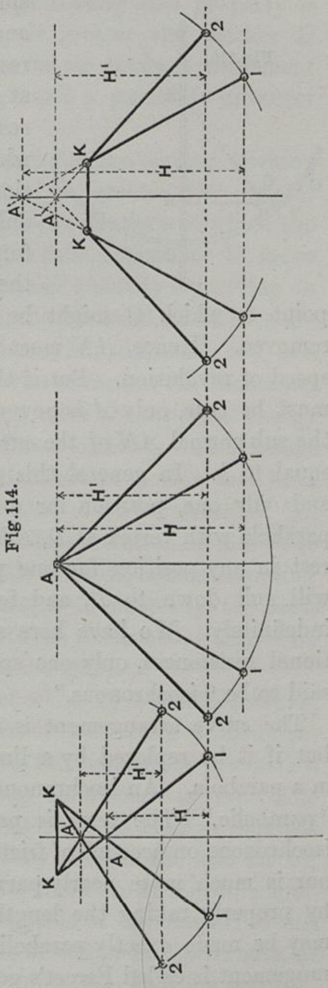


Fig. 114.

In the centre figure the arms are jointed to the spindle so that their centres of rotation are in the axis, in the two others they are jointed to a cross-piece KK, but differently arranged in the two cases. In all three, as explained in the preceding article, the height H is measured to A, the real or ideal intersection of the arms with the axis of rotation.

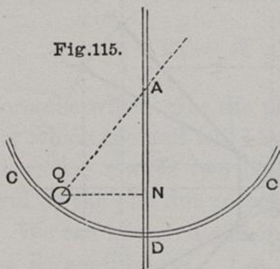
Suppose the arms to move from position 1 to position 2 in the figure; H diminishes to H', but the amount of diminution is different in the three cases: in the right-hand figure it is greatest, and in the left-hand least. Indeed in the latter case where the arms are crossed it is possible, by making KK long enough, to change the diminution into an increase. (Ex. 4, p. 295.)

If then, by an increase in the speed, the arms move into a new position, the speed required for equilibrium does not remain the same but increases, so that, when the adjustment has been effected between the energy and the work, the speed is increased, instead of being the same as before. Conversely, after adjustmen to suit a diminished speed, the speed actually attained is diminished. Thus the effect of the variation in H is to widen the limits within which the speed can vary.

140. *Parabolic Governors.*—A governor may be constructed in which H does not vary at all.

In Fig. 115 Q is a ball resting on a curve CC attached to a vertical spindle. The curve lies in a vertical plane, and D is the lowest point. When at rest the ball can only be in equilibrium at D, but, if the spindle revolve, it may rest at another point, the position of which depends on the speed of revolution. If the curve be a circle we have only the pendulum governor in a different form, for, drawing the normal QA and the perpendicular QN, A will be a point to which Q might be attached by a string and the curve removed. Hence, AN must be equal to h, the height due to the speed of revolution. But if the curve be not a circle the same thing must be true, only A is now not a fixed point; hence in every case the sub-normal AN of the curve at the point of equilibrium must be equal to h. In general this geometrical condition determines one, and only one, position for a given speed; but if the curve be a parabola with vertex at D, AN will be constant, and therefore Q will rest in any position for one particular speed, but for lower speeds will roll down to D, and for higher speeds will move upwards indefinitely. We have here a governor for which, neglecting frictional resistances, only one speed is possible. Such a governor is said to be "isochronous."

Fig.115.



The curve arrangement is inconvenient for constructive reasons, but if it be replaced by a linkwork mechanism the ball still moves in a parabola. An isochronous governor is therefore often said to be "parabolic." The term is preferable, for no governor is actually isochronous on account of frictional resistances. A pendulum governor is much more nearly parabolic when the arms are crossed, and by properly taking the length of the cross-piece (Ex. 4, p. 295) it may be made exactly parabolic for small displacements. This arrangement is called Farcot's governor from the name of the inventor.

141. *Stability of Governors.*—If the curve CC be not a parabola H will diminish or increase as the ball Q moves outwards. Take the first case and suppose Q in equilibrium at a certain point when the speed of revolution has a given value. Let Q now be moved up or down, then, if released, it will not remain at rest, but will return towards its original position and oscillate about it, or in other words the equilibrium of Q is stable. A governor possessing this property is described as "stable," and its stability is greater the quicker H diminishes. Similarly when H increases for an outward movement of the balls the governor is "unstable," and a parabolic governor may properly be described as "neutral."

A certain degree of stability is necessary for the proper working of a governor, and the amount required is greater the greater the frictional resistances. For assuming the revolutions at which the machine is intended to work to be n, the balls commence to move outward at the speed $n+x$, where x is a small quantity depending on the frictional resistance. After starting, the frictional resistances are not increased, but on the contrary will somewhat diminish; and, in a neutral governor, the balls therefore move outwards with increasing speed until by alteration of the regulating valve the supply of energy is diminished and the speed of the machine lessened. This change however requires time, and besides the balls are in motion and have to be stopped. The consequence is that they move outwards too far, and the supply of energy being too small the revolutions diminish to $n-x$, the speed necessary to move the balls inwards, notwithstanding the frictional resistance. Thus the motion is unsteady, the balls oscillating, and the speed fluctuating, between limits wider than $n \pm x$ without ever settling down to a permanent regime.

The oscillation of the balls may be checked by a suitable brake, but it is preferable to employ a governor possessing a moderate degree of stability; the tendency to move the balls then diminishes as soon as the balls move, and they stop before moving far. The greater the frictional resistances the greater is the change required to enable the balls to return at once if they have moved too far for equilibrium. An important characteristic therefore of a good centrifugal governor is that the stability be capable of adjustment to suit the frictional resistances. Certain forms of compound governors, as for example that known as the "cosine," fulfil this condition and can, probably, be made more perfect than the simple pendulum governor.

All such mechanisms are however imperfect in principle, for they cannot come into operation till a certain change of speed has actually existed for a perceptible length of time. Where the changes of resistance are sudden and violent the best governor will scarcely prevent violent fluctuations in speed. In screw vessels, where this difficulty is much felt, it has been proposed to employ an auxiliary engine rotating against a uniform resistance; any difference of speed of which and the screw shaft immediately shifts the regulating valve.

In the "cup governor," invented by Dr. Siemens,* a regulator and an hydraulic brake are combined. A cup containing water rotates within a cylindrical casing; at low speeds the water remains within the cup, but as soon as the speed exceeds a certain limit centrifugal action causes it to pour over the edge of the cup into the space between the cup and the casing. A set of vanes attached to the cup rotate between fixed vanes attached to the casing, and break up the descending water, which re-enters the cup by an orifice in the bottom. There is then a great resistance to the motion of the cup which absorbs surplus energy. Some other forms of governor will be considered hereafter.

SECTION III.—STRAINING ACTIONS ON THE PARTS OF A MACHINE.

142. *Transmission of Stress in Machines.*—We have seen (Art. 94, p. 202) that a mechanism becomes a machine if certain links are added capable of changing their form or size, and so producing forces which tend to move the mechanism combined with other forces which resist the motion. Each link so added exerts equal and opposite forces on the elements it connects, and for the pair of forces the general word "Stress" may be used, which has been already employed in Article 1 in the case of the bars of a framework structure.

When the machine is at rest the forces, being all in pairs, balance each other, and have no tendency to move the machine as a whole. For example in the direct-acting vertical engine represented in Fig. 1, Plate I., p. 119, the driving link is the steam, pressing with equal force, one way on the cylinder cover, and the other way on the piston; the working link is the resistance to turning of the crank shaft, which exerts equal and opposite forces, one way on the crank, the other way on the frame which carries the crank-shaft bearings. The steam pressure and the working resistance may each be de-

* *Phil. Trans.*, 1866.

scribed as a "Stress." The forces which make up the stress are transmitted from the piston through the connecting rod to the crank, and, in the opposite direction, from the cylinder cover through the frame to the crank shaft. The horizontal pressure of the cross-head on the guide bars is in like manner balanced by the equal horizontal thrust of the connecting rod on the crank pin, combined with the moment of the working resistance.

So in every machine, when at rest, or moving slowly and steadily, the stress is transmitted from the driving pair to the working pair, not only through the moveable parts of the machine, but in the opposite direction, through the framing; and this is a circumstance which must be always borne in mind in designing the framing. Thus, in our example, the steam cylinder and crank-shaft bearing must be rigidly connected by a frame strong enough to withstand the total steam pressure and, in addition, the bending due to the lateral pressure on the guide bars.

We have here one of the simplest examples of the transmission of stress; whether in a machine or in a structure it always takes place in a closed circuit.

If the driving pair and the working pair are the same, and acted on by the same stress, the whole state of stress is the same for all the mechanisms which are derived by inversion from the same kinematic chain. All such mechanisms are therefore statically as well as kinematically identical; it is only when we consider machines in motion, or the straining actions due to gravity, that it is necessary to consider which link (if any) is fixed to the earth. For example, the only difference between the direct-acting engine of Fig. 1, and the oscillating engine of Fig. 4, Plate I., is that the working pair is BA in the first and BC in the second. So again, in Plate III., the only difference between the water wheel of Fig. 2 and the horse gear of Fig. 3 is in the nature of the driving link, which in the first case is gravity acting on the falling water, and in the second a living agent.

A striking example of the balance of forces in a machine occurs in the hydraulic rivetting machines. Here the working pair is a small hydraulic cylinder and its ram, between which the rivet is compressed. This cylinder is suspended from a crane by chains, and can be moved into any position, as it communicates with the accumulator (Part V.) by a flexible pipe. Any portable machine, however, is an example of the same kind: machines which require foundations have no com-

T

plete frame apart from the solid ground which connects their parts together.

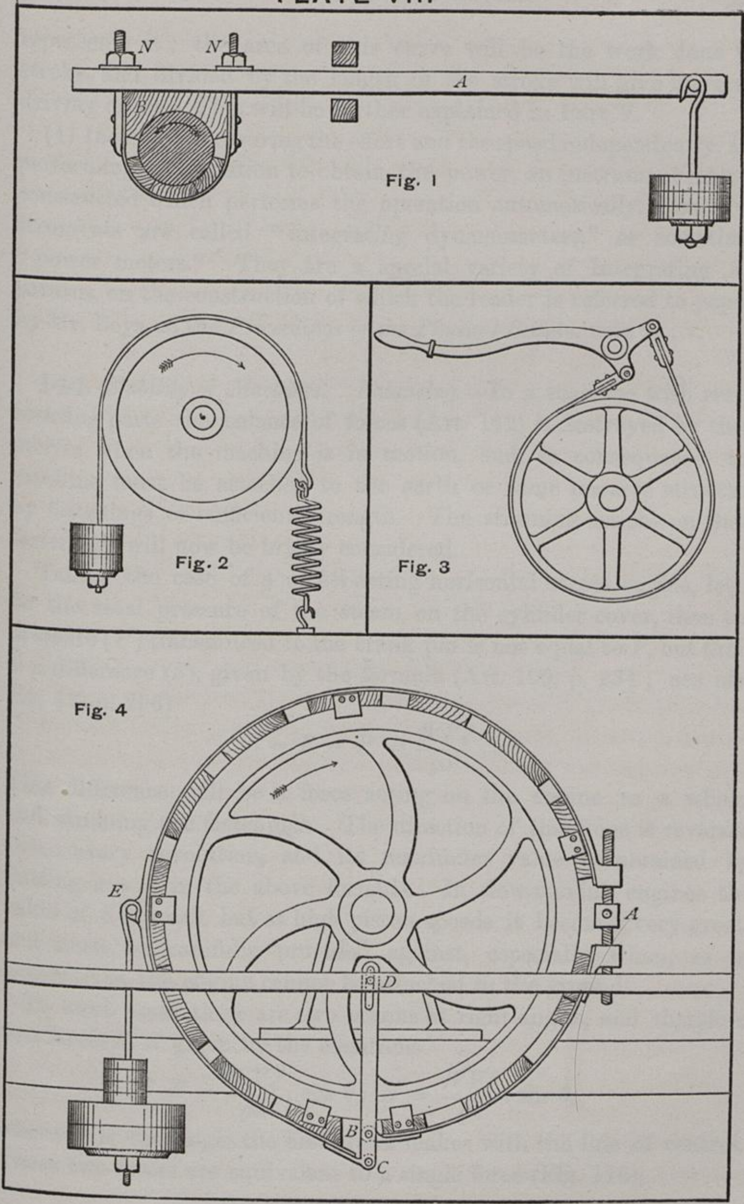
143. *Dynamometers.*—It is often a question of great practical importance to determine by direct experiment the power which is being expended in driving a machine. Instruments for effecting this are called Dynamometers; they are of very various construction, and only a few simple cases can be mentioned here. The most common are those in which the instrument measures the driving effort, while the speed is independently determined, and the power thence obtained, as in Art. 97, page 207.

(1) In Fig. 4, Plate III., a common "transmission dynamometer" is represented. A shaft transmitting the power is divided into parts, and bevel wheels *B*, *D* attached to each. A lever *A*, turning about an axis concentric with the shaft, has a weight suspended from it, which is found by trial just to balance the driving couple, transmitted through the bevel wheels *C* attached to *A* and gearing with *BD*. The whole forms the train described on page 152. Here the driving couple is measured in the act of transmission, and the revolutions of the shaft being known the power can be found.

(2) In Fig. 1, Plate VII., a "friction dynamometer" is represented in one of the various forms in which it is applied. *A* is a lever from which a weight is suspended; *B* is a block fixed to *A*, which rests on a revolving drum. A strap passes below the drum, and is tightened by the nuts *N*, *N*, till the friction just balances the weight, which by trial is made to balance the driving couple tending to turn the shaft. Here the driving couple and, consequently, the power are determined as in the preceding case, from which it only differs in the way in which the power is employed. Instead of being transmitted to the machine it is all absorbed by a friction brake which replaces it for the time being. A modification is shown in Fig. 2, in which the strap passes over a wheel and is tightened by a suspended weight, the difference between which and the tension of a spring balance measures the driving couple. (See Appendix.)

(3) In both the preceding cases the driving effort and the speed of the driving pair are constant, but in the indicator universally employed to measure the power of steam and other heat engines, we find an example in which both vary. The driving effort is now measured for each position of the piston and a curve drawn which

PLATE VII.



To face page 290.

represents it; the area of this curve will be the work done per stroke, and divided by the length of the stroke will give the mean driving effort. This will be further explained in Part V.

(4) Instead of measuring the effort and the speed independently, and performing a calculation to obtain the power, an instrument may be constructed which performs the operation automatically. Such instruments are called "integrating dynamometers," or sometimes "power meters." They are a special variety of Integrating Apparatus, on the construction of which the reader is referred to papers by Mr. Boys, in the *Proceedings of the Physical Society*, vols. iv., v.

144. *Stability of Machines. Balancing.*—In a machine with reciprocating parts the balance of forces (Art. 142) is destroyed by their inertia when the machine is in motion, and, in consequence, the machine must be attached to the earth or some massive structure by fastenings of sufficient strength. The straining actions on these fastenings will now be briefly considered.

Taking the case of a direct-acting horizontal steam engine, let P be the total pressure of the steam on the cylinder cover, then the pressure (P') transmitted to the crank pin is not equal to P, but there is a difference (S), given by the formula (Art. 109, p. 234; see also Ex. 13, p. 296)

$$S = P - P' = \frac{WV_0^2}{ga^2} \cdot x.$$

This difference will be a force acting on the engine as a whole, and straining the fastenings. The direction of this force is reversed twice every revolution, and its maximum value is obtained by putting $x = a$ in the above formula. In slow-moving engines the value of S is small, but at high piston speeds it becomes very great, and must be carefully provided against, especially when, as in locomotives, the engine cannot be attached to the ground.

In most cases there are two cranks at right angles, and therefore two forces S, S' given by the equations

$$S = \frac{WV_0^2}{ga} \cdot \cos\theta; \; S' = \frac{WV_0^2}{ga} \cdot \sin\theta,$$

where θ is the angle the first crank makes with the line of centres. These two forces are equivalent to a single force (Fig. 116),

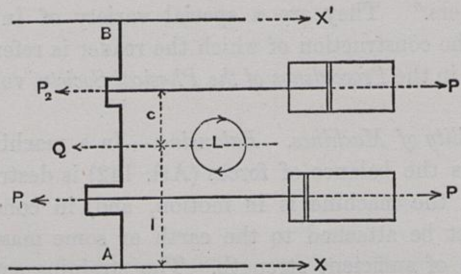
$$Q = S + S' = \frac{WV_0^2}{ga}(\sin\theta + \cos\theta),$$

acting midway between them, and a couple

$$L = (S - S')c = \frac{WV_0^2}{ga} \cdot c(\cos\theta - \sin\theta),$$

where c is the distance apart of the centre lines of the cylinders. The total effect therefore is the same as that of a single alternating

Fig.116.



force combined with an alternating couple, which tends to turn the engine as a whole about a vertical axis. The maximum values are

$$Q_0 = \frac{WV_0^2}{ga}\sqrt{2}\,;\ L_0 = \frac{WV_0^2}{ga}\sqrt{2}c,$$

and they are each reversed twice in every revolution.

In locomotives this action produces dangerous oscillations at high speeds, and must therefore be counteracted by the introduction of suitably placed balance weights, so as to neutralize both the force and the couple.

Fig.117.

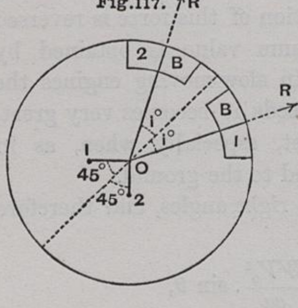


Fig. 117 shows a projection on a vertical plane of the two driving wheels and their cranks. On each wheel a balance weight is placed, occupying a segment between two or more spokes. The centre of gravity of each weight is in a radius nearly, but not exactly, opposite the nearer crank, the angle of inclination to the bisector being an angle i somewhat less than 45°. If B be the weight, r the radius of the circle in which its centre of gravity lies,

$$R = \frac{BV_0^2 r}{ga^2}$$

is its centrifugal force; and by rightly taking the values of B and i the horizontal components of these forces derived from the two balance weights, may be made to counteract both the force and the couple (Ex. 10, p. 296). In practice the weights are fixed approximately by a formula derived in this way, and the final adjustment is performed by trial. The engine is suspended by chains, and its oscillations, when perfectly adjusted, are very small even at very high speeds. (See Appendix.)

In high-speed marine engines similar forces arise, of great magnitude, which must add considerably to the strain on the fastenings, but no attempt is commonly made to balance them.

When the speed of a machine is excessive, reversal of stress must be avoided (see Exs. 17, 18, p. 244), and the greatest care is necessary that the axis of rotation of each rotating piece passes through its centre of gravity, and coincides with one of the axes of inertia of the piece (Art. 132). The magnitude of the forces which arise, in case of any error, may be judged of from the results of Exs. 13, 16, pp. 296, 297. The vibrations due to these forces will, however, in some cases be greatest at some particular speed—depending on the natural period of vibration of the frame of the machine—which could only be determined by trial. (Ch. XVI.)

In similar machines the forces due to inertia will be in a fixed proportion to the weight of the pieces, when the revolutions vary inversely as the square root of the linear dimensions of the machine.

145. *Straining Actions on the Parts of a Machine due to their Inertia.*—Another important effect of the inertia of a piece is to produce straining actions upon it. An important example is that of a ring rotating about its centre: the centrifugal force produces a tension on the ring which may be thus determined.

Suppose Fig. 121, p. 305, to represent the ring. Let the velocity of periphery be V, the weight W, and the radius r, then the centrifugal force on the small portion BB' of length z is

$$S = W \cdot \frac{z}{2\pi r} \cdot \frac{V^2}{gr}.$$

Resolve this in a given direction and sum the resolved parts, as in the article to which this figure refers, then the total is

$$P = W \cdot \frac{2r}{2\pi r} \cdot \frac{V^2}{gr} = \frac{W}{\pi r} \cdot \frac{V^2}{g}.$$

The stress to which this gives rise is evidently

$$q = \frac{W}{2\pi r A} \cdot \frac{V^2}{g} = w \cdot \frac{V^2}{g}.$$

where A is the sectional area of the ring and w is the weight of unit volume. The result here obtained is of great importance; it shows that the "centrifugal tension" of a revolving ring is independent of the radius for a given speed of periphery. Hence the result also applies to every point of a flexible element, such as a belt, whatever be the form of the surfaces over which it is stretched. In high-speed belts the tension is considerably increased by this cause, and additional strength has to be provided (Ex. 12, p. 293).

Another example of the straining actions due to inertia occurs in the motion of a rod, the ends of which describe given curves. Shearing and bending are produced, and at high speeds the magnitude of the stress thus arising is very great. Two common examples are given on page 296, but the limits of this work do not permit us to pursue the subject.

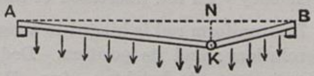
In similar machines the intensity of the stress occasioned by the straining actions we are here considering will be the same if the revolutions vary inversely as the linear dimensions of the machine.

146. *Virtual Machines.*—It has already been pointed out (Art. 94) that a machine may be regarded as a mechanism with two additional links applied as straining links, or, what is the same thing, a frame with one straining link (Art. 43). Further, as also remarked in the article cited, the external forces on any structure may be regarded as a set of straining links. It follows then that if in any framework or other structure one of its parts suffer a change of form or size of any kind, the rest remaining rigid, we shall have a machine in which the driving links exert a known stress and the working link is the bar in question. The principle of work then enables us to determine the stress on the bar, for the stress ratio must be the reciprocal of the velocity ratio. A machine thus formed may be called a "virtual machine," its movements being only supposed for the purpose of the calculation, not actually existing. It is especially in applying this method that we find in treatises on statics the principle of work employed under the title "principle of virtual velocities."

We must content ourselves with a single example of this method.

AB (Fig. 118) is a beam supported at the ends and loaded uniformly. Imagine the beam broken at K, and the pieces united by a stiff hinge, the friction of which is exactly equal to the bending moment M, then if the hinge be supposed gradually to yield under the weight, so that the joint K descends through the small space $KN(=y)$,

Fig.118.



$$\text{Energy exerted} = \tfrac{1}{2}yw(AK + BK).$$

$$\text{Work done} \quad = M(i_1 + i_2) = M\left(\frac{y}{AK} + \frac{y}{BK}\right),$$

where i_1, i_2 are the angles AK, BK make with the horizontal.

Equating the two

$$M\left(\frac{1}{AK} + \frac{1}{BK}\right) = \tfrac{1}{2}w(AK + BK),$$

which gives the known value (p. 44),

$$M = \tfrac{1}{2}w \,.\, AK \,.\, BK.$$

The advantage of this method is that it leads directly to the required result, without the introduction of unknown quantities which require to be afterwards eliminated.

EXAMPLES.

1. In Ex. 1, page 218, suppose the gun to weigh 35 tons, what additional powder will be required to provide for recoil? *Ans.* 1 lb. nearly.

2. Two vessels of displacements 8,000 and 5,000 tons are moving at 6 knots and 4 knots respectively. One is going north and the other south-west; find the energy of collision. *Ans.* 11,700 foot-tons.

3. Find the height of a governor revolving at 75 revolutions per 1′. *Ans.* 6·24″.

4. Find the dimensions of a Farcot governor to revolve at 40 revolutions per 1′, with the arms inclined at 30° to the vertical, and to be parabolic for small displacements. *Ans.* Height of governor = 22″. Length of arms = 34″. Length of cross piece to which arms are attached = $8\frac{1}{2}$″. More generally, if θ be the inclination, l the length of the arms, the length of the cross-piece is $2l \,.\, \sin^3\theta$.

5. In a simple governor revolving at 40 revolutions per 1′ find the rise of the balls in consequence of an increase of speed to 41 revolutions. Also find the weight of ball necessary to overcome a frictional resistance of 1 lb., the linkwork being arranged so that the slider rises at the same rate as the balls. *Ans.* Rise of balls = 1·1″. Weight of each ball = 5 lbs.

6. The balls of a governor weigh 5 lbs. each and it is loaded with 50 lbs. The linkwork is such that the slider rises and falls twice as fast as the balls. Find the height for a speed of 200 revolutions per 1′, and, if the speed be altered 2 per cent., find the tendency to move the regulating apparatus. How much is this tendency increased

by the loading? If the engine is required to work at three fourths its original speed, by how much should the load on the governor be diminished? *Ans.* Height = $9''\cdot7$. Tendency = 2·2 lbs. (increased 11 times).

7. A uniform rod is hinged to a vertical spindle and revolves at a given number of revolutions; find its position. Deduce the effect of the weight of the arms of a governor on its height. *Ans.* Height of rod $= \frac{3}{2} \cdot g/A^2$. Height of governor is increased in the ratio $1 + \frac{1}{2}n : 1 + \frac{1}{3}n$ where n is the ratio of the weight of the arm to the weight of the ball.

8. In Ex. 6, p. 133, find the ratio in which the bending moment at each point is affected by the inertia of the rod.

Every point of the rod describes relatively to the engine a circle and the centrifugal force of any portion of the rod = 18·6 times the weight. In the lowest position the centrifugal force acts with gravity, and so in this position the bending action is the same as if the weight of the material of the rod were 19·6 times its true weight.

9. In a horizontal marine engine with two cranks at right angles distant 8 feet from one another, weight of reciprocating parts attached to each crank 10 tons, revolutions 75 per minute, stroke 4 feet. Find the alternating force and couple due to inertia. *Ans.* Alternating force = 54·2 tons. Alternating couple = 216·8 foot-tons.

10. An inside cylinder locomotive is running at 50 miles per hour, find the alternating force and couple. Also find the magnitude and position of suitable balance weights, the diameter of driving wheels being 6 feet, the distance between centre lines of cylinders 2′ 6″, stroke 2′, weight of one piston and rods 300 lbs. Horizontal distance apart of balance weights 4′ 9″. Diameter of weight circle 4′ 6″. *Ans.* Alternating force = 7,871 lbs. Alternating couple = 9,839 foot-lbs. $B = 106\cdot5$ lbs. $i = 27\frac{3}{4}°$.

11. A fly-wheel 20 feet diameter revolves at 30 revolutions per 1′. Assuming weight of iron 450 lbs. per cubic foot, find the intensity of the stress on the transverse section of the rim, assuming it unaffected by the arms. *Ans.* 96 lbs. per sq. inch.

12. A leather belt runs at 2,400 feet per 1′, find how much its tension is increased by centrifugal action, the weight of leather being taken as 60 lbs. per cubic foot. *Ans.* 20·5 lbs. per square inch.

13. If r be the radius of the circle described by the centre of gravity of a rotating body, h the height due to the revolutions (page 282), show that the centrifugal force is

$$R = W\frac{a}{h}.$$

Obtain the numerical result (1) for a wheel weighing 100 lbs. with centre of gravity one sixteenth of an inch out of centre, revolving at 1000 revolutions per minute, (2) for a piece weighing 10 lbs. revolving at 300 revolutions per minute in a circle 1 foot diameter. *Ans.* (1) 178 lbs. (2) 154 lbs.

NOTE.—The formulæ of Art. 144 can all be expressed most simply in terms of h.

14. In Question 8 suppose the connecting rod of uniform transverse section, find how much the bending moment upon it due to its weight is increased by the effect of inertia.

Here the bending moment is greatest (very approximately) when the crank is at right angles to the connecting rod, and the forces due to inertia then consist (also very approximately) of a set of forces perpendicular to the rod, and varying as the distance from the crosshead pin. At the crank pin we have simply the centrifugal force due to the revolutions and length of crank. Thus the curve of loads is a straight line (p. 68) whence, proceeding by the methods of Chap. III., we find for the maximum moment

$$M = \frac{Wl}{9\sqrt{3}} \cdot \frac{a}{h},$$

where l is the length of rod, a the length of crank, h the height due to the revolutions. In the numerical example the effect of inertia is about $9\frac{1}{2}$ times that of the weight W.

15. A body rotates about an axis OE, lying in a principal plane through its centre of gravity G, and inclined to a principal axis OG at an angle θ. Show that the moment of the centrifugal forces about O is

$$L = W\frac{k^2 - k'^2}{h} \cdot \sin\theta \cdot \cos\theta,$$

where h is the height due to the revolutions, and k', k are the radii of gyration about OG, and a line through O, perpendicular to OG in the plane GOE, respectively. Deduce the height of a compound revolving pendulum.

16. A disc rotates about an axis through its centre at 1000 revolutions per minute. The disc is intended to be perpendicular to the axis, but is out of truth by $\frac{1}{100}$th of the radius: find the centrifugal couple. *Ans.* If r be the radius in inches the couple in inch-lbs. is

$$L = \frac{Wr^2}{13 \cdot 1}.$$

17. In question 10 find the alternate increase and diminution of the pressure of the driving wheel on the rail due to the inertia of the balance weight. *Ans.* 5,900 lbs.

NOTE.—This force of more than $2\frac{1}{2}$ tons produces great straining actions on both the wheel and the rails.

18. The power of a portable engine is tested by passing a strap over the fly-wheel, which is 4 feet 6 inches diameter, fixing one end and suspending a weight from the other. The weight is 300 lbs., and the tension of the fixed end is found by a spring balance to be 195 lbs: what is the power when running at 160 revolutions per minute. *Ans.* 7·8 h.p.