PART IV.—STIFFNESS AND STRENGTH OF MATERIALS.

147. *Introductory Remarks.*—The straining actions which tend to cause a body or a structure to separate into parts A and B in the manner explained in Part I. are counteracted by the mutual action between the parts at each point of the real or ideal surface which divides them. In other words (see Art. 1), a STRESS exists at each point of the surface, the elements of which are A's action on B and B's action on A. If we consider the total amount of the stress, these elements each form one element of the straining actions on A and B respectively; but for our present purpose it is needful to consider, not the total amount, but the intensity of the stress. This in general varies from point to point, and at each point is measured by the stress per unit of area on any small area enclosing the point.

Either element (say A) may be regarded either as A's action *on* B, or as the resistance which A offers to the action *of* B, in other words stress may be regarded in two aspects, either as the cause tending to produce separation into parts, or as the resistance to such separation. It is under the first aspect that we shall chiefly regard stress, generally employing the word resistance when we wish to express the second idea. Stress then may be described as the straining action on the ultimate particles of a body. Conversely a straining action as defined in Ch. II. may also be described as the "resultant stress" on the section we are considering.

If the stress exceeds a certain limit, separation into parts occurs, and this limiting intensity of stress varies for different materials and measures the Strength of the material.

Accompanying the tendency to separation into parts we invariably find changes of dimension in the body and each of its parts, for no body in nature is absolutely rigid. Such changes are called STRAINS, and are of two kinds, changes of volume and changes of figure, or, in other words, changes of size and changes of shape. Changes of size in any dimension are measured by the ratio of the change to the original dimension considered; changes of shape consist in the alteration of relative angular position or distortion of the parts considered, and are measured by the absolute magnitude of the alterations in question. In most cases which concern us, both kinds of change take place together and are of exceeding smallness.

The strains produced in solid bodies by the action of forces depend on the nature of the material and on the kind of stress.

Bodies are either solid or fluid. A fluid may be defined as material which offers no resistance to change of shape, but only to change of volume, especially diminution of volume, so that any distorting stress, however small, will cause indefinite change of shape if sufficient time be allowed. On the other hand a solid body will resist a distorting stress for an indefinite time, provided that stress be not too great. In a fluid body at rest only one kind of stress can exist, namely, a pressure equal in all directions; hence often called "fluid" stress.

There are two extreme conditions in which a solid body may exist, the Elastic state and the Plastic state. Elasticity is the power a body possesses of returning to its original shape and dimensions after the forces which have been applied to it are removed. All bodies possess this property to a greater or less extent, and most (perhaps all) possess it to a great degree of perfection if the strains to which it has been exposed are not too great. Even so unlikely a material as soft clay is elastic if the force applied to it is very small. This may be shown by suspending a long filament, formed by forcing clay through a small orifice, by one end and twisting the other, to which an index is attached: on release the index returns to its original position.* In perfectly elastic material the recovery of size and shape on removal of the forces is complete, unless the tempera-

* See Robison's *Mechanical Philosophy*, vol. I., page 375. The original observation is said to have been made by Coulomb. Though frequently quoted it does not appear to have been verified.

ture has meanwhile varied: and the materials of construction may be regarded as approximately satisfying this condition, provided a certain limit stress be not overpassed. This is called the Elastic Strength of the material. It is also described as the "limit of elasticity."

When, on the other hand, the forces applied to the body are comparatively great, the material in many cases approaches the other extreme condition, the plastic state. In this state any forces causing a distorting stress beyond a certain limit, and so applied that disruption does not occur, will produce indefinite distortion, so that the material behaves like a fluid. Thus soft clay, lead, copper, or even malleable iron may be moulded into different shapes or drawn out into wire. In intermediate cases a body may exhibit the properties of the elastic and the plastic states combined.

We commence by studying matter in the perfectly elastic state. There are two different kinds of elasticity, — Elasticity of Volume and Elasticity of Figure. A fluid possesses the first kind only, since by definition it has no power of resisting change of shape: the second is characteristic of solids. In general a change of dimensions involves both a change of size and a change of shape, so that both kinds of elasticity are called into play together. In perfectly elastic material the strain produced by a given stress is always proportional to the stress, being found by multiplying the stress by a co-efficient or "modulus" of elasticity, depending on the kind of stress and the nature of the material. This property having been discovered by Robert Hooke is known as Hooke's Law. Further, if the stress be relaxed in the slightest degree the strain diminishes, that is, in perfectly elastic material, the elastic forces are completely "reversible" (p. 205).

The magnitude of the stress produced by the action of given forces upon a body depends very much on whether they are applied all at once or are supposed to be at first very small and gradually to increase to their actual amounts. The next four chapters will be limited to the action of a gradually applied load on perfectly elastic material. The experimental part of the subject is placed in the last chapter (Ch. XVIII.), but should be referred to constantly as required.

CHAPTER XII.

SIMPLE TENSION, COMPRESSION, AND BENDING OF PERFECTLY ELASTIC MATERIAL.

SECTION I.—TENSION AND COMPRESSION.

148. *Simple Tension.*—The effect of forces acting on a bar has already been explained in Chapter II. to consist in the production of certain straining actions which we called Tension, Compression, Bending, Shearing, and Twisting, and we now go on to consider the changes of form and size which the bar undergoes and the stress produced at each point on the supposition that the material of the bar is perfectly elastic.

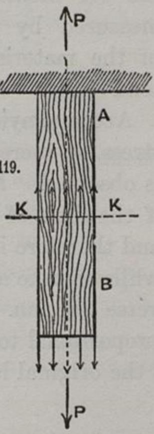


Fig. 119.

Let AB (Fig. 119) be a bar subjected to the action of equal and opposite forces applied at the ends in the same straight line. At any transverse section KK there will be a tendency to separate into two parts A, B, which is counteracted by a mutual action between the parts at each point of the section, which, in accordance with our previous definitions, is called the Tensile Stress at the point. The total amount of the stress will be P; but the intensity will depend on the area of the section (A), so that P/A is the mean intensity of stress or the stress per unit of area. The stress may be the same at all points of the section. We then say it is uniformly distributed, and the intensity at all points $= P/A$.

In order that the intensity of the stress may be the same at every point of every transverse section of the bar, it is *theoretically* necessary that the load P should be applied in a uniformly distributed manner all over the end B. Then if the material is perfectly homogeneous each elementary portion of KB will be strained alike, and the uniformly distributed load at B will be balanced by a uniformly distributed stress over any section KK. In such a case the line of action of the resultant of the applied load P passes through the centre of gravity or centre of position of the transverse section KK. Unless it does so the equilibrium of the portion KB is not possible by means of a uniformly distributed stress over the section. But from experience it appears that for uniformity of stress it is not absolutely necessary for the load to be applied in this distributed manner. It may be applied for instance by pressure on a projecting collar; and yet *if the line of application of the load traverses the centre of gravity of the sectional area*, the material, if homogeneous, will so yield as *practically* to produce at a section a little distant from the place of application of the load a stress of uniform intensity. This is a particular case of a principle which will be further referred to hereafter.

If the applied load is increased, the stress on the section is proportionately increased, until at last the material yields under it and the bar breaks. If $W =$ breaking load, the corresponding stress measured by W/A is a quantity which depends on the nature of the material. If we call it f, then the breaking or ultimate load $= Af$.

Accompanying the application of the load producing a tensile stress, an increase of length and diminution of transverse dimension is observed. In metallic bodies the alterations are exceedingly small if the limit of elasticity is not exceeded (see Table II., page 437), and therefore in estimating the stress on the section it is not worth while to take account of the slight alteration in the area of the transverse section. Under the same load the change of length is proportional to the length. If x be the total change of length, and l the original length, then the extension per unit of length is

$$e = \frac{x}{l}.$$

On account of the smallness of e it is immaterial whether l is taken as the original or altered length of a metallic bar.

As already stated (Art. 147), it is usual to restrict the word *strain* to mean the alteration of the dimension and form which bodies undergo and to use the word *stress* when referring to the elastic forces which accompany the strain. Thus e is a measure of the tensile strain produced in the bar, whilst p is a measure of the accompanying tensile stress. Since by Hooke's law the extension of the bar is proportional to the force producing it, it follows that the strain is proportional to the accompanying stress. Thus p and e may be connected by some constant the value of which depends on the nature of the material. We may write

$$p = Ee,$$

in which E is called the modulus of elasticity of the material, which, when the stress p is expressed in pounds per square inch, has for wrought iron a value of about 28,000,000.

Putting for e its value x/l, we have the general relation,

$$\frac{p}{E} = \frac{x}{l}.$$

The transverse strain, that is, the contraction per unit of transverse dimension, is from one third to one fourth the longitudinal strain.

149. *Work done in Stretching a Rod.*—Having found the relation between the tensile stress and strain, we will now consider how much work must be done in order to stretch it.

Let a load of gradually increasing amount be applied to the bar, the bar will stretch equal amounts for equal increments of load: or the elongation of the bar will for all loads be proportional to the load. This may be represented graphically. Suppose the load P' produces the extension shown, greatly exaggerated, by B_0B' (Fig. 120), and we set off an ordinate $B'N'$ to represent P' on some scale, and do that for any number of loads, taking, for example, BN to represent P, which produces the extension $B_0B = x$; then all the points N will lie on the sloping line passing through B_0. Having done this, the area of the triangle B_0BN will represent the quantity of work done on the bar in stretching it the amount $B_0B = x$. Thus

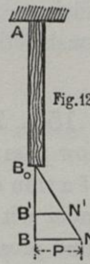


Fig. 120.

$$\text{Work done} = \tfrac{1}{2}Px.$$

The energy thus exerted is stored up in the stretched bar, and may be recovered if the bar is allowed under a gradually diminished load to contract. In the perfectly elastic bar the contraction will be exactly the same as the extension, and there will be no loss of energy in stretching it. In other words the elastic forces are "reversible." But if the elasticity is imperfect, some of the energy expended in stretching the bar is employed in producing molecular changes, as for example, change of temperature. On contraction this amount of energy will not be restored.

We can express the work done in stretching the bar otherwise. For P put its value $= pA$, and for x its value $= pl/E$. The substitution of these values of P and x will give

$$\text{Work done} = \tfrac{1}{2}pA \cdot \frac{pl}{E} = \frac{p^2}{E}\frac{Al}{2} = \frac{p^2}{E} \times \tfrac{1}{2}\text{ volume.}$$

Thus the work required to produce a given stress p is proportional to the volume, or, what is the same thing, to the weight, of the bar.

If the stress produced is increased up to the elastic limit, or, as it is often called, the *proof stress*, so that $p = f$, then $\frac{f^2}{E} \cdot \frac{\text{Volume}}{2}$ expresses the greatest amount of work which can be done on, and stored in the bar without injuring it or impairing its elasticity.

This is called the *resilience* of the bar. The quantity f^2/E, the value of which depends on the nature of the material, is called the *modulus of resilience*, and, as we shall see hereafter, furnishes a measure of the resistance of the material to impact in those cases in which the limits of elasticity are not exceeded (Chap. XVI.). A table of coefficients of strength and elasticity for materials commonly used in construction will be found at the end of Chapter XVIII.

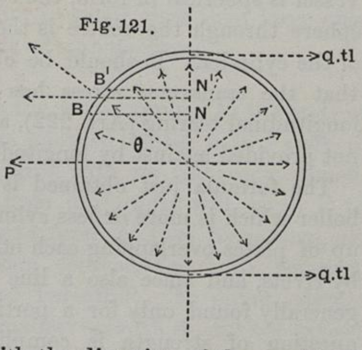
150. *Thin Pipes and Spheres under Internal Fluid Pressure.*—We now pass on to consider an important case of simple tension: that of a thin cylindrical shell subjected to internal fluid pressure. A cylinder with rigid ends and a sphere are cases of a vessel under internal fluid pressure which tends to preserve its form. The equilibrium in these two cases is stable, for if the vessel suffers deformation the internal pressure tends to make it recover its original true form. Vessels the sides of which are flat tend, by bulging, to assume these forms, and the tendency must be resisted by staying the surfaces in some way. If, as generally happens, there is acting also an external fluid pressure less than the internal, then, in what follows, the intensity of the internal pressure must be taken to be the excess of the internal over the external pressure.

Let p be the intensity of the fluid pressure in pounds per square inch, d the diameter, t the thickness of the shell, and l the length of the cylinder. Suppose in some way that the ends are maintained perfectly rigid, and for convenience let them be flat. There are two principal ways in which the strength of the shell can be estimated.

First, consider the tendency to tear asunder longitudinally, parallel to the axis of the cylinder. Imagine the cylinder divided into two parts by a plane passing through the axis of the cylinder. On each half cylinder there is a pressure P due to the resultant fluid pressure on that half which tends to produce a separation at the section imagined. The separation is prevented by the resistance to tearing which the metal of the shell offers, calling into action a uniform tensile stress at the two sections made by the imaginary plane through the axis of the cylinder.

Let q = intensity of tensile stress produced; then the area over which the stress acts being $2tl$, the total resistance to tearing is $q \times 2tl$, which must also be the tendency to tear $= P$.

Fig. 121.



In a transverse section take two points B, B' (Fig. 121) near together. The surface of the shell, $BB' \times l$, is acted upon by a normal pressure p per unit of area. The pressure $p \,.\, BB' \,.\, l$ may be taken to act in a radius drawn to the middle point of BB', making an angle θ with the direction of the resultant force P. The resolved part of this pressure in the direction of P

$$= pl \,.\, BB' \,.\, \cos\theta = pl \,.\, NN',$$

NN' being the projection of BB' on the plane of section. Summing up the pressures on all the small arcs BB', composing the semicircle, we obtain the total separating force,

$$P = pl \,.\, \Sigma NN' = p \,.\, l \,.\, d,$$

$$\therefore\ 2qtl = pld,$$

$$\text{or } q = \frac{pd}{2t};$$

thus the tensile stress is directly proportional to the diameter, and inversely proportional to the thickness of the cylindrical shell. For greatest accuracy d should be taken as the mean of the internal and external diameters. The formula just obtained is true only when the thickness is small compared with the diameter. If t is large, the stress is not uniform over the section; the formula will then give the mean stress if d be understood to mean the internal diameter.

We next consider the tendency for the cylinder to tear across a transverse section. The total pressure on each end of the cylindrical shell is the separating force, and the resistance to separation is due to the tensile stress, q' suppose, called into action over the annular area $\pi d \,.\, t$ of the transverse section.

$$\therefore \pi dt \,.\, q' = \frac{\pi}{4} d^2 p\,; \text{ or } q' = \frac{pd}{4t}.$$

This is just half the stress on the longitudinal section. If the vessel is spherical in form, the stress produced on all sections of the sphere through the centre is the same as at the transverse section of the cylinder. It should be observed that we have here assumed that the transverse stress has no influence on the resistance to longitudinal tearing (Art. 222), and that the pressure on the ends is not provided against by longitudinal stays.

The formula just obtained is used to estimate the strength of a boiler which is more or less cylindrical; but since the boiler is made up of plates overlapping each other, connected together at the edges by rivets, and since also a line of rivets in a longitudinal section is generally found only for a portion of the length of the boiler, the question of strength is complicated. But a longitudinal section through the greatest number of rivet holes is the weakest section, and if for q we write f, where f is a co-efficient of strength to be determined from experience, the value of it depending, amongst other things, on the form of joint, then the formula

$$p = \frac{2ft}{d}, \text{ or } t = \frac{pd}{2f}$$

may be used as a semi-empirical formula to determine the greatest pressure which can be employed in a given boiler, or the thickness of metal required to sustain a given pressure. The value of the co-efficient for iron boilers with single rivetted joints is about 4,000 lbs. per square inch, or, when double rivetted, as is usual in large boilers, 5,500. With steel the value is about one-third greater.

151. *Remarks on Tension.*—The results obtained in the present section are, strictly speaking, only applicable when the piece of material considered is of uniform transverse section, but they nevertheless may be used when the transverse section is variable, provided the rate of variation be not too great and the other conditions mentioned are strictly fulfilled. The intensity of the stress is then different at different parts of the bar, varying inversely as the transverse section, and in determining the elongation this must be taken into account.

In many cases of tension the effect of the weight of the tie and other circumstances introduces an additional stress, the amount of which is often imperfectly known. This is allowed for either by making a certain addition to the theoretical diameter or by the use of a factor of safety adapted to the particular case. On the other hand it also often happens, as in the case of ropes for example, that the strength of the material is greater in small sizes than large ones for reasons connected with the mode of manufacture.

152. *Simple Compression.*—When the forces applied to the ends of a bar act in a direction towards one another the bar is in a state of *compression.* If the bar is long compared with its transverse dimensions, then any slight disturbance from uniformity will cause it to bend sideways under the compressive force, and we have then, *not* simple compression, but compression compounded with bending, an important case to be considered hereafter. To obtain simple compression the ratio of length to smallest breadth should not exceed certain limits which depend on the nature of the material, viz., cast iron 5 to 1, wrought iron 10 to 1, steel 7 to 1. Further, it is necessary that the material be perfectly homogeneous and that the line of action of the load should be in the axis of the bar. Then the results we have obtained for simple tension apply to this case of simple compression

$$p=\frac{P}{A},$$

and the strength of the column is given by $P=Af$, where f is the co-efficient of strength. The compression x which the column undergoes is connected with the stress by the equation

$$p=E\frac{x}{l}.$$

The modulus of elasticity E would, in a perfectly elastic body, be the same as for tension. In actual materials it sometimes appears to be less; but within the elastic limit only slightly less.

EXAMPLES.

1. A rod of iron 1 inch in diameter and 6 feet long is found to stretch one sixteenth inch under a load of $7\frac{1}{2}$ tons. Find the intensity of stress on the transverse section and the modulus of elasticity in lbs. and tons per square inch.

Stress = 21,382 lbs. = 9·55 tons.

Modulus of elasticity = 24,631,855 lbs. = 10996·4 tons.

2. What should be the diameter of the stays of a boiler in which the pressure is 30 lbs. per square inch, allowing one stay to each $1\frac{1}{2}$ square feet of surface and a stress of 3,500 lbs. per square inch of section of the iron? *Ans.* $1\frac{1}{2}$ inches.

3. In example 1 find the work stored up in the rod in foot-pounds. *Ans.* $43\frac{3}{4}$.

4. If in the last question the rod were originally 2″ diameter and half its length were turned down to a diameter of 1″. Compare the work stored in the rod with the result of the previous question.

Ratio = $\frac{5}{8}$.

5. In Example 1 assume the given load of $7\frac{1}{2}$ tons to be the proof load; find the modulus of resilience. *Ans.* 18·56 in inch-lb. units.

6. Find the thickness of plates of a cylindrical boiler 4′ 2″ diameter to sustain a pressure of 50 lbs. per square inch, taking the co-efficient of strength of plate at 4,000 lbs. *Ans.* $\frac{5}{16}$″.

7. A spherical shell 4′ diameter $\frac{1}{4}$″ thick is under internal fluid pressure of 1000 lbs. per square inch. Find the intensity of stress on a section of the sphere taken through the centre. *Ans.* 48,000 lbs. per square inch.

8. Find the necessary thickness of a copper steam pipe 4″ diameter for a steam pressure of 100 pounds above the atmosphere, the safe stress for copper being taken as 1000 lbs. per square inch. *Ans.* ·2″.

9. A circular iron tank, diameter 16 feet, with vertical sides $\frac{1}{2}$″ thick, is filled with water to a depth of 12 feet: find the stress on the sides at the bottom. How should the thickness vary for uniform strength throughout? *Ans.* 1024 lbs. per square inch.

10. What length of iron suspension rod will just carry its own weight, the stress being limited to 4 tons per square inch, and what will be the extension under this load? *Ans.* 2,700 feet.

11. The end of a beam 10″ broad rests on a wall of masonry; if it be loaded with 10 tons what length of bearing surface is necessary, the safe crushing stress for stone being 150 lbs. per square inch. *Ans.* 15″.

12. Find the diameter of bearing surface at the base for a column carrying 20 tons, the stress allowed being as in the last question. *Ans.* 20″ nearly.

13. Compare the weight of the shell of a cylindrical boiler with the weight of water it contains when full. *Ans.* Ratio $=55p/f$.

Section II.—Simple Bending.

153. *Proof that the Stress at each Point varies as its Distance from the Neutral Axis.*—The nature of the straining action producing bending has been sufficiently explained in the third section of Chapter II., and we shall now consider the kind of stress which results on the ultimate particles of a solid bar of uniform transverse section and of perfectly elastic material. The bar is supposed symmetrical about a plane through its geometrical axis, and the bending is supposed to take place in this plane which may be called the Plane of Bending.

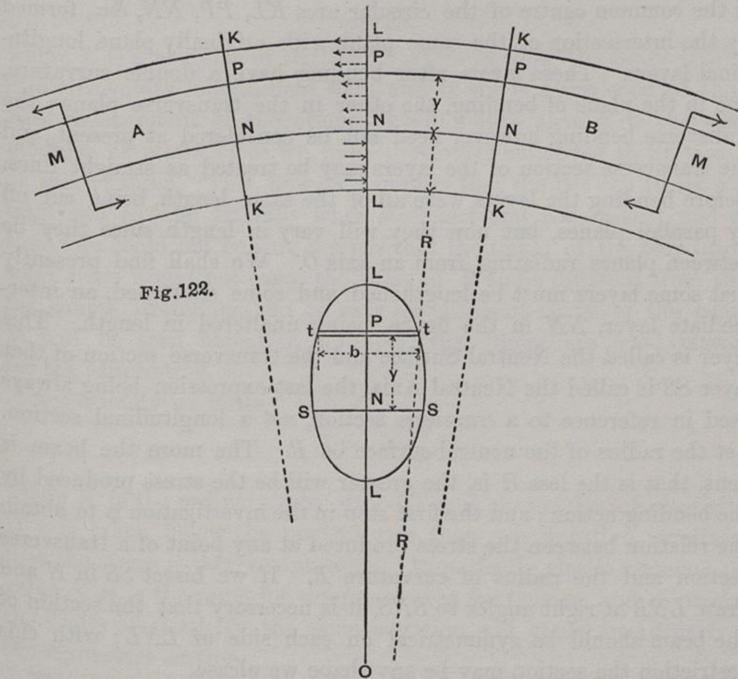


Fig.122.

In the first instance the bending is supposed to be "simple," that is, it is not combined with shearing as is most often the case in practice, but is due to a uniform bending moment (see Art. 21). The curvature of the beam is then uniform, that is to say, it is bent into a circular arc. The investigation consists of three parts.

Fig. 122 shows a longitudinal section AB and a transverse section LL through the centre of the beam; by symmetry it follows that if the bending moment be applied to both ends in exactly the same way, that transverse section, if plane before bending, will be still plane after bending, for there is no reason for deviation in one direction rather than another. It will be seen presently that if the bending moment be applied to the ends of the beam in a particular way all transverse sections will be in the same condition, and we may therefore assume that not only the central section, but any other sections KK we please to take, will remain plane notwithstanding the bending of the beam. All such sections, if produced, will meet in a line the intersection of which by the plane of bending will be a point O which is the common centre of the circular arcs KL, PP, NN, &c., formed by the intersection of the same plane with originally plane longitudinal layers. These layers after bending have a double curvature, one in the plane of bending, the other in the transverse plane; the transverse bending however need not be considered at present, and the transverse section of the layers may be treated as straight lines. Before bending the layers were all of the same length, being cut off by parallel planes, but now they will vary in length since they lie between planes radiating from an axis O. We shall find presently that some layers must be lengthened and some shortened, an intermediate layer, NN in the figure, being unaltered in length. This layer is called the Neutral Surface and the transverse section of that layer SS is called the Neutral Axis, the last expression being always used in reference to a *transverse* section, *not* a longitudinal section. Let the radius of the neutral surface be R. The more the beam is bent, that is the less R is, the greater will be the stress produced by the bending action; and the first step in the investigation is to obtain the relation between the stress produced at any point of a transverse section and the radius of curvature R. If we bisect SS in N and draw LNL at right angles to SNS, it is necessary that the section of the beam should be symmetrical on each side of LNL; with this restriction the section may be any shape we please.

Now consider any layer PP of the beam between the planes LL and KK which is at the distance y from the neutral surface NN or neutral axis SNS. This layer will be curved to a circle whose radius is $R+y$, and it must undergo an alteration of length from NN which it had before bending, to PP which it now has. Thus the alteration of length per unit of length, that is, the strain $e=\dfrac{PP-NN}{NN}$, but since arcs are proportional to radii $\dfrac{PP}{NN}=\dfrac{R+y}{R}$,

$$\therefore \text{the strain } e=\frac{PP-NN}{NN}=\frac{y}{R}.$$

If the layer we are considering is taken below the neutral surface, the strain, which will then be compression, will be given by the same expression $e=y/R$, e and y both being negative.

Accompanying the longitudinal strain just estimated there must be a longitudinal stress proportional to the strain. Let p be the intensity of that stress, then

$$p=Ee,$$

where E is a modulus of elasticity. If we imagine the beam divided into elementary longitudinal bars, and if we imagine each of those bars independent of the others, it will follow that E is the same modulus of elasticity as we have previously employed in Section I. of this chapter. This, however, implies that the bar can freely contract and expand laterally when stretched and compressed, and we therefore could not be sure *a priori* that the union of the bars into a solid mass would not cause the value of E to be different from that for simple stretching, and to vary for different layers of the beam. It will be seen hereafter, however, that there are good reasons for the assumption.

Accordingly we write

$$p=E\,.\,\frac{y}{R},$$

where E is the ordinary (also called Young's) modulus of elasticity. If y is taken below the neutral axis then p is negative, signifying that the stress is now compressive. In perfectly elastic material the value of E is the same for compression as for tension, and so, within the limits of elasticity, the same equation will apply for all parts of the transverse section.

Thus the stress at any point of the transverse section of the bar is proportional to its distance from the neutral axis.

154. *Determination of Position of Neutral Axis.*—The second step in the investigation is to find the position of the neutral axis. That position is deduced by dividing the beam into two portions, A and B, by a section LL, and considering the horizontal equili-

brium of either portion, say B. The external forces acting transversely to the beam balance one another, but being vertical have no resultant in the horizontal direction of the length of the beam.

We have next to take account of the internal molecular forces which act at the section LL. Above the neutral axis the action of LA is a tendency to pull B to the left; but below the neutral axis, the tendency is to thrust B to the right. In order that it may remain in equilibrium, and not move horizontally, it is necessary that the total pull should equal the total thrust; or the total horizontal force at the section must be zero. To estimate the horizontal force, consider the force acting on a thin strip of the transverse section, of breadth b, and thickness t, distant y from the neutral axis. The thrust or pull on this elementary strip $= p \,.\, b \,.\, t$.

Summing the forces on all the strips composing the sectional area, we must have

$$\Sigma p \,.\, bt = 0;$$

but $p = Ey/R$ where E and R are the same for all strips of the section.

$$\therefore \frac{E}{R} \,.\, \Sigma bt \,.\, y = 0.$$

That is to say, the sum of the products of each elementary area into its distance from the neutral axis must be zero.

This can be true only if the axis passes through the centre of gravity of the section; for it is the same thing as saying that the moment of the area about the neutral axis is to be zero.

155. *Determination of the Moment of Resistance.*—The third and last step in the investigation is to obtain the connection between the bending moment applied, and the stress which is produced by it. Again, considering either portion, AL or BL, of the beam, say AL, the external forces on A produce a bending moment or couple, M, which has to be resisted by the internal stresses called into action at the section K; so that the total moment of these stresses must be equal to M. The moment of the resisting stresses, being a couple, may be estimated about any axis with the same result. For convenience we will estimate it about the neutral axis of the section.

Let us again consider the elementary strip of area bt, distant y

from neutral axis, on which the intensity of stress is p, the force, pull, or thrust, on this strip being pbt. The moment of the force $= p \,.\, bt \,.\, y$. Seeing that forces on all elementary strips, whether pull or thrust, all tend to turn the piece AL the same way, the total moment of the stresses will be found by summing all terms, $p \,.\, bty$, for the whole area of the section.

$$\therefore M = \Sigma p \,.\, bty.$$

Since $p = Ey/R$, substitute, and remember that E/R is the same for all strips, then

$$M = \frac{E}{R}\Sigma b \,.\, t \,.\, y^2.$$

In this formula the area of each strip has to be multiplied by the square of its distance from the neutral axis and the sum of the products taken. This, or an analogous sum, is of constant occurrence in mechanics, and has a name assigned to it. Σbty is the simple moment of an area about an axis. Σbty^2 may be called the moment of the second degree, but the common name is the *Moment of Inertia;* because a similar sum (differing only from this in involving the mass) occurs in dynamics under that name. To distinguish the two cases area-moment and mass-moment, the former is sometimes called the geometrical moment of inertia.

Let I denote the moment of inertia, so that $I = \Sigma bty^2$, the value of which for any form of section can be obtained by geometry, then

$$M = \frac{E}{R}I, \text{ or } \frac{M}{I} = \frac{E}{R},$$

thus connecting the curvature of the beam with the moment producing it. Having previously found $p/y = E/R$, we can now connect the moment with the stress by writing

$$\frac{p}{y} = \frac{M}{I}.$$

This equation may be employed to determine the strength of a beam to resist bending. The limit of strength is reached when either the greatest safe tensile stress on one side of the neutral axis, or the greatest safe compressive stress on the other side of the neutral axis is called into action. Thus in the equation $p/y = M/I$ we must put $p = f_1$, the co-efficient of strength under tension, or $p = f_2$, the co-efficient of strength under compression; and for y, either

y_1, the distance of the most remote point on the stretched side, or y_2, the distance of the most remote point on the compressed side, so that

$$M = \frac{f_1}{y_1}I, \text{ or } \frac{f_2}{y_2}I.$$

The strength of the beam, or maximum moment of resistance to bending, is measured by the least of these quantities.

y_1 or y_2 is readily determined from geometry, the form of the section of the beam being given. It may be most conveniently expressed as a fraction of the depth of the beam. Thus y_1 or y_2 may be put $= qh$, where the co-efficient q has different values. In a rectangular section $q = \frac{1}{2}$, in a triangular section $q = \frac{1}{3}$ or $\frac{2}{3}$, and so on.

Next to express the value of I. It will be found that whatever be the form of the section, I may always be written $= nAh^2$, A being the area of the section of the beam, h the depth in the direction of bending, and n a numerical co-efficient, the value of which depends on the form of the section.

For a rectangular section,

$$n = \tfrac{1}{12}, \text{ so that } I = \tfrac{1}{12}Ah^2,$$

„ elliptical or circular „

$$n = \tfrac{1}{16} \quad „ \quad I = \tfrac{1}{16}Ah^2,$$

„ triangular „

$$n = \tfrac{1}{18} \quad „ \quad I = \tfrac{1}{18}Ah^2,$$

and so on.

Therefore assuming q and n known, we can write

$$M = \frac{f}{qh}nAh^2 = f\frac{n}{q}\,.\,Ah,$$

a formula which shows that for sections in which n/q is the same, the moment of resistance to bending is proportional to the product of the area and depth of the beam. Sections with the same n and q are said to be of the *same type*. They are often, but not correctly, said to be *similar*.

In estimating the numerical value of M, care must be taken with the units. It is generally advisable to use the inch unit throughout.

156. *Remarks on Theory of Bending.*—In the foregoing theory of simple bending it is supposed

(1) That the bar is homogeneous and of uniform transverse section and perfectly elastic;

(2) That sections plane before bending are plane after bending, for which it is theoretically necessary that the bending moment should be uniform, and applied at the ends of the bar in a particular way;

(3) That longitudinal layers of the beam expand and contract laterally in the same way, as if they were disconnected from each other (see pp. 303, 401).

These assumptions are not obvious *a priori*, and require justification, which at the present stage of the subject we are not in a position to give: for the present it may be stated that if the material be homogeneous and perfectly elastic, the equations hold good even though the transverse sections and the curvature vary and however the bending moment is applied. The *strength* of the material, however, is not generally the same as if the layers were disconnected, and co-efficients of strength require therefore to be determined by special experiment on transverse strength (Art. 217).

157. *Calculation of Moments of Inertia.*—We have frequently to deal with beams of complex section, in which case to determine I it is convenient to divide the section up into simple areas, the I of each of which is known, and the total moment of inertia of the section will be the sum of these I's. In employing this process we require to know the relation between the moments of inertia of an area about two axes parallel to one another, one being the neutral axis. We make use of a general theorem which may be thus proved.

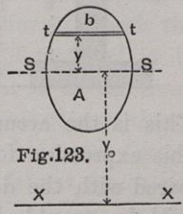


Fig. 123.

Let A be an area of which we know the moment of inertia about the neutral axis, SS (Fig. 123), and we require to know the moment of inertia about any parallel axis, XX, distant y_0 from SS. Dividing the area into strips of breadth b, and thickness t.

Moment of Inertia required $I = \Sigma b \,.\, t \,.\, (y + y_0)^2$.

$$= \Sigma bty^2 + 2y_0 \Sigma bt \,.\, y + y_0^2 \Sigma b \,.\, t.$$

Now Σbty^2 = moment of inertia about neutral axis, $\Sigma bt \,.\, y = 0$, because the neutral axis passes through the centre of gravity of the section, and Σbt = Area A.

$$\therefore I = I_0 + Ay_0^2.$$

The moment of inertia of an area about any axis is, therefore, determined by adding to the moment of inertia of the area about a parallel axis through the centre of gravity the product of the area into the square of the distance between the two axes.

This theorem, together with previously quoted values of I_0, will enable us to determine the following results, which will be useful in application to beams—

Rectangle about its base, ... $I = \frac{1}{3}Ay^2$.

Triangle ,, ,, ... $I = \frac{1}{6}Ay^2$.

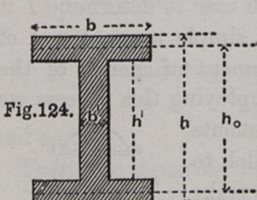
Triangle about a parallel to its base through vertex, $I = \frac{1}{2}Ay^2$.

Many other forms will divide up into rectangles or triangles, or both; for example, the moment of inertia of a trapezoid about the neutral axis may be readily determined by taking, for the area above the neutral axis, the I for a rectangle about one end, and triangles about the base. For the area below, a rectangle about one end and triangles about the vertex, and add the results.

158. *Beams of I Section with Equal Flanges.*—The case of a beam of I section is very important.

First, suppose the flanges of equal breadth and thickness, and the web of uniform thickness b', the depth being h', b being the breadth of the flange, and h the whole depth of the beam. The moment of inertia of the section may be taken as the difference of the moments of inertia of two rectangles (see Fig. 124).

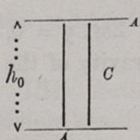
Fig.124.



$$I = b\tfrac{1}{12}h^3 - \tfrac{1}{12}(b - b')h'^3.$$

This is the accurate value of I, and when the flanges are thick this expression for I must be used; but if the flanges are thin compared with the depth, a very close approximation can be obtained with less trouble by supposing each flange to be concentrated in its centre line, and taking for the depth of the beam the distance h_0 to the centre of flanges.

If A = area of each flange and C = area of web,



then $I = A\frac{h_0^2}{4} + A\frac{h_0^2}{4} + \frac{1}{12}Ch_0^2 = \frac{h_0^2}{2}\left(A + \frac{C}{6}\right)$.

Putting $p = f$ and $y = \frac{1}{2}h_0$, in the formula $\frac{p}{y} = \frac{M}{I}$,

$$M = \frac{f}{\frac{1}{2}h_0}\frac{h_0^2}{2}\left(A + \frac{C}{6}\right) = fh\left(A + \frac{C}{6}\right).$$

This shows that, area for area, the web has only one-sixth the power resisting bending that the flanges have.

We previously deduced an approximate expression for the strength of an I beam, viz.,

$$M = Hh = fhA \text{ (see Art. 27),}$$

in which the effect of the web in resisting bending was neglected, the whole of the bending action being supposed to be taken by the flanges. The present formula shows the amount of the error involved in that assumption. In using this approximation when h the effective depth is reckoned from centre to centre of the flanges, two errors are made, one in supposing the resistance to bending of the web neglected, and the other, often much greater, in supposing the mean stress on the flange equal to the maximum, hence it is better to take for the effective depth

$$h = \frac{h_0^2}{h'},$$

where h' is the outside depth and h_0 the depth from centre to centre of flanges.

159. *Ratio of Depth to Span in I Beams.*—The formula just obtained for the moment of resistance of a beam of I section shows that the greater the depth of the beam and the thinner the web the stronger will the beam be for the same weight of material, or in other words that the best distribution of material is as far away from the neutral axis as possible. The practical limitation to this is that a certain thickness of web is necessary to hold the flanges together and give sufficient power of resistance to lateral forces and to the direct action of any part of the load which may rest on the upper flange. Hence the weight of web rapidly increases as the depth increases, and a certain ratio of depth to span is best as regards economy of material (see Ex. 17, page 325). This is especially important in large girders in which economy of material is the primary consideration. In smaller beams the proper ratio of depth to span is generally in great measure a question of stiffness, a part of the subject to be considered in Chapter XIII. The moment of resistance of I sections of practical proportions is generally about double that of a rectangular section of equal area. The straining actions on the web will be considered in Ch. XV.

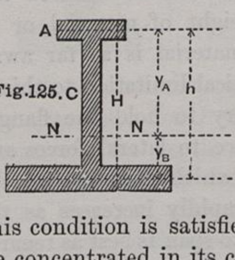
160. *Proportions of I Beams for Equal Strength.*—Materials in general are not equally strong under tension and compression, so that a beam whose section is symmetrical above and below the neutral axis will yield on one side before the material on the other side of the neutral axis has reached its limiting stress. Accordingly we might obtain a more economical distribution of material if we were to take some from the stronger side and put it on the weaker, so that the limiting tensile on one side and the limiting compressive stress on the other may be produced simultaneously. The section of the beam will be different above and below the neutral axis, which will not now be at the centre of depth of the beam, but in such a position that the distances to the top and bottom of the beam are in the proportion of the greatest allowed stresses to one another. The neutral axis in all cases must pass through the centre of gravity of the section.

Let f_A, f_B be the co-efficients of strength under compression and tension respectively, y_A, y_B distances of the most strained layer from neutral axis, then the beam will be strongest when

$$\frac{y_A}{f_A} = \frac{y_B}{f_B} = \frac{y_A + y_B}{f_A + f_B} = \frac{h}{f_A + f_B}.$$

For simplicity of calculation we will consider a beam (Fig. 125) in which the web is of uniform thickness throughout the depth, and so of rectangular section, and each flange also of rectangular section, and determine the relation which should hold between the areas of flanges and web for maximum strength of beam, and the moment of resistance to bending where this condition is satisfied. We will further suppose each flange to be concentrated in its centre line.

Fig.125.



Let A = area of compressed flange, B = area of stretched flange, C = area of web. Since the neutral axis is at the centre of gravity of the section, we obtain, by taking moments about that axis,

$$A \cdot y_A + C\frac{y_A - y_B}{2} = By_B;$$

or, substituting the previously given values of y_A and y_B,

$$Af_A + C\frac{f_A - f_B}{2} = Bf_B.$$

Supposing f_A and f_B known, A, B, and C must be such as to satisfy this relation. We have some liberty of choice between these quantities, and frequently find one of the flanges omitted, so producing a beam of T or ⊥ section.

In a cast-iron beam, where the resistance to compression is greater than for tension, the compressed flange A may be omitted. Putting $A = 0$ we get $C = \frac{2f_B}{f_A - f_B}B$, and supposing $\frac{f_A}{f_B} = 4$; $C = \frac{2}{3}B$, or $B = 1\frac{1}{2}C$. In a wrought-iron beam on the other hand f_A/f_B is about $\frac{3}{5}$, and the stretched flange B is the area to be omitted. Putting $B = 0$, we find $A = \frac{f_B - f_A}{2f_A}C = \frac{1}{3}C$.

Otherwise we may assume the depth and thickness of the web to be given (Art. 159), then the equation

$$Af_A + C\,.\frac{f_A - f_B}{2} = Bf_B,$$

furnishes a relation between the areas of the flanges. For example, in cast iron, if we assume $f_A = 4f_B$, we find

$$B = 4A + \tfrac{3}{2}C.$$

Having decided on the proportions between the parts of the section we can now calculate the moments of inertia and resistance. Still considering the flanges concentrated in their centre lines,

$$I = Ay_A^2 + By_B^2 + \tfrac{1}{3}C\,.\frac{y_A}{h}\,.\,y_A^2 + \tfrac{1}{3}C\,.\frac{y_B}{h}\,.\,y_B^2.$$

$$= Ay^2_A + By_B^2 + \tfrac{1}{3}C\,.\frac{y_A^3 + y_B^3}{h},$$

a result which admits of ready calculation. Further

$$\frac{M}{I} = \frac{f_A}{y_A} = \frac{f_B}{y_B} = \frac{f_A + f_B}{h},$$

whence we obtain

$$M = (f_A + f_B)\frac{I}{h}.$$

The calculation just now made is one which has been frequently given in dealing with beams of I section,* but in applying it to actual examples it should be remembered that the results are obtained on the supposition that the flanges are concentrated in their centre lines, and are consequently only approximate when the coefficients f_A, f_B mean the intensities of the stress at those centre lines, *not* at the surface of the beam where the stress is greatest. If, for example, F_A be the maximum stress on the flange A

$$F_A = f_A \cdot \frac{y_A + \frac{1}{2}t_A}{y_A},$$

where t_A is the thickness of the flange. The difference is especially great in the case of the larger flange of cast-iron beams, and the true ratio of maximum compressive and tensile stress is much less than it appears in the preceding article. On the other hand, in extreme cases, such as we are now considering, the stress may not be uniformly distributed along a line parallel to the neutral axis.

Extensive experiments were made on cast-iron beams by Hodgkinson, with the object of determining the best proportions between the flanges, with the result that rupture always took place by tearing asunder of the lower flange, unless it was at least six times the size of the compressed flange. This proportion is rarely adopted in practice, from the difficulties of obtaining a sound casting, and the necessity of having sufficient lateral strength. Nor is it certain that the proportions which are best for resisting the ultimate load are also best in the case of the working load; it is, in fact, probable that a smaller proportion is better even on the score of strength. If we take $f_A = 2\frac{1}{2}f_B$, instead of $4f_B$, we find

$$B = 2\tfrac{1}{2}A + \tfrac{3}{4}C,$$

which agrees more closely with practice. The ratio of maximum compressive and tensile strength is in this case about 2, which, according to some authorities, is the ratio of *elastic* strengths in the two cases.

In wrought-iron beams the areas of the flanges are usually equal, and this is correct if the elastic strength, and not the ultimate strength, is regarded as fixing the proper proportions, and if there be sufficient provision against the yielding of the top flange by lateral flexure. Small-sized beams of this kind are rolled in one piece, while large girders are constructed of iron or steel

* See Rankine's *Civil Engineering*, page 257.

plates and angle irons, rivetted together. Some of the forms they assume are shown in Plate VIII., Ch. XVIII.

In making calculations respecting girders, approximate methods may be used for preliminary tentative calculations, but should be checked by a subsequent accurate determination of the neutral axis and moment of inertia. A previous reduction of the section to an equivalent solid section is required when, as is often the case, all parts of the section do not offer the same elastic resistance to the stress applied to them, either because they are not sufficiently rigidly connected or from the material being different. This is especially the case in determining the resistance to the longitudinal bending of a vessel occasioned by the unequal distribution of weight and buoyancy already considered in Chapter III. On this important question the reader is referred to a treatise on Naval Architecture by Mr. W. H. White. In many cases of built-up girders the shearing action which generally exists has considerable influence, a matter for subsequent consideration (Ch. XV.). The effect of the weight of the girder itself has been considered in Ch. IV. (See also Ex. 13, p. 324, and Ex. 11, p. 372.)

161. *Beams of Uniform Strength.*—A beam of uniform strength is one in which the maximum stress is the same on all sections. For beams of the same transverse section throughout this can only be the case when the bending moment is uniform, but, by properly varying the section, it is possible to satisfy the condition however the bending moment vary. For this purpose we have only to consider the equation

$$M = f \cdot \frac{n}{q} \cdot Ah,$$

which must now be satisfied at all sections. Suppose

$$A = kbh,$$

where k is a numerical factor depending on the type of section, then

$$M = f \cdot \frac{nk}{q} \cdot bh^2.$$

All sections of the beam being supposed of the same type we have only to make Ah or bh^2 vary as M, that is as the ordinates of the curve of bending moments. The principal cases are—

(1) Depth uniform. Here the breadth must vary as the bending moment, whence it is clear that the curve of moments may be taken as representing the half plan of the beam.

(2) Sectional Area uniform. Here the depth must vary as the bending moment, that is, the curve of moments may be taken to represent the elevation or half elevation of the beam.

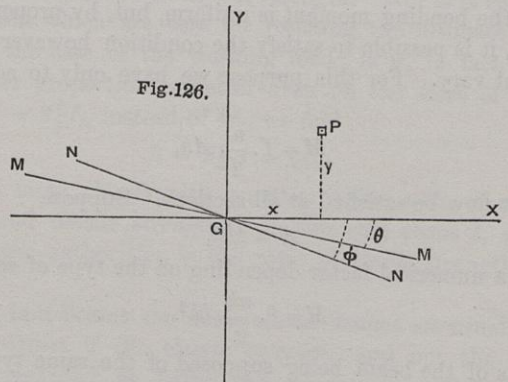
(3) Breadth uniform. Here the elevation or half elevation of the beam must be a curve, the co-ordinates of which are the square roots of the co-ordinates of the curve of moments.

(4) Ratio of breadth to depth constant. Here the half plan and half elevation are each curves, the ordinates of which are the cube roots of the ordinates of the curve of moments.

The first, third, and fourth of these cases are common in practice with some modifications occasioned by the necessity of providing strength at sections of the beam where the bending moment vanishes, as it usually does at one or both ends.

162. *Unsymmetrical Bending.*—It occasionally happens that the plane of the bending moment is not a principal plane of the beam, as for example when a vessel heels over, the plane of longitudinal bending will not coincide with the plane of symmetry of the vessel which is obviously the plane of the masts. The neutral axis does not now coincide with the axis of the bending couple, though in other respects the theory of bending still holds good.

Fig. 126.



In Fig. 126 let *MM* be the axis of the bending moment, *M* inclined at an angle θ to the principal axis of inertia *GX*, *GY* of the plane section. Then the couple *M* may be resolved into two components $M \cos \theta$ and $M \sin \theta$, each of which will produce stress at any point

P as if the other did not exist. Let p be the stress, x, y the co-ordinates of P referred to the axes GX, GY, the moments of inertia about which are I_1, I_2, then

$$p = \frac{M \,.\, \cos\theta \,.\, y}{I_1} + \frac{M \,.\, \sin\theta \,.\, x}{I_2}.$$

The position of the neutral axis NN is found by putting $p = 0$, then the angle ϕ which it makes with GX is given by

$$\tan\phi = -\frac{y}{x} = \frac{I_1}{I_2} \,.\, \tan\theta.$$

This equation shows that the neutral axis is parallel to a line joining the centres of the circles into which the beam would be bent by the component couples supposed each to act alone.

The neutral axis being thus determined and laid down on the diagram the points can be found which lie at the greatest distance from that axis. At these points the stress will be greatest, and if X, Y be their co-ordinates, still referred to the axes GX, GY, the moment of resistance will be determined by the equation

$$f = M\left\{ \frac{Y \,.\, \cos\theta}{I_1} + \frac{X \,.\, \sin\theta}{I_2} \right\}.$$

For a different method of expressing the moment of resistance see Rankine's Applied Mechanics, p. 314.

EXAMPLES.

1. A bar of iron 2″ diameter is bent into the arc of a circle 372′ diameter. Find in tons per square inch, 1st, the greatest stress at any point of the transverse section; 2nd, the stress on a line parallel to the neutral axis half an inch from the centre, E being taken = 29,000,000. *Ans.* Maximum stress = 5·8. Stress at ½″ from centre = 2·9.

2. Find the diameter of the smallest circle into which the bar of the last question can be bent; the stress being limited to 4 tons per square inch. *Ans.* Diameter = 540 feet.

3. Find the position of the neutral axis of a trapezoidal section; the top side being 3″, bottom 6″, and depth 8″. Also find the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 3·56 inches from bottom. Ratio of stresses 5 to 4.

4. A cast-iron beam is of I section with top flange 3″ broad and 1″ thick and bottom flange 8″ broad and 2″ thick; the web is trapezoidal in section ½″ thick at top and 1″ at bottom; total outside depth of beam 16″. Find the position of the neutral axis and the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 4·81 inches from bottom. Ratio of stresses 3 to 7.

5. A wrought iron beam of rectangular section is 9″ deep, 3″ broad, and 10 feet long. Find how much it will carry loaded in the centre, allowing a co-efficient of 3 tons per square inch. Also deduce the load the same beam will bear when set flatways. *Ans.* When upright load = 4·05 tons. When set flatways load = 1·35 tons.

6. A piece of oak of uniform circular section is 16″ diameter and 12 feet long. It is supported at the two ends and loaded at a point 5 feet from one end. How great may the load be, allowing a stress of $\frac{1}{2}$ ton per square inch? *Ans.* Load may be 5·74 tons.

7. In Example 5 suppose the same weight of metal formed into a beam of I section, each flange being equal to the web; what load will the beam carry? *Ans.* Load may then be 9·45 tons.

8. Find the moment of resistance to bending of the section given in Example 4, the co-efficient for tension being 1 ton per square inch. *Ans.* I = 798 inch units. Moment of resistance to bending = 166·4 inch-tons.

9. Suppose the skin and plate deck of an iron vessel to have the following dimensions at the midship section, measured at the middle of the thickness of the plates. Find the position of the neutral axis and moment of resistance to bending. Breadth 48′ and depth of vertical sides 24′, the bilges being quadrants of 12′ radius. Thickness of plate $\frac{5}{8}$″ all round, and co-efficient of strength 4 tons in compression. *Ans.* Neutral axis 14″ above centre of depth. Moment of resistance to hogging = 40,000 ft.-tons.

10. What should be the sectional area of a ⊤ beam of wrought iron to carry 4 tons uniformly distributed? Span 20′, depth of beam 10″ Co-efficient for compression 3 tons, and for tension 5 tons? *Ans.* Area = 13·7 square inches.

11. If, in the last question, the flange is made equal to the web instead of being proportioned for equal strength, show that to carry the same load the beam must be about one quarter heavier.

12. In Example 8 find the moments of inertia and resistance on the supposition that the flanges are concentrated at the centre lines, and thus by comparison with previous results show the amount of the error involved in the assumption. *Ans.* Moment of inertia = 861·5 inch units. Moment of resistance = 227 inch-tons.

13. Show that the limiting span (Art. 41) of a beam of uniform transverse section is

$$L = \lambda \, . \, \frac{8n}{Nq},$$

where N is the ratio of span to depth, and the rest of the notation is the same as on pages 90 and 314. Obtain the numerical result for a wrought iron beam of rectangular section, taking λ from Table II., Ch. XVIII., and supposing $N = 12$. *Ans.* $L = 336$ ft.; in an ordinary I section the result would be doubled. For the case of large girders see page 372.

14. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight when supported at the ends, show that when the stress allowed is 4 tons per square inch $l = \sqrt{224\,d}$.

15. If I_1, I_2 be the moments of inertia of two plane areas A_1, A_2, about their neutral axis which are supposed parallel at distance apart z, show that the moment of inertia of their sum or difference about their common neutral axis is $I = I_1 \pm I_2 + z^2 \, . \, \dfrac{A_1 A_2}{A_1 \pm A_2}$.

Apply this formula to the trapezoidal section of Question 3. *Ans.* $I = 185$ inch units nearly.

16. Find the moment of resistance to bending of a beam of I section, each flange consisting of a pair of angle irons $3\frac{1}{2}'' \times \frac{1}{2}''$ rivetted to a web $\cdot 37''$ thick and $16''$ deep between them. Assuming it 24 feet span, find the load it would carry in the middle, using a co-efficient of 3 tons per square inch. *Ans.* $M = 288$ inch-tons. $W = 4$ tons.

17. If it be assumed that for constructive reasons the thickness of web of an I beam with equal flanges must be a given fraction of the depth, show that for greatest economy of material the sectional area of the web should be equal to the joint sectional area of the flanges. Prove that in this case $M = \frac{1}{3} f \,.\, Sh$. (See p. 372.)

18. In a cast-iron beam of I section of equal strength for which $f^A = 2\frac{1}{2} f_B$; if it be assumed that for constructive reasons the thickness of the web should be a given fraction of the depth, show that for greatest economy of material the large flange, the web, and the small flange should be in the proportion 25, 20, 4. Prove also that the moment of resistance is given by the same formula as in Question 17 supposing $2/f = 1/f_A + 1/f_B$.

19. A beam of rectangular section of breadth one half the depth is bent by a couple the plane of which is inclined at 45° to the axes of the section. Find the neutral axis, and compare the moment of resistance to bending with that about either axis. *Ans.* Ratio $= 2\sqrt{2}/3$ and $\sqrt{2}/3$.

20. If a beam be originally curved in the form of a circular arc of radius R_0, instead of being straight, show that the neutral axis does not pass through the centre of gravity of the section. In a rectangular section of depth h show that the deviation is, approximately,

$$z = \frac{h^2}{12R_0}.$$

21. In the preceding question if R_0 is large show that the equations of bending are

$$\frac{p}{y} = E\left(\frac{1}{R_0} - \frac{1}{R}\right) = \frac{M}{I}.$$

Reference.

For the graphical determination of moments of inertia the reader is referred to the treatises cited on page 82.

CHAPTER XIII.

DEFLECTION AND SLOPE OF BEAMS.

163. *Deflection due to the Maximum Bending Moment.*—It is not only necessary that a beam should be strong enough to support the load to which it is subjected, it is also necessary that its changes of form should not be too great, or in other words, that it should be sufficiently stiff, and we next proceed to determine under what conditions this will be the case.

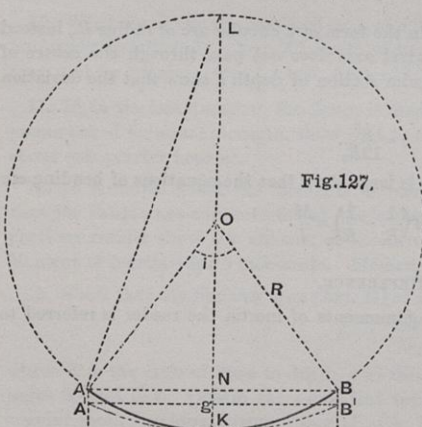


Fig.127.

The question is simplest when the beam is bent into an arc of a circle, we have then

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R}=\text{constant}.$$

Two cases may be especially mentioned--

(1) Depth uniform. We then have p constant, that the beam is of uniform strength. (See Case 1 of Art. 161.)

(2) Sectional area uniform. We then have, since

$$M=\frac{E}{R}I=n\,.\,\frac{E}{R}\,.\,Ah^2,$$

the depth of the beam varying as the square root of the bending moment, as in Case 3 of the same article.

Let l be the length of the beam, i the angle its two ends make with one another, then since i is also the angle subtended by the beam at the centre

$$i = \frac{l}{R} = \frac{Ml}{EI}.$$

If the beam be supported at the ends i is twice the angle which the ends make with the horizontal, an angle called the Slope at the ends. Let AB be the beam (Fig. 127), O the centre of the circle into which it is bent, KL the diameter of the circle through K the middle point of the beam. Then KN is the deflection which is given by a known proposition of Euclid

$$KN \,.\, NL = AN^2.$$

Hence remembering that the diameter of the circle is very large * we have, if δ be the deflection,

$$\delta = \frac{l^2}{8R} = \frac{Ml^2}{8EI}.$$

This formula gives the deflection in any case where the curvature is uniform.

When the transverse section is uniform the curvature varies. Unless the bending moment be likewise uniform, the deflection curve is not then a circle AKB, but for the same maximum bending moment a flatter curve $A'KB'$. Thus the deflection is less than that calculated by the above formula, which may be described as the "deflection due to the maximum moment." The actual deflection may conveniently be expressed as a fraction of that due to the maximum moment. It is possible to construct the deflection curve graphically by observing that the curvature at every point is proportional to the bending moment. We have then only to strike a succession of arcs with radii inversely proportional to the ordinates of the curve of bending moment. It is however more convenient to proceed by an analytical method.† The fraction is least when the beam is least curved, which is evidently the case when it is loaded in the middle, and we shall show presently that it is then two-thirds, while, when uniformly loaded, it is five-sixths.

* For clearness it is made small in the figure.

† Readers who have no knowledge of the Calculus may pass over the next four articles.

164. *General Equation of Deflection Curve.*—It was shown above that

$$i = \frac{M}{EI} \cdot l.$$

If the bending moment vary, then we must replace l by an element of the length ds and i by the corresponding element of the angle; we shall then have an equation

$$\frac{di}{ds} = \frac{M}{EI},$$

which by integration will furnish i. It will generally be convenient to reckon i from a horizontal tangent and it then means the slope of the beam at the point considered. To perform the integration it is in most cases necessary to suppose the slope of the beam small, as it actually is in most important cases in practice, and we may then replace ds the element of arc by dx, the corresponding element of a horizontal tangent AN (Fig. 128) taken as axis of x, whence

$$\frac{di}{dx} = \frac{M}{EI} \text{ approximately,}$$

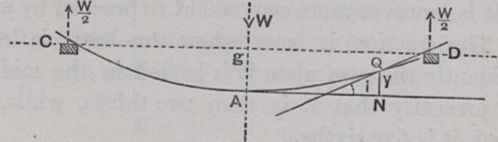
an equation which can generally be integrated because M is usually a function of x.

The deviation y of any point Q of the beam from the straight line AN can now be found since $dy/dx = i$, from which we further obtain the fundamental equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI},$$

which applies to all cases where the bending of the beam is occasioned by a transverse load. We shall first give some elementary examples of the determination of the deflection and slope of a beam and then consider the question more generally.

Fig.128.



165. *Elementary Cases of Deflection and Slope.—Case I.* Suppose a beam supported at the ends and loaded in the middle.

In Fig. 128 CD is the beam resting on supports at C, D, and loaded in the middle with a weight W. Take the centre A as origin and the horizontal tangent at A as axis of x, then if l be the whole length

$$\frac{d^2y}{dx^2}=\frac{M}{EI}=\frac{\frac{W}{2}\left(\frac{l}{2}-x\right)}{EI}.$$

$$\therefore i=\frac{dy}{dx}=\frac{\frac{W}{2}\left(\frac{l}{2}x-\frac{1}{2}x^2\right)}{EI}$$

is the slope of the beam at Q, no constant being required since i is zero when $x=0$.

If $x=l/2$ we get the slope at the ends of the beam

$$i_1=\frac{Wl^2}{16EI}.$$

Integrating a second time

$$y=\frac{\frac{W}{2}\left(\frac{1}{4}lx^2-\frac{1}{6}x^3\right)}{EI}.$$

As before no constant is required because $y=0$ when $n=0$.

If now we put $x=l/2$ we get the elevation of D above AN or, what is the same thing, the depression of A below the level of the supports. This is called the Deflection of the beam ; if we denote it by δ,

$$\delta=\frac{\frac{W}{2}\left(\frac{1}{16}l^3-\frac{1}{48}l^3\right)}{EI}=\frac{Wl^3}{48EI},$$

a result which we may also write

$$\delta=\frac{2}{3}\cdot\frac{M_0l^2}{8EI}=\frac{2}{3}\cdot\delta_0,$$

where M_0 is the maximum moment and δ_0 the deflection due to it.

Case II. Let the beam be supported at the ends and loaded uniformly with w pounds per foot run. It will be sufficient to give the results, which are obtained in precisely the same way, remembering that the bending moment is now $\frac{1}{2}w(a^2-x^2)$ where a is the half span. We have

$$i_1=\frac{wa^3}{3EI}=\frac{Wl^2}{24EI};\ \delta=\frac{5}{24}\cdot\frac{wa^4}{EI}=\frac{5}{384}\cdot\frac{Wl^3}{EI}.$$

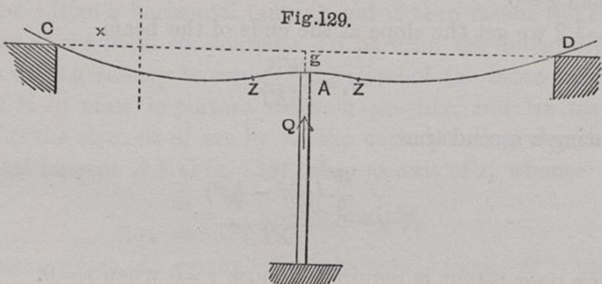
The value of δ may be expressed as in the previous case in terms of the deflection due to the maximum moment. We have $\delta = \frac{5}{6} \cdot \delta_0$.

166. *Beam propped in the Middle.*—When a beam is acted on by several loads the deflection and slope due to the whole is the sum of those due to each load taken separately. An important example is

Case III. Beam supported at the ends and propped in the middle, uniformly loaded. (Fig. 129.)

Here the deflection of the beam is the difference between the downward deflection due to the uniform load and the upward deflection due to the thrust Q of the prop. Hence we write down at once for the deflection at the centre,

Fig.129.



$$\delta = \frac{5}{384} \cdot \frac{Wl}{EI} - \frac{Ql^3}{48EI},$$

an equation which may be used to determine the load carried by the prop when its length is given, and conversely.

First suppose the centre of the beam propped at the same level as the supports, then $\delta = 0$, and

$$Q = \frac{5 \times 48W}{384} = \frac{5}{8}W,$$

so that the prop in this case carries five-eighths of the weight of the beam, the supports C, D only carrying three-eighths. Each supporting force is $\frac{3}{16}wl$, l being as before the whole length of the beam; hence the bending moment at a point distant x from C is given by the formula

$$M = \frac{3}{16}wlx - \frac{1}{2}wx^2 = \frac{1}{2}wx(\frac{3}{8}l - x),$$

from which it appears that the beam is bent downwards until a point Z is reached, such that

$$CZ = \frac{3}{8}l = \frac{3}{4}AC.$$

Here the bending moment is zero, that is, Z is a "point of contrary flexure" or "virtual joint." (Compare Art. 38.)

Beyond Z the beam is bent upwards, and at the centre A we get, by putting $x = \frac{1}{2}l$,

$$-M_0 = \tfrac{1}{32}wl^2.$$

The case here discussed is also that of a beam one end of which is fixed horizontally and the other supported at exactly the same level.

Let us next inquire what will be the effect of supposing the centre of the beam propped somewhat out of the horizontal line through the supports at the ends. Let us suppose δ to be $1/n^{\text{th}}$ the deflection of the beam when the prop is removed, then

$$\frac{1}{n} \cdot \frac{5}{384} \cdot \frac{Wl^3}{EI} = \frac{5}{384} \cdot \frac{Wl^3}{EI} - \frac{Ql^3}{48EI},$$

that is

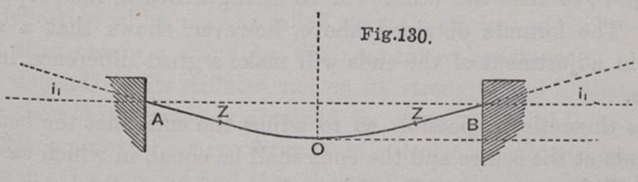
$$Q = \tfrac{5}{8}W\left(1 - \frac{1}{n}\right),$$

a formula which gives the load on the prop. If, for example, $n = 5$, $Q = \frac{1}{2}W$, or if $n = -5$, $Q = \frac{3}{4}W$; thus if the centre of the beam be out of level, by as much as one-fifth the deflection when the prop is wholly removed, the load on the prop will vary between $\frac{1}{2}W$ and $\frac{3}{4}W$, a result which shows the care necessary in adjustment to obtain a definite result.

167. *Beam fixed at the Ends.—Case IV.* Uniformly loaded beam, with ends fixed at a given slope.

In Fig. 130 AB is a uniformly loaded beam, with the ends A, B fixed not horizontally but for greater generality at a slope i. Here

Fig.130.



the central part of the beam will be bent downwards and the end parts upwards; at Z, Z there will be virtual joints; let $OZ = r$, then taking O as origin the bending moment at any point between O and Z is

$$M = \tfrac{1}{2}w(r^2 - x^2),$$

a formula which will also hold for points beyond Z, as can be seen from Art. 38, or proved independently. We have then

$$\frac{d^2y}{dx}=\frac{\frac{1}{2}w(r^2-x^2)}{EI};$$

$$i=\frac{\frac{1}{2}w(r^2x-\frac{1}{3}x^3)}{EI}.$$

No constant is required, because i is zero at O. Let a be the half span OA, or OB, then putting $x=a$, we get for the slope at the ends

$$i_1=\frac{\frac{1}{2}w(r^2a-\frac{1}{3}a^3)}{EI},$$

a formula from which r can be determined if i_1 be given. If $r=a$, we get the case where the ends are free; let the slope then be i_0, we have

$$i_0=\frac{wa^3}{3EI} \text{ as before (p. 329).}$$

Now, assume the actual slope to be $1/n^{\text{th}}$ of this, we get

$$\frac{1}{n}\cdot\frac{wa^3}{3EI}=\frac{\frac{1}{2}w(r^2a-\frac{1}{3}a^3)}{EI};$$

that is,

$$r^2=\tfrac{1}{3}a^2\left(1+\frac{2}{n}\right).$$

If the ends are fixed exactly horizontal, then

$$r^2=\tfrac{1}{3}a^2,$$

and by substitution we find for the bending moment at the centre and the ends

$$M_0=\tfrac{1}{6}wa^2;\ M_A=M_B=\tfrac{1}{3}wa^2.$$

If the ends were free, the bending moment at the centre would have been $\frac{1}{2}wa^2$, so that the beam will be strengthened in the proportion 3 : 2. The formula obtained above, however, shows that a small error in adjustment of the ends will make a great difference in the results.

It is theoretically possible so to adjust the ends that the bending moments at the centre and the ends shall be equal, in which case the beam will be strongest. For this we have only to put

$$\tfrac{1}{2}wr^2=\tfrac{1}{2}w(a^2-r^2),$$

that is,

$$r^2=\tfrac{1}{2}a^2,$$

whence by substitution we get

$$n=4;$$

that is, the ends should be fixed at one fourth the slope which they have when free, and the strength of the beam will then be doubled.

By proceeding to a second integration the deflection of the beam can be found. In particular when the ends of the beam are horizontal it can be shown that the deflection is only one fifth of its value when the ends are free.

The graphical representation of the bending moments in Cases III., IV., is easily effected, as in Fig. 42, p. 86.

168. *Stiffness of a Beam.*—The stiffness of a beam is measured by the ratio of the deflection to the span. In practice, the deflection is limited to 1 or 2 inches per 100 feet of span when under the working load; that is, the ratio in question is $\frac{1}{600}$th to $\frac{1}{1200}$th. It appears from what has been said that if M_0 be the maximum moment the deflection is given by

$$\delta = k \cdot \frac{M_0 l^2}{8EI},$$

where k is a fraction, varying from two-thirds to unity, depending on the way in which the beam is loaded. Hence the greatest moment which the beam will bear consistently with its being sufficiently stiff is

$$M_0 = \frac{8E\delta}{kl} \cdot \frac{I}{l}.$$

If we express I as usual in terms of the sectional area and depth, we get

$$M_0 = s\frac{n}{k}A\frac{h^2}{l},$$

where s is a co-efficient depending on the material and on the admissible deflection which may be called the "Co-efficient of Stiffness."

We thus obtain a value for the moment of resistance of a beam which depends on its stiffness, not on its strength, and if that value be less than that previously obtained for strength (p. 314), we must evidently employ the new formula in calculating dimensions. On comparing the two, we find that they will give the same result if

$$\frac{sh}{kl} = \frac{f}{q}; \text{ or } \frac{h}{l} = \frac{fk}{qs};$$

that is to say, for a certain definite ratio of depth to span, and if there is no other reason for fixing on this ratio, it will be best to choose the value thus determined. The two formulæ then give the same result. In large girders a greater depth is generally desirable, then the strength formula must be used; while in small beams it may often be convenient or necessary to have a smaller depth, and then the stiffness formula must be employed.

169. *General Graphical Method.*—The foregoing simple examples of the determination of the deflection and slope of a beam are perhaps those of most practical use, but, by the aid of graphical processes, there is no difficulty in generalizing the results which are of considerable theoretical interest. We can, however, afford space only for a hasty sketch.

The general equations given in Art. 164 show that the angle (i) between two tangents to the deflection curve of a beam is proportional to the area of the curve of bending moments intercepted between two ordinates at the points considered. Starting from the lowest point of the deflection curve, let us now imagine a curve drawn, the ordinate of which represents that area reckoned from the starting point, then that curve will represent the slope of the beam at every point, and may therefore properly be called the "Curve of Slope." But referring again to the general equations we see that the ordinate of the deflection curve reckoned upwards from the horizontal tangent at the lowest point, is connected with the slope in the same way as the slope with the bending moment, and is consequently proportional to the area of the curve of slope. Thus it appears, on reference to Chapter III., that the curves of Deflection, Slope, and Bending Moment are related to each other in the same way as the curves of Bending Moment, Shearing Force, and Load. The five curves, in fact, form a continuous series each derived from the next succeeding by a process of graphical integration.

We now see that any property connecting together the second three quantities must also be true for the first three. For example, we know, from the properties of the funicular polygon, that two tangents in the curve of moments intersect in a point vertically below the centre of gravity of the area of the corresponding curve of loads (see Arts. 31, 35). It must therefore be true that two tangents to the deflection curve intersect vertically below the centre of gravity of the corresponding area of the curve of moments, a useful property, which can be proved directly without much difficulty.

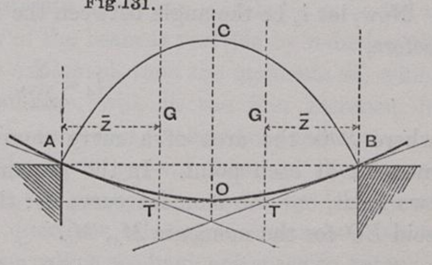
The deflection curve of a beam may therefore be constructed in the same way that the funicular polygon is constructed in Art. 35, the perpendicular distance (H) of the pole from the load line in the diagram of forces being made equal to EI. To do this we have only to divide the moment curve into convenient vertical strips and regard each as representing a weight. Set down these ideal weights as a vertical line and choose a pole at a distance from the line equal to EI, measured (on account of the largeness of E) on a scale less in a given ratio. Now, construct the polygon and draw its closing line, the intercept multiplied by the scale ratio is the deflection of the beam. A parallel to the closing line in the diagram of forces gives the slopes at the extremities of the beam which correspond to the supporting forces of the loaded beam in the original case.

We have hitherto supposed the beam to be of uniform stiffness throughout; if not, let the quantity EI, which is now variable, be E_0I_0 at some datum section. Reduce the ordinates of the curve of moments in the proportion E_0I_0 to EI, then the reduced curve is to be employed in the way just described for the original curve.

170. *Examples of Graphical Method. Theorem of Three Moments.*—Let us now take some examples.

Case I.—Symmetrically loaded beam, of flexibility also symmetrical about the centre. Let ABC (Fig. 131) be the curve of moments, reduced if necessary, AOB the deflection curve; both curves, of course, will be symmetrical about the centre vertical, then from what has been said, tangents at A, B to the deflection curve intersect the tangent at O in points T vertically below the centres of gravity of the two equal areas ACO, BCO. Hence if S be the area of the whole curve of moments, $\bar{z}$ the horizontal distance of either point T from the nearer end,

Fig.131.

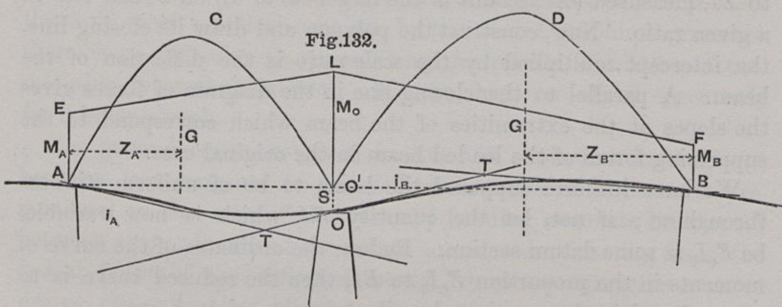


$$i_0 = \frac{S}{2EI}\ ;\ \delta = \bar{z} \,.\, i_0 = \frac{S \,.\, \bar{z}}{2EI}$$

must be the slope of the ends of the beam and its deflection.

Case II. Beam continuous over several, spans loaded in any way. (Fig. 132.) Let ACO', BDO' be the moment curves due to the load on two spans AO', BO' of a beam AOB, continuous over three supports A, O, B, of which the centre O is somewhat below the level of A, B. Being continuous, there will be bending moments at A, O, B, which are represented in the diagram by AE, $O'L$, BF. Joining EL, FL, the actual bending moment at each point of the beam will be

Fig.132.



represented by the intercept between the line ELF and the curves of moments due to the load and corresponding supporting forces. (See Art. 38.) The curve AOB is the deflection curve, AT, BT are the tangents at A, B and TOT is the tangent at O, intersecting AT, BT in the points T.

Now, let i_A be the angle between the tangents at O and A, then, as before,

$$i_A = \frac{S}{EI},$$

where S is the area of a curve representing the actual bending moment at each point. In the present case S is the difference of two areas, one the moment curve for the load, the other the trapezoid EO' for the moments M_A, M_0.

$$\therefore S = A - \frac{M_A + M_0}{2} \cdot l_A,$$

where A is the area of the moment curve ACO' and l_A is the span AO'. Let the horizontal distance from A of the common centre of gravity of the two curves be x; then, as before, x is also the horizontal distance of T from A, and

$$y_A = \frac{Sx}{EI}, \text{ as before}$$

To find x, let z_A be the horizontal distance of the centre of gravity of ACB from A, then

$$Sx = Az_A - M_A l_A \cdot \frac{l_A}{2} - \frac{M_0 - M_A}{2} \cdot l_A \cdot \tfrac{2}{3} l_A ;$$

$$= Az_A - \tfrac{1}{6} M_A \cdot l_A^2 - \tfrac{1}{3} M_0 \cdot l_A^2.$$

We have thus found y_A the distance of A from the tangent through O; and y_B, the corresponding distance of B, is written down by change of letters.

Assuming now the depression of O, the centre of the beam, below the level of the two other supports to be δ, it appears from the geometry of the diagram that

$$\frac{y_A - \delta}{l_A} = \frac{-y_B + \delta}{l_B} ;$$

or

$$\frac{y_A}{l_A} + \frac{y_B}{l_B} = +\delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right);$$

hence dividing the values of y_A, y_B by l_A, l_B respectively, and adding

$$A\frac{z_A}{l_A} + B \cdot \frac{z_B}{l_B} - \tfrac{1}{3} M_0(l_A + l_B) - \tfrac{1}{6} M_A l_A - \tfrac{1}{6} M_B l_B = \delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right) EI.$$

This equation connects the bending moments at three points of support of a continuous beam, the centre support being below the end supports by the small quantity δ. It can readily be extended to the case where the flexibility of the beam is variable by reducing the moment curves as previously explained, then the moments M, which are the results of the calculation, will, in the first instance, be reduced, and can afterwards be increased to their true values.

The above equation is the most general form of the famous Theorem of Three Moments, originally discovered by Clapeyron, which is always employed in questions relating to continuous beams —a somewhat large subject, on which we have not space to enter.

171. *Resilience of a Bent Beam.*—The work done in bending a beam by a uniform bending moment M is evidently $\frac{1}{2}Mi$, where i is the angle which the two ends of the beam make with each other, as in Art. 163; hence by substitution for i we find for the work U,

$$U = \frac{M^2}{2EI} \cdot l ;$$

and if the bending moment vary,

$$U = \int \frac{M^2}{2EI} \cdot dx.$$

An important case is when the beam is of uniform strength, then we have

$$p = \frac{My}{I} = \text{constant} = \frac{M_0 y_0}{I_0},$$

where the suffix 0 refers to a datum section. Then

$$U = \frac{M_0^2}{2EI_0} \int \frac{I}{I_0} \cdot \frac{y_0^2}{y^2} \cdot dx.$$

Assuming now the section (A), though varying, to remain of the same type,

$$\frac{I}{I_0} = \frac{Ay^2}{A_0 y_0^2}.$$

If, therefore, we call V the volume of the beam,

$$U = \frac{M_0^2}{2EI_0} \cdot \frac{V}{A_0} = \frac{p^2}{2E} \cdot \frac{I_0}{A_0 y_0^2} V.$$

With the notation of Art. 155 this gives

$$U = \frac{p^2}{E} \cdot \frac{n}{2q^2} \cdot V,$$

For the resilience we have only to change p into f, the proof strength. It thus appears that in beams of uniform strength with transverse sections of the same type the resilience is proportional to the volume, and less than that of a stretched or compressed bar, as might have been foreseen from general considerations. The ratio of reduction is $q^2 : n$, being $3 : 1$ in rectangular sections, $4 : 1$ in elliptic sections. When the beam is not of uniform strength the ratio of reduction must be greater for the same type of section. The reduction is of course least in I sections of uniform strength.

The function U is of great importance in the theory of continuous beams and other similar structures, the relative yielding of the several parts of the structure being always such that this function is less than it would be for any other distribution of stress and strain. It may be called the Elastic Potential, and when known all the equations necessary to determine the distribution of stress may be found by simple differentiation. (See Appendix.)

EXAMPLES.

1. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight with a deflection of 1 inch per 100 feet of span, show that

$$l = \sqrt{233d^2}.$$

Compare this result with that of Ex. 14, p. 324, and state what formula is to be used when both stiffness and strength are required.

2. Find the ratio of depth to span in a beam of rectangular section loaded in the middle, assuming stress $= 8{,}000$, $E = 28{,}000{,}000$, deflection $= \dfrac{\text{span}}{1200}$. *Ans.* $\dfrac{1}{17{\cdot}5}$.

3. A beam is supported at the ends and loaded at a point distant a, b from the supports with a weight W, show that the depression of the weight below the points of support is $\dfrac{Wa^2b^2}{3EI(a+b)}$.

4. In the last question deduce the work done in bending the beam, and verify the result by direct calculation. (See Art. 20.)

5. A dam is supported by a row of uprights which take the whole horizontal pressure of the water. The uprights may be regarded as fixed at their base at the bottom of the water, while their upper ends at the water level are retained in the vertical by suitable struts sloping at 45°, the intermediate part remaining unsupported. Find the bending moment at any point of the upright, and show that the thrust on the struts is about two sevenths the horizontal pressure of the water.

6. A timber balk 20 feet long of square section supports 160 square feet of a floor, find the dimensions that the deflection of the floor, when loaded with 60 lbs. per square foot, may not exceed $\frac{1}{2}$ inch.

7. A shaft carries a load equal to m times its weight (1) distributed uniformly, (2) concentrated in the middle. Considering it as a beam fixed at the ends, find the distance apart of bearings for a stiffness of $\dfrac{1}{1200^{\text{th}}}$. *Ans.* If l be the distance apart in feet, d diameter in inches, then for a wrought iron or steel shaft

$$(1)\ l = 10{\cdot}5\sqrt[3]{\frac{d^2}{m+1}}; \qquad (2)\ l = 8{\cdot}3\sqrt[3]{\frac{d^2}{m+\frac{1}{2}}}.$$

8. A beam originally curved, as in Ex. 21, p. 325, is fixed at one end and loaded in any way. If i be the change of slope at any point and X, Y the displacements parallel to axes of x, y of the point consequent on any load, prove that

$$\frac{di}{ds} = \frac{M}{EI}; \quad \frac{dX}{dy} = -i; \quad \frac{dY}{dx} = i.$$

Apply these formulæ to find the straining actions at any point of one of the rings of a chain of circular links.

CHAPTER XIV.

TENSION OR COMPRESSION COMPOUNDED WITH BENDING CRUSHING BY BENDING.

172. *General Formula for the Stress due to a Thrust or Pull in combination with a Bending Moment.*—The bars of a frame and the parts of other structures are often exposed, not only to a pull or thrust alone, or to a bending action alone, but to the two together; and the total stress at any point of a transverse section is then the sum of that due to each taken separately. That is to say, if H be the thrust, reckoned negative if a pull, M the bending moment, the stress at any point distant y from the neutral axis of the bending (see Art. 155), reckoned positive on the compressed side, must be given by

$$p=\frac{H}{A}+\frac{My}{I}=\frac{H}{A}\left\{1+\frac{q}{n}\cdot\frac{M}{Hh}\right\},$$

the notation being as in the article cited.

This formula shows how the effect of a thrust or pull is increased by a bending action: it has many important applications, some of which we shall now briefly indicate.

173. *Strut or Tie under the Action of a Force parallel to its Axis in cases where Lateral Flexure may be neglected.*—*Case I.* Bar under the action of a force in a principal plane parallel to its axis.

Let z be the distance from the axis of the line of action of the force, then

$$M=Hz;\ p=\frac{H}{A}\left(1+\frac{q}{n}\cdot\frac{z}{h}\right).$$

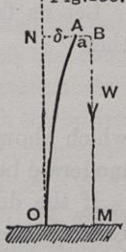
For example, let the section be circular, then $n=\frac{1}{16}$, $q=\frac{1}{2}$, and we find

$$p=\frac{H}{A}\left(1+\frac{8z}{h}\right),$$

from whence it appears that a deviation from the axis of $\frac{1}{16}$th the diameter of a rod increases the effect of a thrust or pull 50 per cent. Similarly it can be shown that if the line of action of the force lie outside the middle fourth of the diameter of a circular section, or the middle third of a rectangular section, the maximum stress will be more than double the mean, and at certain points the stress will be reversed. In designing a structure, then, the greatest care must be exercised that the line of action of a thrust or pull lies in the axis of the piece which is subjected to it; to effect which, the joints, through which such straining actions are exerted, must be so designed that the resultant stress at the joint is applied at the centre of gravity of the section of the piece. This is a condition which cannot always be satisfied, and allowance in any case must be made for errors in workmanship. In practical construction it is the joints which require most attention, being most often the cause of failure. In frames which are incompletely braced the friction of pin joints causes the line of action of the stress to deviate from the axis. (See Ch. XVIII.)

The effect is increased in the case of a thrust and diminished in the case of a pull by the curvature of the piece, which increases or diminishes z. Fig. 133 shows the axis of a column, under the action of a weight W, suspended from a short cross piece of length a. The column bends laterally, as shown in an exaggerated way in the figure. The inclination of AB to the horizontal is so small that the difference between the actual and the projected length of AB may be disregarded; the bending moment at O is therefore $W(a+\delta)$, where δ is the lateral deviation AN of the top of the pillar. This deviation we will in the first instance suppose small compared with a, and then determine the condition that this may actually be the case. Neglecting it, the axis of the pillar is bent by the uniform bending moment Wa into a circular arc of radius R, and as in Art. 163

Fig. 133.



$$\delta\ .\ 2R=l^2,$$

substituting for R its value (Art. 155) we get

$$\delta=\frac{Ml^2}{2EI}=\frac{Wal^2}{2EI};$$

whence we find

$$\frac{\delta}{a}=\frac{Wl^2}{2EI}.$$

The condition, then, that the lateral deviation should be small is that W should be much less than $2EI/l^2$, and if this condition be satisfied the stress will not be much increased beyond that indicated by the formula given above. The very important cases in which W is large will be treated presently.

In the case of a pull this restriction on the use of the formula need not be attended to, the effect of the deviation being to diminish the stress.

174. *Effect of a Thrust on a Loaded Beam.—Case II.* Uniformly loaded beam supported at the ends and subject to compression.

Let the load be W and the thrust H, then

$$p=\frac{H}{A}\left\{1+\frac{q}{n}\cdot\frac{\frac{1}{8}Wl}{Hh}\right\}.$$

For example, let the section be rectangular, then $q=\frac{1}{2}$, $n=\frac{1}{12}$, and we find

$$p=\frac{H}{A}\left\{1+\frac{3l}{4h}\cdot\frac{W}{H}\right\}.$$

Let us further suppose the ratio of depth to span one sixteenth, then

$$p=\frac{H}{A}\cdot\left(1+12\frac{W}{H}\right)=\frac{W}{A}\left(12+\frac{H}{W}\right),$$

which shows how greatly the effect of a thrust is increased by a moderate bending moment.

If the deflection be supposed 1 inch in 100 feet then H will in consequence produce an additional bending action at the centre equal to $Hl/1200$, which will be equivalent to an addition to W of $H/150$. For safety H ought not to exceed $3W$, and the stress due to the bending action of the uniform load on the beam will then be increased about 25 per cent. by the effect of the thrust. This calculation shows why it is often necessary to support a beam at points not too far apart by suitable trussing even when support is not required to give sufficient stiffness. Theoretically a proper "camber" given to the beam will counteract the bending action, and, conversely, a small accidental deflection will increase it.

175. *Remarks on the Application of the General Formula.*—The formula given above in Art. 172 is much used in questions relating to the stability of chimneys, piers, and other structures in masonry and brickwork. The stress on horizontal sections of such structures varies uniformly or nearly so, and the formula then shows where the stress is greatest and also where it becomes zero, tension usually not being permissible. It must be borne in mind however that the bending is frequently unsymmetrical, so that the axis of the bending moment will not coincide with the neutral axis of the bending stress on the section (Art. 162). The stability of blockwork and earthwork structures is a large subject which will not be considered in this treatise.

176. *Straining Actions due to Forces Normal to the Section.*—The reasoning of this section shows that when a structure is acted on by forces some or all of which have components normal to a given section, the straining actions due to the normal components will in general depend on the relative yielding of the several parts of the section (Art. 42). These normal components however can always be reduced to a single force, acting through any proposed point in the section, and a couple, and if the point be properly chosen according to the nature of the structure at the section that single force will be a simple thrust or pull; thus in the cases we have mentioned the point is the centre of gravity of the section. Having done this the couple will be so much addition to the bending action. An important example of this is the case of a vessel floating in the water in which the horizontal longitudinal component of the fluid pressure generally produces bending, the arm of the bending couple being the distance of the intersection of the line of action of the resultant with the section considered, from the neutral axis of the "equivalent girder."

177. *Maximum Crushing Load of a Pillar.*—When the compressing force is sufficiently great it produces a strong tendency to bend the pillar even though there be no lateral force. We have already seen that the condition that this shall not be the case is that W shall be small compared with the quantity $2EI/l^2$, and we now proceed to inquire the effect produced when W has a larger value. All these cases come under the head of what is called Crushing by Bending, and are very common and important in practice.

As in the case of the deflection of a beam the question is much more simple when the pillar bends into an arc of a circle, which it will do in various cases explained in Art. 163. The case which we select is that in which the sectional area remains constant and the thickness varies. Such a pillar is of uniform strength when very slightly bent, and when more bent the weakest point is at the base. As the breadth becomes great at the summit this form could not be practically applied without modification, but the conclusions derived by considering it may be applied with slight modifications to the cases which occur in practice. *

When the load is applied exactly at the centre the elevation of such a pillar is a semi-ellipse with vertex at the summit; when not exactly at the centre the ellipse is truncated. For the present purpose it is not necessary to consider this point further, as the form is not intended for practical application.

Assuming then the form of the bent pillar to be a circular arc we have as before

$$\delta = \frac{Ml^2}{2EI},$$

but we have now, since we cannot neglect δ,

$$M = W(a + \delta).$$

Hence by substitution we find

$$\delta = \frac{W(a + \delta)l^2}{2EI},$$

where I is the moment of inertia at the base, from which we find

$$\delta = \frac{a}{\dfrac{2EI}{Wl^2} - 1}.$$

This result shows that the pillar bends laterally more and more

* The case where the thickness is uniform has been considered by Dr. Young in his *Natural Philosophy* (see Young's works, Peacock's edition, p. 139), who shows that the outline is a circular arc, as follows at once from Art. 161. The compressive stress however near the summit of the pillar is then very great.

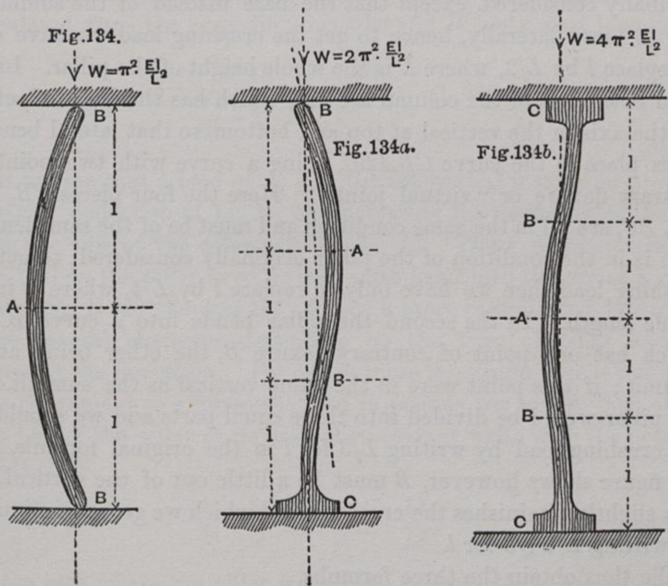
as W increases, and breaks with some value of W which we will find presently by substitution in the formula of Art. 172.

First, however, observe that if $a = 0$, that is, if the line of action of the load pass through the centre of the pillar at its summit, then $\delta = 0$ unless the denominator of the fraction be also zero, that is, unless

$$W = 2 \cdot \frac{EI}{l^2}.$$

The interpretation of this is, that if W be less than the value just given the pillar will not bend at all, but if disturbed laterally will return to the upright position when the disturbing force is removed. If W have exactly that value then, when put over into any inclined position the pillar will remain there in a state of neutral equilibrium, while the smallest increase of W above this limit will cause the

Fig.134.



pillar to bend over indefinitely and so break. Thus the foregoing equation may be regarded as giving the crushing load of the pillar under certain conditions to be defined more exactly presently.

If the pillar had not bent into the arc of a circle as has been just supposed, we should have arrived at exactly the same formula except that the co-efficient 2 is replaced by a not very different number depending on the circumstances of the particular case. If the transverse section be uniform then the pillar bends into a curve of sines and we must replace 2 by $\pi^2/4$ or 2·47, thus obtaining

$$W = \frac{\pi^2}{4} \cdot \frac{EI}{l^2},$$

a formula which having been first obtained by Euler is known as Euler's Formula. It applies directly to a column fixed firmly in the ground and entirely free at the upper end; it can however easily be modified to suit the cases more common in practice where the ends of the column are constrained to lie in the same vertical line. There will be three such cases shown in Figs. 134, 134*a*, 134*b*.

In the first the ends of the pillar are rounded and it bends laterally in the curve BAB; each half AB is then in the position of the pillar originally considered, except that the base instead of the summit is free to move laterally, hence to get the crushing load we have only to replace l by $L/2$, where L is the whole height of the pillar. In the third both ends of the column are flat, which has the effect of retaining the axis in the vertical at top and bottom, so that lateral bending takes place in the curve $CBABC$, being a curve with two points of contrary flexure or "virtual joints." Here the four pieces CB, BA, AB, BC, are all in the same condition and must be of the same length; each is in the condition of the pillar originally considered; to get the crushing load then we have only to replace l by $L/4$, where L is the whole length. In the second the pillar bends into a curve $BABC$ which has one point of contrary flexure B, the other being at the summit; if this point were in the same vertical as the summit then the pillar would be divided into three equal parts and we should get the crushing load by writing $L/3$ for l in the original formula. As the figure shows however, B must be a little out of the vertical, and this slightly diminishes the crushing load which we get approximately by writing $L/2\sqrt{2}$ for l.

We thus obtain the three formulæ,

$$W = \pi^2 \cdot \frac{EI}{L^2};\ W = 2\pi^2 \cdot \frac{EI}{L^2};\ W = 4\pi^2 \cdot \frac{EI}{L^2},$$

for the three cases in question with a uniform section. If the pillar be bent into a circle as described above, then π^2 is to be replaced by 8.

178. *Manner in which a Pillar crushes. Formula for Lateral Deviation.*—The value of W here found is the maximum load which a pillar, free to deflect laterally, can sustain under any circumstances; but, in order that it may actually be sustained, the pillar must be perfectly straight, the material must be perfectly homogeneous, and the line of action of the load must be exactly in the axis. These conditions cannot be accurately satisfied, and consequently a lateral deflection is produced, which increases indefinitely as the load approaches the theoretical maximum. This may be expressed by supposing that a is not zero, but some known quantity depending on the degree of accuracy with which the conditions are satisfied, and which may be called the "effective" deviation; since, when the pillar is straight and homogeneous, it will be the actual deviation of the line of action of the load from the axis. Let W_0 be the theoretical maximum load as calculated from the preceding formulæ and W the actual load, then

$$\delta = \frac{a}{\dfrac{W_0}{W} - 1} = a \cdot \frac{W}{W - W_0} \text{ (p. 344.)};$$

thus we see that a load of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ the theoretical maximum produces a lateral deflection of $1a$, $2a$, $3a$, increasing the deviation of the load from the axis of the column to $2a$, $3a$, $4a$. These numbers are only exact when the pillar is so formed as to bend into the arc of a circle, when this is not the case they follow a more complicated law of the same general character depending on the type of pillar and the nature of the deviation. For our purpose the simple case is sufficient. It is convenient to express the load in pounds per square inch of the area (A) of the pillar at its base, then we may write with the notation of Art. 155

$$p_0 = \frac{W_0}{A} = \pi^2 \cdot nE \cdot \frac{h^2}{L^2};$$

for the case where the pillar is rounded at both ends, the number π^2 being replaced by $2\pi^2$ or $4\pi^2$ in the two other cases of the last article. Similarly writing $p = W/A$ for the actual load on the pillar, we get by substitution

$$\delta = a \cdot \frac{p}{p_0 - p}, \text{ or } a + \delta = a \cdot \frac{p_0}{p_0 - p}.$$

The deviation is accompanied by an increase in the maximum stress (f) on the transverse section, which is given by the formula

$$f = \frac{H}{A}\left(1 + \frac{q}{n} \cdot \frac{M}{Hh}\right) \qquad \text{(p. 340)},$$

from which we get, replacing H by W and M by $W(a+\delta)$,

$$f = p\left(1 + \frac{qa}{nh} \cdot \frac{p_0}{p_0 - p}\right),$$

a result which shows that f increases indefinitely as p approaches p_0, so that the pillar must break before the theoretical maximum is reached, however small the original deviation is. The greatest value of f must be the elastic strength, for as soon as this is past an additional lateral deviation at the most compressed part will occur, sooner or later accompanied by rupture.

The formula may be written in the more convenient form,

$$\left(\frac{f}{p} - 1\right)\left(1 - \frac{p}{p_0}\right) = \frac{qa}{nh},$$

in which it is worth while to observe that the right-hand side is unity for the deviation necessary to produce double stress when the pillar is so short that no sensible augmentation of the deviation is produced by lateral bending. In materials like cast iron which have a low tenacity, very long pillars give way by tension on the convex side; the formula then becomes

$$\left(\frac{f'}{p} + 1\right)\left(1 - \frac{p}{p_0}\right) = \frac{qa}{nh},$$

where f' is the tensile stress at the elastic limit. The two formulæ give the same result if

$$p = \frac{f - f'}{2}.$$

For loads greater than this the first formula applies, and for small loads the second. In pillars flat, but not fixed at the ends, without capitals f' may be zero.

179. We thus see that if a pillar were absolutely straight and homogeneous it would crush, by direct compression if p_0 were greater than f, and by lateral bending if p_0 were less than f, the crushing load being the least of these two quantities; but that the smallest deviation will be augmented by lateral bending, so that the actual crushing load will be less than the least of these quantities. Experience confirms this conclusion. When a long pillar is loaded we do not find that it remains straight till a certain definite load p_0 is reached, and then suddenly bends laterally. We find, on the contrary, that a perceptible lateral deflection is produced by a small load, which gradually increases as the load is increased, till rupture takes place, showing, as we might anticipate, that some small deviation existed originally. And as that deviation evidently depends upon accidental circumstances it is impossible, from imperfection of data, to find the actual crushing load of a pillar for those proportions of height to thickness, for which its effect is greatly augmented by a small deviation. The augmentation is on the whole greatest when

$$f = p_0 = \pi^2 \, . \, n \, . \, E \, . \, \frac{h^2}{L^2};$$

that is, when

$$\frac{L}{h} = \sqrt{\frac{\pi^2 n E}{f}}.$$

This gives, by taking the values of E and f from Table II., Ch. XVIII.

Wrought Iron,	$L = 38\sqrt{\pi^2 n} \, . \, h = 30h$	(Circular Section).
Soft Steel,	$L = 29\sqrt{\pi^2 n} \, . \, h = 23h$	,,
Hard Steel,	$L = 23\sqrt{\pi^2 n} \, . \, h = 18h$	,,
Cast Iron,	$L = 20\sqrt{\pi^2 n} \, . \, h = 16h$	,,

In the case of cast iron there is a difficulty in determining the value of f, but if we suppose that the elasticity of the material is not greatly impaired at half the ultimate crushing load, we get the value given. The case of timber is exceptional, and will be referred to further on. For pillars fixed or half-fixed at the ends the number π^2 is to be replaced by $4\pi^2$ or $2\pi^2$ as before.

Let us assume this condition satisfied, and let us imagine the pillar loaded with three fourths the theoretical maximum crushing load, then by substitution we find, $qa/nh = \frac{1}{3} \, . \, \frac{1}{4}$, or since $n/q = \frac{1}{8}$ for a circular section,

$$\frac{a}{h} = \frac{1}{96},$$

from which it will be seen how small a deviation will cause the pillar to crush under three fourths the theoretical maximum load, when the proportion of height to thickness is that just given. With a pillar of double this height deviation has little influence, and with a pillar of one third this height lateral flexure has little influence on the resistance to crushing.

On the whole, then, it would seem that the most rational way of designing pillars would be to calculate the theoretical maximum load, and then adopt a factor of safety depending on the value of the deviation found from the above formula; it is obvious that in some cases a much larger deviation may be considered likely than in others. For the case of thin tubes see Ch. XVIII.

180. *Gordon's Formula.*—The greater part of our experimental knowledge respecting the strength of pillars is due to Hodgkinson.* His results show that in cast-iron pillars with flat ends, the length of which exceeds 100 diameters, the theoretical maximum is closely approached, while with shorter lengths the strength falls off considerably, as might be expected. In other respects the theoretical laws are approximately fulfilled, the principal difference being that columns with one or both ends rounded are somewhat stronger relatively to columns with flat ends than theory would indicate, an effect which may be partly due to imperfect fixing of the ends. Various empirical formulæ have been given to express the results of experiment on the crushing of pillars. That which has been most used is commonly known as Gordon's. It is so constructed as to agree in form with the theoretical formulæ in the extreme cases in which those formulæ give correct results. As modified by Rankine, only replacing r^2, the square of the radius of gyration, by nh^2, in the notation of this work the formula is

$$\frac{W}{A} = \frac{f}{1 + \dfrac{l^2}{cnh^2}},$$

which becomes, when l/h is small,

$$W = Af,$$

and when l/h is large,

$$W = \frac{cnfAh^2}{l^2} = cf \cdot \frac{I}{l^2};$$

while for intermediate values it gives intermediate results.

* *Phil. Trans.*, 1840, Part II. An abridgment is given in Hodgkinson's work on Cast Iron, cited at the end of Chapter XVIII.

If we compare this last with Euler's formula for a column with flat ends, we get

$$c = 4\pi^2 \frac{E}{f},$$

and this may be called the "theoretical" value of the constant c. The values actually used for c are somewhat different, being deduced from such experiments as have been made, and the results for different forms of section are not always consistent. Rankine gives

VALUE OF CONSTANTS.

	Value of f.	Value of c.
Wrought Iron, . . .	36,000	36,000
Cast Iron, ...	80,000	6,400
Dry Timber, ...	7,200	3,000

These values refer to struts fixed at the ends and to the crushing load. If one end be rounded, the value of c must be divided by 2, and if both ends are rounded, by 4. A large factor of safety must be employed, for reasons already sufficiently indicated.

Rankine's formula has been very extensively tested for the case of wrought columns of large size of various transverse sections, constructed of riveted plates, and has been found to give good results.*

In the case of timber Hodgkinson found, from a limited number of experiments on struts of oak and red pine of small dimensions, a formula which agrees with the formula for the theoretical maximum crushing load when the value of E in that formula is taken as about 900,000 lbs. per square inch. It is possible that the low lateral tenacity of this material increases its flexibility under a heavy crushing load. The formula gives a crushing stress greater than the direct resistance to crushing of the material when L is less than $20h$, which seems hardly probable, and the lower values given by Gordon's formula appear preferable. In the case of steel the value of f may be expected to be increased and the value of c diminished in the ratio of the direct resistance to crushing of steel and wrought iron respectively.

Calculations made by Gordon's formula may be tested by calculating the deviation a by the formula on p. 348; the magnitude of this will be to some extent a measure of the safety of the proposed load.

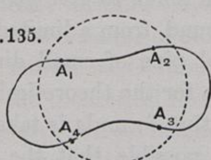
* "Minutes of Proceedings of the Institution of Civil Engineers," vol. liv.

In all cases of struts of large size subject to a heavy load, special care is necessary in considering all the circumstances—if a deflection be occasioned by the unsupported weight of the strut itself, or if, as is often the case, it be constructed of riveted plates, a large margin of safety is desirable. So also in pieces forming part of a machine in which a bending action may be produced by inertia and friction, or which are subject to shocks, the simple thrust alone is often a very imperfect measure of the stress to which they are subject.

Returning to the case of a long slender column we observe that the resistance to crushing depends solely on the stiffness and not on the strength being proportional to the modulus of elasticity. Hence a long column is stronger when made of wrought iron than when made of cast iron, although with short columns the reverse is true. It appears from Gordon's formula that for a ratio of length to diameter of about $26\frac{1}{2}$ the two materials are equally strong. In very long columns steel is not stronger than iron, for its modulus of elasticity is not very different; in shorter lengths however the greater resistance to direct crushing of steel gives it an advantage.

181. *Collapse of Flues.*—There are other cases of crushing by bending. An important one is that of the yielding of a thin tube under *external* fluid pressure. The strength of a tube under external fluid pressure is as different from that of a tube under internal pressure as the strength of a bar under compression is different to its strength under tension.

Fig. 135.



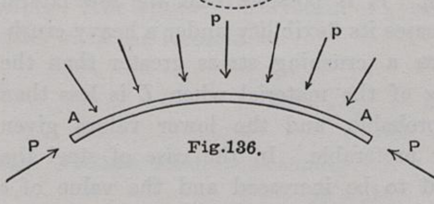


Fig. 136.

A tube perfectly uniform in thickness made of perfectly homogeneous hard material and subject to perfectly uniform normal pressure externally, would theoretically maintain its form until it yielded by the direct crushing of the material. But when the pressure exceeds a certain limit the tube is in a state of unstable equilibrium, and any deviation from perfect accuracy in the above conditions will cause the tube to yield by collapsing, the collapsing being accompanied by bulging. If the tube is very long it will collapse in the manner shown in Fig. 135, the circumference dividing itself up into four arcs two of which are concave outwards and the other two convex. A want of exactness in the construction will in practice generally prevent the collapsing from being symmetrical. Each portion of tube between the points A is under the action of forces applied at the ends towards one another, which crush it by lateral bending just as a long column is crushed. Just before collapsing, each segment AA (Fig. 136), of length s say, will be under the action of a thrust P suppose, applied at the ends tangentially. Equilibrium is maintained by fluid pressure of intensity p on the convex side. When the pressure exceeds a certain limit the equilibrium is unstable, some accidental circumstance determining the position of the point A of contrary flexure, and the consequent length s of any arc.

The thrust per inch length of the tube may be taken as approximately proportional to p. Thus if $t =$ thickness of tube, we may expect that the collapsing pressure would be given by a formula like that which expresses the crushing load of a long slender rod of rectangular section, namely, $p = k't^3/s^2$ where k' is an unknown co-efficient. All other things being equal, the diameter alone varying, the length s of an arc AA would be proportional to the diameter of the tube d, and, under those circumstances, the collapsing pressure would probably vary with t^3/d^2. But the length of the tube, as well as the diameter, influences the value of s. In all practical cases, as in all those on which experiments were made, the ends of the tube are rigidly constructed, and very much support the tube in the neighbourhood from collapsing; thus the proximity of the ends has an important effect in determining the length of the arcs into which the circumference divides itself. If the length of the tube is decreased a limit will be reached below which the tube on collapsing divides itself up into six arcs, three concave and three convex, as shown in Fig. 137. Then the length of each arc will bear a smaller proportion to the diameter than in the long tube. A still shorter tube will, when it collapses, divide it into eight arcs, and so on. Thus the length s is in some way dependent on the length of the tube. The correctness of this reasoning is borne out by experiments made by Fairbairn and others. In Fairbairn's experiments the tubes were made of riveted wrought-

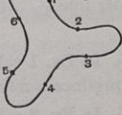


Fig. 137.

iron plates. The ends were made rigid by a strong stay placed within the tube, keeping the ends apart. The tube thus constructed was placed in a larger cylinder of wrought iron and external pressure was applied by forcing water in. The pressure being gradually increased the tube will at last suddenly collapse, making a noise which indicates the instant of the occurrence. The results of the experiments showed that the collapsing pressure may be approximately expressed by the formula

$$p = k\frac{t^2}{ld},$$

the dimensions being all in inches, the co-efficient $k = 9{,}672{,}000$. This formula must not be used for extreme cases nor for tubes of thickness less than $\frac{3}{8}$ inch.

Since a short tube is so much stronger than a long one, we have an explanation of the advantage of riveting a T iron ring around a boiler furnace tube, which amounts to a virtual shortening of the length of the tube. Other formulæ have been proposed, some of which represent the results of experiment more closely, but the materials at present available do not admit of the construction of a satisfactory formula. *

EXAMPLES.

1. Find the thickness of metal of a cast-iron column fixed at the ends, 1 foot mean diameter, 20 feet high, to carry 100 tons. Factor of safety, 8. *Ans.*—Thickness 1″.

2. Find the crushing load of a wrought-iron pillar 3″ diameter, 10 feet high, free at the ends. *Ans.*—Crushing load = 66,218 lbs. = 30 tons nearly.

3. If in last question the pillar were of rectangular section of breadth double the thickness, what sectional area would be required for equal strength? *Ans.*—Sectional area = 9·4 square inches instead of 7 square inches as before.

4. Find the collapsing pressure, according to Fairbairn's formula, of a cylindrical boiler flue $\frac{7}{16}$″ thick, 48″ diameter, and 30 feet long. *Ans.*—Collapsing pressure = 107 lbs.

5. In Ex. 1 calculate the deviation of the line of action of the load from the axis to produce a maximum stress of 10,000 lbs. per square inch. *Ans.*—1·8″.

6. In Ex. 2 calculate the deviation to produce a maximum stress of 9,000 lbs. per square inch with a load of 11,000 lbs. or of 22,000 lbs. *Ans.*—1·2″ or $\frac{1}{12}$″.

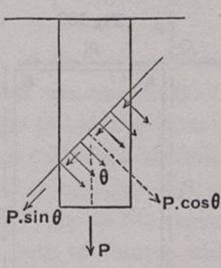
* See a paper by Professor W. C. Unwin, *Minutes of the Proceedings of the Institution of Civil Engineers*, from which the preceding remarks are partly taken. Some other cases of crushing by bending will be given in the Appendix.

CHAPTER XV.

SHEARING AND TORSION OF ELASTIC MATERIAL.

182. *Distinction between Tangential and Normal Stress.—Equality of Tangential Stress on Planes at Right Angles.*—In the cases we have hitherto considered of simple tension, compression, and bending, the stress on the section under consideration has been at all points normal to the section. But we may take our section inclined at any angle to the stress, and the mutual action is then not normal to the section. The particles on each side of the section partly act on one another in the direction of the section itself, and so constitute a stress analogous to friction, resisting the sliding of one portion relatively to the other. Such a stress is called *tangential* or *shearing stress*, being the stress called into action by shearing.

Fig.138.



Let us return to the case of the stretched bar carrying a load P (Fig. 138). On a transverse section of the bar only a normal stress is produced. Now suppose we take an oblique section, whose normal makes an angle θ with the axis of the bar, and let us resolve the force P into two components, one perpendicular and the other parallel to the section. The normal component $P\cos\theta$ tends to produce a direct separation at the section, producing a tensile stress similar in character to that on a transverse section, but of less intensity.

If A = area of transverse section of bar, then $A\sec\theta$ = area of oblique section; the intensity of the normal stress

$$pn=\frac{P\cos\theta}{A\sec\theta}=\frac{P}{A}\cos^2\theta=p\cos^2\theta,\text{ where }p=\frac{P}{A}.$$

The other component $P \sin\theta$ produces a tangential or shearing stress of intensity

$$p = \frac{P \sin\theta}{A \sec\theta} = p \sin\theta \cos\theta.$$

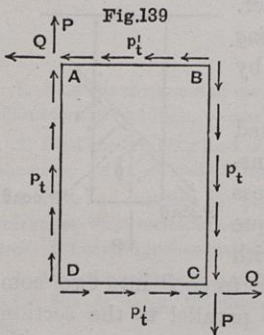
Similarly if the bar is subjected to a compressive instead of a tensile load.

Many materials which offer great resistance to direct compression yield by sliding across an oblique plane. Now p_t is a maximum when $\theta = 45$, this is therefore approximately the angle of separation. The same maximum stress, the value of which is $p/2$, occurs on another plane sloping the other way at an angle of 45°. We sometimes find fracture to occur across two oblique planes; sometimes across one only.

If in $p_t = p \sin\theta \cos\theta$ we change θ into $90 + \theta$, p_t has the same value; so that the intensity of the tangential stresses on two planes at right angles to one another is the same. This is true generally in all cases of stress, as will be seen presently.

183. *Tangential Stress equivalent to a Pair of Equal and Opposite Normal Stresses. Distorting Stress.*—In the example we have just considered we have both shearing and normal stress; but there are cases in which there is only a shearing stress. Let $ABCD$ (Fig. 139) be a rectangular plate of thickness t. Over the surfaces BC and AD suppose a tangential stress to be applied of intensity p_t. Calling b and a the length of the sides of the plate, the total amount of the tangential stress on each side is

Fig.139



$$P = p_t \,.\, bt.$$

To prevent the turning of the plate, suppose the forces P balanced by the application of an uniform stress over the surfaces BA and DC, of intensity p'_t. The amount of the force on each of these sides,

$$Q = p'_t \,.\, a \,.\, t.$$

Since equilibrium is produced, the moment of the couple P must be equal to the moment of the couple Q.

$$\therefore p_t \cdot bt \cdot a = p'_t \cdot at \cdot b;$$

$$\text{or } p_t = p'_t:$$

that is, the intensity of the stress is the same on BA as on AD.

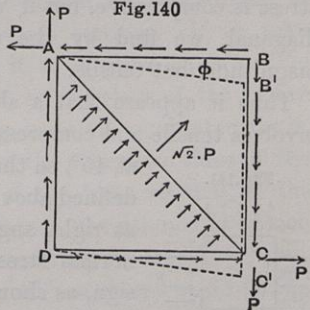
Shearing therefore cannot exist along one plane only. It must be accompanied by a shearing stress of equal intensity along a plane at right angles. Such a pair of stresses unaccompanied by normal stress constitute a Simple Distorting Stress, so called because it distorts the elements of the body.

Let us now assume, for simplicity, the plate to be square (Fig. 140). The effect of the forces is to produce a change of form, which, in perfectly elastic bodies, is exactly proportional to the shearing force which produces it. The square $ABCD$ becomes a rhombus $AB'C'D$, the angle of distortion ϕ being proportional to the stress p_t. We may write

$$p_t = C\phi,$$

where the co-efficient C is a kind of Modulus of Elasticity, but of a different nature from that previously employed. The volume of the elastic body A is in general practically unaltered. Under the action of the forces it has simply undergone a change of form or figure, and the co-efficient C which connects the change of form with the stress producing it, is a co-efficient of elasticity of figure. It is sometimes called the *modulus of transverse elasticity*, but preferably the *co-efficient of rigidity.*

Fig.140



The ordinary (Young's) modulus of elasticity E connects the stress and strain in a bar when it undergoes changes both of volume and figure. The co-efficient of rigidity C for metallic bodies is generally less than $\frac{2}{5}E$, and for wrought-iron bars may be taken as 10 to $10\frac{1}{2}$ millions.

Let us now take a section of the square plate (Fig. 140) along one of the diagonals and consider the forces which act on the two sides of the triangular upper portion. Resolve these forces parallel and perpendicular to the diagonal. The components of the two P's along the diagonal balance one another, and there will be no tendency for this triangular portion to slide relatively to the other; that is to say, there is no shearing stress on the diagonal section. But the other components, perpendicular to the diagonal, cause the upper triangular portion to press on the lower with a force

$$2\frac{P}{\sqrt{2}}=\sqrt{2}\,.\,P.$$

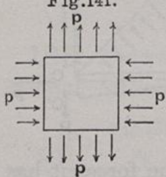
If we divide this force by the area of the diagonal section over which it is distributed, we obtain the intensity of this normal stress,

$$p_n=\frac{\sqrt{2}\,.\,P}{\sqrt{2}\,.\,at}=p_v$$

On the diagonal section AC which we have been considering, this stress is compressive, but if we take the section along BD, the other diagonal, we find by the same reasoning a stress of the same magnitude, but tensile.

Thus it appears that a shearing stress on any plane necessarily involves tensile and compressive stresses of equal intensity on planes at 45°, so that a simple distorting stress, which was defined above as a pair of shearing stresses on planes at right angles, may also be defined as a pair of normal stresses of equal intensity and of opposite sign, as shown in Fig. 141.

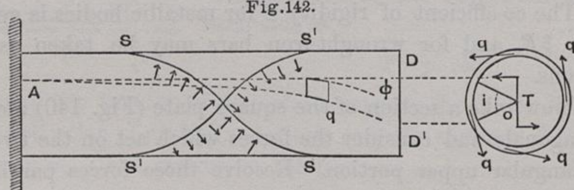
Fig.141.



We now proceed with various examples of this kind of stress, commencing with the case of torsion. Torsion was mentioned as one of the five simple straining actions to which a bar as a whole may be exposed. It is produced by a pair of equal couples applied at the ends of the bar, the axis of the couples being the axis of the bar.

When we consider the nature of the elastic forces called into

Fig.142.



action amongst the particles of the bar, Torsion reduces to a case of Shearing. To understand this, we will begin with a simple case. Imagine a thin tube (Fig. 142) with one end fixed, and the other acted on by a uniform tangential stress of intensity q. Let t be the thickness and d the mean diameter of the tube, then

$$\text{Sectional area of tube} = \pi dt \text{ approximately ;}$$
$$\text{Total shearing force} = q\pi dt \text{ ;}$$

and since the force on each unit of area of the section acts approximately at the same distance from the centre of the tube, the total twisting moment $= q\pi dt \times \frac{1}{2}d = \frac{1}{2}q\pi d^2t$. This twisting moment is balanced by the resistance to turning offered at the fixed end. At any transverse section KK of the tube there will be produced a uniform stress of intensity q.

Let us now consider a small square traced on the surface of the tube, with two sides on two transverse sections. If we take the square small enough we may treat it as a plane square. To balance the shearing stress q, which acts on the sides of the square lying in the transverse planes, a shearing stress of equal intensity is, as explained above, called into action on the other two sides of the square, in the direction of the length of the tube, so that, if the tube were cut by longitudinal slits, the power of resistance to torsion would be as effectually destroyed as if it were cut by transverse slits. But if we make spiral slits at an angle of 45°, as shown at SS in Fig. 142 ; supposing the slits indefinitely fine, and no material removed, the strength of the tube to resist torsion in the direction shown would not be impaired. The material of the tube would then be divided into spirally-bent ribands, which would be in tension along their length, and in compression laterally, the ribands being caused to press against one another. Along a second set of spirals such as $S'S'$, longitudinal compression and lateral tension exist ; the lateral forces are indicated in both cases by arrows in the figure.

So much for the state of stress induced in the tube by the torsion. Next as to the change of form which accompanies the stress. The square will be distorted into a rhombus. A straight line AD, drawn on the surface parallel to the axis of the tube passing through the centre of the square, will be twisted into a spiral AD', the angle of the spiral being the angle of distortion of the square. Let θ be that angle, then

$$q = C\theta, \text{ where } C \text{ is the co-efficient of rigidity.}$$

The effect of this is that, relatively to the end A, the end D is twisted round through an angle $DOD' = i$ suppose, called the angle of torsion.

In circular measure $i = \frac{\text{arc } DD'}{r}$ (r = radius of tube). Also arc $DD' = l\theta$, θ being a small angle. Therefore $i = l\theta/r$. Since also $\theta = q/C$, we have the angle of torsion $i = ql/Cr$, in terms of the stress. From this we may express the angle of torsion in terms of the twisting moment producing the torsion.

184. *Torsion of a Shaft.*—We now pass on to the consideration of the torsion of a solid cylindrical shaft. First, let us imagine the shaft to be made up of a number of concentric tubes exactly fitting one another, and let us further imagine that at the end of each tube a suitable twisting moment is applied, so that each tube is twisted round through exactly the same angle. This effect will be produced by applying over the section at the end of each elementary tube a tangential stress, which is proportional to the radius of the tube. If we make $q/r = q_1/r_1$, where q_1 and r_1 refer to the outside tube, then the angle of torsion will be the same for all the tubes, and they will not tend to turn relatively to one another, but all together. We may then suppose them united together again in a solid mass. If the stress applied be proportional to the distance from the centre, the shaft will twist just as if it were a set of tubes, each being subjected to the same stress and strain as if it were an independent tube.

Now in the actual case of the twisting of a solid shaft, all portions from the outside inwards to the centre must turn through the same angle, and hence the shearing stress at any point of the section of the shaft must be proportional to its distance from the centre. This is true except very near the point of application of the twisting moment. Suppose, for example, the twisting moment is applied by means of a wheel keyed on the shaft, then in the immediate neighbourhood of the key-way, the stress will not be as stated, but at a short distance along the shaft the stress distributes itself in the manner described. This is another instance of the general principle already employed in the case of stretching and bending.

The total resistance to torsion of the solid shaft is the sum of the twisting moments of all the concentric tubes into which it may be imagined to be divided. Thus

$$T = \Sigma 2\pi r^2 tq\,;\ \text{in which } q = r\,.\,\frac{q_1}{r_1}.$$

$$\therefore\ T = \Sigma_0^{r_1} 2\pi r^2 tr \frac{q_1}{r_1} = \frac{q_1}{r_1}\Sigma_0^{r_1} 2\pi rt\,.\,r^2,$$

that is, the product of the sectional area of each tube multiplied by the distance squared of the area from the axis of the shaft must be taken and summed. The result is called the Polar Moment of Inertia, which may be written I. Its value is $\frac{1}{2}\pi r_1^4$. Thus

$$T = \frac{q_1}{r_1} I = \frac{q_1}{r}\frac{\pi}{2} r_1^4 = \frac{\pi}{2} q_1 r_1^3.$$

It is not to be supposed that the strength of a shaft of any section to resist torsion is proportional to the polar moment of inertia of that section. In non-circular sections the stress is generally greatest not at the points farthest away from the centre, but more often at those which are nearest the centre. The cases of a rectangle, an ellipse and various other forms have been investigated by M. St. Venant who has obtained the annexed results.*

Relative Strengths of Shafts of the same Sectional Area.	
FORM OF SECTION.	STRENGTH.
Circular, - - - - - - - -	1
Square, - - - - - - - -	·8863
Rectangle with sides in the ratio $n : 1$, -	$\sqrt{\frac{2}{n+1/n}} \times$ ·8863
Ellipse with axes in the ratio $n : 1$, - -	$\sqrt{n}$ $(n < 1)$

Dropping the suffixes, taking r to be the outside radius, we can write the moment of resistance to torsion of the shaft,

$$T = \tfrac{1}{2}\pi f r^3,\ \text{or}\ \tfrac{1}{16}\pi f d^3;$$

where f is the co-efficient of strength of the material to resist shear-

* Diagrams and particulars with respect to M. St. Venant's results will be found in Sir W. Thomson's *Treatise on Natural Philosophy*, 1st ed., vol. 1, p. 545.

ing. Thus the strength under torsion is proportional to the cube of the diameter. The formula shows that, assuming f to be the same in each case, the strength of a shaft to resist a twisting moment is double its strength to resist a bending moment. Since $i = ql/Cr$ we can eliminate q, and thus obtain

$$i = \frac{2}{\pi C} \cdot \frac{l}{r^4} \cdot T.$$

185. *Diameter of Shaft to transmit a Given Power.*—Having determined the diameter of shaft required to take a given twisting moment we are now able to obtain a solution of the practical question, What diameter of shaft is required to transmit a given horse-power at a given number of revolutions per minute?

Let $T_0 =$ *mean* twisting moment transmitted in inch-tons, then $T_0 \times 2\pi N =$ work transmitted per minute in inch tons, where $N =$ revolutions per minute of shaft.

Let HP denote the horse-power to be transmitted, then

$$T_0 \times 2\pi N = \frac{33000 \times 12}{2240} H.P.$$

$$\therefore T_0 = \frac{33000 \times 12}{2240 \times 2\pi} \frac{H.P.}{N}.$$

Now the shaft must be strong enough to take not only the mean but the maximum twisting moment.

We may express the maximum in terms of the mean by writing $T = KT_0$, where K is a co-efficient whose value is different in different cases and $T =$ maximum twisting moment, but

$$T = \frac{\pi}{16} fd^3 \text{ or } d^3 \frac{16T}{\pi f}.$$

$$\therefore d^3 = \frac{16 \times 33000 \times 12}{2\pi^2 \times 2240} \frac{K}{f} \frac{H.P.}{N},$$

and

$$d = 5{\cdot}233 \sqrt[3]{\frac{K}{f} \frac{H.P.}{N}}.$$

The value of f depends in some measure on the fluctuation to which the twisting moment is subject, but under ordinary circumstances should not exceed $3\frac{1}{4}$ tons per square inch (Art. 221) for wrought iron, or, probably, about 5 tons for steel, and $2\frac{1}{4}$ tons for cast iron. The value of K, the ratio of maximum to mean twisting moment, depends on the circumstances discussed in Chapter X. We may assume it equal to $1\frac{1}{2}$ under ordinary circumstances, allowing a small addition for the bending due to the weight of the shaft. On substitution we obtain for wrought iron

$$d = 4\sqrt[3]{\frac{H.P.}{N}}.$$

This formula agrees closely with the best practice in screw-propeller shafting.

When the amount of bending to which the shaft is subject is considerable, as in the case of crank shafts, the diameter determined by this formula is too small. It will be seen hereafter that when all the forces acting on the shaft are known, a value of K can be calculated which gives the effect of bending. If we assume $K = 2$, the co-efficient 4 in the above formula will be replaced by 4·5, and this agrees closely with practice in the crank shafts of marine screw engines. In other cases a still larger value may be necessary.

In the formula for the angle of torsion

$$i = \frac{ql}{Cr};$$

if we replace q by its working value for wrought iron (7,200 lbs.), C by 10,500,000 lbs., and i by the circular measure of 1°, we find

$$l = 12.7d,$$

showing that under the working stress the shaft twists through 1° for each $12\frac{3}{4}$ diameters in its length. For many purposes this is much too small, and the dimensions of a shaft then depend on stiffness, not on strength, as in the case of beams (Art. 168). The greatest angle of torsion permissible depends in great measure on the irregularity of the resistance, and no general rule can therefore be laid down for it. If the angle of torsion be given and the length, the diameter will depend on the fourth root of the twisting moment, as shown by the formula of Art. 184. In this, as in other cases where dimensions depend on stiffness, not on strength, steel has no advantage over iron, because the co-efficients of elasticity of the two materials are the same, or nearly so. A hollow shaft is both stronger and stiffer than a solid shaft of the same length and weight.

186. *Distance apart of Bearings.*—The distance apart of the bearings of a shaft depends on the stiffness necessary to resist the bending due to the weight of the shaft itself, and of any pulleys or wheels upon it, together with the tension of belts and other similar forces. If the total load be equivalent to m times the weight of the shaft itself uniformly distributed, the length between bearings for a wrought iron or steel shaft d inches diameter will be given approximately for a stiffness of $\frac{1}{1200}$th by Ex. 7, p. 339.

When, as in screw propeller shafting, the bearings are liable to get out of line, too great stiffness in a shaft will produce great straining actions upon it.

187. *Web of a Beam of I Section.*—Torsion is one of the few cases in practice where a simple distorting stress occurs alone and not in combination with other kinds of stress. It generally happens that a normal stress is combined with it; such, for example, is the case in the web of a beam of I section, to which we next proceed to direct our attention. Taking a transverse section, the normal stress at a point distant y from the neutral axis is given by the formula

$$\frac{p}{y}=\frac{M}{I},$$

and is therefore the same for the same values of M and I, whether the web be thin or thick, while it will be shown presently that the tangential stress is greater the thinner the web, and becomes the most important element when the web is thin.

Let us suppose, for simplicity, the flanges equal, and also that the beam is supported at the ends and loaded in the centre with a weight W.

As we have previously seen, the flanges will sustain the greater

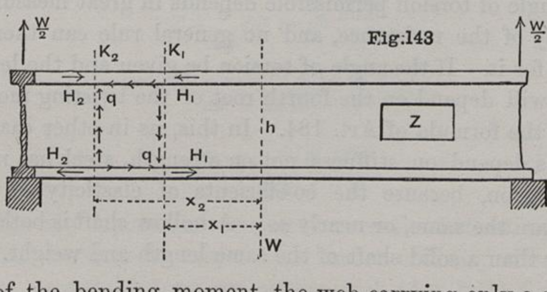


Fig:143

portion of the bending moment, the web carrying only a small portion of it, $\frac{1}{6}$, if the area of the web equals the area of each flange. For simplicity, let us imagine the flanges to take the whole of the bending. Let K_1 and K_2 (Fig. 143) be two transverse sections of the beam at distances x_1 and x_2 from the centre of the beam, $2a$ being the span of the beam, the bending moment at the first section,

$$M = \tfrac{1}{2}W(a - x_1) \text{ and at the 2nd } M_2 = \tfrac{1}{2}W(a - x_2).$$

Now, supposing the flanges to take the whole of the bending the stress H produced on the flanges is given by the formula

$$Hh = M. \quad \text{Thus at } K_1 \text{ we have } H_1 = \frac{W(a - x_1)}{2h},$$

$$\text{and at } K_2 \text{ we have } H_2 = \frac{W(a - x_2)}{2h},$$

and similar forces on the bottom flange only reversed in direction. There will thus be a resultant force $H_1 - H_2$ tending to push the portion K_1K_2 of the flange to the left,

$$H_1 - H_2 = \frac{W(x_2 - x_1)}{2h}.$$

This force is balanced by the resistance of the web to shearing along the line of junction with the flange.

Since H_1H_2 is proportional to the length of K_1K_2, the shearing force per unit of length of web $= W/2h$. If we suppose t to be the thickness of the web, the intensity of the shearing stress will be

$$q = \frac{W}{2ht}.$$

Thus, considering the portion of the web between the sections K_1 and K_2 apart by itself, we see that on the upper and lower horizontal edges of it, where it joins the flanges, it is subject to a shearing stress of intensity q. Now, to balance this stress there must act on the vertical sides KK a shearing stress of equal intensity q. Now, the shearing force for the vertical sections KK is $\tfrac{1}{2}W$. Supposing the web to be of rectangular section and of height h, then, assuming the whole of the shearing force to be borne by the web, the intensity of the shearing stress on the vertical sections is

$$q = \frac{W}{2ht}.$$

Therefore the assumption that the flanges take the whole of the bending moment is equivalent to supposing the web to take all the shearing. Assuming this, we see that the shearing stress, being uniformly distributed over the vertical section, will be accompanied by an equal shearing stress on any horizontal section. When considered alone, the effect of these shearing stresses on planes at right angles to one another is to produce tensile and compressive stresses on the web in directions making an angle of 45° with the horizontal and vertical planes; and thus the web may be superseded by an indefinite number of diagonal bars inclined at an angle of 45°, thus forming a lattice girder.

If the web is designed so as to be strong enough only to withstand the shearing stress, replacing q by f the co-efficient of strength against shearing f, we find

$$t = \frac{W}{2hf}.$$

The influence of the normal stress due to bending will be considered in the next chapter. Its effect is greatly to increase the strain on the web (see Art. 202), which besides will in most cases exhibit weakness on account of the compressive stress in one of the diagonal directions. If the distance between the flanges is great, the web will be liable to yield by buckling or lateral flexure (see page 317). To prevent this, the web must be stiffened by angle irons rivetted on it. But the girder would then be made heavy, and it is therefore more economical to make large girders with openwork diagonal bracing.

We have in this investigation supposed the beam loaded in the middle, so that the shearing force is uniform throughout the length of each half, and the problem was thus simplified. But the same principles apply if the load be distributed in any manner. The shearing force will then vary from point to point along the beam.

188. *Distribution of Shearing Stress on the Section of a Beam.*—In beams of other types it is still true that the central parts of the beam are subject to shearing, but the total amount of the shearing stress being the same, its intensity is much less, because it is distributed over a greater area. The intensity at the centre of the beam is found as follows for a beam of uniform transverse section.

Suppose the beam supported at the ends and loaded in the middle as before, and take section K_1K_1', K_2K_2'. Let NN be the neutral surface, SS the neutral axis (as in Fig. 122, Art. 153). Above the neutral surface the beam is compressed and below it it is stretched by equal forces. Let these forces be H_1 for the section K_1K_1, and H_2 for the section K_2K_2; then, reasoning as before, the shearing force in the neutral surface must be $H_1 - H_2$, and the intensity of that shearing stress, if b_0 be the breadth at the centre,

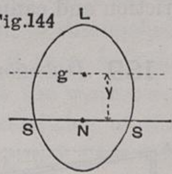
$$q = \frac{H_1 - H}{NN \cdot b_0}.$$

Now to find H we have, in the notation of Art. 154,

$$H = \Sigma bt \cdot p = \frac{M}{I} = \Sigma bt \cdot y = \frac{M}{I} \cdot kA\bar{y},$$

where kA is the area of that part of the section A which lies above the neutral axis (SLS in Fig. 144), and $\bar{y}$ is the distance of its centre of gravity (g) from that axis. The same result will be obtained if we take that part of the area which lies below the axis. We now have, as before, by substitution,

Fig.144



$$H_2 - H_1 = \frac{W(x_2 - x_1)}{2} \cdot \frac{k \cdot A\bar{y}}{I},$$

whence, as usual, replacing I by nAh^2, we find

$$q = \frac{W}{2n} \cdot \frac{k \cdot \bar{y}}{b_0 h^2}.$$

The total shearing stress on the section is $\frac{1}{2}W$, and therefore the mean intensity is

$$q_0 = \frac{W}{2A}.$$

Thus we obtain the ratio of the shearing stress on the neutral surface to the mean shearing stress on the whole transverse section.

$$\frac{q}{y_0} = \frac{A}{b_0 h} \cdot \frac{k \cdot \bar{y}}{nh}.$$

In the present case where the beam is loaded in the middle the shearing stress is the same at all points of the neutral surface, but in other methods of loading this will not be the case. The formula however in all cases gives the ratio in question correctly, which will be found to be greater than unity. In fact it is not difficult to see that the shearing stress must be greatest at the neutral surface, and must diminish to zero as we approach the external surface of the beam. The formula then gives the maximum shearing stress on the section.

Let us for example take a rectangular section, then

$$A = b_0 h : k = \tfrac{1}{2} : n = \tfrac{1}{12} : \bar{y} = \tfrac{1}{4}h.$$

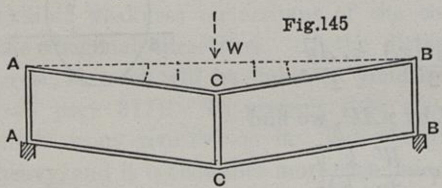
$$\therefore \frac{q}{q_0} = \frac{12}{8} = \frac{3}{2},$$

so that the greatest shearing stress is $1\frac{1}{2}$ times the mean. In like manner in a circular section it may be shown to be as 4 : 3. Other cases are given in the examples at the end of this chapter.

In all cases where a bar is subject to shearing and the sides of the bar are free from tangential stress, the stress on the transverse section will be increased in this way. In pin joints where the pin is an easy fit the only tangential stress on the sides of the pin will be due to friction and cannot be relied on.

189. *Deflection due to Shearing.*—A certain part of the deflection of a beam is due to the distortion of its central parts.

Fig.145



Returning to the beam of I section, loaded in the middle, suppose the flanges hinged at the centres, and let vertical stiffening pieces AA, BB, CC, be rigidly connected to the web but hinged to the flanges, then distortion of the web takes place as shown in a very exaggerated way in the figure (Fig. 145), causing a deflection δ of the beam such that

$$\frac{\delta}{\frac{1}{2}l} = i = \frac{q}{C} = \frac{W}{2htC},$$

where C as before is the co-efficient of rigidity, and q the shearing stress is expressed as before.

$$\therefore \ \delta = \frac{Wl}{4htC} = \frac{ql}{2C}.$$

For wrought iron take $q = 9,000$ for the working load and $C = 9,000,000$, then

$$\delta = \frac{l}{2000},$$

which is about half the working deflection due to bending in ordinary cases.

This calculation however greatly exaggerates the deflection due to shearing even in a beam of I section, for the web cannot in general be so thin as to give a stress of 9,000 lbs. per square inch, and the effect is much less for a uniformly distributed load. Nevertheless in beams of this class the deflection due to shearing is a sensible part of the whole, the more so as in rivetted girders the union of the parts seldom renders them completely rigid. This is the principal reason why large girders show a considerably smaller modulus of elasticity when the deflection is calculated in the usual way than solid bars. In bars this part of the deflection is insensible, the distorting stress being small.

190. *Weakening of Beams by Insufficient Resistance to Longitudinal Shearing of the Web.*—If the central part of a beam be cut away as shown at Z in Fig. 143, the strength of the beam will be diminished and its deflection increased. This will be true even if there be only a narrow longitudinal slot at the neutral surface, but the weakening is the greater the more material is cut away, the condition of the beam in an extreme case becoming that of an N girder (Art. 25) without diagonal bracing. Imperfect union of the parts of the web along either a longitudinal or vertical section will have the same effect in a less degree. Wooden ships not unfrequently exhibit weakness due to this cause, and to counteract it diagonal riders of iron are introduced to take part of the shearing force. The ordinary formula for resistance to bending cannot be applied in such cases.

191. *Joints and Fastenings.*—Among the most important cases of shearing are those which occur in joints and fastenings of all kinds. Such questions are generally very complex, considered as purely theoretical problems, and the direct results of experience are always required at every step to interpret and confirm theoretical conclusions.

When two pieces butt against each other the pressure is transmitted by contact only, and fastenings are therefore required not for transmission of stress but merely to retain the pieces in their relative positions. With tension it is otherwise; it is still necessary to have surfaces which press against one another, and these can only be obtained by the introduction of fastenings which transmit stress laterally, and are therefore subject to shearing and bending. The parts of a joint should be so proportioned as to be of equal strength. One of the simplest examples is that of a pin joint connecting two bars in tension as in a suspension chain with bar links. Fig. 1. (Plate VIII.) shows a pair of bars of rectangular section connected together by links C and D united as shown by pins passing through eyes at their extremities. In suspension chains there are generally four or five bars placed side by side, but the principle is the same in any case. The pull on the chain is balanced by the resistance to shearing of the pins, which have besides to resist bending. Let d be the diameter of the pins, b the breadth, t the thickness of one of the bars, t' the thickness, b' the breadth of the links which for equality of strength, that is to say, of sectional area, will be connected by the equation

$$2b't' = bt.$$

Let f be the co-efficient of strength for tension, then $\frac{4}{5}f$ (Art. 224) will be the co-efficient for shearing, whence remembering that the maximum shearing stress exceeds the mean in the ratio $4:3$ as shown above,

$$P = btf = 2\pi\frac{d^2}{4} \cdot \tfrac{3}{5}f = \frac{3\pi}{10}fd^2.$$

According to this estimate the area for shearing should be five-thirds the area for tension, but the true ratio is probably not so great: the calculation supposes that the sides of the pin are subject to normal stress alone, whereas the tangential stress due to friction must be considerable. Besides the strength of iron such as is used for pins is greater than that of plates. As the calculation applies only to stress within the elastic limit, it is impossible to test it by experiment. In practice the areas are made nearly equal when nothing else is considered except resistance to shearing. When, however, such a joint is actually pulled asunder it frequently gives way in quite a different manner before shearing commences. Imagine a cylinder pressed down into a semicircular hollow which it very exactly fits, and let the material be elastic and soft compared with the cylinder, then, reasoning as in Art. 115, p. 249, it appears that the stress between the surfaces will be given by the equation

$$p = p_0 . \cos\theta,$$

and if P be the pressing force, l the length,

$$p_0 . \tfrac{1}{4}\pi dl = P \text{ or } p_0 = \frac{4P}{\pi dl}.$$

If the pin fits the eye exactly the pressure will follow this law so long as the tension is small. As the tension increases, however, the pressure becomes more uniformly distributed over the semicylinder, because the eye-hole tends to contract laterally as the links of a chain of rings would do under tension. The other extreme supposition would be to suppose it uniformly distributed, then

$$p_0 \, . \, dl = P \text{ or } p_0 = \frac{P}{dl}.$$

The actual pressure will be intermediate between these two values. If p_0 be too great the metal crushes under the pressure. The theoretical limit to p_0 will be considered hereafter (Art. 222); for the present it will be sufficient to say that the experiments of Sir C. Fox* have shown that the curved area should be at least equal to the sectional area under tension, that is to say we ought to have

$$\tfrac{1}{2}\pi dl = bt = \tfrac{1}{10} 3\pi d^2.$$

To satisfy these conditions we must have for the ordinary case where the thickness of the eye is the same as that of the rest of the bar

$$d = \tfrac{2}{3}b : t = \tfrac{2}{5}b \text{ approximately.}$$

The first of these gives the diameter of pin recommended by Sir C. Fox and other authorities; the second gives the greatest thickness of link for which this diameter gives sufficient resistance to shearing, but the thickness in actual examples of suspension links is generally considerably less. The pin has also to resist bending, but of small amount in the present example. The sides and end of the eye are subject to tension, but it is not uniformly distributed, the question being similar to that of a thick hollow cylinder under internal fluid pressure. The mode in which the eye crushes and then fractures transversely by tension, is shown in Plate VIII., and further described in Chapter XVIII.

In rivetted joints the question is further complicated by the friction between the plates united by the rivets. On the subject of joints and fastenings the reader is referred to Prof. W. C. Unwin's work cited on page 134.

EXAMPLES.

1. Find the diameter of a shaft for a twisting moment of 1000 inch-tons; stress allowed being $3\frac{1}{2}$ tons per square inch. *Ans.* Diameter = 11·3″.

2. From the result of the previous question deduce the diameter of a shaft to transmit 5000 H.P. at 70 revolutions per minute. Maximum twisting moment $= \frac{3}{2}$ the mean. *Ans.* 15·7″.

* Proceedings of the Royal Society, vol. xiv., p. 139.

3. The angle of torsion of a shaft is not to exceed 1° for each 10 feet of length. What must be the diameter for a twisting moment of 100 inch-tons—modulus of transverse elasticity, 10,500,000?

Compare the result with the diameter determined from consideration of strength, taking a co-efficient of $3\frac{1}{4}$ tons. *Ans.* Diameter determined from consideration of stiffness $= 6{\cdot}2''$. Diameter from consideration of strength $= 5{\cdot}2''$.

4. Show that the resilience of a twisted shaft is proportional to its weight.

$$\textit{Ans.}\ \text{Resilience} = \tfrac{1}{2}Ti = \frac{f^2}{C} \times \frac{\text{Volume}}{4}.$$

5. Compare the strengths of a solid wrought iron shaft and hollow steel shaft of the same external diameter, assuming the internal diameter of the hollow shaft half the external, and the co-efficient for steel $1\frac{1}{2}$ times that for iron.

6. The external diameter of a hollow shaft is double the internal. Compare its resistance to twisting with that of a solid shaft of the same weight and material.

$$\textit{Ans.}\ \text{Strength is greater in the ratio } \frac{5\sqrt{3}}{6} = 1{\cdot}443.$$

7. A pillar, whose sectional area is $1\frac{1}{3}$ square feet, is loaded with two tons. Find in lbs. per square inch the intensity of the tangential stress on a plane inclined at 15° to the axis of the pillar. *Ans.* Tangential stress $= 5{\cdot}18$ lbs.

8. In a single rivetted lap joint, the pitch of the rivets being three diameters or six times the thickness of the plates, find, 1st, the mean stress on the reduced area; 2nd, the shearing stress on the rivets; and, 3rd, the mean direct stress between rivet and plate: the tension of the joint being 4 tons per square inch of the original area, and the friction between the two surfaces of the plate in contact neglected.

Ans. Mean tension on reduced area - $= 6$ tons.
Shearing stress on rivet - - $= 7{\cdot}6$ tons.
Mean direct stress $\dfrac{4 \times \text{pitch} \times \text{thickness}}{\text{diameter} \times \text{thickness}} = 12$ tons per sq. in.

9. In a beam of I section with flanges and web which may be considered as rectangles, the thickness of each flange is one sixth the outside depth of the beam, and the breadth twice the thickness. The thickness of the web is half that of the flanges: find the ratio of maximum to mean shearing stress on the section. *Ans.* $\frac{18}{7}$.

10. In the last question find the fraction of the whole shearing force which is taken by the web. *Ans.* 80 per cent.

11. If the sectional area of the web of a flanged girder be proportional to the shearing force and the r^{th} power of the depth; find the most economical ratio of span to depth and the limiting span.

If the web be C and each flange A, as on page 317, the whole sectional area is $C + 2A = S$ and the moment of resistance to bending is

$$M = fh(\tfrac{1}{2}S - \tfrac{1}{3}C).$$

Assuming now $C = c\,.\,h^r$, where c is constant,

$$\frac{M}{fh} + \tfrac{1}{3}ch^r = \tfrac{1}{2}S,$$

and therefore, for a given value of M, S is least when

$$M = \tfrac{1}{2} \cdot \frac{r}{r+1} \cdot fSh : \quad C = \frac{3S}{2(r+1)}.$$

In a girder with lattice web the same formula for M holds good, but $S = C(r+1)$.

If now $F = f'C$, where F is the shearing force and f' is a co-efficient much less than the resistance to shearing on account of the necessary stiffening (Art. 187),

$$M = \tfrac{1}{3} \cdot r \frac{f}{f'} \cdot Fh,$$

a formula which will give the required ratio (N) for any given load. If the load be uniformly distributed

$$N = \frac{4r}{3} \cdot \frac{f}{f'}.$$

It is probable that in most cases $r = 2$ nearly, but that the value of f/f' will vary, according to the type of girder, from 2 to 4, being greatest for a continuous web.

The limiting span of a girder of uniform section is readily shown to be

$$L = \frac{4r}{r+1} \cdot \frac{\lambda}{N}. \qquad \text{(Comp. Ex. 13, p. 324.)}$$

The weight of a smaller girder of the same type is found as in Ch. IV.

On the influence of size on the strength of vessels, see papers by Mr. John and the late Mr. Froude in the *Transactions of the Institutions of Naval Architects* for 1874.

12. Show that the weight in lbs. of a shaft to transmit a given horse power at a given number of revolutions is

$$W = 21{,}000 \cdot \frac{K \cdot HP}{N\lambda} \cdot \frac{l}{d},$$

the value of λ being given as in Ch. XVIII, the proper co-efficient of resistance to shearing being used. The rest of the notation is explained on page 362.

The distance to which power can be transmitted by shafting with a given loss by friction is given by Ex. 18, p. 272, when the angle of torsion is immaterial, but in practice is generally limited by the necessity of having sufficient stiffness. The bending and twisting of shafts is considered in Chapters XVII., XVIII.

CHAPTER XVI.

IMPACT.

192. *Preliminary Remarks. General Equation of Impact.*—Hitherto the forces applied to the body or structure under consideration have been imagined to have been originally very small, and to have increased gradually to their actual amount. This is seldom exactly the case in practice, while it frequently happens that the load is applied all at once, or that it has a certain velocity at the instant it first comes in contact with the body. Such cases may all be included under the head of IMPACT, and will form the subject of the present chapter.

When a body in motion comes into contact with a second body against which it strikes, a mutual action takes place between them, which consists of a pair of equal and opposite forces, one acting on the striking body, the motion of which it changes, the other on the body struck which it in general moves against some given resistance. Certain changes of figure and dimension, or, in other words, strains are likewise produced in both bodies, in consequence of the stress applied to them.

The simplest case is where the impact is direct and the resistance to motion has some definite value, as, for example, where a pile is driven by the action of a falling weight. Here let R be the resistance which the pile offers to be driven; that is to say, the load which, resting steadily on the pile, would just cause it to commence to sink; let W be the falling weight, h the height from which it falls, x the space through which the pile sinks in consequence of the blow; then the mutual action between the pile and the weight at the instant of impact consists of a pair of equal and opposite forces R.

The whole height through which the weight falls is $h+x$, and the space through which the resistance is overcome is x; hence, equating energy exerted and work done, we have

$$W(h+x)=Rx.$$

This equation shows that any resistance, however great, can be overcome by any weight, however small; and also, that the force of the blow, as measured by the space the pile is driven, is proportional to its energy. We have however assumed that the whole energy of the blow is employed in driving the pile, whereas some of it will always be expended in producing vibrations and in damaging the head of the pile and the bottom of the weight. As the pile is driven deeper, the resistance to being driven increases and at length becomes equal to the crushing stress of the material: the pile then sinks no farther, the whole of the energy of the blow being wasted in crushing.

This last is also the case of impact of a flying shot against a soft plastic substance, which exerts during deformation a definite force uniform or variable which brings the weight to rest in a certain space. Suppose V the velocity of the shot, x the space, and R the mean resistance which the substance offers, then the kinetic energy of the shot is $WV^2/2g$, while the work done is Rx, equating which

$$W.\frac{V^2}{2g}=Rx.$$

Here the whole energy of the blow is spent in producing changes of figure in the body struck; but if the striking body had been soft, and the body which is struck hard and immoveable, the energy of the blow would have been employed in producing change in the shape of the striking body. Thus we may write down as the general equation of impact—

Energy of blow = Work done in overcoming the resistance to movement of the body struck.
+ Work done in internal changes in the striking body
+ Work done in internal changes in the body struck.

Which of these three terms is the most important will depend on the relative magnitude of the resistance to movement, and the crushing stress of the materials of the two bodies. If either body have a sensible motion after impact, the corresponding kinetic energy must be taken account of in writing down the equation, as will be seen farther on.

193. *Augmentation of Stress by Impact in Perfectly Elastic Material.*—We now proceed to apply the equation to the case which most immediately concerns us, namely, that of impact on perfectly elastic material, including in this the effect of a load which is applied all at once.

We will suppose a structure or piece of material of any kind resting on immoveable supports, and struck by a body harder than itself, so that we may neglect all changes produced in the striking body. Generally in both bodies there will also be produced vibrations, of the nature of those constituting sound, which absorb a certain amount of energy, but this we shall neglect. The whole energy of the blow then is supposed expended in straining the structure, or piece of material, struck by the blow.

Now the effect of impact is to produce a mutual action S, which represents a force applied to the structure at some definite point. In consequence of this the structure suffers deformation, and the point of application moves through a space x. The resistance to deformation is proportional to x, because the limit of elasticity is not exceeded; it therefore commences by being zero, and increases gradually till the velocity of the striking body is wholly destroyed. The mean value of the resistance is therefore one half its maximum value. During the first part of the period occupied by the impact the mutual action S is greater than the resistance, and during the second part less, as will be explained fully presently; but, when the maximum strain has been produced, the mean value during the whole period must be exactly equal to the mean resistance, the weight and the structure being at rest. The state of rest is only momentary, for the strained structure will immediately, in virtue of its elasticity, commence to return to its original form; but, for the moment, a strain has been produced, which is a measure of the effect of the blow, and which must not exceed the powers of endurance of the material.

Let now R be the maximum resistance, and let the blow consist in the falling of a weight W, through a height h above the point where it first comes in contact with the structure; then $h + x$ is the whole height fallen through, and it follows from what has been said that

$$W(h + x) = \tfrac{1}{2}Rx.$$

The resistance R may also be described as the "equivalent steady load," being the load which, if gradually applied at the point of impact, would produce the same stress and strain which the structure actually experiences. We most conveniently compare it with W by supposing that we know the deflection δ which the structure would experience if the striking weight W were applied as a steady load at the point of impact; we then have, since the limits of elasticity are not exceeded,

$$\frac{x}{\delta} = \frac{R}{W}.$$

Substituting the value of x we get

$$\frac{R^2}{W^2} = \frac{2R}{W} + \frac{2h}{\delta}.$$

Let the height h be n times the deflection δ, then solving the quadratic, the positive root of which alone concerns us,

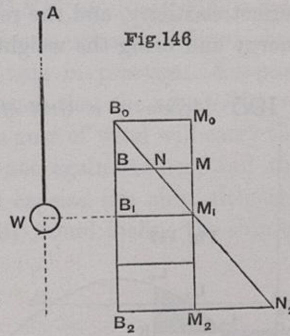
$$R = W(1 + \sqrt{2n + 1}),$$

an equation which shows how the effect of a load is multiplied by impact.

194. *Sudden Application of a Load.*—A particular case is when $h = 0$, then $R = 2W$. So that if a load W is suddenly applied to a perfectly elastic body, from rest, not as a blow, it will produce a pressure just twice the weight. This case is so important that we will consider a special example in detail.

Let a long elastic string be secured at A. If a gradually increasing weight be applied the string will stretch, and the weight descend. Let the load required to produce any given extension be represented by the ordinates of the sloping line B_0NN_2 (Fig. 146). Next, instead of applying a gradually increasing load, let a weight W represented by B_0M_0 be applied all at once to the unstretched string. The string will of course stretch, and the weight descend. When it has reached B (Fig. 146) the tension of the string pulling upwards, being represented by BN, will be less than W acting downwards. Moreover, in the descent B_0B_1 an amount of energy has been exerted by the weight represented by the area of the rectangle B_0M_0MB.

Fig.146



At the same time the work which has been done in stretching the string is represented by the area of the triangle B_0NB. The excess of energy exerted over work done has been employed in giving velocity to the descending weight, and is stored as kinetic energy in the weight.

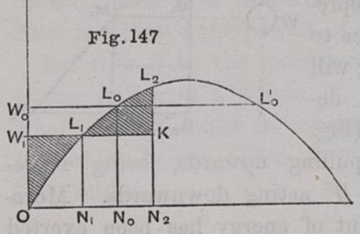
On reaching B, the tension of the string is just equal to the weight, but the stretching does not cease here. The weight has now its greatest velocity, which corresponds to an amount of kinetic energy represented by the triangle $B_0M_0M_1$. Although any further extension of the string causes the upward pull of the string to be greater than the weight W, yet the weight will go on descending until the energy that it has exerted is equal to the work done in stretching the string; then the kinetic energy will be exhausted and the weight will be brought to rest. This will occur when the area of the triangle $B_0N_2B_2$ equals the area of the rectangle $B_0M_0M_2B_2$, that is when $B_2N_2 = 2B_2M_2$, or $B_0B_2 = 2B_0B_1$.

We thus see that the tension of the string produced by the sudden application of the load is twice that due to the same load steadily applied.

The string will not remain extended so much as B_0B_2, for now the upward pull of the string, exceeding the weight, will cause it to rise. On reaching B_1 it will have the same velocity upwards that it had on first reaching B_1 downwards. This will carry it to B_0, from which it will again fall, and so on. Practically, the internal friction due to imperfect elasticity, and the resistance of the air, will soon absorb the energy and bring the weight to rest at B_1.

195 *Action of a Gust of Wind on a Vessel.*—Another interesting example of the way in which the sudden application of a load augments its effect is furnished by the case of a vessel floating upright in the water and acted on by a sudden gust of wind, a question which, though not strictly belonging to this part of the subject, involves exactly the same principle.

Fig. 147



First, suppose no wind pressure, but that a gradually increasing couple is applied to heel the vessel.

If along a horizontal line (Fig. 147) angles of heel be marked off, such as ON, and for those points ordinates such as NL are set up to represent on some convenient scale the magnitude of the couple required to produce that angle of heel, a curve OL will be obtained, which we have already (p. 198) called the curve of *Statical Stability* of the ship.

Now suppose a steady wind pressure to be gradually applied. It will produce on the masts and sails a definite moment, on account of which the ship will incline to a certain angle, such that the ordinate of the curve of stability corresponding to that angle will represent the moment of the wind pressure. So long as the wind is constant, she will remain inclined at that angle. Next suppose the same wind pressure to be suddenly applied all at once, as by a gust to the ship floating upright at rest. The ship will heel over, and until she is inclined to some extent the wind moment will be greater than the righting moment, and the excess will cause the ship to acquire an angular velocity. Accordingly, when she arrives at the angle of heel corresponding to the moment of wind pressure on the stability curve, she does not come to rest, but inclines farther, until the energy exerted by the wind pressure is all taken up in overcoming the righting moment through the angle of inclination. The work thus done is represented by the area of the curve of stability standing above the angle of heel reached.

Let OW_1 represent the magnitude of the wind moment. The ship will incline until the area $OL_2N_2 =$ area OW_1KN_2, or area $OW_1L_1 =$ area L_1L_2K; that is, if the moment of wind pressure remains undiminished as the ship heels, which will hardly be true in practice. Suppose the moment of wind pressure OW_0 to be such that the area $OW_0L_0 =$ the area $L_0L_2L_0'$. In this case the sudden gust of wind will carry the ship to such an angle ON_0' that she will not again return; and the smallest additional pressure of wind will capsize the ship, although that same wind pressure applied gradually would incline the ship to the angle ON_0 only.

196. *Impact at High Velocities. Effect of Inertia.*—Returning to the general case of impact against a perfectly elastic structure (Art. 193), let us now take the other extreme case in which the height through which the weight falls is great compared with the deflection

δ due to the same weight gradually applied; then, since n is great, our equation becomes

$$R = W.\sqrt{2n} = W\sqrt{\frac{2h}{\delta}},$$

which may be written in either of the forms

$$R = \sqrt{\frac{2W}{\delta}} \cdot \sqrt{Wh} \quad (1); \quad \text{or } x = \sqrt{2h\delta} \qquad (2.)$$

The first form shows that the stress produced by the impact is proportional to the square root of the energy of the blow, and the second, that the deflection occasioned by the fall of a given weight is proportional to the square root of the fall, or, what is the same thing, to the velocity of impact. These results are exact when the impact is horizontal, and the last has been verified by experiment. It is to be remembered that the limits of elasticity are supposed not to be exceeded: when a rail or carriage axle is tested by a falling weight, as is very commonly done, the energy of the blow is generally much in excess, and the piece of material suffers a great permanent set, the resistance is then approximately constant instead of increasing in proportion to the deflection. The effect of the blow is then more nearly directly proportional to its energy. It will be seen presently how small a blow matter is capable of sustaining without injury to its elasticity.

The effect of a blow, on a structure or piece of material as a whole, is diminished, on account of its inertia, by an amount which is greater the greater the velocity of impact, but which varies according to the relative mass and stiffness of its parts. In the act of yielding the parts of the body are set in motion, and the force required to do this is frequently greater than the crushing strength of the materials, so that a part of the energy of the blow is spent in local damage near the point of impact.

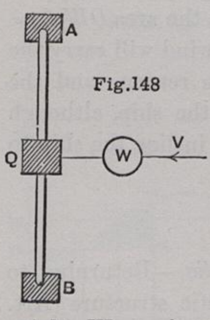


Fig.148

Figure 148 shows a narrow deep bar AB, the ends of which rest in recesses in the supports, which prevent them from moving horizontally, but do not otherwise fix them. The bar carries a weight Q in the centre, against which a second weight W moving horizontally strikes with velocity V. The bar being very flexible horizontally, the weight Q at the first instant of impact moves as it would do if free; that is, the two weights move onwards together with a common velocity v fixed by the consideration that the sum of the momenta of the two weights is the same before and after impact, so that

$$WV = (W + Q)v.$$

The energy of the two weights after impact is

$$(W + Q)\frac{v^2}{2g} = \frac{W^2}{W + Q}\cdot\frac{V^2}{2g},$$

showing that the energy of the blow has been diminished in the proportion $W : W + Q$. The loss is due to the expenditure of energy in damage to the weights.

If now, instead of a weight Q attached to the centre of a flexible bar, we suppose the bar less flexible and of weight Q, the effect of the blow is diminished by the same general cause, but not to the same extent: the diminution cannot be calculated exactly, but may be estimated by replacing Q in the preceding formula by kQ, where k is a fraction to be determined by experiment. In a series of elaborate experiments made by Hodgkinson on bars struck horizontally by a pendulum weight, it was found that k was $\frac{1}{2}$.

We are thus led to separate the energy of a blow into two parts:

$$E_1 = \frac{W^2}{W + kQ}\cdot\frac{V^2}{2g}: \quad E_2 = \frac{k.WQ}{W + kQ}\cdot\frac{V^2}{2g}.$$

The first of these strains the structure or piece of material as a whole, and the second does local damage at the point of impact. Hence the great difference which exists between the effect of two blows of the same energy, one of which is delivered at a low, and the other at a high velocity. At high velocities most of the energy is expended in local damage; at low velocities most is expended in straining the structure as a whole.

If the body which is struck be in motion, instead of resting on immoveable supports, as in Fig. 148, the energy of the blow will be diminished. This case has been considered in Ch. XI., p. 280, where it is shown that the energy of the collision is

$$E = \frac{WQ}{W + Q}\cdot\frac{V^2}{2g},$$

where V is the relative velocity of the bodies. Of this a part—represented, as before, by replacing Q by kQ—is spent in local damage and the rest in straining the structure as a whole.

The exceptional case where, as in the collision of billiard balls, the limit of elasticity is not exceeded at the point of impact, need not be here considered. The energy of local damage is, then, not wholly dissipated in internal changes: a part is recovered during the restitution of form which occurs in the second part of the process of impact, and increases the action on the structure as a whole. In ideal cases the whole may be thus recovered, but, in practice, a portion is always employed in producing local vibrations, and finally dissipated by internal friction.

197. *Vibrating Loads. Synchronism.*—The load on a structure may vary from time to time, continuously, or otherwise, and its effect will then, in general, be greatly augmented. Some simple examples will now be considered.

Returning to the case of the weight suspended from an elastic string (Fig. 146, p. 377); suppose in the first instance the weight at rest, then the corresponding extension (δ) is B_0B_1 in the figure and the position of the weight is B_1. Next imagine the weight raised vertically and suddenly released, it will oscillate about B_1 as a mean position. In any position B the tension of the string is represented by BN and the weight by BM, so that NM represents an unbalanced force which draws the weight downwards when it is above B_1, and upwards when below. Now NM is proportional to BB_1, and the weight therefore moves under a force always proportional to its distance from B_1.

This kind of motion is known as a "simple harmonic motion"; we have already had an example in the case of a piston moving in its cylinder; for in Ch. IX., p. 235, it was shown that the force necessary to move the piston varies as the distance from the centre of the stroke. In fact Fig. 99, p. 234, may be taken to represent the motion, the velocity of the weight in any position being represented by QN. From the formulæ given in the article cited it is easy to show that the time of a double vibration of the weight is given by

$$t = 2\pi\sqrt{\frac{\delta}{g}},$$

being the same as that of the small oscillations of a pendulum of length B_0B_1. It is dependent only on the elasticity of the string and the magnitude of the weight, not on the extent of the vibration.

The vibrations of any structure may be distinguished into general and local, that is into the vibrations of the structure as a whole, and the vibrations of its parts. All such vibratory motions are of the same general character, as in the simple case just described they take place in certain definite times depending on the inertia and elasticity of the structure and its parts.

Next suppose the weight (Fig. 146) oscillating about B_1, and let B be the extreme upward position. At the instant when the weight is at B imagine a small downward force P applied; the effect of this will be that the weight descends to a position B_2 before coming to rest, such that $B_1B_2 > BB_1$, instead of being equal to BB_1, as would otherwise be the case. Then suppose P removed, the weight will rise to a point as much above B as B_1B_2 is greater than BB_1. Again suppose P applied, then the weight will descend below B_2, and this process may be continued indefinitely. Thus it appears that a load P, however small, if applied and removed at intervals, *corresponding to the natural period of vibration of the weight W*, will produce a vibration of continually increasing extent, thus augmenting indefinitely the tension of the string, which will soon break, however small the original load W and its fluctuation P. If the weight P be applied as before at B, but removed and replaced at a different interval, the vibration will still augment, in the first instance, but the augmentation will be limited, and will be succeeded by a diminution, and so on indefinitely.

In the foregoing simple example numerical results could readily be obtained if necessary; in actual structures and machines the circumstances are much more complex, and calculations are therefore generally difficult, but the same general principles hold good. Whenever the load on a structure fluctuates the stress due to it is greater than that which corresponds to the maximum load: and the augmentation is greater the more nearly the period of fluctuation approaches the period of vibration of the whole structure, or of that part of it immediately affected by the load. Vibrations of the same period are often described as "synchronous."

As examples of a fluctuating load may be mentioned—

(1) When a company of soldiers march in regular time over a suspension bridge vibrations of the flexible structure are set up which are constantly augmented by synchronism. On a girder bridge the augmentation would be comparatively small, the period of vibration of the bridge being generally very different.

(2) In certain torpedo boats the vibration due to the action of the screw is excessive at one particular speed. This is an effect of synchronism between the revolutions of the screw and the period of bending vibrations of the boat in a horizontal plane.

(3) When a ship rolls broadside on to a series of equal waves the rolling is increased by the action of the waves, and is greatest when the period of the waves is equal to the period of rolling of the ship in still water.

One case of a fluctuating load can be completely worked out without much difficulty, and the result has been applied to various purposes. This is where the load fluctuates according to the harmonic law already considered for an elastic string. The calculation cannot be given here, but it may be mentioned that it is in this way that the late Mr. Froude arrived at his well known conclusions respecting the rolling of ships amongst waves.*

198. *Impact when the Limits of Elasticity are not Exceeded. Resilience.*—The effect of impact on perfectly elastic material may also be dealt with by considering the amount of energy stored up in the body in consequence of the deformation which each of its elementary parts have suffered. We have already seen that when a piece of material is subjected to a simple uniform longitudinal stress of intensity p, the amount of work U done by the stress is

$$U = \frac{p^2}{2E} \times \text{Volume}.$$

Let w be the weight of a unit of volume of the material, and W the weight of the body considered, then we may write our equation

$$U = W \, . \, H,$$

where H is a certain height given by

$$H = \frac{p^2}{2Ew},$$

and the whole elastic energy of the body may be measured by this height, which is the distance through which the body must fall to do an equivalent amount of work.

If for p we write f the elastic strength of the material, then we obtain what we have already called the Resilience of the body, and H becomes what we may call the "height due to the resilience," which,

* *Transactions of the Institution of Naval Architects*, vol. ii.

for each material, has a certain definite value, given in feet in Table II., Ch. XVIII., for various common materials.

Now in cases of impact where the limit of elasticity is not exceeded, the whole energy of the blow is spent in straining the material or structure, and hence that energy must not, in any case, exceed the resilience. Thus, on reference to the table, it will be seen that in ordinary wrought iron the height is given as 2 ft. 2 in., from whence it follows that in the most favourable case a piece of iron will not stand a blow of energy greater than that of its own weight falling through twenty-six inches, without being strained beyond the elastic limit. If the parts of the body are subject to torsion, about 50 per cent. may be added to these numbers, but, on the other hand, they are subject to large deductions on account of the inequality of distribution of stress within the body. Only a portion of the body is subjected to the maximum stress, the rest is strained to a less degree, and consequently has absorbed a less amount of the energy of the blow. Thus, for example, a beam or circular section, even though it be of "uniform strength" (Art. 161), has only one fourth the resilience of a stretched bar of the same weight, because it is only the particles on the upper and lower surfaces which are exposed to maximum stress, the central parts having their strength only partially developed.

We now draw two very general and important conclusions.

(1) When a body or structure is exposed to a blow exceeding that represented by its own weight falling through a very moderate height, a part, or the whole, is strained beyond the elastic limit.

(2) When a body or structure is not of uniform strength throughout, the excess of material is a cause of weakness.

On reference to Table II., Ch. XVIII., it will be seen that exceptions occur to the first principle in the case of the hardest and strongest steel, and in wood and some other substances of organic origin of low specific gravity; but, as a rule, the property of ductility or plasticity is essential to resistance to impact. Bodies which do not possess it are generally brittle. In good ductile iron and soft steel the non-elastic part of the resistance to impact will be seen hereafter to be at least 1000 times the elastic part, assuming both equally developed through all parts of the material. These remarks apply to a single blow; the effect of repetition will be considered hereafter.

As an example of the application of the second principle we may mention the bolts for armour plates invented by the late Sir W. Palliser. In these bolts the shank is turned down to the diameter of the base of the thread so as to be of equal strength throughout. (See Ex. 4, p. 308.)

EXAMPLES.

1. A hammer weighing 2 lbs. strikes a nail with a velocity of 15 feet per 1 inch driving $\frac{1}{8}$ inch, what is the mean pressure overcome by the nail? *Ans.* 673 lbs.

2. If the load on a stretched bar is suddenly reversed so as to produce compression, show that the stress will be trebled.*

Energy stored in stretched bar will on the release of the load be employed in compression, and in addition the load will be exerted through a distance = original extension + compression. The two together must be equal to the work done in compressing the bar.

Note—Such sudden reversal as is here supposed rarely if ever occurs in practice.

3. A load of 1000 lbs. falls through 1″ before commencing to stretch a suspending rod by which it is carried. If the sectional area of the rod is 2 sq. in., length 100″, and modulus of elasticity 30,000,000, find the stress produced.

Stress = 17,828 lbs. per sq. in.

4. A load of 5000 lbs. is carried by the rod of the preceding question, and an additional load of 2000 lbs. is suddenly applied; what is the stress produced?

Stress = 4500 lbs. per sq. in.

5. A beam will carry safely 1 ton with a deflection of 1 inch; from what height may a weight of 100 lbs. drop without injuring it, neglecting the effect of inertia? *Ans.* 11·2 inches.

6. The maximum stability of a vessel is 4000 foot-tons. The curve of stability is represented sufficiently approximately by a triangle, such that the angle of maximum stability is $1/n$ the angle of vanishing stability. Find the moment which, applied suddenly and of uniform amount to the ship upright and at rest, would just capsize her.

Area OCD = area DAE. $OB = N.ON$, and $AN = 4000$;

$$CD = \frac{OB}{N}\frac{OC}{4000}, \text{ and } DE = OB\frac{4000 - OC}{4000}.$$

g the areas in terms of OC and OB we get

$$OC \text{ the capsizing moment} = \frac{4000\sqrt{n}}{1 + \sqrt{n}}.$$

* This result which appears little known was pointed out to the writer by Mr. Hearson. Some examples on impact will be found in Prof. Alexander's treatise on Applied Mechanics, part I.

CHAPTER XVII.

STRESS, STRAIN, AND ELASTICITY.

Section I.—Stress.

199. *Ellipse of Stress.*—Stress consists, as we have said (Art. 147), in a mutual action between two parts, into which we imagine a body divided by an ideal section. If the section be plane, and if the stress be uniform, the intensity and direction of the stress at each point of the section are the same at all points of a given section, and, for a given point, depend only on the position of the plane. In a fluid the intensity is the same for all planes, and the direction is normal to the plane. In simple tension and compression the direction of the stress is the same for all planes, but its intensity varies, becoming zero for planes parallel to the stress. In shearing the intensity is the same for all planes perpendicular to a third given plane, but the direction varies: on one pair of planes it is normal, on another tangential.

We now proceed to consider stress more generally, and we shall first examine the effect of combining together a pair of simple longitudinal stresses, the directions of which are at right angles and the intensities of which are given. Let the plane of the paper be parallel to the directions of the stresses, and let us consider a piece of material of thickness unity. If the stress be uniform, the size and shape of the piece are immaterial. Let us then imagine a rectangular block *ABCD* (Fig. 149) with sides perpendicular to the stresses p_1, p_2. On the faces *AB*, *CD* a stress, of intensity p_1, and of total amount $p_1 \cdot AB$ will act; while on *BC* and *AD* there will be a stress of intensity p_2, and of total amount $p_2 \cdot BC$. Divide now the rectangle by a diagonal plane *AC*; there will be a stress on that plane, which it is our object to determine in direction and magnitude. Let θ be the angle which the normal to the plane makes with the direction of p_1; by determining rightly the ratio of the sides of the rectangle this angle may be made what we please. Proceeding as in Art. 81, we find for the normal stress

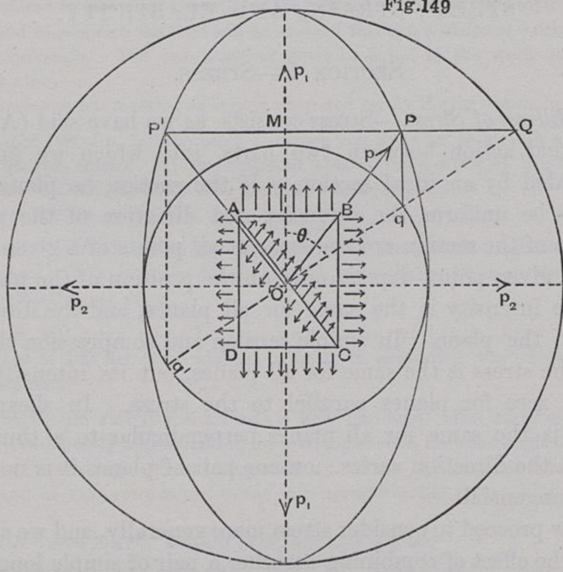
$$p_n = p_1 \cdot \cos^2\theta + p_2 \cdot \sin^2\theta,$$

and for the tangential stress

$$p_t = (p_1 - p_2) \sin\theta \cdot \cos\theta.$$

The resultant stress might be found in direction and magnitude by

Fig.149



compounding these results, but it is better to proceed by a graphical construction. On the perpendicular set off OQ to represent p_1 and Oq to represent p_2; also draw the ordinate QM and qP parallel to p_1 to meet it in P. Then

$$OM = OQ \cdot \cos\theta = p_1 \cdot \frac{AB}{AC};$$

$$PM = Oq \cdot \sin\theta = p_2 \cdot \frac{BC}{AC}.$$

Whence it follows that the intensity of the stress on AC due to p_1 is represented by OM, and that due to p_2 by PM. If then we join OP we shall obtain the resultant stress on AC in direction and magnitude. It is easily seen that P lies on an ellipse of which p_1, p_2 are the semi-axes. This ellipse is called the Ellipse of Stress.

If the pair of stresses p_1, p_2 have opposite signs, then $Oq = p_2$ must be set off on the opposite side of O, and OP' the radius vector of the ellipse lies on the other side of OM, but in other respects the construction is unaltered. When p_1, p_2 are equal the ellipse becomes a circle; if they have the same sign the stress is the same in all directions in magnitude and direction like fluid pressure; if they have opposite signs, as in the chapter on Torsion, the intensity is the same, but the angle of inclination $P'OQ$, called the "obliquity" of stress, is variable, being always equal to QON.

200. *Principal Stresses. Axes of Stress.*—We now propose to show that any state of stress in two dimensions (Art. 204) may always be reduced to a pair of simple stresses such as we have just considered.

For, drawing the same figure as in the last article, let us inquire the effect of replacing p_1, p_2 by other stresses of any magnitude and in any directions. Whatever they be, they evidently must have given tangential and normal components, of which, reasoning as in the last chapter, we know that the tangential must be equal and opposite.

Let the equal tangential components be p_t, and the normal components p_n and p'_n. Consider the equilibrium of the triangular portion ABC (Fig. 150), and let us determine under what conditions it is possible that the stress on AC should be a normal stress only, without any tangential component. Resolve parallel to BC; then, if p be that normal stress,

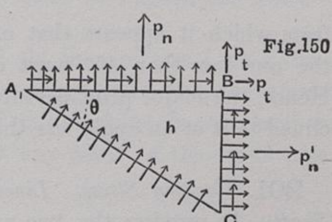


Fig. 150

$$p \cdot AC \cdot \cos\theta = p_t \cdot BC + p_n \cdot AB;$$

or

$$p - p_n = p_t \cdot \tan\theta.$$

Similarly resolving parallel to AB,

$$p - p'_n = p_t \cdot \cot\theta,$$

whence, subtracting one equation from the other,

$$p_n - p'_n = p_t . (\cot\theta - \tan\theta) = 2p_t . \cot 2\theta ;$$

or

$$\tan 2\theta = \frac{2p_t}{p_n - p'_n}.$$

This equation always gives two values of θ at right angles, showing that two planes at right angles can always be found on which the stress is wholly normal. The magnitude of the stress on these planes is found by multiplying the equations together, when we get the quadratic

$$(p - p_n)(p - p'_n) = p_t^2,$$

the roots of which, p_1, p_2, are the stresses required. Having determined p_1, p_2, the ellipse of stress can now be constructed by the method of the last article.

Every state of stress in two dimensions then can always be represented by an ellipse, the semi-axes of which are called Principal Stresses, and their directions the Axes of Stress.

The particular case in which p'_n is zero is one of constant occurrence in practical applications. If q be the shearing stress, the equations may then be written

$$p_n \tan 2\theta = 2q \quad (1); \qquad p(p - p_n) = q^2 \qquad (2).$$

Of the roots of the quadratic the greater has the same sign as that of p_n, and the other the opposite. Also, we find by dividing the two equations for p by one another,

$$\tan^2\theta = \frac{p - p_n}{p} = \frac{q^2}{p^2},$$

from which it appears that of the two values of θ furnished by (1) the one less than 45° must correspond to the greater value of p. Hence, the major principal stress is of the same kind as p_n, and inclined to it at an angle less than 45°.

201. *Varying Stress. Lines of Stress. Bending and Twisting of a Shaft.*—In proving the two very important propositions just given, we have assumed (1) that the stress was uniform, throughout the region including the portion of matter we have been considering; (2) that gravity or any other force acting not on the bounding surface, but on each particle of the interior, may be neglected. It is however to be observed that by taking the portion of matter

small enough, both these suppositions may be made, in general, as nearly true as we please: the first, because any change of stress must be continuous, and therefore becomes smaller the less the distance between the points we consider; the second, because any internal force is proportional to the volume, while any force on the boundary of a piece of material is proportional to the surface of the piece. Now the volume of a body varies as the cube, and the surface as the square of its linear dimensions, and it follows that the internal force vanishes in comparison with the stress on the boundary when the dimensions diminish indefinitely. Hence these propositions are still true as respects the state of stress at any given point of a body, even though the stress be variable, and notwithstanding the action of gravity. When however we consider the variation of stress from point to point, gravity must be considered. Thus, for example, in the case of a fluid the action of gravity does not prevent the pressure from being the same in all directions, but it does cause the pressure to vary from point to point.

When the stress varies from point to point, both the intensity and the direction may vary; thus, for example, in a twisted shaft the intensity of the stress at any point varies as the distance from the axis, and the direction of the stress varies according to the position of the point, the principal stresses making an angle of 45° with the axis of the cylinder. The axes of stress in this case always touch certain lines which give, at each point they pass through, the direction of the stress at that point. These lines are called Lines of Stress; in a simple distorting stress, or, in other cases where the principal stresses are of opposite signs, one is a Line of Thrust, the other a Line of Tension.

In a twisted shaft of elastic material the lines of stress are spirals traced on a cylinder passing through the point considered, the spirals being inclined at 45° to the axis. If the shaft be bent as well as twisted, the maximum normal stress at any point of the transverse section is given by the equation

$$p_n = \frac{M}{\frac{1}{4}\pi r^3} \qquad \text{(Art. 155)},$$

where M is the bending moment and r the radius. The shearing stress at the external surface due to a twisting moment T is given by

$$q = \frac{T}{\frac{1}{2}\pi r^3} \qquad \text{(Art. 184)}.$$

Combining these two together we get, by solving the quadratic for the principal stresses,

$$p = \frac{M \pm \sqrt{M^2 + \tau^2}}{\frac{1}{2}\pi r^3},$$

which gives the principal stresses at that point of the shaft where the stress is greatest. The maximum stress is the same as would be given by a simple twisting moment equal to $M + \sqrt{M^2 + T^2}$, which is sometimes called the simple equivalent twisting moment. The minor principal stress ought, however, also to be considered in calculations respecting strength, as will be seen hereafter.

The lines of stress here are spirals of variable pitch angle.

202. *Straining Actions on the Web of an I Beam.*—Let us now return to the case of an I beam with a thin web, in which the web resists nearly the whole of the shearing force F, and the flanges nearly the whole of the bending moment M. The intensity of the shearing stress q is approximately

$$q = \frac{F}{ht}.$$

where h is the depth and t the thickness. The intensity of the normal stress at a point distant y from the neutral axis is

$$p_n = \frac{M}{I} \cdot y.$$

The principal stresses and axes of stress are given by the equations

$$p(p - p_n) = q^2\,; \ \tan 2\theta = \frac{2q}{p_n}.$$

From this it appears that, even when the web is very thin so that it carries a very small fraction of the total bending moment, it cannot be treated as resisting shearing alone, and if it is so treated will be the most severely strained part of the beam. Let us, for example, suppose the flanges to be subject to a stress of 4 tons per sq. inch at a given section, and the web to a shearing stress also of 4 tons per sq. inch: then at points in the web near the flanges, say, for example, at a distance from the centre, of three fourths the half depth of the beam, the normal stress will be 3 tons per sq. incn. Putting these values in the formula, we get the quadratic equation

$$p(p - 3) = 16\,;$$

whence

$$p = 5{\cdot}77, \text{ or } - 2{\cdot}77,$$

a result which shows that the web is much more severely strained than the flanges. The lines of stress are found from the equation for θ. By a graphical method it is possible to draw the lines of stress approximately. As to this the reader is referred to a treatise by Mr. Chalmers, cited on page 82.

203. *Remarks on Stress in General.*—We have hitherto been considering only the stress on planes at right angles to a certain primary plane, to which we have supposed the stress on every plane to be parallel. In most practical questions relating to strength of materials this is sufficient, since, though stress frequently exists on the primary plane, it is usually normal and of relatively small intensity. Thus, for example, in a steam boiler there is stress on the internal and external surface of the boiler due to the pressure of the steam and the atmosphere; but it is of small amount compared to the stress on planes perpendicular to the surface. We therefore content ourselves with a statement without demonstration of corresponding propositions in three dimensions.

(1) Any state of stress at a point within a solid may always be reduced to three simple stresses on planes at right angles.

(2) The resultant stress on any plane due to the action of three simple stresses at right angles to each other is always represented in direction and magnitude by the radius vector of an ellipsoid.

The first of these propositions may be regarded as the last step in a process of analysis, by which we reduce all external forces acting on a structure of any kind: *first*, into a set of forces acting on each piece of the structure; and *second*, into forces acting on each of the small elements of which we may imagine that piece composed; and *lastly*, into three forces at right angles acting upon the element, of which one in practical cases is usually small. All questions in Strength of Materials, then, ultimately resolve themselves into a consideration of the effects of forces so applied.

One method of conceiving the effect of three such forces is to imagine each separated into two parts, one of which is the same for all, being the mean value of the three; while the other is compressive for one and tensile for the two others, or *vice versa.* In isotropic matter (Art. 207) the first set produces change of volume only, and may be called the "volume-stress," or, as no other stress can exist in fluid bodies at rest, a "fluid" stress. The second is a distorting stress, consisting of three simple distorting stresses tending to produce distortion in the three principal planes.

EXAMPLES.

1. A tube, 12 inches mean diameter and ½ inch thick, is acted on by a thrust of 20 tons and a twisting moment of 25 foot-tons. Find the principal stresses and lines of stress.

Taking a small rectangular piece with one side in the transverse section, we find one face acted on by a normal stress of 1·06 tons per square inch due to the thrust, and a tangential stress of 2·66 tons due to the twisting. Substituting these values for p_m, p_i, and observing that the stress on the other face is wholly tangential, we find from the quadratic

Major principal stress = 3·24 (thrust);
Minor principal stress = 2·18 (tension).

Lines of stress are spirals, the lines of tension inclined at 50½° to the axis, and the lines of thrust at 39½°.

2. A rivet is under the action of a shearing stress of 4 tons per square inch, and a tensile stress, due to the contraction of the rivet in its hole, of 3 tons per square inch. Find the principal stresses.

Ans. Major principal stress = 5·8 tons (tension).
Minor principal stress = 2·77 tons (thrust).

3. The thrust of a screw is 20 tons; the shaft is subject to a twisting moment of 100 foot-tons, and, in addition, to a bending moment of 25 foot-tons, due to the weight of the shaft and its inertia when the vessel pitches. Find the maximum stress and compare it with what it would have been if the twisting moment had acted alone. Shaft 14 inches diameter.

Ans. Major principal stress = 2·9, Ratio = 1·32.
Minor principal stress = 1·6.

4. A half-inch bolt, of dimensions given in Ex. 6, page 271, is screwed up to a tension of 1 ton per square inch of the gross sectional area. Assuming a co-efficient of friction of ·16, find the true maximum stress on the bolt while being screwed up. *Ans.* Principal stresses = 1·95 and ·3 tons.

5. It has been proposed to construct cylindrical boilers with seams placed diagonally instead of longitudinally and transversely. What is the object of this arrangement, and what is the theoretical gain of strength? *Ans.* Increase of strength = 26½ per cent.

6. A thick hollow cylinder is under the action of tangential stress, applied uniformly all over its internal surface in directions perpendicular to its axis, the cylinder being prevented from turning by a similar stress, applied at the external surface. Find the principal stresses and lines of stress. *Ans.* The principal stresses are equal and opposite, forming a simple distorting stress, of intensity varying inversely as the square of the distance from the centre. Lines of stress equiangular spirals of angle 45°.

7. In Ex. 9, page 372, suppose the beam so loaded that the maximum stress due to bending is 4 tons per square inch, and the total shearing force divided by the sectional area of the web also 4 tons per square inch: find the principal stresses at points immediately below the flanges. *Ans.* Principal stresses $4\frac{1}{2}$ and $1·9$ tons per square inch.

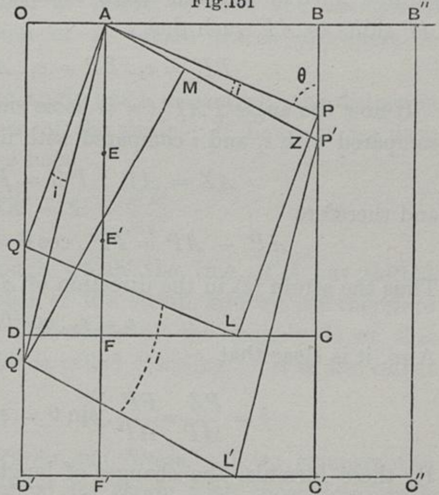
8. In any state of stress at a point in a body show that the sum of the normal stresses on three planes at right angles is the same however the planes be drawn.

SECTION II.—STRAIN.

204. *Simple Longitudinal Strain. Two Strains at Right Angles.*— We now go on to consider the changes of form and size which are produced by the action of stress. Such changes, it has already been said, are called Strains.

In uniform strain every set of particles lying in a straight line must still lie in a straight line, and two lines originally parallel must still be parallel. The lengths of all parallel lines are altered in a given ratio $1 + e : 1$, where e is a quantity, in practical cases very small, which measures the strain in the direction of the line considered. Two sets of parallel lines, however, will not in general remain at the same inclination to each other, nor will their lengths alter in the same ratio. Thus the sides of a cube remain plane, and opposite sides are parallel, but the parallelopiped is not generally rectangular, and its sides are not equal.

Fig.151



The simplest kind of strain is a simple longitudinal strain in which all lines parallel to a fixed plane in the body are unaltered in length, while all lines perpendicular to that plane remain so: that is to say, a simple change of length, the breadth, and thickness remaining unaltered.

Fig. 151 shows an extensible band $OBCD$, in which OB is fixed, while CD moves to $C'D'$, the breadth being in the first instance unaltered, and the length altered so that

$$CC' = e_1 . BC.$$

If any line AEF be traced in the band parallel to BC, the points EF will shift to $E'F'$ positions in the same line, such that

$$EE' = e_1 . AE \ : \ FF' = e_1 . AF.$$
$$E'F' = (1 + e_1)EF;$$

for since the strain is uniform the change of length of all parts of the band is the same. If, however, we draw a line QL inclined at an angle θ to BC, that line will shift to $Q'L'$, a position such that QL has not increased in so great a ratio, and is not inclined to BC at the same angle as before. We are about to determine the actual change of length and angular position of QL by finding that of a parallel AP drawn through A. It has been already remarked that parallel lines in uniform strain must suffer the same strain. Now AP shifts to AP' such that

$$PP' = e_1 . BP = e_1 . AP . \cos\theta.$$

If now the angle PAP' ($= i$) be so small that i^2 may be neglected compared with i, and i compared with unity,

$$AZ = AP \ : \ P'Z = PP' . \cos\theta;$$

and therefore

$$AP' - AP = PP' . \cos\theta = e_1 . AP . \cos^2\theta.$$

Thus the strain (e) in the direction of AP is

$$e = e_1 . \cos^2\theta.$$

Also, it is clear that

$$i = \frac{PZ}{AP} = \frac{PP'}{AP} . \sin\theta = e_1 . \sin\theta . \cos\theta.$$

By these formulæ the changes of length and angular position of all lines in the band are determined.

Next draw a line AQ perpendicular and equal to AP, and let AQ' be the position into which it moves in consequence of the strain; we find for e', the extension of AQ,

$$e' = e_1 . \sin^2\theta;$$

while the angle QAQ' is

$$i' = e_1 . \sin\theta . \cos\theta = i.$$

Imagine now the square AQL completed; this square, in con sequence of the strain, will have its sides altered in length by the quantities e, e', and will have suffered a distortion given by

$$2i = 2e_1 . \sin\theta . \cos\theta.$$

In this way the effect of a simple longitudinal strain is completely determined, for we can calculate the changes taking place in any portion of the band we please.

Next suppose the band to suffer a second simple longitudinal strain e_2 in the direction of the breadth, and observe that since the strains are very small, the effect of e_1, e_2 taken together must be the sum of those due to each taken separately; then we find for the change of length and position of any line AP,

$$e = e_1 . \cos^2\theta + e_2 . \sin^2\theta;$$
$$i = (e_1 - e_2) \sin\theta . \cos\theta,$$

results which may be applied as before to show the changes of dimension and the distortion of a square traced anywhere in the band.

We have here regarded the angle i as a measure of the distortion a square suffers in consequence of the strain. If, however, we drop $Q'M$ perpendicular to AP', we have

$$AQ'M = 2i = \frac{AM}{AQ'}.$$

Now AM is the space through which the line $A'Q'$ has shifted parallel to itself in consequence of the strain, and we see therefore that the angle i also gives a measure of the magnitude of this shifting. By some writers this is called "sliding." It is also called "shearing strain."

205. *Comparison between Stress and Strain.*—If we compare the equations we have just obtained for strain with those previously obtained in Art. 199 for stress, we find them identical; and hence it appears that, so long at least as the strains are very small, all propositions respecting stress must also be true, *mutatis mutandis*, with respect to strain. Thus, for example, a simple distortion must be equivalent to a longitudinal extension accompanied by an equal longitudinal contraction; and, again, every state of strain can be reduced to three simple longitudinal strains at right angles to each other, and represented by an ellipsoid of strain. The simple strains are called Principal Strains, and their directions Axes of Strain. Strain, like stress, generally varies from point to point of the body: but the relations here proved still hold good at each point, and we have Lines of Strain just as we previously had Lines of Stress.

SECTION III.—CONNECTION BETWEEN STRESS AND STRAIN.

206. *Equations connecting Stress and Strain in Isotropic Matter.*—So far we have merely been stating certain conditions which stress must satisfy in order that each element of a body may be in equilibrium, and certain other conditions which strain must satisfy if the body is continuous. We now connect the two by considering the way in which stress produces strain, which differs according to the nature of the material.

We first consider perfectly elastic material (see Art. 147), and suppose that material to have the same elastic properties in all directions, in which case it is said to be isotropic. Metallic bodies are often not isotropic, as will be seen hereafter (Ch. XVIII.). Suppose a rectangular bar under the action of a simple longitudinal stress p_1, then there results (Art. 148) a longitudinal strain e_1 given by

$$p_1 = Ee_1,$$

where E is the corresponding modulus of elasticity. Accompanying the longitudinal extension we find a contraction of breadth that is a lateral strain of opposite sign of magnitude $1/m^{\text{th}}$ the longitudinal strain, where m is a coefficient. The contraction in thickness will be equal, because the material is supposed isotropic. Hence the effect of the simple longitudinal stress p_1 is to produce three simple longitudinal strains at right angles,

$$e_1 = \frac{p_1}{E}; \ e_2 = -\frac{p_1}{mE}; \ e_3 = -\frac{p_1}{mE}.$$

Next remove p_1, and in its place suppose a simple stress p_2 applied in the direction of the breadth of the bar; we have by similar reasoning the three strains

$$e_1 = -\frac{p_2}{mE}; \ e_2 = \frac{p_2}{E}; \ e_3 = -\frac{p_2}{mE}.$$

And similarly removing p_2 and replacing it by p_3 acting in the direction of the thickness,

$$e_1 = -\frac{p_3}{mE};\ e_2 = -\frac{p_3}{mE};\ e = \frac{p_3}{E}.$$

These three sets of equations give the strains due to p_1, p_2, p_3, each acting alone; and we now conclude that if all three act together we must necessarily have

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE},$$

with two other symmetrical equations.

Hence it appears that the effect of three principal stresses, and consequently of any state of stress whatever on isotropic matter, is to produce a strain, the axes of which coincide with the axes of stress, and in which the principal strains are connected with the principal stresses by the equations just written down.*

207. *Elasticity of Form and Volume.*—The value of the constant m may be found directly by experiment, though with some difficulty, on account of the smallness of the lateral contraction which it measures; but it may also be found indirectly, by connecting it with the co-efficient employed in the last chapter to measure the elasticity of torsion. For if we subtract the second of the three equations just obtained from the first, we get

$$e_1 - e_2 = (p_1 - p_2)\frac{m+1}{mE},$$

or

$$p_1 - p_2 = \frac{m}{m+1} \cdot E(e_1 - e_2).$$

Now referring to Arts. 31, 33 we find

$$p_t = (p_1 - p_2)\sin\theta \,.\, \cos\theta,$$
$$2i = 2(e_1 - e_2)\sin\theta \,.\, \cos\theta,$$

where p_t is the tangential stress on a pair of planes inclined at angle θ to the axes, and $2i$ is the distortion of a square inclined at that angle to the axes of strain. Since now the axes of strain coincide with the axes of stress, we must have

$$\frac{p_t}{2i} = \frac{p_1 - p_2}{2(e_1 - e_2)} = \frac{1}{2}\frac{m}{m+1} \cdot E,$$

* The form in which these equations are given is due to Grashof. For practical application it is more convenient than any other.

an equation which, compared with Art. 183, shows that the co-efficient of rigidity C must be

$$C=\frac{1}{2}\frac{m}{m+1}\cdot E.$$

Experiment shows that in metallic bodies C is generally somewhat less than $\frac{2}{5}E$, whence it follows that m lies between 3 and 4. In the ordinary materials of construction the comparison cannot generally be made with exactness, because such bodies are rarely exactly isotropic. The value of m for iron is about $3\frac{1}{2}$.

Again, if we add together the three fundamental equations, we find

$$E(e_1+e_2+e_3)=\left(1-\frac{2}{m}\right)(p_1+p_2+p_3).$$

Now the volume of a cube, the side of which is unity, becomes when strained $(1+e_1)(1+e_2)(1+e_3)$, and therefore the volume strain is $e_1+e_2+e_3$ when the strains are very small. Hence, if we separate the stress into a fluid stress N and a distorting stress (Art. 204), we have

$$N=\frac{m}{3(m-2)}\cdot E\times\text{Volume Strain},$$

and the co-efficient

$$D=\frac{m}{3(m-2)}E$$

measures the elasticity of volume. The two constants C and D, which measure elasticity of distinctly different kinds, may be regarded as the fundamental elastic constants of an isotropic body. The ordinary Young's modulus E involves both kinds of elasticity.

208. *Modulus of Elasticity under various circumstances. Elasticity of Flexion.*—When the sides of a bar are free the ratio of the longitudinal stress to the longitudinal strain is the ordinary modulus of elasticity E; but the equations above given show that, when the sides of the bar are subject to stress, the modulus will have a different value. For example, let the bar be forcibly prevented from contracting, either in breadth or thickness, by the application of a suitable lateral tension, $p_2(=p_3)$, then e_2, e_3 are both zero, and

$$Ee_1=p_1-\frac{2p_2}{m};\ 0=p_2-\frac{p_1+p_2}{m},$$

whence we obtain for the magnitude of the necessary lateral stress

$$p_2 = \frac{p_1}{m-1},$$

and for the corresponding extension of the bar

$$Ee_1 = \frac{m^2 - m - 2}{m^2 - m} \cdot p_1.$$

Hence the modulus of elasticity is now

$$A = \frac{m(m-1)}{(m+1)(m-2)} \cdot E.$$

This constant A is what Rankine called the direct elasticity of the substance: it is of course always greater than E. For $m=4$, $A=\frac{6}{5}E$; for $m=3$, $A=\frac{3}{2}E$.

If the bar be free to contract in thickness, but not in breadth, we have p_3 and e_2 zero, and the equations become

$$Ee_1 = p_1 - \frac{p_2}{m}; \quad 0 = p_2 - \frac{p_1}{m}; \quad Ee_3 = 0 - \frac{p_1 + p_2}{m},$$

whence we find

$$Ee_1 = p_1 \cdot \frac{m^2 - 1}{m^2},$$

so that the value of the modulus of elasticity is $\frac{m^2}{m^2-1}E$. In a similar way if p_2, p_3 have any given values the modulus can be found.

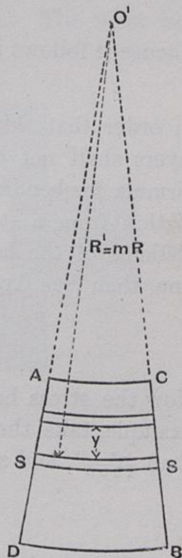


Fig.152

It will now be convenient to examine an important point already referred to in the theory of simple bending, that is to say the assumption (Art. 153) that the modulus of elasticity E was the same as in the case of simple tension, notwithstanding the lateral connection of the elementary bars, into which we imagined the whole beam split up. If these elementary bars were prevented from contracting freely, as they would do if separated from each other, the modulus could not be the same. In fact, however, there is nothing in their lateral connection which prevents them from doing so. Figure 152 shows, on a very exaggerated scale, the form assumed by a transverse section $ACBD$ originally rectangular, cutting a series of longitudinal sections originally parallel to the plane of bending in the straight lines shown. Assuming the upper side stretched as in Fig. 122, page 309, these lines all radiate from a centre O' above the beam, which bends transversely, while the originally straight horizontal layers are cut in arcs of circles struck from the same centre. The upper side of the beam contracts and the lower side expands, and reasoning exactly in the same way as we did when we derived the principal formula,

$$p=\frac{Ey}{R}, \qquad \text{(Art. 153)}$$

we find a corresponding formula for the transverse curvature,

$$p=m\frac{Ey}{R'},$$

whence it follows immediately that

$$R'=mR.$$

In order that this transverse curvature of the originally horizontal layers shall not be inconsistent with the reasoning by which the formula for bending is obtained, all that is necessary is that the deviation from a straight line shall be small as compared with the distance of the layer from the neutral axis. Let x be that deviation, then (see Art. 163) if b be the breadth,

$$x=\frac{b^2}{8R^1}=\frac{b^2}{8mR}=\frac{b^2 \cdot p}{8mEy}.$$

Now the stress being within the elastic limit p/E is very small, for example, take the case of wrought iron, for which p/E is not more than $\frac{1}{1200}$th, and suppose $m=4$,

$$x=\frac{b^2}{38{,}400 \cdot y_1},$$

where y_1 is the greatest value of y, say $\frac{1}{2}h$, where h is the depth, thus

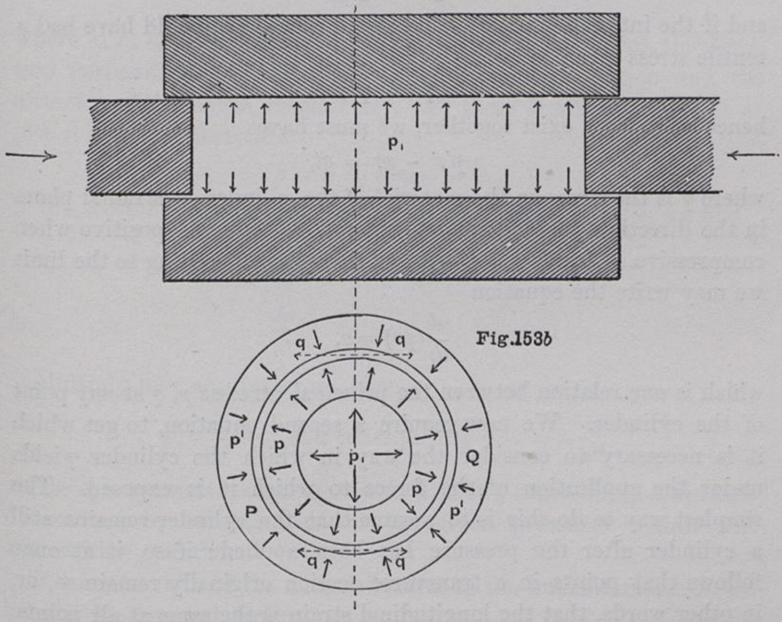
$$x=\frac{b^2}{19{,}200h}.$$

It is obvious that x must be always very small compared with y, except very near the neutral axis, and unless b be very large compared with h. When then a beam is bent *within* the limit of elasticity, the lateral connection of the parts cannot have any sensible influence on its resistance to bending, unless its breadth be great as compared with its depth. The case of a broad thin plate has not been hitherto dealt with theoretically. Beyond the limit of elasticity the lateral connection of the parts may greatly increase the resistance to bending, but this is a matter for subsequent consideration.

209. *Thick Hollow Cylinder under Internal Pressure.*—The equations connecting stress and strain in combination with suitable equations expressing the continuity of the body and the equilibrium of each of its elements are theoretically sufficient to determine the distribution of stress within an elastic body exposed to given forces, and in particular to determine the parts of the body exposed to the greatest stress, and the magnitude of such stress. The most im-

Fig. 153*a*

Fig. 153*b*



portant cases hitherto worked out, in addition to those considered in preceding chapters, are the torsion of non-circular prisms and the action of internal fluid pressure on thick hollow cylinders and spheres. For M. St. Venant's investigations on torsion we must refer to Art. 188, page 360, and the authorities there cited. We shall only consider the comparatively simple case of a homogenous cylinder.

Fig. 153*a* shows a longitudinal section of a hollow cylinder open at the ends, which are flat: the cylinder contains fluid which is acted on by two plungers forced in by external pressure so as to produce an internal fluid pressure p_1. Fig. 153*b* shows the same cylinder in transverse section: imagine a cylindrical layer of thickness t, this thin cylinder will be acted on within and without by stress which symmetry shows must be normal; let these stresses be p and p', and the internal and external radii of the thin cylinder be r and r'. Now, if p' the external pressure had existed alone, a compressive stress q would have been produced on the material of the cylinder given by the equation (see Art. 150)

$$p'r' = qt;$$

and if the internal pressure had existed alone, we should have had a tensile stress given by

$$pr = qt;$$

hence when both exist together, we must have

$$p'r' - pr = qt,$$

where q is the stress on the material of the cylinder on a radial plane in the direction perpendicular to the radius reckoned positive when compressive. Clearly $t = r' - r$, and therefore proceeding to the limit we may write the equation

$$\frac{d}{dr}(pr) = q,$$

which is one relation between the principal stresses p, q at any point of the cylinder. We now require a second equation, to get which it is necessary to consider the way in which the cylinder yields under the application of the forces to which it is exposed. The simplest way to do this is to assume that the cylinder remains still a cylinder after the pressure has been applied: if so, it at once follows that points in a transverse section originally remain so, or, in other words, that the longitudinal strain is the same at all points. It is not to be supposed that there is anything arbitrary about this assumption: no other, apparently, can be made if the ends of the cylinder are free, the pressure on the internal surface exactly uniform, and the cylinder be homogenous and free from initial strain. For when this is the case, there is no reason why the cylinder should be in a different condition in one part of its length than in another. If the ends are not free, or if the pressure is greater in the centre, the middle of the cylinder will bulge, but not otherwise.

It is also clear that the total pressure on a transverse section must be zero because the ends are free, and hence it is natural to suppose that it is also zero at every point of the transverse section, an assumption which we shall presently verify.

The equations connecting stress and strain therefore become

$$Ee_1 = p - \frac{q}{m};$$

$$Ee_2 = q \quad \frac{p}{m};$$

$$Ee_3 = 0 - \frac{p+q}{m},$$

where e_1, e_2, e_3 are the strains in, the direction of the radius, the direction perpendicular to the radius in the transverse section, and, the direction of the length, respectively. Of these the last is constant, as just stated, and therefore

$$p + q = \text{const.} = 2c_1$$

is the second equation connecting p, q. Substituting for q, we find

$$\frac{d}{dr}(pr) + p = 2c \;;$$

or

$$\frac{dp}{dr} + 2p = 2c_1.$$

Multiply by r and integrate, then

$$p = \frac{c_2}{r^2} + c_1, \text{ and consequently } q = c_1 - \frac{c_2}{r^2},$$

where c_2 is a constant of integration. The two constants c_1, c_2 are now determined by consideration of the given pressures within and without the cylinder.

If n be the ratio of the external radius to the internal radius R, we have at the internal surface

$$\left.\begin{matrix} p = p_1 \\ r = R \end{matrix}\right\} \quad \therefore\ p = c_1 + \frac{c_2}{R^2};$$

and at the external surface

$$\left.\begin{matrix} p = 0 \\ r = nR \end{matrix}\right\} \quad \therefore\ 0 = c_1 + \frac{c_2}{n^2R^2};$$

from which two equations we get

$$c_1 = -\frac{p_1}{n^2-1}, \text{ and } c_2 = \frac{p_1 n^2}{n^2-1} \cdot R^2.$$

Substituting these values in the equation for q,

$$q = -\frac{p_1}{n^2-1}\left\{1 + n^2 \cdot \frac{R^2}{r^2}\right\};$$

the negative sign in this formula indicates that the stress is tensile, as we might have anticipated. The formula shows that the stress decreases from $\frac{n^2+1}{n^2-1} \cdot p_1$ at the internal surface to $\frac{2p_1}{n^2-1}$ at the external surface. The mean stress is obtained from the equation (Art. 150.)

$$q_0(nR - R) = p_1 R;$$

hence the maximum stress is greater than the mean in the ratio $n^2 + 1 : n + 1$, and it is clear that it can never be less than p_1.

Verification of Preceding Solution.—The radial strain (e_1) and the hoop strain (e_2) are given by the above equations in terms of the stress. Now these changes of dimension are not independent, but are connected by a certain geometrical relation which it is necessary to examine in order to see whether it is satisfied by the values we have found.

Returning to the diagram, suppose the internal radius of the ring BQ to increase from r to s, and the external radius from r' to s'; then

$$2\pi s = 2\pi r(1 + e_2),$$

$$2\pi s' = 2\pi r'\left(1 + e_2 + t\frac{de_2}{dr}\right),$$

$$\therefore \quad s' - s = (r' - r)(1 + e_2) + r't\frac{de_2}{dr};$$

or since the thickness of the ring changes from t to $(1 + e_1)t$,

$$1 + e_1 = 1 + e_2 + r'\frac{de_2}{dr},$$

$$e_1 = \frac{d}{dr}(e_2 r).$$

This relation must always hold good, in order that the rings after strain may fit one another, and should therefore be satisfied by our results. On trial it will be found that it is satisfied, and we conclude that the solution we have obtained satisfies all the conditions of the problem, and is therefore the true and only solution, subject to the conditions already explained. For further remarks on this question, see Appendix.

210. *Strengthening of Cylinder by Rings. Effect of great Pressures.*—The stress within a thick hollow cylinder under internal fluid pressure may be equalized, and the cylinder thus strengthened by constructing it in rings, each shrunk on the next preceding in order of diameter. For a cylinder so constructed will be in tension at the outer surface and compression at the inner surface before the pressure is applied, and therefore after the pressure has been applied will be subjected to less tension at the inner and more tension at the outer surface than if it had been originally free from strain. It is theoretically possible to determine the diameters of the successive rings so that the pressure shall be uniform throughout. The principle is important, and frequently employed in the construction of heavy guns.

When the limit of elasticity is overpassed the formula fails, and the distribution of stress becomes different. If the pressure be imagined gradually to increase until the innermost layer of the cylinder begins to stretch beyond the limit, more of the pressure is transmitted into the interior of the cylinder, so that the stress becomes partially equalized. If the pressure increases still further, the tension of the innermost layer is little altered, and in soft materials longitudinal flow of the metal commences under the direct action of the fluid pressure. The internal diameter of the cylinder then increases perceptibly and permanently. This is well known to happen in the cylinders employed in the manufacture of lead piping, which are exposed to the severe pressure necessary to produce flow in the lead. The cylinder is not weakened but strengthened, having adapted itself to sustain the pressure. Cast-iron hydraulic press cylinders are often worked at the great pressure of 3 tons per sq. inch, a fact which may perhaps be explained by a similar equalization.

EXAMPLES.

1. When the sides of a bar are forcibly prevented from contracting, show that the necessary lateral stress is given by

$$p_2 = Be,$$

where $B = \dfrac{mE}{m^2 - m - 2}$. This constant B is what Rankine called the "lateral" elasticity of the substance.

2. With the notation of the preceding question and of Art. 106, prove that

$$C = \frac{A - B}{2}.$$

3. In a certain quality of steel $E = 30{,}000{,}000$; $C = 11{,}500{,}000$: find the elasticity of volume and the values of A and B, assuming the material to be isotropic. *Ans.* $m = 3\frac{2}{7}$; $D = 25{,}400{,}000$.

4. The cylinder of an hydraulic accumulator is 9 inches diameter. What thickness of metal would be required for a pressure of 700 lbs. per square inch, the maximum tensile stress being limited to 2,100 lbs. per square inch? Also, find the tensile stress on the metal of the cylinder at the outer surface. *Ans.* Thickness = 1·84″; Stress = 1,400 lbs. per square inch.

5. If the cylinder in the last question were of wrought iron, proof resistance to simple tension 21,000 lbs. per square inch, at what pressure would the limit of elasticity be overpassed? $m = 3{\cdot}5$. (See Art. 223, page 428.) *Ans.* 6400.

6. Find the law of variation of the stress within a thick hollow sphere under internal fluid pressure. By a process exactly like that for the case of the cylinder (page 404) it is found that the equation of equilibrium is

$$\frac{d}{dr}(pr^2) = 2qr.$$

The equation of continuity is the same as that for a cylinder (Art. 209), and the equations connecting stress and strain are now

$$Ee_1 = p - \frac{2q}{m};$$

$$Ee_2 = q - \frac{p+q}{m}.$$

We can now by elimination of q, reduction, and integration obtain

$$p = c_1 + \frac{c_2}{r^3};$$

$$q = c_1 - \frac{c_2}{2r^3};$$

the constants being found as in the cylinder.

7. The cylinder of an hydraulic press is 8 inches internal and 16 inches external diameter. If the pressure be 3 tons per sq. inch find the principal stresses at the internal and external circumference.

Ans. At inner circumference $\begin{cases} \text{Major Stress} = 5 \text{ (Tension).} \\ \text{Minor Stress} = 3 \text{ (Thrust).} \end{cases}$

At outer ,, $\begin{cases} \text{Major Stress} = 2 \text{ (Tension).} \\ \text{Minor Stress} = 0. \end{cases}$

8. In the last question find the "equivalent simple tensile stress" (p. 428), assuming $m = 3{\cdot}5$. *Ans.* 5·86 and 2 tons.

CHAPTER XVIII.

MATERIALS STRAINED BEYOND THE ELASTIC LIMIT. STRENGTH OF MATERIALS.

211. *Plastic Bodies*—If the stress and strain to which a piece of material is exposed exceed certain limits its elasticity becomes imperfect, and ultimately separation into parts takes place. We proceed to consider what these limits are in different materials under different circumstances: it is to this part of the subject alone that the title "Strength of Materials" is, strictly speaking, appropriate.

Reference has already been made (Art. 147) to a certain condition in which matter may exist, called the Plastic state, which may be regarded as the opposite of the Elastic state, which has been the subject of preceding chapters. In this condition the changes of size of a body are very small, as before; but if the stress be not the same in all directions the difference, if sufficiently great, produces continuous change of shape of almost any extent. Some materials are not plastic at all under any known forces, but many of the most important materials of construction are so, more or less, under great inequality of pressure.

Fig.154

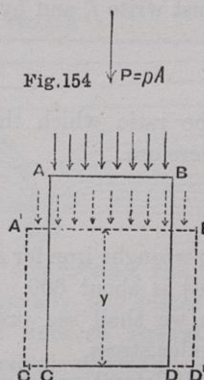


Fig. 154 shows a block of material which is being compressed by the action of a load P applied perfectly uniformly over the area AB. Let the intensity of the stress be p, then so long as p is small the compression is small and proportional to the stress; but when it reaches a certain limit the block becomes visibly shorter and thicker.

This limit depends on the hardness of the material, and the value of p may be called the "co-efficient of hardness." In an actual experiment the friction of the surfaces between which the block is compressed holds the ends together, so that it bulges in the middle, as in Fig. 158, p. 419, which represents an experiment on a short cylinder of soft steel. In the ideal case the sectional area remains uniform, changing throughout inversely as the height, as expressed by the equation

$$Ay = A_1 y_1,$$

where A is the area and y the height of the block.

In a truly plastic body p the intensity of the stress remains constant, and therefore the crushing load P varies as A, that is inversely as y. This is the same law as that of the compression of an elastic fluid when the compression curve is an hyperbola, and we therefore conclude (Art. 90) that the work done in crushing is

$$U = Py \cdot \log_\varepsilon r = pAy \log_\varepsilon r = pV \log_\varepsilon r,$$

where r is the ratio of compression and V the volume. Certain qualities of iron and soft steel will endure a compression of one-fourth or even of one-half the original height, and amounts of energy are thus absorbed which are enormous compared with the resilience of the metal. To illustrate this, suppose that plasticity begins as soon as the limit of elasticity f is overpassed, then for p we must write f, and by Art. 96 the resilience for a volume V is

$$\text{Resilience} = \frac{f^2}{2E} \cdot V.$$

The ratio which the work just found bears to the resilience is therefore

$$\text{Ratio} = \frac{2E}{f} \cdot \log_\varepsilon r.$$

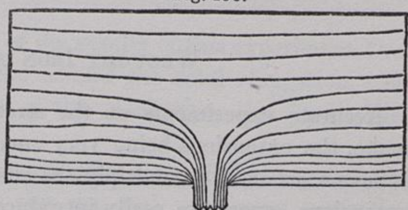
In wrought iron for a compression of one fourth the height ($r = 1{\cdot}333$) this is about 800. The actual ratio must be much greater, because, as we shall see presently, the hardness of the material increases under stress.

If lateral pressure of sufficient magnitude be applied to the sides of the block, the longitudinal force being removed, the effect is elongation instead of compression, contraction of area instead of expansion. The magnitude of the lateral pressure is found by imagining a tension applied both longitudinally and laterally of equal intensity. Such a tension has no tendency to alter the form of the block, being equivalent to fluid pressure, but it reduces the lateral pressure to zero, while it introduces a longitudinal tension of the same amount, which has the same value as the longitudinal compression of the preceding case. We see then that in every case a certain definite difference of pressure is required to produce change of shape in a plastic body, the direction of the change depending on the direction of the difference. The work done is found by the same formula as before, r meaning now the ratio of elongation.

In the process of drawing wire the lateral pressure is applied by the sides of the conical hole in the draw-plate, which are lubricated to reduce friction, and the force producing elongation in the wire is the sum of the tensile stress applied to draw the wire through the hole and the compressive stress on the sides. The work done is given by the same formula as before, p being now the sum in question.

212. *Flow of Solids.*—When a plastic body changes its form the process is exactly analogous to the flow of an incompressible fluid, which indeed may be regarded as a particular case. In the solid the distorting stress at each point at which the distortion is going on has a certain definite value which in the fluid is zero. The experimental proof of this is furnished by the experiments of M. Tresca, of which Fig. 155 shows an example. Twelve circular plates of lead are placed one upon another in a cylinder, which has a flat bottom with a small orifice at its centre. The pile of plates being forcibly compressed, the lead issues at the orifice in a jet, and the originally flat plates assume the forms shown in the figure. The lines of separation, indicating the position of particles of the metal originally in a transverse section, are quite analogous to the corresponding lines in the case of water issuing from a vessel through an orifice in the bottom. Tresca's experiments were very extensive, and showed that all non-rigid material flowed in the same way. Lead ap-

Fig. 155.



proaches the truly plastic condition; the difference of pressure necessary to make it flow being always about the same. Tresca ascribes to it the value of 400 kilogrammes per square centimetre, or about 5,700 lbs. per square inch; * but it is probably subject to considerable variations.

The manufacture of lead pipes, the drawing of wire, and all the processes of forging, rolling &c., by which metals are manipulated in the arts, are examples of the Flow of Solids.

213. *Preliminary Remarks on Materials. Stretching of Wrought Iron and Steel.*—Materials employed in construction may roughly be divided into three classes. The first are capable of great changes of form without rupture, and, when possessing sufficient strength to resist the necessary tension, may be drawn into wire. This last property is called ductility, and this word may be used to describe the class which we shall therefore call Ductile Materials. The second, being incapable of enduring any considerable change of this kind, may be described as Rigid Materials. The third are in many cases not homogenous, but may be regarded as consisting of bundles of fibres laid side by side, they may therefore be described as Fibrous Materials; they are generally of organic origin.

We shall commence with the consideration of ductile materials, and more especially of

Wrought Iron and Steel.

Accurate experiments on the stretching of metal are difficult to make, the extensions being very small and the force required great. If levers are used to multiply the effect of a load or to magnify the extensions, errors are easily introduced. If the levers are dispensed with, a great length of rod is necessary and a heavy load the manipulation of which involves difficulties. The best modern testing machines operate by hydraulic pressure, and the elongations are measured by micrometers. The experiment we select for description was made by Hodgkinson on a rod of wrought iron ·517 inch diameter,

* The co-efficient employed by Tresca, and called by him the "co-efficient of fluidity," is half that used in the text. It is the magnitude of the distorting stress necessary to produce flow.

49 feet 2 inches long, loaded by weights placed in a scale pan* suspended from one end. The load applied was increased by equal increments of 5 cwt. or 2667·5 lbs. per square inch of the original sectional area of the bar; each application of the load being made gradually, and the whole load removed between each. At each application and removal the elongation was measured so as to test the increment of elongation, both temporary and permanent, occasioned by each load. If the rod were perfectly elastic the temporary increments should be equal and the permanent elongations (usually called "sets") zero.

STRETCHING OF A WROUGHT IRON ROD, 49 FEET 2 INCHES LONG.

Load.		Elongation in Inches.	Increment of Elongation.	Permanent Set.
2667·5 × 1	2667·5	·0485	·0485	
,, × 2	5335	·1095	·061	
,, × 3	8003	·1675	·058	·0015
,, × 4	10,670	·224	·0565	·002
,, × 5	13,338	·2805	·0565	·0027
,, × 6	16,005	·337	·0565	·003
,, × 7	18,673	·393	·056	·004
,, × 8	21,340	·452	·059	·0075
,, × 9	24,008	·5155	·0635	·0195
,, × 10	26,675	·598	·0825	·049
,, × 11	29,343	·760	·162	·1545
,, × 12	32,010	1·310	·550	·667

The annexed table shows part of the results of this experiment, the first column giving the load, the second the total elongation, the third the successive increments of the elongation, the fourth the total permanent set.

On examining the table we see that, after some slight irregularities at the commencement due to the material not being perfectly homogeneous, the increments of elongation are nearly constant till we reach the eighth load of 21,340 lbs. per square inch, after which the increments show a rapid increase. Further, the permanent set,

* Being one of the best of its kind of old date this experiment has often been quoted. For the original description see the *Report of the Commissioners appointed to enquire into the Application of Iron to Railway Structures.* For a notice of some important experiments on stretching recently made the reader is referred to the Appendix.

which at the commencement is very minute and increases very slowly, at the same point shows a sudden increase indicating that the observed increase is almost wholly due to a permanent elongation of the bar, the temporary increase following approximately the same law as before. Notwithstanding this the bar is not torn asunder till a much greater load is applied. The table shows the results up to a load of 32,000 lbs. per square inch, but rupture did not occur till a load of 53,000 lbs. was applied. The extension at the same time increased to nearly 21 inches, being more than forty times its amount at the elastic limit.

We conveniently represent the results graphically by setting off the elongations as abscissæ along a base line with corresponding ordinates to represent the stress, thus obtaining a curve of "Stress and Strain" (Fig. 156). The curve will be seen to be nearly straight up to a stress of 22,000 lbs. and then to bend sharply, becoming nearly straight in a different direction. A curve of permanent set may also be constructed which is seen to follow the same general law.

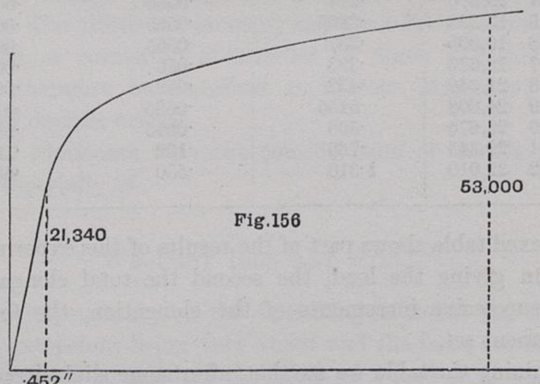


Fig.156

This experiment may be taken as a type of many hundreds of such experiments which have been made on iron and the softer kinds of steel, showing that in these materials a tolerably well-defined limit exists, within which the material is nearly perfectly elastic (compare Art. 127); the small deviations are more due to the want of perfect homogeneity in the bar than to actual defect in the elasticity. They usually diminish greatly if the experiment be tried a second time on the same bar. The position of the limit of elasticity and the value of the modulus of course vary. Some examples will be given presently.

Accompanying the increase of length of the bar we find a contraction of area; within the elastic limit, however, this is so small as to escape observation. Outside the limit it becomes visible, consisting in the first instance of a more or less uniform contraction at all or nearly all points, followed by a much greater contraction at one or sometimes two points where there happens to be some local weakness.* Within the elastic limit the density of the bar diminishes, but by an amount so small that the fact is rather known by reasoning than determined by experiment. Outside the limit there is a permanent diminution which is perceptible, though still very small.

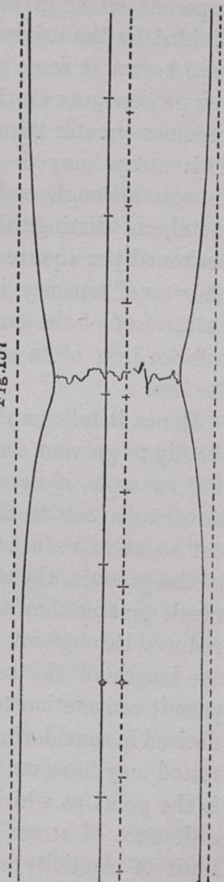


Fig.157

Thus beyond the elastic limit the bar draws out, changing its form like a plastic body without sensible change of volume. The bar finally tears asunder at the most contracted section, as shown by the annexed figure (Fig. 157) representing an experiment by Mr. Kirkaldy on a bar of iron 1 inch diameter, in which the contraction of area was 61 per cent., and the elongation 30 per cent., ultimate strength 58,000 lbs. per square inch of original area, 146,000 lbs. per square inch of fractured area. The contraction of section in good iron and soft steel is 50 or 60 per cent.

214. *Real and Apparent Tensile Strength of Ductile Metals.*—Thus the process of stretching an iron bar beyond the limit of elasticity till it breaks is an example of the "flow of solids," the iron behaving to a certain extent like a plastic body. There is, however, this

* On this point see *Preliminary Experiments on Steel by a Committee of Civil Engineers*, London, 1868. On account of the uncertainty of the amount of contraction at various points, the ultimate extension is an imperfect measure of the ductility of the iron, even when the pieces are of the same length and sectional area.

difference, that a constantly increasing stress is necessary to produce continuous flow, which increase is supplied partly by increase of the stretching load, partly by the contraction of area. The actual stress at each instant on the contracted area is much greater than the apparent stress given in the table, which is merely the total load divided by the original sectional area. Hence when contraction has once begun at some point of local weakness it continues there, and the process goes on till the stress per square inch of the reduced area becomes greater than the metal will bear, when fracture takes place. This stress may to a certain extent be regarded as a measure—though a rough and imperfect one — of the true tenacity of the metal, as distinguished from the "apparent" tenacity which is reckoned per square inch of the original area. For many purposes the "true" tenacity, in good iron more than double the apparent, is a better test of the quality of the iron than the actual breaking load, inferior iron often showing a high apparent tenacity but contracting far less.

Hence it follows that if the form of the piece be such as partly or wholly to prevent contraction the apparent strength will be increased. For example, if two pieces of the same bar be taken and one turned down to a certain diameter, while in the other narrow grooves are cut so as to reduce the diameter to the same amount at the bottom of the grooves, the strength of the grooved piece will be found to be much greater than that of the piece the diameter of which has been reduced throughout, and this can only be explained by observing that the length of the reduced part of the grooved bar is insufficient to permit contraction to any considerable extent. This is a point to be noticed in considering experimental results.* The form of the specimen tested may have much influence. Further, since the limit of elasticity is the point at which flow commences, and since the flow is due to difference of stress, it follows that the same causes must raise the limit of elasticity, and thus we are led to the conclusion that there are two elements constituting strength in a material, first, tenacity and, secondly, rigidity. In some materials, such as these we are now considering, the tenacity is much greater than the rigidity, and in them the limit of elasticity will depend on the rigidity, and will have different positions according to the way the stress is applied.

* See *Experiments on Wrought Iron and Steel*, by Mr. Kirkcaldy, p. 74. 1st edition. Glasgow, 1862.

It will lie much higher, and the apparent strength will be much greater when lateral stress is applied to prevent contraction.

215. *Increase of Hardness by Stress beyond the Elastic Limit.*—In clay and other completely plastic bodies a certain definite difference of pressure is sufficient to produce flow: in iron, copper, and probably other metals, however, as we have just seen, this is not the case, the metal acquiring increased rigidity in the act of yielding to the pressure. Thus the effect of stress exceeding the elastic limit is always to raise the limit, whether the stress be a simple tensile stress or whether it be accompanied by lateral pressure. All processes of hammering, cold rolling, wire drawing, and simple stretching have this effect. If a bar be stretched by a load exceeding the elastic limit and then removed, on re-application of a gradually increasing load we do not find a fresh drawing out to commence at the original elastic limit, but at or near the load originally applied.* If the load be further increased drawing out re-commences. Hence, whenever iron is mechanically "treated" in any way which exposes it to stress beyond the elastic limit, contraction is prevented and the apparent strength is increased: for example iron wire is stronger than the rod from which it is drawn; when an iron rod is stretched to breaking the pieces are stronger than the original rod. It is not certain that the real strength of materials is always increased by such treatment; perhaps in some cases the contrary, for we know that the modulus of elasticity and specific gravity are somewhat diminished.† On the other hand there are cases in which the increase of strength is greater than can be accounted for in this way. On annealing the iron it is found to have resumed its original properties, a circumstance which indicates that the increased rigidity is due to a condition of constraint which is removed by heating the metal till it has assumed a completely plastic condition. In considering the effect of impact, the diminution of ductility occasioned by the application of stress beyond the elastic limit is a most important fact to be taken into account (see Art. 226). Working

* Styffe *On Iron and Steel*, p. 68.

† The raising of the limit of elasticity by mechanical treatment of various kinds has long been known: in the case of simple stretching the effect appears to have been first noticed by Thälen in a paper, a translation of which will be found in the *Philosophical Magazine* for September, 1865.

iron or steel hot has generally the effect of increasing both its strength and its ductility.

216. *Compression of Ductile Material.*—In a perfectly elastic material compression is simply the reverse of tension, the same changes of dimension being produced by the same stress, but in the reverse direction. Also in a plastic body a given difference of stress produces flow, whether the stress be tensile or compressive; hence in ductile metals we should expect to find the modulus of elasticity and the limit of elasticity nearly the same in compression as in tension. These conclusions are borne out by experiment. In the case of wrought iron and steel experiments on the direct compression of a bar are more difficult to carry out than experiments in tension, the bars are necessarily of limited length, and must be enclosed in a trough to prevent lateral bending; minute accuracy is therefore hardly attainable. A considerable number have, however, been made, from which it appears that the modulus of elasticity and the limit of elasticity are nearly the same in the two cases.*

EXPERIMENT BY SIR W. FAIRBAIRN ON A BLOCK ·72 INCH DIAMETER OF SOFT BESSEMER STEEL.

TOTAL LOAD $= P$.	HEIGHT OF BLOCK $= y$.	CRUSHING STRESS $p = \frac{Py}{Ay_1}$.
0	·997	0
16·7	·92	37·8
20·1	·865	42·9
23·3	·797	45·9
26·3	·731	47·4
29·5	·672	48·9
32·6	·613	49·4
35·8	·574	50·6
39·3	·535	51·9
41·0	·505	50·8

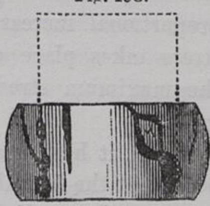
REMARKS.—The apparent ultimate tensile strength of this steel was 36 tons, its limit of elasticity 22 tons per square inch. Modulus of elasticity 30,300,000 lbs. Ratio of contraction ·41. Real tensile strength 88·5 tons.

* Perhaps the best set of experiments are those made by the "Committee of Civil Engineers." See their report already cited, p. 7-13.

The metal yields beyond the limit by a process of flow of the same character as in tension, but expanding laterally instead of contracting. This is especially seen in experiments made by the late Sir W. Fairbairn in 1867, and somewhat earlier by Mr. Berkeley, on the compression of short blocks of steel. In both, the blocks were pieces of round bars, of height somewhat greater than the diameter, and the results were very similar.

The annexed table gives the results of one of Sir W. Fairbairn's experiments. Column 1 gives the actual load laid on; column 2 the corresponding height of the block, both given directly by the experiments. Column 3 is calculated by dividing the product of load and height by the original sectional area and height, and represents the crushing stress per square inch of the mean sectional area. If the block did not bulge in the centre on account of friction holding its ends together (Fig. 158), this would be the actual crushing stress, which, however, must in fact be less. The table shows that after a compression of about one-third the crushing stress remains nearly constant at about 50 tons per square inch. The experiment terminated at a compression of one-half. This kind of steel then is perfectly elastic up to 22 tons per square inch, is partially plastic between 2 and 50, and behaves as a plastic body under a difference of stress of 50 tons per square inch.

Fig. 158.



In ductile materials fracture takes place under compression by longitudinal cracks as shown in Fig. 158, which represents an experiment on a different quality of steel. The amount of compression which different materials will bear is very different according to their malleability; it is generally difficult to fix upon the ultimate strength, as it depends on the mode in which the experiment is made. In iron and steel it is somewhat less than the apparent tensile strength.

The compression of iron blocks has been less thoroughly studied than that of steel, but it is known that the results are similar although the strength and the ultimate ratio of compression are much less. Set becomes sensible at about 10 tons per square inch, and the ultimate strength is from 40 to 50,000 lbs. per square inch if lateral flexure be prevented.

217. *Bending beyond the Elastic Limit.*—Since wrought iron and steel are nearly perfectly elastic when the stress applied is not too great, it follows that the formula already obtained for the moment of resistance to bending of a bar must be true for these materials so long as the stress does not exceed the elastic limit determined by tension experiments of the kind just described. Experience fully confirms this conclusion, for the deflection obtained by experiment agrees well with that found from formulæ previously given with the same value of the modulus. As soon, however, as the maximum stress exceeds this limit, it is no longer true that the stress at different points of the transverse section varies as the distance from the neutral axis. It does not increase so fast, because the extension and compression at points near the surface is not accompanied by a proportional increase of stress. Hence, a partial equalization of stress takes place over the transverse section, and consequently the maximum stress for a given moment of resistance is not so great.

Again, it has been repeatedly explained in the earlier part of this book that the lateral connection of the several layers into which we imagine a beam divided has no influence on the stress produced by bending so long as the limit of elasticity is not exceeded. But when the limit is passed, the connection between those layers which are most stretched and compressed with those layers which have not yet lost their elasticity prevents their contraction and expansion, and so raises the limit of elasticity in accordance with the general principle explained in Art. 215. Thus, the limit of elasticity lies higher, and the apparent strength is greater in bending than in tension. In Fairbairn's experiment quoted above the same steel was tested in tension, compression, and bending. The elastic limit in bending was 30 tons, in tension 22 tons. The magnitude of the difference will depend on the form of transverse section, and on the ductility of the material. According to Mr. Barlow it may reach 50 per cent. in a rectangular section.* The case of çast iron will be referred to farther on.

Putting aside the effect of lateral connection, it may be interesting to make a calculation of the effect of equalization, by supposing that under a bending moment very slowly and steadily applied beyond the elastic limit, the metal behaves like a truly plastic material

* *Phil. Trans.*, 1855-57.

throughout the transverse section, so that the stress is uniform. Referring to the formula on page 313, we have

$$\Sigma pybt = M,$$

in which we must now, instead of assuming that p varies as y, suppose p a constant. Then

$$M = 2p \, . \, A\bar{y},$$

where A is the area of the part of the section which lies on either side of the neutral axis and $\bar{y}$ the distance of its centre of gravity from that axis. For the same value of the modulus this gives a moment of resistance in a rectangular section 50 per cent. greater than if the material had been elastic. How far any apparent increase of strength due to equalization or lateral connection may be regarded in practice is uncertain. A failure of elasticity must have taken place at certain points in order that there may be any increase at all, and in cases where the load is frequently reversed the bar must be weakened. (See Art. 225.)

Cast Iron and other Rigid Materials.

218. *Stretching of Cast Iron.*—The phenomena attending rupture by tension of cast iron are essentially different from those described above for the case of ductile metals. This will be sufficiently shown by an experiment, also made by Hodgkinson, on a bar of this material 50 feet long, 1·159 inch diameter. The experiment was made in the same way as that already described on the wrought-iron rod,* and the results are shown in the annexed table. The first four loads were applied as before, by increments of 5 cwt., here equivalent to 531 lbs. per square inch; the whole load, after measurement of the elongation, being completely removed, and the permanent set measured. After the fourth load the increment was 10 cwt., and this was carried on till the bar broke at a stress of 16,000 lbs. per square inch.

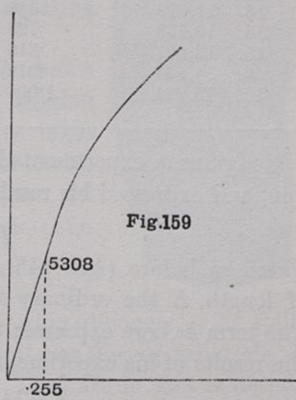


Fig.159

* *Report of Commissioners on the Application to Railway Structures*, p. 51.

The third column as before shows the increments of elongation, which, after a stress of 5,308 lbs. per square inch, or $\frac{1}{3}$ the breaking load, has been reached, show a gradual increase till actual rupture occurs. The results of the experiments are graphically exhibited in the annexed diagram (Fig. 159) of stress, strain, and permanent set. The form of the curve is different from that of wrought iron, showing no point of maximum curvature, because in this material the bar does not draw out.

STRETCHING OF A CAST-IRON BAR 50 FEET LONG, 1·159 INCH DIAMETER.

LOAD IN LBS. PER SQUARE INCH.		ELONGATION IN INCHES.	INCREMENT OF ELONGATION.	PERMANENT SET.
1.	531	·024	·024 × 2 = ·048	Perceptible.
2.	1,062	·0495	·0255 × 2 = ·051	·0015
3.	1,592	·0735	·024 × 2 = ·048	·002
4.	2,123	·09828	·0247 × 2 = ·0514	·0045
5.	3,185	·1485	·0503	·0105
6.	4,246	·200	·0515	·0155
7.	5,308	·255	·055	·022
8.	6,370	·313	·058	·028
9.	7,431	·374	·061	·037
10.	8,493	·435	·061	·046
11.	9,554	·504	·069	·056
12.	10,616	·572	·068	·067
13.	11,678	·648	·076	·0795
14.	12,739	·728	·080	·095
15.	13,801	·816	·088	·1115
16.	14,863	·912	·096	·132
17.	15,924	1·000	·088	—

Hodgkinson experimented on a large variety of different kinds of iron, and expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

where, as before (Art. 148), p is the stress, e the extension per unit of length, E the ordinary modulus of elasticity, and k a constant. The term ke here expresses the defect of elasticity of the bar. From the results of his experiments we find the average values

$$E = 14,000,000\ ; \quad k = 209.$$

Cast iron, however, is a material of variable quality, and the value of these constants may have a considerable range. Up to one third the breaking load it may be regarded as approximately perfectly elastic, but the limit is by some authorities placed much higher.

219. *Crushing of Rigid Materials.*—In the ductile metals the effects of compression are nearly the reverse of those of extension, as has been sufficiently shown in previous articles, but in cast iron this is by no means the case. Hodgkinson experimented in this question with great care and accuracy, testing pieces of iron of exactly the same quality under compression and tension to enable a comparison to be made. The bars were enclosed in a frame and tested by direct compression. Hodgkinson expressed his results by a formula, which may be written

$$p = Ee(1 - ke),$$

the symbols having the same meanings as before, and the values may be taken as

$$E = 13{,}000{,}000\ ;\quad k = 40.$$

The smaller value of k indicates that the elasticity under compression is much less imperfect under the same stress. Short cylinders of the metal were also crushed, and the crushing load found to be five times the tensile strength or more.

It thus appears that in compression cast iron is six times stronger than in tension, and this is true not merely of the ultimate resistance, but in great measure also of the elastic resistance, for the elasticity of the metal is not sensibly impaired until one third the crushing load is reached.

Fig. 160.

The manner in which crushing occurs is shown in the accompanying figure; instead of bulging out like a ductile metal, oblique fracture takes place on a plane inclined at 45° or rather less to the axis, being (approximately) the plane on which the shearing stress is a maximum (Fig. 160).

Great resistance to compression, as compared with tension, and sudden fracture by shearing obliquely or by splitting longitudinally are characteristics of all non-ductile materials, of which cast iron may be taken as a type. They are in fact materials the tenacity of which is much less than the rigidity.

220. *Breaking of Cast-Iron Beams.*—When a cast-iron bar is bent till the tensile stress at the stretched surface exceeds one third the tensile strength of the material, the defective elasticity of the metal causes a partial equalization of stress on the transverse section as in the case of wrought iron. Besides this, the elasticity being much more perfect under compression than under tension, the equalization is greater on the stretched side than on the compressed side, and the neutral axis moves towards the compressed edge of the beam. For both these reasons the moment of resistance to bending is greater for a given maximum tensile stress than it would be if the material were perfectly elastic. Thus it follows that if the co-efficient in the ordinary formula for bending be assumed equal to the tensile strength of the material, the calculated moment of resistance will be less than the actual moment of rupture of the beam by an amount which is greater for a rectangular section than for an I section. The discrepancy is found by experiment to be very great, the calculated moment for a rectangular section being less than one half, while for an I section it is about equal that found by experiment. The causes just pointed out only partially account for this, especially as Mr. Barlow's experiment cited above appears to show that no considerable deviation of the neutral axis takes place, and it is probable that the lateral connection of the several layers of the beam has (near the breaking point) a sensible influence on the strength of the parts of the beam exposed to tensile stress, a question we shall return to farther on.

SHEARING AND TORSION. COMPOUND STRENGTH.

221. *Shearing and Torsion.*—We now pass on to cases where the ultimate particles of the material are subject not to a simple longitudinal stress, but to stress of a more complex character. The simplest case is that of a simple distorting stress where the stress consists of a pair of shearing stresses (Fig. 140) on planes at right angles, or what is the same thing (Art. 183) of a pair of equal and opposite longitudinal stresses (Fig. 141) on planes at right angles. Examples of this kind of stress occur in shearing, punching, and twisting. Experiments on shearing are subject to many difficulties and are often not conducted in such a way as to satisfy the conditions necessary for uniformity of distribution of stress on the section. Moreover they necessarily give the ultimate resistance only without reference to the limit of elasticity. The whole process of shearing and punching is very complex, being at the commencement of the operation usually accompanied by a flow of the metal similar to that already referred to. Thus, when a hole is punched in a thick plate the punch sinks deep into the plate before the actual punching takes place, the metal being displaced by lateral flow, and the piece ultimately punched out being of less height than the thickness of the plate.*

Separation takes place in the first instance by the formation of fine cracks inclined at 45° to the plane of shearing. In soft materials the surfaces slide past each other and separate, but in harder materials there is a strong tendency to the formation of an oblique fracture. In wrought iron and steel the ultimate resistance to shearing is probably about three fourths the ultimate resistance to tension of the same material. The question of a theoretical connection between the elastic strengths in the two cases is considered further on.

Experiments on torsion are not numerous, and many of those which exist are not experiments on simple twisting, but on a combination of bending and twisting. Such experiments would be of great value if accompanied by corresponding experiments on simple twisting and bending made on similar pieces of material. It is known however that in the ductile metals the elastic resistance to torsion is less than the resistance to tension. A series of experiments on torsion made by Prof. Thurston give some interesting results.† Curves are drawn the abscissæ of which represent angles and the ordinates twisting moments, and the form of these curves shows that in some cases defective homogeneity causes a great deficiency in the elasticity at small angles of torsion. In general, however, the curves closely resemble the ordinary curve of stress and strain, already given for a stretched bar, being nearly straight up to a certain point and then curving towards the axis.

In twisting, as in bending, after passing the elastic limit, the stress at each point of the section, instead of varying as the distance from the centre as it must do in perfectly elastic material, varies much

* On this subject see M. Tresca's paper cited above, and two articles in the Journal of the Franklin Institute.

† See Paper on *Materials of Machine Construction*, read before the American Society of Civil Engineers, 1874. No diameters are given, except for the woods, so that the stress corresponding to the limit of elasticity cannot be found.

more slowly so as to become partially equalized. Hence the twisting moment corresponding to a given maximum stress is greater than it would be if the elasticity were perfect. In the case where the equalization is perfect it is easy to show that the twisting moment is increased in the proportion 4 : 3, a result first given in 1849 by Prof. J. Thomson. The curves given by Thurston show that in many cases an approximately constant twisting moment was reached indicating that nearly complete equalization must have existed. On the case of cast iron see Art. 223.

222. *Theories of Compound Strength.*—A simple distorting stress is included in the more general case of three simple longitudinal stresses of any magnitudes acting on planes at right angles. To this, indeed, all cases of stress can be reduced, and if we knew the powers of resistance of a material to three such stresses simultaneously, all questions relating to strength of materials could (at least theoretically) at once be answered. Unfortunately experiments fitted to decide the question have not hitherto been made, and in consequence hypotheses have explicitly or implicitly been resorted to.

First, it is often tacitly supposed that the powers of resistance of a material to a simple longitudinal stress are unaffected by the existence of a lateral stress. For example, if a material bears 10 tons per square inch under a simple stretching force, it is assumed that when formed into a pipe and exposed to internal fluid pressure it would also bear 10 tons on the square inch if the pipe were homogeneous and free from joints, notwithstanding the fact that the material is exposed to stress (Art. 150) tending to tear it transversely as well as longitudinally. It is, however, far from probable that this can be the case, at any rate as regards the elastic strength. In ductile materials, the limit of elasticity of which depends as we have seen on rigidity, any lateral force must raise or lower the elastic limit according as it acts in the same direction as the longitudinal stress or in the opposite direction.

Secondly, it may be supposed that the maximum elongation or contraction of a material in a given direction must be a certain definite quantity, irrespective of any elongation or contraction in any other direction. This theory leads to results which are more probable than the preceding, and as it has been much employed by Continental writers we shall give some examples.

Let us take a piece of wrought iron and imagine that when exposed to a simple stretching force its limit of elasticity corresponds to a stress of 10 tons per square inch, accompanied by an elongation of $\frac{1}{1200}$th of its length. The second theory asserts that the maximum admissible elongation is still $\frac{1}{1200}$th, even though the sides of the bar be acted on by any force, the effect of which will be that quite a different longitudinal stress will be required to produce that elongation.

The relations between stress and strain are expressed by the equations (Art. 206)

$$Ee_1 = p_1 - \frac{p_2 + p_3}{m};$$

$$Ee_2 = p_2 - \frac{p_1 + p_3}{m};$$

$$Ee_3 = p_3 - \frac{p_2 + p_1}{m}.$$

The first theory supposes that p_1 can never exceed 10 tons, and the second that e_1 can never exceed $\frac{1}{1200}$th (or Ee_1 10 tons), whatever p_2, p_3 are. In the case of a thin pipe under internal fluid pressure $p_3 = 0$ (nearly), $p_2 = \frac{1}{2}p_1$ (Art. 150); thus assuming $m = 4$ we have on the second theory

$$10 = p_1 - \frac{p_1}{8}, \text{ or, } p_1 = 11{\cdot}43,$$

so that the material will bear under these circumstances a stress of 11·43 tons per square inch as safely as it bears 10 tons under simple tension, and this value, therefore, may be assumed for the co-efficient in the formula which gives the corresponding internal pressure. In like manner in the case of a thin sphere the material will bear a stress of $13\frac{1}{3}$ tons per square inch, being an increase of 30 per cent.

223. *Connection between the Co-efficients of Strength for Shearing and Tension.*—On either theory the resistance to a simple distorting stress may be found in terms of the resistance to simple tension, for such a stress consists (p. 358) of a pair of equal and opposite simple stresses of equal intensity. In the first case the resistances to tension and shearing ought to be equal, in the second since, writing $p_2 = p_1$, we find

$$Ee_1 = p_1 + \frac{p_1}{m},$$

$$\text{or, } p_1 = \frac{m}{m+1} Ee_1,$$

it follows that the resistance to shearing is $\frac{m}{m+1}$ or about four fifths the resistance to tension, a result on the whole borne out by experience. It should be remarked that the theory only professes to give a connection between the elastic resistances in the two cases, the equations only holding good for perfectly elastic material, which, moreover, must be supposed isotropic. The ultimate resistance to torsion of cast iron is much greater than its resistance to tension, which is probably due to the same causes as in the case of bending.

Now, rigid materials on this theory are imagined to give way to longitudinal compression, when the lateral expansion produced by the compression is the same as would be produced by a simple tensile stress; from which it appears that the elastic resistance to compression should be from three to four times the elastic resistance to tension, as may easily be supposed to be the case.

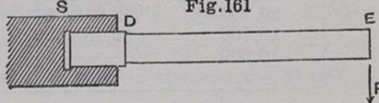
A third theory, more easily conceivable *a priori*, is to suppose that each material is capable of enduring, without injury to its elasticity, a certain definite change of volume and a certain definite change of shape. We thus have two co-efficients of elastic strength analogous to the two fundamental constants which express the other elastic properties of isotropic matter. On this theory, if the resistance to a simple distorting stress in any plane be independent of the existence of any other kind of stress whether fluid or otherwise, as in fact is the case before the limit is reached, it would follow that this resistance must be one half the resistance to a longitudinal stress. It is probable that some theory of this kind may ultimately prove to be the true one; but, in the absence of the necessary experimental data, the second theory may be provisionally assumed, as its results have not as yet been contradicted by experience. It is applied by first finding the principal stresses as in Ch. xvii., and then deducing the principal strains as just now explained. The greatest of these strains multiplied by E may be described as the "equivalent simple tensile stress," and should not exceed the limit prescribed by the strength of the material.

Repetition and Impact.

224. *Wöhler's Experiments on Alternate Bending and Twisting.*—In bodies which satisfy the definition of perfect elasticity a load within the elastic limit produces no permanent change, unless perhaps some thermodynamic effect, and it follows from this that after removal the body is completely uninjured, so that the load may be repeated indefinitely. Experience confirms this conclusion. The balance spring of a watch bends and unbends more than a million times a week for years together: and the parts of a machine if originally sufficiently strong, remain so to all appearance for an indefinite time. But, if the load be beyond the elastic limit, permanent changes are produced, and there is every reason to believe that a slow deterioration of strength, due perhaps to some kind of internal abrasion, is ultimately destructive. The most definite information on this point is furnished by the experiments of M. Wöhler published in 1870. Bars were loaded in various ways and the load wholly or partially removed: the process was repeated till the bar broke: the number of repetitions necessary for this purpose being counted was found to depend, first, on the maximum stress and, secondly, on the fluctuation of stress.

First suppose the stress alternately tensile and compressive of equal intensity. Wöhler tried this both in bending and in twisting. Figure 161 represents a round bar DE, with one end enlarged and fitted into a socket in a revolving shaft S. At the free end E a load P was applied, which produced at D, the point of maximum bending, a stress of intensity found by the usual formula. The shaft being set in motion the piece of material was bent alternately backwards and forwards once in each revolution. A number of pieces being tried successively with gradually diminishing loads, the revolutions necessary to produce fracture were found to increase as shown by the annexed table for the case of wrought iron. The pieces broken were exactly similar, and we therefore find a regular increase in the number of revolutions necessary to produce fracture as the stress diminishes. It is already very large at 18,700 lbs. per square inch, and at 16,600 the piece cannot be broken at all. We may therefore place the resistance to alternate bending of this kind of iron at about 17,000 lbs. per square inch, while for cast steel of various qualities it was found to range from 25,000 to 30,000, and for copper 10,400. These results do not differ much from the limit of elasticity of the materials in question

Fig. 161



as determined in the usual way by experiments on tension. Indeed we have here the most satisfactory definition of the limit of elasticity. If we attempt to define the elastic limit as the stress at which the material ceases to possess the properties of a perfectly elastic body, we are embarrassed by the small and variable deviations which we find under almost any load, which only gradually and at very different loads under different circumstances pass into the large differences characteristic of the non-elastic state. The resistance to unlimited alternate stress however is a definite quantity which, so far as we know, is independent of the causes which produce these variations.

ALTERNATE BENDING OF A BAR OF AXLE IRON FURNISHED BY THE PHŒNIX COMPANY IN 1857.

STRESS IN LBS. IN SQ. INCH.	REVOLUTIONS.	REMARKS.
33,300	56,430	The last of these pieces was unbroken after more than 132 million revolutions.
31,200	99,000	The ultimate tensile strength of this iron was 47,000 lbs. per square inch and the elongation about 20 per cent.
29,100	183,145	
27,000	479,490	
25,000	908,800	
23,000	3,632,588	
20,800	4,918,000	
18,700	19,187,000	
16,600	——	

Similar experiments were made with a different apparatus on alternate twisting. They were less extensive, but led to the important conclusion that the strength of the qualities of steel for which they were tried was four fifths that of the same steel under alternate bending. From this it is inferred that the proof resistance to shearing is four fifths the proof resistance to tension, as required by a theory of strength already referred to. (See Art. 223.)

225. *Influence of Fluctuation of Stress.*—It had already been shown by Prof. J. Thomson, in a paper published in 1848,* that twisting or bending a bar beyond its elastic limit in one direction must increase its powers of resistance to a second strain in the same direction, and diminish it to a strain in the opposite direction.

* *Cambridge and Dublin Mathematical Journal.*

Accordingly we find that when a bar is strained in one direction only its powers of resistance to unlimited repetition are greatly increased. Wöhler made very extensive experiments on stretching, bending, and twisting of pieces of iron and steel to a given maximum stress, the load being wholly or partially removed at each repetition. The number of repetitions necessary for fracture was found to vary, not only according to the magnitude of the maximum stress, but also according to the fluctuation. It was greater when the load was only partly removed than when it was wholly removed. Some results are given in the annexed table, which shows the limits between which the stress varied when fracture was just not produced by unlimited repetition.

RESISTANCE TO UNLIMITED REPETITION OF BENDING.

NATURE OF FLUCTUATION.	FLUCTUATION OF STRESS.			
	IRON.		STEEL.	
Alternating, ...	+17,000 ;	−17,000	+29,000 ;	−29,000
Load wholly removed,	31,000 ;	0	50,000 ;	0
Load partially removed,	45,000 ;	25,000	83,000 ;	36,500

REMARK.—The ultimate tensile strength of the iron was 47,000, and of the steel, 106,000.

Any greater fluctuations with the given maximum stress, or any greater maximum stress with the given fluctuation, produced fracture. Experiments on stretching and twisting led to similar results, and it should be especially noticed that in cases of unlimited repetition the resistance to stretching is the same as the resistance to bending, but the resistance to twisting less. In the case of cast iron the resistance to stretching with complete removal of load was found to be 10,400, but no experiments on bending or twisting were made.

Thus it appears that the ultimate strength of a material is very different according to the fluctuation in the load to which it is exposed; the same iron, which will bear only 17,000 lbs. per square inch when bent alternately backwards and forwards, will bear 31,000 when bent in one direction only, and 45,000 when the stress varies between 25,000 and 45,000. Several formulæ have been devised to represent the results of the experiments, of which one will now be given.* Let p_0 be the ultimate tensile strength of a material and Δ the fluctuation, then the actual ultimate strength under unlimited repetition will be

$$p = \tfrac{1}{2}\Delta + \sqrt{p_0(p_0 - \tfrac{3}{2}\Delta)}.$$

When $\Delta = 2p$ we get the case of alternate stress with which we commenced, where $p = \frac{1}{3}p_0$, and when $\Delta = p$ we have the case of repeated stress in one direction with complete removal at each repetition. The formula gives the same results as the experiments in the extreme cases, and may be expected to be approximately correct in intermediate cases.

226. *Impact.*—In Wöhler's experiments the load was applied without shock. In cases of impact also there is reason to believe that within the limit of elasticity a material will bear unlimited repetition. Thus in Hodgkinson's experiments on beams struck by a pendulum weight, it was found that if the blow produced less than one third the ultimate deflection, the beam would sustain more than 4,000 blows without apparent injury, a plate of lead being introduced to prevent local damage.

In most cases of impact, however, the elastic limit is exceeded, and the destructive effect of repetition is then much greater than when the load is gradually applied. In the ductile metals the resistance to impact is at first very great, as has already been sufficiently explained; but every time the limit of elasticity is overpassed the hardness of the metal is increased, so as to make it less able to resist the second blow. This may be illustrated by a diagram in which OQQ' is a curve of stress and strain, Q the original elastic-limit, $Q'N$ the stress produced by the first blow, so that the area $OQQ'N$ represents the energy of that blow. The

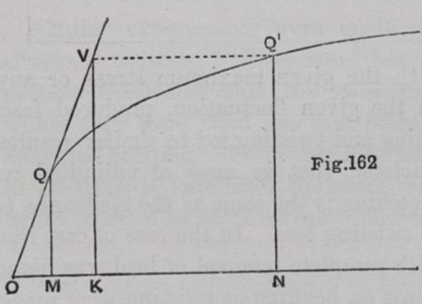


Fig.162

* *Elements of Machine Design*, by Prof. W. C. Unwin, p. 25.

effect of the blow is to raise the limit from the stress QM to the stress $Q'N$ nearly. Hence the curve of stress and strain now becomes OVS, where V is the new limit, and the material will only bear a blow the energy of which is the triangle OVK, without the original stress $Q'N$ being exceeded. Thus by constant repetition of blows, which originally only produced a stress not much exceeding the elastic limit, a much greater stress may be produced. It is believed that this is in the main the explanation of the destructive effect of repeated blows and continuous severe vibration: pieces of material exposed to which are found to have a short life. The effect may be further augmented by synchronism (Art. 197, p. 382).

Co-efficients of Strength and Factors of Safety.

227. *Factors of Safety and Co-efficients of Working Strength.*—Before we can apply theoretical formulæ to the determination of the dimensions of actual structures and machines, it is necessary to know the value of the co-efficients of strength to be used, and this is always a matter which requires great care and attention to the circumstances under which certain dimensions are found to be sufficient by long practical experience. In the first instance it depends on the ultimate strength of the material, and may be expressed by dividing that quantity by a Factor of Safety. But the ultimate strength varies as we have seen, and the word "factor of safety" is used with various meanings.

The primary meaning of the expression is the divisor necessary to provide a margin of strength for unknown contingencies such as the following.

(1.) The ultimate strength of a piece of material is uncertain, for two pieces of material of the same description and manufacture are not always equally strong. The liability to variation is much greater in some materials than others, for example in cast iron than in wrought iron. The strength of stone varies so much that, in carrying out any important work, experiments are frequently made on the stone to be employed in it.

(2.) The piece of material may be subject to corrosion or other influence, which in course of time diminishes its strength.

(3.) Errors of workmanship are unavoidable, and in some instances may greatly increase the stress to which the material is exposed. This, for example, is the case in pillars, the factor of safety for which must always be greater than for other parts of a structure.

(4.) The magnitude of the load and its mode of application is generally more or less uncertain. This however may be provided for by assuming a maximum load.

The factor required to provide for contingencies such as these may be called the "real" factor of safety, but by an addition to its value it may be made to provide against contingencies which can if necessary be exactly foreseen and calculated. Assuming all the forces acting on a structure to be known it is possible to find the stress on each part of it, but the calculation may be too complex to be often used, or its result may be known approximately under similar circumstances. Hence it often happens that the dimensions of a piece are determined by a formula involving only part of the straining forces which act on it, and the rest are provided for by an increased factor of safety. Thus the real stress on the metal of a screw bolt, when the effect of screwing up is taken into account, is double the total tension per square inch of the gross sectional area. If that bolt be used for a cylinder cover exposed to steam pressure the total tension will be much greater than that due to the pressure of the steam. These two circumstances taken together may be taken into account by the use of a factor of safety three or four times greater than the real one. Such cases are common in practice, but the factor to be used must then be determined by comparison with good examples under similar circumstances.

Again, it is necessary that a piece should be stiff enough as well as strong enough, and when formulæ for strength are used in such cases it is often necessary to employ very large and very arbitrary factors of safety. Here however the difficulty arises from an erroneous method of calculation.

228. *Values of Co-efficients.*—In parts of machines subject to alternating straining actions we know by Wöhler's experiments that the ultimate strength is somewhat less than the elastic strength under simple tension, being for wrought iron and soft steel about one third the ultimate tensile strength. The load on such parts will rarely be applied without shock, the effect of which cannot precisely be determined. In ordinary cases it will be sufficient to

treat this case as if the load were suddenly applied by using a further divisor of 2. We thus obtain the working strength by using a total factor of safety of 6. For wrought iron this gives a co-efficient of 4 tons, or 9,000 lbs. per square inch, which is known by experience to give sufficient strength where all the straining actions are taken into account. In long struts a factor of 8 or more must be used for reasons already sufficiently explained, and this is also necessary where, as has been the case, till lately with steel, the material is not completely reliable. For timber the usual factor is 10. The co-efficient for shearing and torsion is to be taken provisionally as four-fifths that for tension and bending, that is for wrought iron $3\frac{1}{4}$ tons per square inch; but from the incompleteness of experimental data it is not certain that this value is not too large.

In structures the fluctuation of the straining actions is in general much less, and the ultimate strength by Wöhler's experiments is much greater. Yet the working strength employed is not very different. In the first place, it is rarely permissible to exceed the elastic limit on account of the permanent deformation which ensues. In the second place, the whole of the straining actions on each piece of the structure, especially the effect of imperfect joints, are rarely included in calculations. For example, the friction of pin joints may, under unfavourable circumstances, add 60 per cent. to the maximum stress on the links of a suspension chain (Ex. 4, p. 440). Hence the working strength for wrought iron rarely exceeds $4\frac{1}{2}$ or 5 tons per square inch. In reckoning the load Rankine recommended that the "dead" load should be divided by 2 and added to the "live" load in order to obtain the effective live load. More recently on the strength of Wöhler's experiments it has been proposed to find the ultimate strength of each piece under the maximum stress and fluctuation of stress to which it is subject, and divide by a constant factor of safety. There can be no doubt that a smaller co-efficient is necessary the greater the fluctuation, but the principle of a constant factor is open to question: it appears to lead either to co-efficients which are smaller than are known to be safe, or else to values above the limit of elasticity.

229. *Fibrous Materials. Ropes.*—Fibrous materials are those which may be regarded as made up of fibres, usually of organic origin, more or less closely united by cohesion or interlacing. The relative movements of the fibres are hindered by forces of the nature of friction, which are much less than the molecular forces to which the tenacity of a homogeneous solid body is due. Hence the strength and stiffness of a piece of material are much less than those of the fibres of which it is made up.

In most kinds of woods the fibres are arranged longitudinally, and the material is therefore especially characterized by its low resistance to division into parts longitudinally. Thus the resistance to longitudinal shearing of fir timber is only 600 lbs. per square inch, whereas its tenacity is about 20 times this amount, approaching that of cast iron. So, again, crushing takes place by longitudinal splitting under a stress little more than half the tenacity. Further, the condition of the material greatly influences the lateral cohesion of the fibres and thus affects its strength and elasticity. In timber which has been artificially dried the elasticity is nearly perfect up to the breaking point, whereas in the green state the elasticity is imperfect and the strength greatly reduced. Hence the importance of seasoning timber so as to be moderately dry.

The ordinary formulæ, however, will apply in all cases where the stress is a simple longitudinal stress, the direction of which is that of the fibres; that is to say, in tension, compression, and ordinary cases of bending. They will only fail when the bending is accompanied by crushing and shearing of considerable intensity, as when short pieces are acted on by transverse forces.

In cloth and similar materials two sets of fibres at right angles are united by interlacing. Resistance to tension is thus obtained with almost complete flexibility.

In ropes of all kinds the fibres are ranged in spiral curves in the process of manufacture, and their tension then produces lateral pressure, the friction arising from which is sufficient for union. The strength of a rope, though very great compared with its weight, is only one third that of the yarn of which it is spun, and on a similar principle the strength of large cables is less than that of the smaller ropes called "hawsers" of which they are made up. The strength of a rope is usually expressed by the formula

$$T = \frac{C^2}{k},$$

where C is the girth of the rope in inches, T the tension in tons, and

k a constant. The old rule in the navy was to take $k=5$ to obtain the breaking weight of a rope, but the table now employed gives $k=3{\cdot}3$, that is, a strength 50 per cent. greater. In small ropes k may be even less. The safe working load is not more than one sixth the breaking load. In iron wire ropes $k=1$, or for ropes above 6 inches girth somewhat more. The strength of wire ropes is more than doubled by the employment of steel. The safe working load may be taken as one fifth their breaking load.

TABLE I.—WEIGHT AND WORKING STRENGTH OF VARIOUS MATERIALS.

MATERIAL.	WORKING STRENGTH.				WEIGHT PER YARD IN POUNDS.			Working Strength in Feet of Material. $=\lambda$	
	Stress in Tons per Square Inch.		Area in Square Inches per Ton.		Per Ton of Stress under Working Load.		Per Square Inch of Area.		
	T.	C.	T.	C.	T.	C.		T.	C.
Cast Iron, ...	1·5	4·5	·667	·222	6	2	9	1120	3360
Wrought Iron, ...	4·5	4·5	·222	·222	2·22	2·22	10	3024	3024
Ordinary Soft Steel,	7	7	·143	·143	1·43	1·43	10	4700	4700
Ordinary Steel Wire,	13	...	·077	...	·77	...	10	9000	...
Copper Wire, ...	4	...	·25	...	2·9	...	11·5	2320	...
Deal, ...	·5	·3	2	3·3	1·5	2·5	·75	4480	2700
Oak, ...	·75	·45	1·33	2·22	1·33	2·22	1	5040	3024
Granite, ...	...	·3	...	3·33	...	11·66	3·5	...	576
Brickwork, ...	...	·06	...	15·6	...	39	2·5	...	160
Hemp Ropes, ...	·6	...	1·67	...	2·5	...	1·5	2700	...
Iron Wire Ropes, ...	2	...	·5	...	2·6	...	5·25	2600	...
Steel Wire Ropes,	5	...	·2	...	1·1	...	5·5	6000	...

TABLE II.—ELASTICITY AND RESILIENCE.

MATERIAL.	ELASTIC STRENGTH.						ELASTICITY (Tons and Inches).		RESILIENCE UNDER TENSION.	
	Stress in Tons per Square Inch.			Strain.			Young's Modulus.	Rigidity.	Foot Pounds per Cubic Foot.	Height in Feet and Inches
	T.	C.	S.	T.	C.	S.				
Cast Iron, ...	3	9	...	·000375	·001125	...	8000	...	185	4″·5
Wrought Iron, ...	9	9	7	·0007	·0007	·0014	13000	5000	1060	2′ 2″
Soft Steel, ...	15	15	12	·0012	·0012	·0024	13000	5200	2900	6′
Hard Steel, ...	25	25	20	·002	·002	·004	13000	5200	8000	16′ 6″
Tempered Steel, ...	50	...	...	...	...	...	15000(?)	...	34500	72′
Strongest Steel Wire,	150	...	...	·0115	...	...	13000	...	276000	577′
Fir, ...	1½	...	...	·0021	...	...	700	35	2150	58′
Oak, ...	2	...	...	·0028	...	...	700	35	4300	86′

TABLE III.—ULTIMATE STRENGTH AND DUCTILITY.

MATERIAL.	Ultimate Strength in Tons per Square Inch. T.	C.	S.	Elongation per Cent.	$\frac{\text{Tension} \times \text{Elongation}}{2}$
Iron Bars, ...	25	22	18	20	250
Iron Plates, ...	22	19	16	10	110
Soft Steel (·15 to ·3 per cent. of carbon),	30	...	$22\frac{1}{2}$	25	375
Medium Steel (·3 to ·5 per cent. of carbon)	35	...	27	15	262
Hard Steel (·5 to ·75 per cent. of carbon),	45	...	...	8	180
Cast Iron, ...	$7\frac{1}{2}$	45	12	...	...
Lead, ...	$1\frac{1}{2}$	...	...	...	...
Sheet Copper, ...	$13\frac{1}{2}$	...	...	...	...
Cast Copper, ...	$8\frac{1}{2}$	...	...	...	...
Oak, ...	$5\frac{1}{2}$	...	1	...	...
Fir, ...	$5\frac{1}{2}$	...	·27	...	...

230. *Tables of Strength.*—For a detailed account of the properties of materials the reader is referred to the authorities cited above and at the end of this chapter. A convenient summary is given in Rankine's *Useful Rules and Tables.* It will be here sufficient to give a few examples.

Table I. gives the weight and working strength of a variety of materials. From what has been said in preceding articles it appears that the working strength varies according to circumstances. Hence the values given in the table may be exceeded, and sometimes greatly exceeded when special care is taken in the selection of material, in the estimation of strains, and in the execution of the work. On the other hand cases occur in which they are too large, and, it may be, greatly too large if due care is not exercised. The first two columns give the safe load in tons per square inch of sectional area, the second two the area necessary to sustain a load of 1 ton, in tension (T) and compression (C) respectively. The next two give the weight of 1 yard length of a bar which will sustain 1 ton, and the numbers therein given are therefore the comparative weights of bars of equal strength. The same comparison is effected in a different way in the last two columns, which give the length in feet of a bar or column the weight of which is equal to the working load on its transverse section. It is on this quantity, which is denoted by λ in Arts. 40, 41, pp. 90, 92, that the limiting dimensions of a structure depend. It is used for this purpose in Ex. 13, p. 324, and Ex. 11, p. 372. It will be observed that weight for weight timber is stronger than wrought iron.

Table II. gives the elastic properties of certain materials in tension (T), compression (C), and shearing (S). It has been sufficiently explained in preceding chapters.

Table III. shows the ultimate strength and ductility of materials in common use. The first three columns give the ultimate resistance to tension, compression, and shearing. The fourth gives the elongation expressed as a percentage of the original length, which, if the length of the pieces experimented on be a constant multiple of the diameter, forms a measure of the ductility. The ultimate strength and ductility of steel vary according to the amount of carbon it contains in such a way that the sum of the two remains nearly constant, other things being equal. Thus in the examples given in the table the sum is about 53. In steel compressed in a fluid state by Sir J. Whitworth's process the constant sum is about one third greater. The last column gives half the product of the ultimate tensile stress and the elongation, a quantity which is sometimes used as a measure of the powers of resistance to impact. The actual amount of work done in stretching a bar till it breaks is much greater than this, as is seen on considering the form of the curve of stress and strain. A more exact measure of the resistance to impact would be furnished by an experiment on a short block such as that described on page 418.

EXAMPLES.

1. Show that the modulus of rupture of a material is 18 times the load which will break a bar of the material 1 inch square and 1 foot long: the bar being supported at the ends and the load applied at the centre.

N.B.—The modulus of the rupture is the value of the co-efficient in the ordinary formula for bending when the load is that found by experiment to break the beam.

2. A balcony, 6 feet long and 4 feet broad, is supported by a pair of cast-iron beams fixed in the wall at one end. The beams are of rectangular section, 2 inches broad, and depth near the wall 4 inches. What load per square foot will the balcony bear, the stress on the iron being limited to 1 ton per square inch? Also, how should the depth vary for uniform strength along the length of the beam?

Ans. Equating the greatest bending moment to the maximum moment of resistance to bending we find the load which the balcony will bear

$$= 41{\cdot}5 \text{ lbs. per square foot.}$$

As to the depth of the beam: for uniform strength $\frac{M}{I}y$ must be constant from which we find that the depth at any point of the beam must be proportional to the distance from the outer end of the beam; so that the lower side of the beam should be a sloping plane.

3. A paddle shaft is worked by a pair of engines with cranks at right angles. Supposing the steam pressure constant, and the resistance of each wheel equal and uniform, and obliquity of connecting rod neglected; compare the co-efficients of strength to be used in calculating the diameter of the paddle and intermediate shafts.

Ans. The uniform moment of resistance of the paddle wheel $= \frac{1}{2}$ the mean turning moment of the two engines. The twisting moment of the paddle shaft, when either crank is on the dead centre, $= \frac{1}{2}$ maximum twisting moment of one engine. At the same instant this is the twisting moment on the intermediate shaft. When the other crank is on the dead centre the twisting moment on intermediate shaft is the same in magnitude, but reversed in direction, and when the two cranks make angles of 45° with the dead centres the twisting of the paddle shaft $= \frac{1}{2}$ the maximum combined twisting moment of the two engines, that is $\sqrt{2}$ times its amount when either crank is on the dead centre; but the twist is in the same direction always. Therefore on the paddle shafts the stress alternates between x and $\sqrt{2x}$, and on the intermediate shaft between x and $-x$.

Hence applying formula

$$p = \tfrac{1}{2}\bar{p} + \sqrt{p_0(p_0 - \tfrac{3}{2}\bar{p})},$$

we have for paddle shaft,

$$\bar{p} = \cdot 414\, x;\ p = 1\cdot 414\, x;\ \therefore \bar{p} = \cdot 292 p;$$

substituting, we obtain

$$p = \cdot 888\, p_0.$$

For intermediate shaft, $\bar{p} = 2x$; $p = x$; $\bar{p} = 2p$; and $p = \frac{1}{3}p_0$.

If the stress on the paddle shaft alternates to zero, by the wheels rolling out of the water, or by the stopping of the engine, then $p = \cdot 6 p_0$.

4. A suspension chain is constructed with bar links united by pin joints; the diameter of the pins is two-thirds the breadth of the link (p. 371). If the bridge vibrate show that the maximum stress on the links may be increased by deviation (p. 341) due to friction of pins (p. 248) in the ratio $1+2f : 1$, where f is the co-efficient of friction.

AUTHORITIES ON STRENGTH OF MATERIALS.

In addition to the works expressly cited in this chapter may be mentioned—

HODGKINSON. *Experimental Researches on the Strength and other Properties of Cast Iron.* Weale. 1846.

WEYRAUCH. *Iron and Steel.* New York. 1877.

BARLOW. *Strength of Materials.* Lockwood.

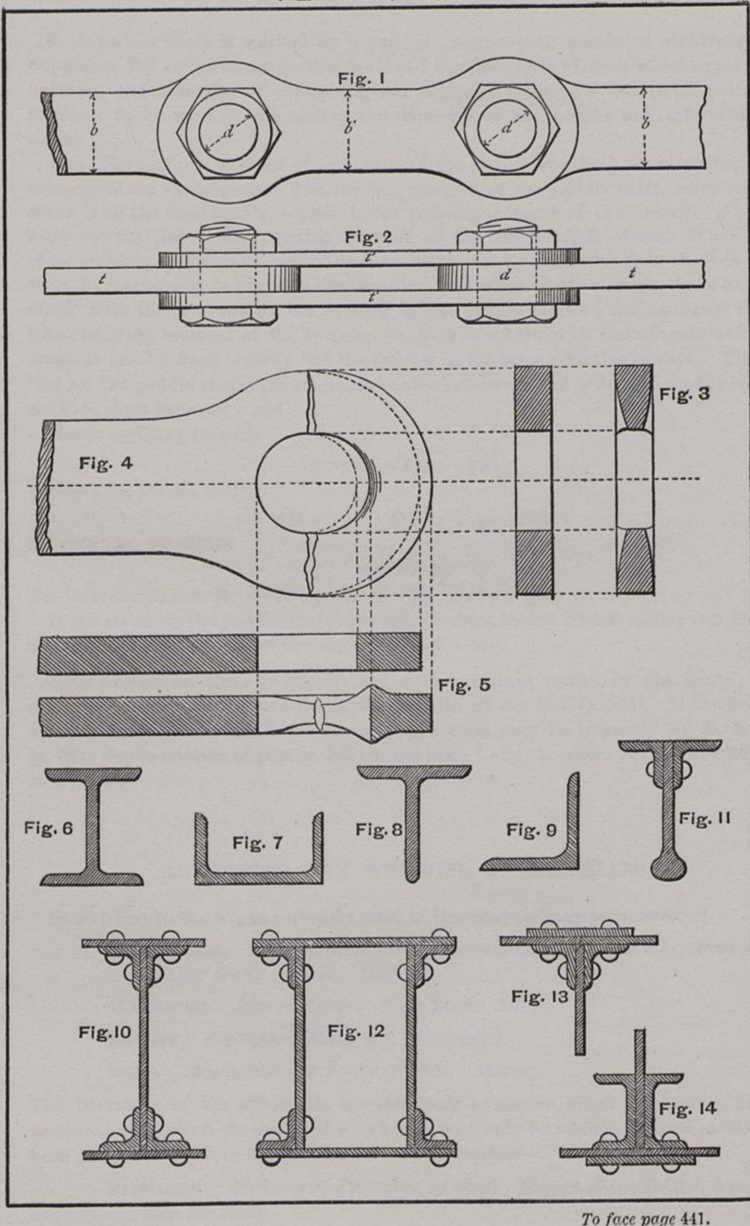
REED. *Shipbuilding in Iron and Steel.* Murray.

The literature of the subject is however very extensive, much information being scattered in various memoirs, of which two need only be mentioned here as having been much employed in the preparation of this treatise—

FAIRBAIRN. *Mechanical Properties of Steel.* Report of the British Association for 1867.

WÖHLER. *Die Festigkeits-Versuche.* Berlin. 1870.

PLATE VIII.



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DESCRIPTION OF PLATE VIII.

To illustrate various questions considered in Chaps. XII. and XV. Plate VIII. has been drawn.

Figs. 1, 2 represent the pin joint connecting two bars in tension, discussed in Art. 191, p. 369. Figs. 3, 4, 5 show the way in which the joint yields when the pins are too small. In Fig. 4 the original dimensions of the eye and eyehole are shown by dotted lines, while the full lines show what they become after yielding. Fig. 3 gives transverse sections of the eye before and after failure, showing the thinning out due to lateral contraction during stretching beyond the elastic limit. After this contraction has reached a certain limit the metal tears asunder, as shown in Fig. 4. The longitudinal section (Fig. 5) shows the corresponding spreading out at the top of the hole due to compression beyond the elastic limit. This lateral expansion is partially prevented in riveted joints, and (p. 410) this may be the reason why direct stress in them is of less importance. The failure of pin joints in this way furnishes a good example of the "flow of solids."

The remaining figures of this plate are intended to give some idea of the manner in which iron girders are constructed. Figs. 6, 7, 8, 9 are transverse sections of "H iron," "channel iron," "tee iron," and "angle iron:" these are rolled in one piece and, in combination with plates, form the materials from which large girders are built up. For small beams such as floor joists H iron or tee iron of the requisite depth and sectional area may be used. Figs. 10, 12 are sections of two of the commonest forms of built-up girders. In the first the web is a single plate to which angle irons are riveted to form the flanges, further strength being obtained by an additional covering plate. The second is similar, but the web consists of a pair of plates, a form known as a "box beam." Fig. 11 is commonly used in shipbuilding as a deck beam or otherwise: a "bulb iron" here forms the web and lower flange, while the upper flange is formed by a pair of angle irons as before. Figs. 13, 14 give examples of girders of more complex construction employed where greater strength is necessary: one flange only is shown in section in each case. For further details the reader is referred to the treatises by Mr. Hutchinson and Sir E. Reed, cited on pages 60 and 440.