PART III.—DYNAMICS OF MACHINES.

CHAPTER VIII.

PRINCIPLE OF WORK.

SECTION I.—BALANCED FORCES (STATICS).

88. *Preliminary Explanations. Definition of Work.* If the principal object of a piece of mechanism be to do some kind of work it becomes a machine. Many mechanisms—as for example clocks and watches—are not, properly speaking, machines; for though work is done during their action, yet the object of the mechanism is not the doing of the work but the measurement of time or some similar operation. Even in these cases, however, the forces in action cannot in general be excluded from consideration, and therefore in all mechanism a study of the manner in which forces are transmitted and modified is essential. This part of the subject is called the Dynamics of Machines.

A body can in general only be moved into a different position or be changed in form or size by overcoming resistances which oppose the change. This process is called doing WORK, and the amount of work is measured by the resistance multiplied by the space through which it is overcome. If there be many resistances, the total work done is the sum of that done in overcoming each resistance separately.

Consider the case of a weight raised vertically. Here the resistance is due to the action of gravity which is overcome by some external force, and the work done is simply the product of the weight and the height through which it is raised. The weight is measured by comparing it with that of a certain quantity of matter called a pound, the weight of which is taken as a unit for measuring forces. This mode of measurement has the disadvantage of giving a different unit for different points on the earth's surface, because the force of gravity varies according to the position of the point, and, for scientific purposes therefore, force is measured by the velocity which, when unbalanced, it produces in a given quantity of matter. In practical applications, however, gravitation measure is preferable, as the variation is very small, and the measure may be made precise when necessary by specifying the place on the earth's surface at which our operations are taking place. The unit of space is generally 1 ft., so that the unit of work is 1 lb. raised through 1 ft., or, as it is generally called, 1 foot-pound. Other units, however, such as, for example, "foot-tons," may also be employed for special purposes.

89. *Oblique Resistance.*—The resistance is here directly opposed to the movement which is taking place; if this be not the case it must be resolved into two components, one along and the other perpendicular to the direction of motion. The second of these is balanced by a constraint to which the motion is subject or by the opposition which the inertia of the body offers to a change in its direction at any finite rate; it is the first alone in overcoming which work is done. In Fig. 84 let R be a resistance applied at a point A which moves through a distance AB in a direction inclined at an angle θ to the direction of the resistance, then the work done is $R.\cos\theta.AB$, but if BN be drawn perpendicular to the direction of R to meet that direction in N,

$$AN = AB.\cos\theta,$$

and therefore the work done is $R.AN$.

Now AN is the distance through which A has moved in the direc-

Fig.84.

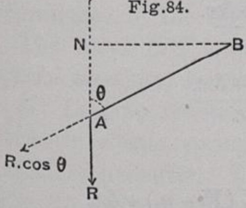
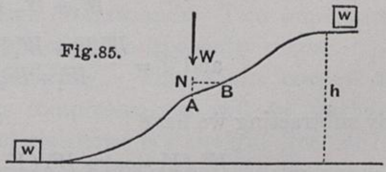


Fig.85.



tion of the resistance, so we obtain another rule for estimating the work done against an oblique resistance. It is equal to the product of the resistance into the distance moved in the direction of the resistance.

Suppose for example that a weight is raised, but that, instead of being lifted vertically, it is moved in any curved path—there being no friction or other resistance than that due to gravity.

Considering any small portion AB of the path (Fig. 85), the resistance being always vertical, the work done is $W.AN$. So the total work of raising the weight is $W.\Sigma AN$ or $W.h$, which is independent of the path described by the lifted weight, but depends simply on the height through which the weight is raised.

If there are a number of weights each of them raised through different heights, the total work done in raising all the weights is the sum of the works done in raising each weight separately; and the direct method of finding the total work is to add the separate results for each weight. But it may be determined by another method thus—

Let W_1, W_2, W_3 &c. be a number of weights which are at heights y_1, y_2, y_3 &c. above a given datum plane. Now suppose they are raised so that they are at heights Y_1, Y_2, Y_3 &c. above the same plane. The total work done in raising the weights will be the sum of the products,

$$W_1(Y_1 - y_1) + W_2(Y_2 - y_2) + W_3(Y_3 - y_3) + \text{\&c.}$$

Now suppose the centres of gravity g and G for the initial and final positions of the weights to be at heights $\bar{y}$ and $\bar{Y}$ above the datum plane.

The centres of gravity g and G are such that if all the weights were collected at either centre, the moment of the collected weights about the plane is equal to the sum of the moments of each separate weight, before being collected, about the same plane. This is mathematically expressed thus

$$\bar{y} = \frac{W_1 y_1 + W_2 y_2 + W_3 y_3 + \text{\&c.}}{W_1 + W_2 + W_3 + \text{\&c.}}$$

$$\text{and } \bar{Y} = \frac{W_1 Y_1 + W_2 Y_2 + W_3 Y_3 + \text{\&c.}}{W_1 + W_2 + W_3 + \text{\&c.}}.$$

By subtracting we have

$$\bar{Y} - \bar{y} = \frac{W_1(Y_1 - y_1) + W_2(Y_2 - y_2) + W_3(Y_3 - y_3) + \text{\&c.}}{W_1 + W_2 + W_3 + \text{\&c.}};$$

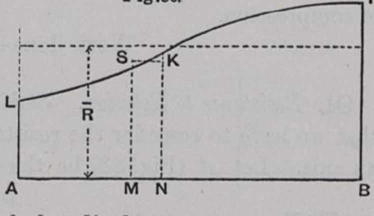
hence the total work done in raising the weights may be expressed

$$=(W_1+W_2+W_3+\&c.)\times(\bar{Y}-\bar{y})$$
$$\text{or } \Sigma W(\bar{Y}-\bar{y}).$$

That is to say, the total work of raising a number of weights is equal to the product of the sum of the weights by the vertical displacement of the centre of gravity of the weights.

90. *Variable Resistance.*—Let us next consider the work required to be done to overcome a variable resistance. The whole distance through which the resistance is overcome must then be divided into a number of parts, each being so small that, for that small space, the magnitude of the resistance may be treated as sensibly uniform. The work of overcoming the resistance through each of the small spaces being thus found, the total work will be the sum. The estimation can generally be most conveniently performed by a graphical construction. We will, for simplicity, take the case in which the direction of action of the resistance is that of the line of motion. Suppose a body moved from A to B against a resistance the magnitude of which varies from point to point in such a way that it is represented by the ordinates of the curve standing above AB. (Fig. 86.) For the small distance MN the resistance will vary slightly, but will have a mean value represented by SM or KN suppose, and the work of overcoming the resistance through the small space MN is $MN \times SM$ or is exactly represented by the area of the curve standing above MN; and so for any other small portion of the displacement of the body. Thus the total work of overcoming the resistance through AB is represented by the whole area $ALTB$ = mean resistance $\bar{R} \times AB$.

Fig.86.



The curve LST is called a curve of resistance. Two important special cases may be mentioned both of which frequently occur.

(1) Let the resistance vary uniformly. This is the case of a perfectly elastic spring which is compressed, as will be further explained hereafter. The curve of resistance is a straight line AST (Fig. 87a) where AB is the compression of the spring, BT the corresponding compressing force R_1. During the compression R is at first

zero and gradually increases to R_1, its value at any intermediate point being graphically represented by the ordinate SN corresponding to the compression AN. The work done is the area of the triangle, that is $\frac{1}{2}R_1.AB$, and the mean resistance $\frac{1}{2}R$.

(2) Let the resistance be inversely proportional to the distance of the point of application from a given point O. (Fig 87*b*.)

Fig.87a.

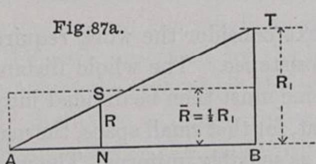
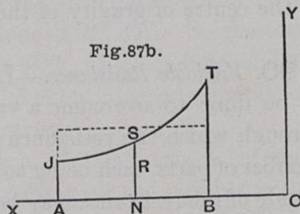


Fig.87b.

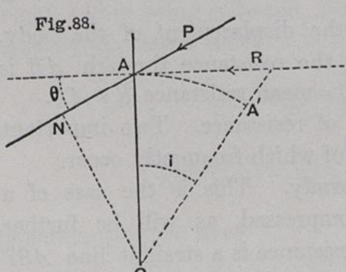


This applies to many cases of the compression of air and other elastic fluids. In the figure $NS = R$ is the resistance and $ON.NS$ is constant, so that the curve of resistance JST is an hyperbola. Let the ratio $OA : OB$ be called r, this is called the ratio of compression; then from the geometry of the hyperbola we know that the area of the curve is equal to the constant rectangle $ON.NS$ multiplied by $\log_e r$, the logarithm being Napierian, or as it is often called "hyperbolic" from this property of the hyperbola. If ON be denoted by V this gives a formula in frequent use for the work done in this kind of compression.

$$\text{Work done} = RV \log_e r.$$

91. *Resistance to Rotation. Stability of a Vessel.*—It often happens that we have to consider the resistance of a body to rotation about an axis. Let A (Fig. 88) be the point of application of a force P which resists the rotation of a body about an axis C perpendicular to the plane of the paper. If the resistance at A be not in the plane of rotation P must be supposed to be the component in that plane; the other component will be parallel to the axis of rotation and need not be considered. Let θ be the angle it makes with the direction of A's motion, then $R = P.\cos\theta$ is the effective resistance, the other com-

Fig.88.



ponent of P merely producing pressure on the axis. As the body turns through an angle i the resistance R will be overcome through the arc AA', and, assuming in the first instance R constant, the work done will be—

$$\text{Work done} = R.AA' = R.CA.i.$$

But, dropping a perpendicular CN on P's direction,

$$CN = CA. \cos \theta$$

$$\therefore \text{ Work done} = P.CN.i = Mi,$$

where M is the moment of the resistance about the axis of rotation. If there be many resistances then the same formula will hold if M be understood to mean the total moment of resistance.

We can readily extend this to the case of a variable moment by the graphical process already described for a linear resistance, the base of the diagram now representing the angles turned through and the ordinates the corresponding moments. As an example take the case of a heavy pendulum swinging about an axis O (Fig. 89a), let g be the centre of gravity, $Og = l$, and let it be swung through the angle i from the vertical, then the moment of resistance is

$$M = W.gN = W.l \sin i.$$

Fig. 89a.

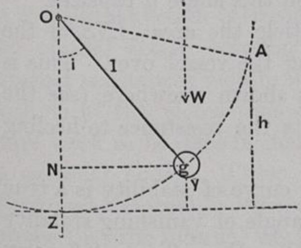
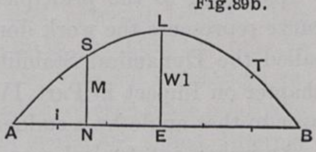


Fig. 89b.



In Fig. 89b draw a curve on the base AB such that the horizontal ordinate AN at every point represents the angle i on the same scale that AB represents two right angles, while the vertical ordinate represents M. This curve will be the curve of resistance, and in the present case is a curve of sines of which the maximum ordinate LE is Wl. The angles being supposed reckoned in circular measure so that $AB = \pi$, the area of the diagram from A up to any point S will represent the work done. We can, however, in this example find this work otherwise, for g rises through the height NZ, and therefore if U be the work

$$U = Wl\,(1 - \cos i).$$

By use of the integral calculus it can be verified that this is also the value of the area ASN.

It is not necessary that the axis of rotation should be fixed in estimating the work done during rotation, provided that the resistance be a couple, for then there is no pressure on the axis. An important example is that of a vessel floating in the water and steadily heeled over by the action of a couple M produced by external agency, or more frequently by shifting the weights on board in such a way that the displacement and trim remain constant. Then for each angle of heel this couple has a certain definite value which can be found either by calculation or by observation of the shift of the weights. The moment of resistance which is equal and opposite to M is called the Statical Stability of the vessel, and the curve of resistance drawn as above described is called the Curve of Stability. The construction of this curve is an important part of the design of the vessel. Such curves, though usually unsymmetrical, often bear a general resemblance to a curve of sines (Fig. 89*b*), the ordinate increases to a maximum which gives the maximum stability and then diminishes to zero at an angle of heel called the "Angle of Vanishing Stability." If the vessel be heeled beyond this angle it capsizes.

According to the principles of this article the area ANS of the curve represents the work done in heeling the vessel over. This is called the Dynamical Stability, and as is shown elsewhere (see the chapter on Impact in Part IV.) represents the resistance to heeling over to that angle by a sudden gust.

An important typical case is when the curve of stability is a true curve of sines. In this case suppose the angle of vanishing stability to be π/k, where k is some given number, then the ordinate S for any angle i is given by the equation

$$S = S_1 \,.\, \sin ki,$$

and the stability is the same as that of a heavy pendulum swinging through k times the angle. The dynamical stability is easily shown to be

$$U = \frac{S_1}{k}\,(1 - \cos ki).$$

92. *Internal and External Work.*—In all that precedes the position of a body has been changed by overcoming external resistances. All forces, however, arise from the mutual action between two bodies or between two parts of the same body, and every change of position must be with reference to some other body which is regarded as fixed. Work, then, consists in a change of relative position of two bodies notwithstanding a mutual action between the two which opposes the change. In raising weights the second body is the earth, but the pair of bodies may be such as occur in mechanism and the mutual action between the two may be due to springs or an elastic fluid, or to the resistance of some body to separation into parts. In scissors, nutcrackers, bellows, and other similar instruments, the elements of the pair are exactly alike and their existence is recognised in popular language.

In reckoning the work done either body may be regarded as fixed, the result must be the same and will be unaffected by any movement of the pieces common to both; thus when air is compressed in a cylinder the work done depends on the pressure of the air and the amount of compression, not on the movements of the cylinder within which the air is contained. In other words the motion to be considered is the motion of the pair as defined in Art. 46, p. 102.

In every case where we have to do with a number of pieces connected in any way, we may distinguish between the resistances due to the mutual action between the pieces themselves and those due to the mutual action between the pieces and external bodies. The internal resistances require work to be done in changing the relative position of the pieces themselves, while the external resistances require work to be done in changing the position of each piece relatively to external bodies. These two kinds of work are called Internal Work and External Work respectively. In two cases we can at once foresee that the internal work will be zero, first when the pieces are disconnected, secondly when they are rigidly connected. Thus for example if a heavy mass of matter be raised, we need only consider the rise of the centre of gravity (Art. 89) if the mass be rigid; but if not, any change of form which occurs ought to be taken into account. In raising ordinary solid bodies and masses of earth the internal work may usually be disregarded.

93. *Energy. Principle of Work.*—Hitherto we have been speaking

of the *resistance* which is being overcome during the process of doing work, let us now fix our attention on the *effort* which overcomes the resistance.

The forces arising from the mutual action between a pair of bodies, when not purely passive like the normal pressure between two surfaces in contact, are of two kinds. The first always oppose the motion of the pair, in other words they are always resistances. Friction between two surfaces is the simplest example of this, and hence such actions are called Frictional Resistances. The second on the other hand promote or oppose the motion of the pair according to the direction in which the motion is taking place, so that a resistance becomes an effort when the direction of motion is reversed. Such actions are conveniently described as Reversible; and systems of bodies, in which they occur, possess, when the parts are suitably disposed, the power of doing work. This power is called ENERGY. As examples of bodies possessing energy may be taken a raised weight, a compressed spring, or steam of high pressure. Change of velocity in a moving body likewise gives rise to efforts and resistances, but this is a matter for subsequent consideration. For the present we suppose all bodies with which we have to do to be in a state of uniform motion, or to move so slowly and steadily that no sensible action of this kind can arise.

Energy is measured by the quantity of work which it is capable of doing, and the process called doing work may also be described as the exertion or expenditure of energy, so that we write

$$\text{Energy exerted} = \text{Work done.}$$

If the effort which is being exerted and the resistance which is being overcome be applied to the elements of the same lower pair, as when a weight is lifted vertically or a spring wound up, the effort and the resistance are equal, and the equation shows that the energy exerted by an effort is the product of the effort and the space through which it is exerted. Thus all the examples given above of the doing of work will also serve as examples of the exertion of energy simply by supposing the direction of motion reversed. In short the exertion of energy and the doing of work are merely different aspects of the same process.

In this case the effort and the resistance may be regarded as applied at the same point, but the equation has a much wider application than this, for it is equally true if the points of application be different, provided only that they are rigidly connected. Thus, for example, if we dig in the ground, the energy we exert at the handle of the spade is—if the spade be perfectly rigid—exactly equal to the work done at the blade. This can be shown to be a necessary consequence of the forces we are considering being balanced, and the equation may be regarded as a concise statement of the conditions of equilibrium of forces applied to a rigid body. It is preferable, however, for our purposes to regard it as the simplest case of a fundamental mechanical principle continually verified by experience. This principle may be called the PRINCIPLE OF WORK.

We have now a means of transferring the power of doing work, that is to say energy, from one place to another: evidently we are not restricted to one piece as in the case of the spade. We may make use of a series of pieces through which energy may be transferred from piece to piece in succession; and if there were no frictional resistances to the relative motion of the pieces, there would be no loss of energy in the process. Thus the principle of work is true when the points of application of the effort and the resistance are mechanically connected in any way. Frictional resistances however absorb a portion of the energy whenever any relative motion occurs which they tend to prevent, and therefore a certain loss always accompanies the transmission of energy. Nevertheless the principle of work still holds good if overcoming friction be reckoned as part of the work done.

It may here be remarked that though frictional resistances are never a source of energy, yet friction may, like normal pressure between surfaces, transmit energy, and hence, in cases where one only of the bodies between which it is exerted belong to the set of bodies we are considering, may be an effort by means of which work is done on the set. Thus, for example, in the case of a shaft driven by a belt, the whole power of the engine is transmitted by friction closure between the belt and the pulleys; and if we consider the shaft alone apart from the rest of the mechanism, the friction may be regarded as the effort which drives the shaft. We cannot however in such cases properly speak of the friction as exerting energy; the source of energy is the steam, or other motive power, and the friction merely transmits it in the same way as the pressure between a connecting rod head and the crank pin transmits energy to the crank shaft.

Nevertheless in both of these cases the phrase "energy exerted" may be used conveniently, though "energy transmitted" would be more precise.

If a piece of material through which energy is transmitted yield under stress applied to it, as in fact it always does, the energy exerted will not be equal to the work done. Either the change of relative position of the several parts of the piece will require work to be done in order to overcome the mutual actions between the parts which resist the change, or, conversely, those mutual actions exert energy during the change. In the first case the work is done at the expense of the energy transmitted; in the second the piece of material is a source of energy which increases the energy transmitted. In perfectly elastic material the mutual actions are reversible, and any energy exerted in overcoming them is stored up in the piece and recovered when the piece resumes its original form, as in the case of a watch spring. (Compare Art. 98.)

94. *Machines.*—A mechanism becomes a machine if we connect together two of its elements by a link capable of changing its form or dimensions, and so moving the mechanism, notwithstanding a resistance applied by a similar link connecting two other elements.

The elements connected may be called the "driving pair" and the "working pair," and these pairs often, though by no means always, have one element common, namely the frame-link of the mechanism. The driving link is the source of energy. As examples, we may take steam which connects the piston and cylinder which form the driving pair in a steam engine, or gravity which, as in Art. 62, is to be conceived replaced by a link exerting the same effort. The working link is gravity in cranes and other hoisting machines, or a piece of material the deformation of which is the object of the machine, as in the case of machine tools.

In addition to the driving and the working links, the force of gravity acts on all the parts of the machine, and frictional resistances have to be overcome; but these are matters for subsequent consideration.

The driving and working pairs are very frequently kinematic pairs of the lower class. Let us suppose them in the first instance sliding pairs. Let the driving pair move through a space x, then the working pair will move through a space y, which is in a certain definite proportion to x depending on the nature of the mechanism. Let P be the driving effort, which, by taking x small enough, can be made as nearly uniform as we please; and let R be the resistance opposing the motion of the working pair, then

$$\text{Energy exerted} = Px; \text{ Work done} = Ry,$$

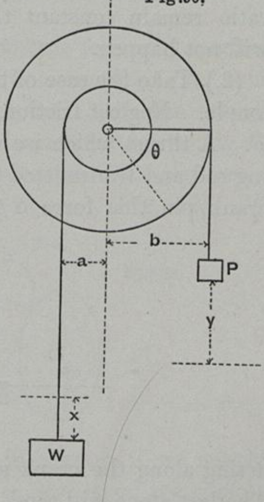
and these must be equal, therefore

$$\frac{P}{R} = \frac{y}{x} = \frac{\text{Velocity of Working Pair}}{\text{Velocity of Driving Pair}};$$

from which it appears that the ratio of the effort to the resistance, or as we may briefly call it, the "force ratio," is the reciprocal of the velocity ratio of the driving and working pairs. In works on mechanics this is also known as the Principle of Virtual Velocities.

If the pairs be turning instead of sliding pairs, then the effort and resistance are moments, and the velocities will be angular; and if one pair be sliding, the other turning, a suitable "radius of reference" must be selected (p. 103) to compare the motions and the forces, but the same principle holds good.

Fig.90.



In the simplest machines, known frequently as the "mechanical powers," we have a 2 or 3-linked chain, so that the driving pair and working pair are identical or very closely connected. But they may be separated by a long train of mechanism and have no common link. In all cases it must be carefully remembered that the effort and the resistance arise from the mutual action between the elements, each consisting of two equal and opposite forces, just as in the straining actions considered in Chapter II. and elsewhere. Either of these as before measures the magnitude of the action.

95. *Verification of the Principle of Work in Special Examples.*—We will now take some examples to illustrate and verify the principle of work, neglecting friction.

(1.) Take the common wheel and axle. Suppose P to be just sufficient to lift the weight W, so that the two forces exactly balance one another. Now let P descend through the distance y (Fig. 90), and W rise through the corresponding distance x.

As P falls it is said to exert energy. Energy exerted $= Py$. This is employed in overcoming the resistance to the rise of the weight W. Work done $= Wx$. The principle of work asserts that Energy exerted = Work done, that is $Py = Wx$.

Suppose the wheel and axle to turn through the angle θ, then $y = b\theta$ and $x = a\theta$. Then in order that the weights P and W may statically balance one another, $Pb = Wa$; from which it follows that $Py = Wx$, verifying the principle of work.

Also, we may write,

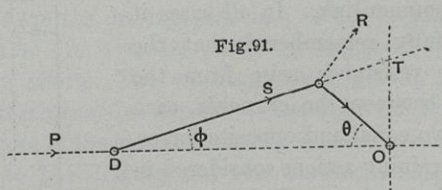
$$\frac{P}{W} = \frac{a}{b} = \frac{x}{y} = \frac{v}{V},$$

where v, V are the velocities of P, W respectively, thus showing that the force ratio is the reciprocal of the velocity ratio.

In this simple example both the force ratio and the velocity ratio remain constant throughout the movement. In general this will not happen.

(2.) Take the case of the mechanism of the steam engine for an example. Neglect friction and let the driving pressure on the piston be P. A thrust which we will call S will be produced along the connecting rod and transmitted to the crank pin as shown in Fig. 91. At the crank pin this force S may be resolved into two components, one

Fig. 91.



acting along the crank arm and the other, R, perpendicularly to it. The last alone will tend to turn the crank, the other component producing only a pressure on the shaft immediately balanced by the pressure of the bearings on the journals of the shaft.

This component R which tends to turn the shaft is called the *crank effort*. If the turning effort on the crank is perfectly balanced at all points of its revolution by some suitable resistance, then the resisting force which must be applied at the crank pin at right angles to the crank arm in order to balance perfectly the pressure of the steam on the piston must be equal and opposite to the component R previously referred to. The force ratio will be P/R. We have, with the notation employed in Chap. V., $S \cos \phi = P$ and $S \sin (\theta + \phi) = R$.

$$\text{Thus } \frac{R}{P} = \frac{\sin (\theta + \phi)}{\cos \phi} = \frac{\sin OBT}{\sin OTB} = \frac{OT}{OB}.$$

That is, the crank effort is to the steam pressure as the intercept OT is to the crank arm OB.

But we have previously shown (see p. 109) that this fraction expresses the velocity ratio of piston to crank pin; hence we have again found in this case that the force ratio is the reciprocal of velocity ratio, and the curve which we previously drew to represent the varying velocity of the piston, the crank pin moving uniformly, will represent also the varying crank effort, the pressure of the steam on the piston being uniform throughout the stroke. So we may call it the Curve of Crank Effort.

(3.) The same thing may be proved to be true for every mechanism, the forces acting on which balance one another. In some cases it may be easier to determine the force ratio than the velocity ratio or *vice versa*. In any case either may be inferred by taking the reciprocal of the other. As an additional example take the case of two pieces driving one another by simple contact (Fig. 92). We have already found the velocity ratio by a direct process (p. 165), but we may also determine it in the following way. When A presses on B there is a resistance R equal and opposite to the pressure,

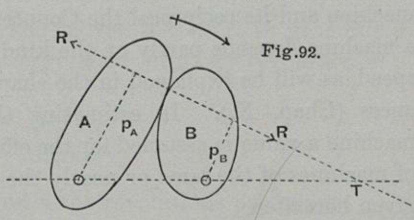


Fig. 92.

and normal to the portions of the surfaces in contact, if we suppose no friction to exist. Drop perpendiculars p_A and p_B on the common normal. Then the moment of the driving pressure R which A exerts on B or the turning moment due to $A = M_A = Rp_A$. Similarly the moment of the resisting force which B exerts on A or the moment of resistance to turning which B opposes to $A = M_B = Rp_B$.

Thus the ratio $\frac{\text{Driving moment}}{\text{Resisting moment}} = \frac{M_A}{M_B} = \frac{p_A}{p_B}$.

But we have previously proved that this fraction is the angular velocity ratio of the piece B to the piece A, and thus we show that the moment ratio is the reciprocal of the angular velocity ratio.

96. *Periodic Motion of Machines.*—One of the most essential characteristics of a machine is the periodic character of its motion. Each part goes through a cycle of changes of position and velocity and returns periodically to its original place. When moving steadily the periods are equal and the velocity of each piece is the same at the beginning and end of each period. That this may be the case it is not necessary that the driving effort should balance the working resistance in every position; on the contrary, this seldom happens; it is sufficient if the mean effort be equivalent to the mean resistance, or as we may otherwise express it

Energy exerted during a period = Work done in the period;

a condition which always governs the action of a machine in steady motion. In reckoning the energy and work the action of gravity on any piece of the machine may be omitted, for, if the piece rise through any height during one part of the period, it will fall through an equal height during another part. The work done consists partly of the work which the machine is designed to do, and partly of frictional resistance to the relative motion of the parts of the machine, or in other words of Useful Work and Waste Work. The ratio of the useful work to the energy exerted is called the Efficiency of the machine and its reciprocal the Counter-Efficiency. The efficiency of a machine depends partly on the kind of machine and partly on the speed, as will be explained in the chapter devoted to frictional resistances (Chap. X.). In estimating the power required to drive a machine a value is assumed for the efficiency derived from experience of machines of the same or nearly the same type. Examples will be given hereafter.

97. *Power. Sources of Energy.*—The sources of energy are—

(1.) Living agents;
(2.) Gravity acting usually by means of falling water;
(3.) Springs and elastic fluids;
(4.) Gunpowder and other explosive agents.

The energy thus derived may be traced further back to the action of heat and chemical affinity, and we may add to the list electric and magnetic forces, but the foregoing is a sufficient statement for our present purpose. A machine which employs such agents directly is called a Prime Mover, or, more briefly, a Motor, but a number of machines may be driven from one prime mover which serves as their source of energy. In general, each source of energy has a motion and an effort peculiar to itself while the work is required to be done at a different place and under different circumstances. A machine, then, is a mechanism which transmits energy and converts it into a form suitable to the work to be done.

The rate at which energy is exerted is called Power; it is this which measures the value of a source of energy and the expense of the work which is being done. The ordinary unit of measurement is the conventional horse-power of 33,000 foot-pounds per minute or 550 per second, a quantity much greater than the working power of an ordinary draught horse on the average of a day's work. The unit of power employed universally on the Continent is somewhat less, being 75 kilogrammetres per second or 32,550 foot-pounds per minute.

In prime movers the effort may generally be regarded as applied at a point which moves with a known mean velocity; then the horse-power is given by the equation

$$\text{H.P.} = \frac{PV}{33{,}000},$$

where P is the mean value of the effort in lbs. and V the mean velocity in feet per minute.

In machines driven from a prime mover the effort is generally a moment M which exerts the energy $M.2\pi$ in every revolution of a driving shaft. We then have

$$\text{H.P.} = \frac{M.2\pi n}{33{,}000},$$

where M is the mean moment and n the revolutions per minute.

98. *Reversibility. Conservation and Storage of Energy.*—The resistance overcome at the working point may be either frictional as in machine tools or reversible as in machines for raising weights. In the second case, if the machine were stopped and set in motion in the reverse direction it would, if friction could be neglected, work equally well, the driving effort and working resistance would be interchanged, and constructive modifications might be required, but otherwise the action is unaltered. This may be described by saying that the machine is Reversible. Many machines actually occur in both their direct and their reversed forms; thus a pump is a reversed hydraulic motor. Hence it appears that in reversible machines the power of doing work, that is to say, energy, is not lost after being exerted, for by reversing the machine it may be employed a second time. Thus it is that we describe the action of reversible machines as a transfer of energy, and are led to conceive of energy as indestructible and independent of the bodies through which it is manifested. No machine, indeed, is completely reversible, for in all cases frictional resistances occur to a greater or less extent, while many machines are completely non-reversible; but we shall see as we proceed that even then energy is not lost but only converted into another form, so that we have in reversible machines the first and most simple example of the great natural law called the Conservation of Energy. The importance of reversibility as a test of maximum efficiency will be seen more fully hereafter.

Again, we can store up energy and use it as required when it is inconvenient to resort to any of the usual sources. For example, by a few turns of the watch key we store energy in the mainspring which is supplied at a regular rate to the watch throughout the day. So the hydraulic accumulator (Part V.) receives energy from the pumping engines and supplies it at irregular intervals to the hydraulic machines which lift weights and move gates in a dockyard or work the guns in a ship of war.

A large part of what follows in the present work is merely a development of what has been said here: in the succeeding chapters of the present division we consider machines comprising solid elements only, while in a future division we shall consider the transmission and conversion of energy by means of fluids.

EXAMPLES.

1. A waggon weighs 2 tons and its draught is $\frac{1}{50}$th of its weight. Find the work done in drawing it up a hill 1 in 20, half a mile long. Find also how long three horses will take to do it supposing each horse to work at the rate of 16,000 foot-pounds per minute.

Work done = 370 ft.-tons. Time occupied = 17′ 15″.

2. A force of 10 lbs. stretches a spiral spring 2″, find the work done in stretching it successively 1″, 2″, 3″, &c., up to 6″. *Ans.* $2\frac{1}{2}$, 10, $22\frac{1}{2}$, 40, $62\frac{1}{2}$ and 90 inch-lbs.

3. Find the H.-P. required to draw a train weighing 200 tons at the speed of 40 miles an hour on a level, the resistance being estimated at 20 lbs. per ton. Find also the speed of the train up a gradient of 1 in 100, the engine exerting the same power. H.-P. required = $426\frac{2}{3}$. *Ans.* Speed up the incline = 18·87 miles per hour.

4. The resistance of H.M.S. "Iris" at 17 knots is estimated at 40,000 lbs., what will be the H.P. required simply to propel the ship. Find also in inch-tons the moment, on each of the twin screw shafts, equivalent to this power, the revolutions being 80 per minute. *Ans.* H.-P. required = 2088. Moment on each shaft = 367 inch-tons.

5. The curve of stability of a vessel is a common parabola, the angle of vanishing stability 70″, and the maximum moment of stability 4,000 ft.-tons. Find the statical and dynamical stabilities at 30°. *Ans.* Statical stability = 3918 ft.-tons. Dynamical stability = 1283 ft.-tons.

6. Verify the principle of work, neglecting friction, in:—(*a*) The differential pulley (Art. 59). (*b*) A pair of 3-sheaved blocks. (*c*) The hydraulic press (Art. 62).

7. From the results in question 3, p. 112, deduce the crank efforts for the given positions of the piston and the mean crank effort, supposing the effective steam pressure on the piston 20 tons and neglecting friction.

Crank effort at quarter stroke in the $\begin{cases}\text{forward stroke} = 18\cdot4 \text{ tons.}\\ \text{backward ,, } = 16\cdot6 \text{ tons.}\end{cases}$ Mean = 12·74 tons.

8. Show that the efficiency of a machine is equal to the velocity ratio divided by the force ratio.

SECTION II.—UNBALANCED FORCES (KINETICS).

99. *Kinetic Energy of a Particle.*—We now proceed to consider the cases in which efforts or resistances arise from the changes of velocity of the parts of a system, which changes thus become a source of energy or require energy in order to produce them. The commonest observation is sufficient to show the importance of such cases: a cannon ball possesses a great power of doing work, and a railway train requires energy to be exerted by the steam to obtain the requisite speed, quite irrespectively of that necessary to maintain the speed when once produced.

First, suppose a weight under the action of gravity only. Unless it be supported by a vertical force exactly equal to the weight it will fall with a gradually increasing velocity. Let it be wholly unresisted, let it start from rest and fall through a height h, then we know that it will acquire a velocity v given by the formula

$$v^2 = 2gh,$$

where g is a number which for velocities in feet per second ranges from 32·117 at the equator to 32·227 at the pole, and having intermediate values at other points on the earth's surface according to the intensity of gravity at the point. The average value 32·2 is usually adopted for this important constant, and the height h is called the "height due to the velocity."

During the whole fall, the weight W of the body has been exerting an effort upon it which overcomes an equal resistance occasioned by the change of velocity which is taking place; thus an amount of energy has been exerted, and an amount of work done equal to Wh. Resistance of this kind is of the reversible kind, for if we imagine the weight, after reaching the ground, projected up again with the same velocity, it will, if wholly unresisted, attain the height from which it originally fell. Hence we describe the weight as possessing energy, and the amount it possesses when moving with velocity v is

$$Wh = \frac{Wv^2}{2g}.$$

Energy due to motion is called Kinetic Energy, to distinguish it from that kind of energy considered previously, which is a consequence of the relative position of the parts of a system, and which is called Potential Energy. The kinetic energy of a body depends on its velocity only, not on the direction of its motion nor on the way in which its motion has been produced; and the energy exerted in changing the motion of a body is always represented by an exactly equivalent increase of kinetic energy, whether this effort be uniform or variable, or whether its direction coincide with the direction of motion or not. To illustrate this, consider the following cases.

(1) Let the body move in a straight line under the action of a force P, in that line let it start with velocity V, and after moving through a space x let its velocity be v, then, it is shown in works on elementary dynamics, that v is given by a formula which may be written

$$Px = \frac{Wv^2}{2g} - \frac{WV^2}{2g}.$$

Now, the left-hand side of the equation is the energy exerted by P, and the right-hand side is the increase of kinetic energy of the body.

If P be a resistance instead of an effort, then work is done at the expense of the kinetic energy which is now diminished. If P be variable we must represent it graphically by a curve as in Art. 90, and it should be especially remarked that the ordinate of the curve of areas deduced as in Art. 31 will, on affixing a suitable scale and measuring the ordinates from a suitable base line, represent the height due to the velocity of the body.

(2) Let the body be constrained by means of a smooth guiding curve to move along a given path by a force P in any direction, then the energy exerted by P is the same as that exerted by the resolved part of P in the direction of motion. But this resolved part accelerates the motion just as if the body moved in a straight line, so that this case is reduced to the last.

(3) The pressure on the guiding curve will be the difference between the normal component of P and the force necessary to change the direction of P's motion. If the two are equal the guiding curve may be removed, and we obtain the case where the body moves freely, as in the case of a projectile in vacuo.

100. *Partially Unbalanced Forces. Principle of Work.*—Again, the effect which is changing the motion of the body may be partly balanced by an external resistance to which the body is subject. If this be the case we can imagine it separated into two parts, a part which is, and a part which is not, balanced. The energy exerted by the first is employed in overcoming the external resistance, while that exerted by the second is employed in increasing the kinetic energy of the body. Or the resistance may be greater than the effort, then the excess is overcome at the expense of the kinetic energy of the body, the velocity of which now diminishes.

In the present treatise we shall use the phrases "energy exerted" and "work done" only in reference to efforts and resistances other than those due to inertia, subject to which convention, we may state the principle of work as applied to cases where the forces are partially unbalanced, as follows—

$$\text{Energy exerted} = \text{Work done} + \text{Change of Kinetic Energy.}$$

In this statement the work done may be greater or less than the energy exerted. In the first case the change of kinetic energy is a decrease, in the second an increase.

Not only does this principle apply to a single body, but—subject to the observations of the preceding section—to a set of bodies mechanically connected in any way, provided that one of them be fixed to the earth; or, in other words, that a body of great mass like the earth be one of the set. When no one of the set predominates over the rest it is necessary to consider further how the kinetic energy should be reckoned: for the present, however, we shall suppose this condition satisfied.

A simple case is that of Atwood's machine. Let the descending weight P be greater than the rising one Q. Neglecting friction, the excess sets the two weights in motion. Let P descend through a distance y, then Q rises through the same distance, and therefore

$$\text{Energy exerted} = Py.$$
$$\text{Work done} \quad = Qy.$$

Let v be the velocity of the two weights; then supposing them to start from rest,

$$\text{Kinetic energy acquired} = (P+Q)\,\frac{v^2}{2g}$$

From principle of work

$$Py = Qy + \frac{(P+Q)v^2}{2g}; \quad \therefore\ v^2 = \frac{P-Q}{P+Q}\,2gy.$$

The law of increase of velocity is, therefore, the same as that of a body falling freely, but the rate of increase is less. This formula is the same as that obtained by other methods, and we have therefore here a verification of the principle of work.

101. *Kinetic Energy of the Moving Parts of a Machine.*—Instead of a single body, suppose we have a system of bodies, and we require to know the total kinetic energy of the system. The direct method is to find the energy of each separate particle of the system and add the results. In the particular case of a rotating rigid body we are able to express the result of the summation in a convenient and simple form. First consider a ring of small section rotating about an axis in the centre perpendicular to its plane. Every portion of the ring will move with the same velocity, v say, and the kinetic energy of the ring may, as before, be written $Wv^2/2g$.

We may express this another way, as follows:—If n be the revolutions per second, and a the radius, $v = 2\pi an$,

$$\therefore \frac{Wv^2}{2g} = W\,.\,\frac{4\pi^2n^2}{2g}\,a^2.$$

If the ring is not complete, but W is the weight of a portion which has the same centre of rotation, the expression will still hold.

Now, suppose we have a body consisting of a number of particles rigidly connected together, rotating about a centre O, at n revolutions per second.

Let the weights of the particles be w_1, w_2, w_3, w_4, etc.,
rotating about O at distances y_1, y_2, y_3, y_4, etc.

By adding together the results for each particle, we obtain for the kinetic energy of the system,

$$\frac{4\pi^2 n^2}{2g}(w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.})$$

Now suppose a is such a radius that

$$a^2 = \frac{w_1 y_1^2 + w_2 y_2^2 + w_3 y_3^2 + \text{etc.}}{w_1 + w_2 + w_3 + \text{etc.}},$$

then substituting, we may write

$$\text{Kinetic energy} = \frac{4\pi^2 n^2}{2g}(w_1 + w_2 + w_3 + \text{etc.})\, a^2 = \frac{4\pi^2 n^2}{2g} W a^2.$$

By this method we are always able to reduce any system of bodies to a ring, which ring is often called the *Equivalent Fly Wheel*, and the radius a is called the *Radius of Gyration*. The quantity Wa^2/g is usually called the *Moment of Inertia*, and denoted by the symbol I.

However numerous the particles are, the expression obtained above will hold, and so will be true if they are sufficient in number to make up a solid body. In a continuous body, the separate weights w_1, w_2, w_3, etc., must be taken indefinitely small and close together to get accurate results, and the results of the summation may be most conveniently arrived at by the use of the calculus. The quantity W/g is called the mass of the body, and but for the introduction of this factor the symbol I would have the same meaning as in Chap. XII. Hence all the results there given may be used here for thin plates simply by multiplication by the mass of a unit of area. In addition, the following simple cases will be sufficient. The fourth is a particular case of the second.

1. Solid cylinder rotating about its axis. $\qquad a^2 = \dfrac{r^2}{2}$
   Radius $= r$.

2. Rectangular parallelopiped rotating about an axis. Diagonal of either end = $2d$. $a^2 = \frac{d^2}{3}$

3. Sphere rotating about a diameter. Radius $= r$. $a^2 = \frac{2r^2}{5}$

4. Rod rotating about an axis perpendicular to it through one end. Length $= l$. $a^2 = \frac{l^2}{3}$

In other cases such as occur in practice, the body is generally too irregular and complex in form to render mathematical formulæ useful; we then apply the rule given in Ch. XII. for plane areas, which by a similar process can readily be extended to solids. That is to say, if I be the moment of inertia of a body about any axis, I_0 that about a parallel axis through the centre of gravity at a distance h,

$$I = I_0 + mh^2,$$

where m is the mass of the body. In applying this rule the body is cut up into portions to which the values just given apply exactly or with sufficient approximation, just as in the chapter cited.

In estimating the kinetic energy of a fly-wheel, which consists of rim, arms, and boss, since the rim is by far the most important part for storing energy, it is generally sufficient to consider it alone. If it be desired to take the remaining parts into account, an addition of about one-third the weight of the arms may be made to the weight of the rim. The combined effect of arms and boss is said to amount to an addition of, on the average, about 8 per cent. to the weight of the rim.

If the body have a motion of translation, combined with a motion of rotation about its centre of gravity, it will be shown hereafter that its total kinetic energy is the sum of that due to the translation and the rotation taken separately, so that the whole can be found by preceding rules. As an example of the use of this principle, consider the case of a ball rolling down an inclined plane, the ball and plane being sufficiently rough that slipping does not take place between them; and suppose the resistance to rolling, called the rolling friction, is insensible. In this case the whole energy due to the descent of the ball is employed in generating kinetic energy in the ball, which will be stored in it by virtue of its two motions of translation and rotation. Let V be the velocity of translation, A the angular velocity, r the radius of sphere; then since no slipping occurs $V = Ar$.

Let the ball descend through a vertical height h, then the energy exerted is Wh, equating which to the kinetic energy stored we obtain

$$Wh = \frac{WV^2}{2g} + \frac{WA^2a^2}{2g},$$

where a = radius of gyration is given by $a^2 = \frac{2}{5}r^2$.

$$\therefore Wh = \frac{WV^2}{2g} + \frac{W}{2g} \cdot \frac{2}{5}r^2 = \frac{7}{5}W\frac{V^2}{2g}.$$

$$\therefore V^2 = \frac{5}{7}2gh.$$

Thus the velocity of the ball will be less than if it simply slid down the plane without rotating in the proportion $\sqrt{5} : \sqrt{7}$.

The total kinetic energy of the moving parts of a machine in any position may be found by drawing a diagram of velocity for that position in the manner explained in Chaps. V. and VI. Each part may be divided into a number of small portions, and the centre of each portion may be laid down on the diagram, as explained on page 125. If now the diagram be imagined to represent a set of particles rigidly connected, of masses equal to those of the particles in question, the moment of inertia of those particles about the pole of the diagram must be the total kinetic energy required; the radius vector of each particle representing the velocity of the corresponding portion.

102. *Conservation of Energy.*—The principle of work may also be stated in another form, which, though not so convenient in practical applications, is much employed by scientific writers. It has already been explained that, when there are no frictional resistances, the power of doing work (energy) exerted in doing a given amount of work is not lost but merely transferred from one place to another (Art. 98), while it appears from the present section that any energy exerted in changing the motion of a body is represented by an exactly equivalent amount of kinetic energy stored up in the moving body; hence it follows that in any dynamical system, which receives no energy from without and supplies none to external bodies, the total amount of energy is always the same if there be no frictional resistances. We express this by the equation

$$\text{Kinetic Energy} + \text{Potential Energy} = \text{Total Energy} = \text{Constant},$$

and call it the principle of the Conservation of Energy. In all actual motions frictional resistances occur which gradually absorb the energy, but we shall find hereafter that this process is accompanied by the generation of heat which is equivalent to the energy absorbed a fact which leads us to conclude that heat is a form of energy, so that the principle still holds good.

103. *Examples.*—Let us now illustrate and verify the principle by some examples.

(1) Suppose a weight suspended by a string and oscillating under the action of gravity, forming the simple pendulum Og (Fig. 89*a*, p. 197), of length l.

Let the pendulum start from the position OA, and when it reaches the position Og let its velocity be v. Let the height of g above the tangent at the lowest point be y, and that of A, h, then we know that

$$v^2 = 2g(h - z),$$

which may be written, if W be the weight,

$$W\frac{v^2}{2g} + Wy = Wh.$$

Here the first term on the left-hand side is the kinetic energy of the weight and the second term Wy the potential energy, that is to say, the power of doing work which the weight possesses, in virtue of its height y above the lowest position it is capable of occupying. The sum of the two is the total energy Wh, and the motion consists in a continual interchange between the kinetic and potential energies. It is, of course, supposed that the resistance of the air is neglected; this is a resistance of the frictional kind, and continually absorbs energy from the weight which is thus at last reduced to rest.

The time of an unresisted double oscillation is shown in works on dynamics to be

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

when the oscillations are small enough to be sensibly isochronous.

Larger oscillations are sensibly slower, as shown by the approximate formula,

$$T = T_0 \left\{ 1 + \frac{\theta^2}{16} \right\} = T_0 \left\{ 1 + \frac{n^2}{52521} \right\},$$

where θ is the angle of swing in circular measure, and n is the same angle in degrees.

(2) The pendulum has been here supposed to be merely a heavy particle attached to the end of a string without weight. Let us next suppose a rigid body, the centre of gravity of which is g, oscillating about a centre O. Let v be the velocity of g, then

$$\text{Kinetic Energy} = W\frac{v^2}{2g} + W\frac{k^2A^2}{2g} \text{ (p. 214)},$$

where k is the radius of gyration about the centre of gravity, and A the angular velocity. If L be the length Og of the compound pendulum, this may be written

$$\text{Kinetic Energy} = \frac{Wv^2}{2g}\left\{ 1 + \frac{k^2}{L^2} \right\}.$$

The potential energy is the same as if the whole weight were concentrated at g; therefore, assuming the pendulum to start from the position OA, as before,

$$\frac{Wv^2}{2g}\left\{ 1 + \frac{k^2}{L^2} \right\} + Wy = Wh.$$

Comparing this with the result previously obtained for the simple pendulum, it is not difficult to see that the motion is identical if

$$l = \sqrt{L^2 + k^2}$$

which is the length of the simple equivalent pendulum.

(3) Take the case of a projectile unresisted by the air. Let A be the point from which the projectile starts with velocity V. If we draw through A a horizontal line AL, from this set up an ordinate $AH = h = V^2/2g$, and then draw a horizontal line HK, this line will be the directrix of the parabola in which the projectile moves. When the projectile has reached any point in its path, which is at a height y from the ground and at which it has the velocity v, the total energy possessed by the projectile $= W\left(y + \dfrac{v^2}{2g}\right)$. This being

equal to that which it had at starting $= W\dfrac{V^2}{2g} = Wh$, $\dfrac{v^2}{2g} = h - y$, and so the projectile will, at every point of its path, have a velocity due to its having fallen from the directrix.

EXAMPLES.

1. The energy of 1 lb. of pebble powder is 70 foot-tons. Find the weight of charge necessary to produce an initial velocity of 1300 feet per second in a projectile weighing 700 lbs., neglecting the recoil of the gun and the rotation of the shot.

Wt. of powder required = 117 lbs.

2. In Example 1 suppose the gun fired at an elevation of 30°, and resistance of the atmosphere neglected, find the kinetic and potential energies of the shot at its greatest elevation. Also deduce the greatest elevation.

Horizontal velocity = velocity at highest point $= 1300\ \dfrac{\sqrt{3}}{2}$,

Kinetic energy at highest point = 6150 ft.-tons,
Potential ,, ,, = 2050 ,,

$\dfrac{\text{Potential energy}}{\text{Wt. of shot}} = 6560{\cdot}6$ feet = maximum elevation.

3 A train is running at 40 miles an hour, find the resistance in pounds per ton necessary to stop the train in 1000 yards on a level. Also find the distance in which the train would be brought up by the same brake power on a gradient of 1 in 100, both when going up and when going down.

Resistance = 39·9 lbs. per ton.
Distance required to bring up the train when ascending the gradient ... ... = 640 yards.
When descending ... ... = 2280 ,,

4. The reciprocating parts of an engine running at 75 revolutions per minute weigh 25 tons, of which parts weighing 20 tons have a stroke of 4 feet and parts weighing 5 tons a stroke of 2 feet. Find the energy stored in the parts, assuming a pair of cranks OP, OQ at right angles and neglecting obliquity of connecting rod.

Velocity of parts attached to crank $P = PN\dfrac{V}{OP}$.

,, ,, ,, $Q = QM\dfrac{V}{OP}$.

Where V is the velocity of the crank pin and PN, PM are perpendiculars on the line of centres.

Assuming weights attached to these cranks each equal W. Then energy stored in these weights together $= \dfrac{WV^2}{2g}(PN^2 + QM^2)\dfrac{1}{OP^2} = \dfrac{WV^2}{2g}$.

In example, total kinetic energy = 40·7 ft.-tons.

5. One weight draws up another by means of a common wheel and axle. The force ratio is 1 to 8 and the velocity ratio is 9 to 1. Find the revolutions per minute after 10 complete revolutions have been performed, neglecting frictional resistances and the inertia of the wheel and axle. Diameter of axle 6 inches.

Revolutions per second = 2·14.

6. In Ex. 1 suppose the gun rifled so that the projectile makes 1 turn in 40 diameters, find the additional powder charge required to provide for the rotation of the shot, the diameter of shot being 12 inches and the radius of gyration $4\frac{1}{2}$ inches.

Additional powder required = ·407 lb.

7. A disc of iron rolls along a horizontal plane with velocity 15 feet per second, and comes to an incline of 1 in 40 on to which it passes without shock. Find how far it will ascend the incline, neglecting friction.

Distance along incline it will run = 209·6 feet.

8. In Ex. 5 suppose the weight of wheel = weight of axle, and the two together = sum of weights, obtain the result, taking account of the inertia of the wheel and axle.

After 10 revs. it will rotate at 1·22 revs. per second.

9. Assuming that when a vessel rolls her dynamical stability is the same as when steadily heeled over (Art. 91), and neglecting that part of her kinetic energy which is due to the motion of her centre of gravity (Art. 101), write down her equation of energy (Art. 103). If the curve of stability be a true curve of sines, show that the vessel will keep time with a pendulum of length l swinging through k times her angle of heel, where

$$k\theta_0 = \pi \; ; \; l = \frac{r}{\sqrt{k}},$$

θ_0 being her angle of vanishing stability and r her radius of gyration.

Note.—The rolling is here supposed unresisted. Observe that the deviation from isochronism is much greater than in a simple pendulum swinging through the same angle, k being always greater than unity.

REFERENCES.

Numerous elementary examples on the application of the Principle of Work will be found in Twisden's *Practical Mechanics*.