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CHAPTER XII.

SIMPLE TENSION, COMPRESSION, AND BENDING OF PERFECTLY ELASTIC MATERIAL.

SECTION I.—TENSION AND COMPRESSION.

148. *Simple Tension.*—The effect of forces acting on a bar has already been explained in Chapter II. to consist in the production of certain straining actions which we called Tension, Compression, Bending, Shearing, and Twisting, and we now go on to consider the changes of form and size which the bar undergoes and the stress produced at each point on the supposition that the material of the bar is perfectly elastic.

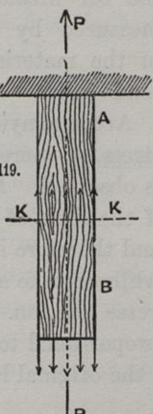


Fig. 119.

Let AB (Fig. 119) be a bar subjected to the action of equal and opposite forces applied at the ends in the same straight line. At any transverse section KK there will be a tendency to separate into two parts A, B, which is counteracted by a mutual action between the parts at each point of the section, which, in accordance with our previous definitions, is called the Tensile Stress at the point. The total amount of the stress will be P; but the intensity will depend on the area of the section (A), so that P/A is the mean intensity of stress or the stress per unit of area. The stress may be the same at all points of the section. We then say it is uniformly distributed, and the intensity at all points $= P/A$.

In order that the intensity of the stress may be the same at every point of every transverse section of the bar, it is *theoretically* necessary that the load P should be applied in a uniformly distributed manner all over the end B. Then if the material is perfectly homogeneous each elementary portion of KB will be strained alike, and the uniformly distributed load at B will be balanced by a uniformly distributed stress over any section KK. In such a case the line of action of the resultant of the applied load P passes through the centre of gravity or centre of position of the transverse section KK. Unless it does so the equilibrium of the portion KB is not possible by means of a uniformly distributed stress over the section. But from experience it appears that for uniformity of stress it is not absolutely necessary for the load to be applied in this distributed manner. It may be applied for instance by pressure on a projecting collar; and yet *if the line of application of the load traverses the centre of gravity of the sectional area*, the material, if homogeneous, will so yield as *practically* to produce at a section a little distant from the place of application of the load a stress of uniform intensity. This is a particular case of a principle which will be further referred to hereafter.

If the applied load is increased, the stress on the section is proportionately increased, until at last the material yields under it and the bar breaks. If W = breaking load, the corresponding stress measured by W/A is a quantity which depends on the nature of the material. If we call it f, then the breaking or ultimate load $= Af$.

Accompanying the application of the load producing a tensile stress, an increase of length and diminution of transverse dimension is observed. In metallic bodies the alterations are exceedingly small if the limit of elasticity is not exceeded (see Table II., page 437), and therefore in estimating the stress on the section it is not worth while to take account of the slight alteration in the area of the transverse section. Under the same load the change of length is proportional to the length. If x be the total change of length, and l the original length, then the extension per unit of length is

$$e = \frac{x}{l}.$$

On account of the smallness of e it is immaterial whether l is taken as the original or altered length of a metallic bar.

As already stated (Art. 147), it is usual to restrict the word *strain* to mean the alteration of the dimension and form which bodies undergo and to use the word *stress* when referring to the elastic forces which accompany the strain. Thus e is a measure of the tensile strain produced in the bar, whilst p is a measure of the accompanying tensile stress. Since by Hooke's law the extension of the bar is proportional to the force producing it, it follows that the strain is proportional to the accompanying stress. Thus p and e may be connected by some constant the value of which depends on the nature of the material. We may write

$$p = Ee,$$

in which E is called the modulus of elasticity of the material, which, when the stress p is expressed in pounds per square inch, has for wrought iron a value of about 28,000,000.

Putting for e its value x/l, we have the general relation,

$$\frac{p}{E} = \frac{x}{l}.$$

The transverse strain, that is, the contraction per unit of transverse dimension, is from one third to one fourth the longitudinal strain.

149. *Work done in Stretching a Rod.*—Having found the relation between the tensile stress and strain, we will now consider how much work must be done in order to stretch it.

Let a load of gradually increasing amount be applied to the bar, the bar will stretch equal amounts for equal increments of load: or the elongation of the bar will for all loads be proportional to the load. This may be represented graphically. Suppose the load P' produces the extension shown, greatly exaggerated, by B_0B' (Fig. 120), and we set off an ordinate $B'N'$ to represent P' on some scale, and do that for any number of loads, taking, for example, BN to represent P, which produces the extension $B_0B = x$; then all the points N will lie on the sloping line passing through B_0. Having done this, the area of the triangle B_0BN will represent the quantity of work done on the bar in stretching it the amount $B_0B = x$. Thus

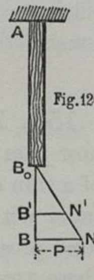


Fig. 120.

$$\text{Work done} = \tfrac{1}{2}Px.$$

The energy thus exerted is stored up in the stretched bar, and may be recovered if the bar is allowed under a gradually diminished load to contract. In the perfectly elastic bar the contraction will be exactly the same as the extension, and there will be no loss of energy in stretching it. In other words the elastic forces are "reversible." But if the elasticity is imperfect, some of the energy expended in stretching the bar is employed in producing molecular changes, as for example, change of temperature. On contraction this amount of energy will not be restored.

We can express the work done in stretching the bar otherwise. For P put its value $= pA$, and for x its value $= pl/E$. The substitution of these values of P and x will give

$$\text{Work done} = \tfrac{1}{2}pA \cdot \frac{pl}{E} = \frac{p^2}{E}\frac{Al}{2} = \frac{p^2}{E} \times \tfrac{1}{2} \text{ volume.}$$

Thus the work required to produce a given stress p is proportional to the volume, or, what is the same thing, to the weight, of the bar.

If the stress produced is increased up to the elastic limit, or, as it is often called, the *proof stress*, so that $p = f$, then $\dfrac{f^2}{E} \cdot \dfrac{\text{Volume}}{2}$ expresses the greatest amount of work which can be done on, and stored in the bar without injuring it or impairing its elasticity.

This is called the *resilience* of the bar. The quantity f^2/E, the value of which depends on the nature of the material, is called the *modulus of resilience*, and, as we shall see hereafter, furnishes a measure of the resistance of the material to impact in those cases in which the limits of elasticity are not exceeded (Chap. XVI.). A table of coefficients of strength and elasticity for materials commonly used in construction will be found at the end of Chapter XVIII.

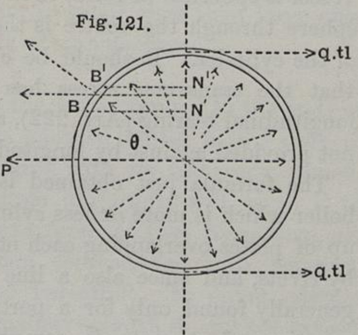
150. *Thin Pipes and Spheres under Internal Fluid Pressure.*—We now pass on to consider an important case of simple tension: that of a thin cylindrical shell subjected to internal fluid pressure. A cylinder with rigid ends and a sphere are cases of a vessel under internal fluid pressure which tends to preserve its form. The equilibrium in these two cases is stable, for if the vessel suffers deformation the internal pressure tends to make it recover its original true form. Vessels the sides of which are flat tend, by bulging, to assume these forms, and the tendency must be resisted by staying the surfaces in some way. If, as generally happens, there is acting also an external fluid pressure less than the internal, then, in what follows, the intensity of the internal pressure must be taken to be the excess of the internal over the external pressure.

Let p be the intensity of the fluid pressure in pounds per square inch, d the diameter, t the thickness of the shell, and l the length of the cylinder. Suppose in some way that the ends are maintained perfectly rigid, and for convenience let them be flat. There are two principal ways in which the strength of the shell can be estimated.

First, consider the tendency to tear asunder longitudinally, parallel to the axis of the cylinder. Imagine the cylinder divided into two parts by a plane passing through the axis of the cylinder. On each half cylinder there is a pressure P due to the resultant fluid pressure on that half which tends to produce a separation at the section imagined. The separation is prevented by the resistance to tearing which the metal of the shell offers, calling into action a uniform tensile stress at the two sections made by the imaginary plane through the axis of the cylinder.

Let q = intensity of tensile stress produced; then the area over which the stress acts being $2tl$, the total resistance to tearing is $q \times 2tl$, which must also be the tendency to tear $= P$.

Fig. 121.



In a transverse section take two points B, B' (Fig. 121) near together. The surface of the shell, $BB' \times l$, is acted upon by a normal pressure p per unit of area. The pressure $p \,.\, BB' \,.\, l$ may be taken to act in a radius drawn to the middle point of BB', making an angle θ with the direction of the resultant force P. The resolved part of this pressure in the direction of P

$$= pl \,.\, BB' \,.\, \cos\theta = pl \,.\, NN',$$

NN' being the projection of BB' on the plane of section. Summing up the pressures on all the small arcs BB', composing the semicircle, we obtain the total separating force,

$$P = pl \,.\, \Sigma NN' = p \,.\, l \,.\, d,$$

$$\therefore\ 2qtl = pld,$$

$$\text{or } q = \frac{pd}{2t};$$

thus the tensile stress is directly proportional to the diameter, and inversely proportional to the thickness of the cylindrical shell. For greatest accuracy d should be taken as the mean of the internal and external diameters. The formula just obtained is true only when the thickness is small compared with the diameter. If t is large, the stress is not uniform over the section; the formula will then give the mean stress if d be understood to mean the internal diameter.

We next consider the tendency for the cylinder to tear across a transverse section. The total pressure on each end of the cylindrical shell is the separating force, and the resistance to separation is due to the tensile stress, q' suppose, called into action over the annular area $\pi d \cdot t$ of the transverse section.

$$\therefore \pi dt \cdot q' = \frac{\pi}{4}d^2p\,; \text{ or } q' = \frac{pd}{4t}.$$

This is just half the stress on the longitudinal section. If the vessel is spherical in form, the stress produced on all sections of the sphere through the centre is the same as at the transverse section of the cylinder. It should be observed that we have here assumed that the transverse stress has no influence on the resistance to longitudinal tearing (Art. 222), and that the pressure on the ends is not provided against by longitudinal stays.

The formula just obtained is used to estimate the strength of a boiler which is more or less cylindrical; but since the boiler is made up of plates overlapping each other, connected together at the edges by rivets, and since also a line of rivets in a longitudinal section is generally found only for a portion of the length of the boiler, the question of strength is complicated. But a longitudinal section through the greatest number of rivet holes is the weakest section, and if for q we write f, where f is a co-efficient of strength to be determined from experience, the value of it depending, amongst other things, on the form of joint, then the formula

$$p = \frac{2ft}{d}, \text{ or } t = \frac{pd}{2f}$$

may be used as a semi-empirical formula to determine the greatest pressure which can be employed in a given boiler, or the thickness of metal required to sustain a given pressure. The value of the co-efficient for iron boilers with single rivetted joints is about 4,000 lbs. per square inch, or, when double rivetted, as is usual in large boilers, 5,500. With steel the value is about one-third greater.

151. *Remarks on Tension.*—The results obtained in the present section are, strictly speaking, only applicable when the piece of material considered is of uniform transverse section, but they nevertheless may be used when the transverse section is variable, provided the rate of variation be not too great and the other conditions mentioned are strictly fulfilled. The intensity of the stress is then different at different parts of the bar, varying inversely as the transverse section, and in determining the elongation this must be taken into account.

In many cases of tension the effect of the weight of the tie and other circumstances introduces an additional stress, the amount of which is often imperfectly known. This is allowed for either by making a certain addition to the theoretical diameter or by the use of a factor of safety adapted to the particular case. On the other hand it also often happens, as in the case of ropes for example, that the strength of the material is greater in small sizes than large ones for reasons connected with the mode of manufacture.

152. *Simple Compression.*—When the forces applied to the ends of a bar act in a direction towards one another the bar is in a state of *compression.* If the bar is long compared with its transverse dimensions, then any slight disturbance from uniformity will cause it to bend sideways under the compressive force, and we have then, *not* simple compression, but compression compounded with bending, an important case to be considered hereafter. To obtain simple compression the ratio of length to smallest breadth should not exceed certain limits which depend on the nature of the material, viz., cast iron 5 to 1, wrought iron 10 to 1, steel 7 to 1. Further, it is necessary that the material be perfectly homogeneous and that the line of action of the load should be in the axis of the bar. Then the results we have obtained for simple tension apply to this case of simple compression

$$p = \frac{P}{A},$$

and the strength of the column is given by $P = Af$, where f is the co-efficient of strength. The compression x which the column undergoes is connected with the stress by the equation

$$p = E\frac{x}{l}.$$

The modulus of elasticity E would, in a perfectly elastic body, be the same as for tension. In actual materials it sometimes appears to be less; but within the elastic limit only slightly less.

EXAMPLES.

1. A rod of iron 1 inch in diameter and 6 feet long is found to stretch one sixteenth inch under a load of $7\frac{1}{2}$ tons. Find the intensity of stress on the transverse section and the modulus of elasticity in lbs. and tons per square inch.

Stress = 21,382 lbs. = 9·55 tons.

Modulus of elasticity = 24,631,855 lbs. = 10996·4 tons.

2. What should be the diameter of the stays of a boiler in which the pressure is 30 lbs. per square inch, allowing one stay to each $1\frac{1}{2}$ square feet of surface and a stress of 3,500 lbs. per square inch of section of the iron? *Ans.* $1\frac{1}{2}$ inches.

3. In example 1 find the work stored up in the rod in foot-pounds. *Ans.* $43\frac{3}{4}$.

4. If in the last question the rod were originally 2″ diameter and half its length were turned down to a diameter of 1″. Compare the work stored in the rod with the result of the previous question.

Ratio = $\frac{5}{8}$.

5. In Example 1 assume the given load of $7\frac{1}{2}$ tons to be the proof load; find the modulus of resilience. *Ans.* 18·56 in inch-lb. units.

6. Find the thickness of plates of a cylindrical boiler 4′ 2″ diameter to sustain a pressure of 50 lbs. per square inch, taking the co-efficient of strength of plate at 4,000 lbs. *Ans.* $\frac{5}{16}$″.

7. A spherical shell 4′ diameter $\frac{1}{4}$″ thick is under internal fluid pressure of 1000 lbs. per square inch. Find the intensity of stress on a section of the sphere taken through the centre. *Ans.* 48,000 lbs. per square inch.

8. Find the necessary thickness of a copper steam pipe 4″ diameter for a steam pressure of 100 pounds above the atmosphere, the safe stress for copper being taken as 1000 lbs. per square inch. *Ans.* ·2″.

9. A circular iron tank, diameter 16 feet, with vertical sides $\frac{1}{2}$″ thick, is filled with water to a depth of 12 feet: find the stress on the sides at the bottom. How should the thickness vary for uniform strength throughout? *Ans.* 1024 lbs. per square inch.

10. What length of iron suspension rod will just carry its own weight, the stress being limited to 4 tons per square inch, and what will be the extension under this load? *Ans.* 2,700 feet.

11. The end of a beam 10″ broad rests on a wall of masonry; if it be loaded with 10 tons what length of bearing surface is necessary, the safe crushing stress for stone being 150 lbs. per square inch. *Ans.* 15″.

12. Find the diameter of bearing surface at the base for a column carrying 20 tons, the stress allowed being as in the last question. *Ans.* 20″ nearly.

13. Compare the weight of the shell of a cylindrical boiler with the weight of water it contains when full. *Ans.* Ratio=$55p/f$.

Section II.—Simple Bending.

153. *Proof that the Stress at each Point varies as its Distance from the Neutral Axis.*—The nature of the straining action producing bending has been sufficiently explained in the third section of Chapter II., and we shall now consider the kind of stress which results on the ultimate particles of a solid bar of uniform transverse section and of perfectly elastic material. The bar is supposed symmetrical about a plane through its geometrical axis, and the bending is supposed to take place in this plane which may be called the Plane of Bending.

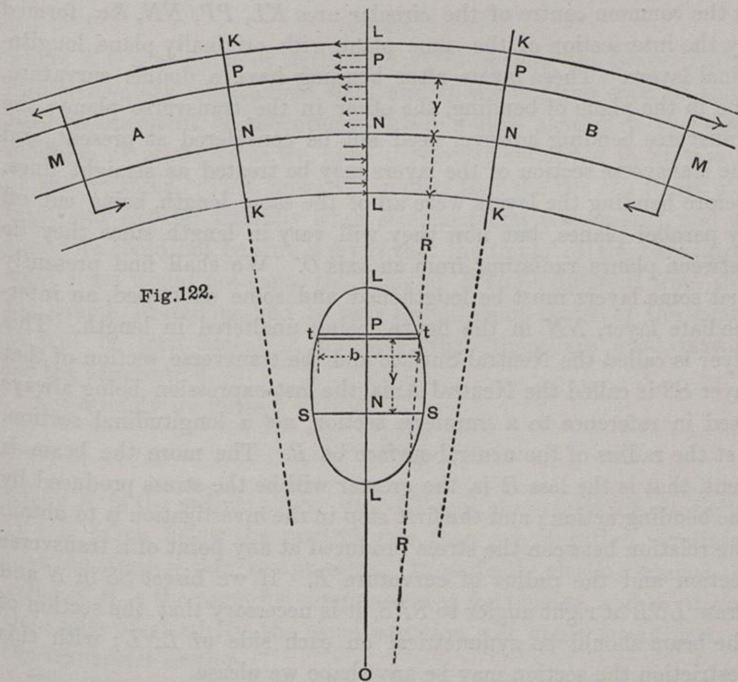


Fig.122.

In the first instance the bending is supposed to be "simple," that is, it is not combined with shearing as is most often the case in practice, but is due to a uniform bending moment (see Art. 21). The curvature of the beam is then uniform, that is to say, it is bent into a circular arc. The investigation consists of three parts.

Fig. 122 shows a longitudinal section AB and a transverse section LL through the centre of the beam; by symmetry it follows that if the bending moment be applied to both ends in exactly the same way, that transverse section, if plane before bending, will be still plane after bending, for there is no reason for deviation in one direction rather than another. It will be seen presently that if the bending moment be applied to the ends of the beam in a particular way all transverse sections will be in the same condition, and we may therefore assume that not only the central section, but any other sections *KK* we please to take, will remain plane notwithstanding the bending of the beam. All such sections, if produced, will meet in a line the intersection of which by the plane of bending will be a point *O* which is the common centre of the circular arcs *KL*, *PP*, *NN*, &c., formed by the intersection of the same plane with originally plane longitudinal layers. These layers after bending have a double curvature, one in the plane of bending, the other in the transverse plane; the transverse bending however need not be considered at present, and the transverse section of the layers may be treated as straight lines. Before bending the layers were all of the same length, being cut off by parallel planes, but now they will vary in length since they lie between planes radiating from an axis *O*. We shall find presently that some layers must be lengthened and some shortened, an intermediate layer, *NN* in the figure, being unaltered in length. This layer is called the Neutral Surface and the transverse section of that layer *SS* is called the Neutral Axis, the last expression being always used in reference to a *transverse* section, *not* a longitudinal section. Let the radius of the neutral surface be *R*. The more the beam is bent, that is the less *R* is, the greater will be the stress produced by the bending action; and the first step in the investigation is to obtain the relation between the stress produced at any point of a transverse section and the radius of curvature *R*. If we bisect *SS* in *N* and draw *LNL* at right angles to *SNS*, it is necessary that the section of the beam should be symmetrical on each side of *LNL*; with this restriction the section may be any shape we please.

Now consider any layer *PP* of the beam between the planes *LL* and *KK* which is at the distance y from the neutral surface *NN* or neutral axis *SNS*. This layer will be curved to a circle whose radius is $R+y$, and it must undergo an alteration of length from *NN* which it had before bending, to *PP* which it now has. Thus the alteration of length per unit of length, that is, the strain $e=\dfrac{PP-NN}{NN}$, but since arcs are proportional to radii $\dfrac{PP}{NN}=\dfrac{R+y}{R}$,

$$\therefore \text{the strain } e=\frac{PP-NN}{NN}=\frac{y}{R}.$$

If the layer we are considering is taken below the neutral surface, the strain, which will then be compression, will be given by the same expression $e=y/R$, e and y both being negative.

Accompanying the longitudinal strain just estimated there must be a longitudinal stress proportional to the strain. Let p be the intensity of that stress, then

$$p=Ee,$$

where E is a modulus of elasticity. If we imagine the beam divided into elementary longitudinal bars, and if we imagine each of those bars independent of the others, it will follow that E is the same modulus of elasticity as we have previously employed in Section I. of this chapter. This, however, implies that the bar can freely contract and expand laterally when stretched and compressed, and we therefore could not be sure *a priori* that the union of the bars into a solid mass would not cause the value of E to be different from that for simple stretching, and to vary for different layers of the beam. It will be seen hereafter, however, that there are good reasons for the assumption.

Accordingly we write

$$p=E\,.\,\frac{y}{R},$$

where E is the ordinary (also called Young's) modulus of elasticity. If y is taken below the neutral axis then p is negative, signifying that the stress is now compressive. In perfectly elastic material the value of E is the same for compression as for tension, and so, within the limits of elasticity, the same equation will apply for all parts of the transverse section.

Thus the stress at any point of the transverse section of the bar is proportional to its distance from the neutral axis.

154. *Determination of Position of Neutral Axis.*—The second step in the investigation is to find the position of the neutral axis. That position is deduced by dividing the beam into two portions, A and B, by a section LL, and considering the horizontal equili-

brium of either portion, say B. The external forces acting transversely to the beam balance one another, but being vertical have no resultant in the horizontal direction of the length of the beam.

We have next to take account of the internal molecular forces which act at the section LL. Above the neutral axis the action of LA is a tendency to pull B to the left; but below the neutral axis, the tendency is to thrust B to the right. In order that it may remain in equilibrium, and not move horizontally, it is necessary that the total pull should equal the total thrust; or the total horizontal force at the section must be zero. To estimate the horizontal force, consider the force acting on a thin strip of the transverse section, of breadth b, and thickness t, distant y from the neutral axis. The thrust or pull on this elementary strip $= p \cdot b \cdot t$.

Summing the forces on all the strips composing the sectional area, we must have

$$\Sigma p \cdot bt = 0;$$

but $p = Ey/R$ where E and R are the same for all strips of the section.

$$\therefore \frac{E}{R} \cdot \Sigma bt \cdot y = 0.$$

That is to say, the sum of the products of each elementary area into its distance from the neutral axis must be zero.

This can be true only if the axis passes through the centre of gravity of the section; for it is the same thing as saying that the moment of the area about the neutral axis is to be zero.

155. *Determination of the Moment of Resistance.*—The third and last step in the investigation is to obtain the connection between the bending moment applied, and the stress which is produced by it. Again, considering either portion, AL or BL, of the beam, say AL, the external forces on A produce a bending moment or couple, M, which has to be resisted by the internal stresses called into action at the section K; so that the total moment of these stresses must be equal to M. The moment of the resisting stresses, being a couple, may be estimated about any axis with the same result. For convenience we will estimate it about the neutral axis of the section.

Let us again consider the elementary strip of area bt, distant y

from neutral axis, on which the intensity of stress is p, the force, pull, or thrust, on this strip being pbt. The moment of the force $= p \,.\, bt \,.\, y$. Seeing that forces on all elementary strips, whether pull or thrust, all tend to turn the piece AL the same way, the total moment of the stresses will be found by summing all terms, $p \,.\, bty$, for the whole area of the section.

$$\therefore\ M = \Sigma p \,.\, bty.$$

Since $p = Ey/R$, substitute, and remember that E/R is the same for all strips, then

$$M = \frac{E}{R}\Sigma b \,.\, t \,.\, y^2.$$

In this formula the area of each strip has to be multiplied by the square of its distance from the neutral axis and the sum of the products taken. This, or an analogous sum, is of constant occurrence in mechanics, and has a name assigned to it. Σbty is the simple moment of an area about an axis. Σbty^2 may be called the moment of the second degree, but the common name is the *Moment of Inertia;* because a similar sum (differing only from this in involving the mass) occurs in dynamics under that name. To distinguish the two cases area-moment and mass-moment, the former is sometimes called the geometrical moment of inertia.

Let I denote the moment of inertia, so that $I = \Sigma bty^2$, the value of which for any form of section can be obtained by geometry, then

$$M = \frac{E}{R}I, \text{ or } \frac{M}{I} = \frac{E}{R},$$

thus connecting the curvature of the beam with the moment producing it. Having previously found $p/y = E/R$, we can now connect the moment with the stress by writing

$$\frac{p}{y} = \frac{M}{I}.$$

This equation may be employed to determine the strength of a beam to resist bending. The limit of strength is reached when either the greatest safe tensile stress on one side of the neutral axis, or the greatest safe compressive stress on the other side of the neutral axis is called into action. Thus in the equation $p/y = M/I$ we must put $p = f_1$, the co-efficient of strength under tension, or $p = f_2$, the co-efficient of strength under compression; and for y, either

y_1, the distance of the most remote point on the stretched side, or y_2, the distance of the most remote point on the compressed side, so that

$$M = \frac{f_1}{y_1}I, \text{ or } \frac{f_2}{y_2}I.$$

The strength of the beam, or maximum moment of resistance to bending, is measured by the least of these quantities.

y_1 or y_2 is readily determined from geometry, the form of the section of the beam being given. It may be most conveniently expressed as a fraction of the depth of the beam. Thus y_1 or y_2 may be put $= qh$, where the co-efficient q has different values. In a rectangular section $q = \frac{1}{2}$, in a triangular section $q = \frac{1}{3}$ or $\frac{2}{3}$, and so on.

Next to express the value of I. It will be found that whatever be the form of the section, I may always be written $= nAh^2$, A being the area of the section of the beam, h the depth in the direction of bending, and n a numerical co-efficient, the value of which depends on the form of the section.

For a rectangular section,

$$n = \tfrac{1}{12}, \text{ so that } I = \tfrac{1}{12}Ah^2,$$

„ elliptical or circular „

$$n = \tfrac{1}{16} \quad „ \quad I = \tfrac{1}{16}Ah^2,$$

„ triangular „

$$n = \tfrac{1}{18} \quad „ \quad I = \tfrac{1}{18}Ah^2,$$

and so on.

Therefore assuming q and n known, we can write

$$M = \frac{f}{qh}nAh^2 = f\frac{n}{q}\,.\,Ah,$$

a formula which shows that for sections in which n/q is the same, the moment of resistance to bending is proportional to the product of the area and depth of the beam. Sections with the same n and q are said to be of the *same type*. They are often, but not correctly, said to be *similar*.

In estimating the numerical value of M, care must be taken with the units. It is generally advisable to use the inch unit throughout.

156. *Remarks on Theory of Bending.*—In the foregoing theory of simple bending it is supposed

(1) That the bar is homogeneous and of uniform transverse section and perfectly elastic;

(2) That sections plane before bending are plane after bending, for which it is theoretically necessary that the bending moment should be uniform, and applied at the ends of the bar in a particular way;

(3) That longitudinal layers of the beam expand and contract laterally in the same way, as if they were disconnected from each other (see pp. 303, 401).

These assumptions are not obvious *a priori*, and require justification, which at the present stage of the subject we are not in a position to give: for the present it may be stated that if the material be homogeneous and perfectly elastic, the equations hold good even though the transverse sections and the curvature vary and however the bending moment is applied. The *strength* of the material, however, is not generally the same as if the layers were disconnected, and co-efficients of strength require therefore to be determined by special experiment on transverse strength (Art. 217).

157. *Calculation of Moments of Inertia.*—We have frequently to deal with beams of complex section, in which case to determine I it is convenient to divide the section up into simple areas, the I of each of which is known, and the total moment of inertia of the section will be the sum of these I's. In employing this process we require to know the relation between the moments of inertia of an area about two axes parallel to one another, one being the neutral axis. We make use of a general theorem which may be thus proved.

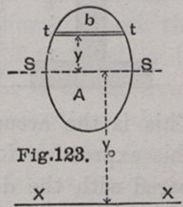


Fig.123.

Let A be an area of which we know the moment of inertia about the neutral axis, SS (Fig. 123), and we require to know the moment of inertia about any parallel axis, XX, distant y_0 from SS. Dividing the area into strips of breadth b, and thickness t.

Moment of Inertia required $I = \Sigma b \,.\, t \,.\, (y + y_0)^2$.

$$= \Sigma bty^2 + 2y_0 \Sigma bt \,.\, y + y_0^2 \Sigma b \,.\, t.$$

Now Σbty^2 = moment of inertia about neutral axis, $\Sigma bt \,.\, y = 0$, because the neutral axis passes through the centre of gravity of the section, and Σbt = Area A.

$$\therefore I = I_0 + Ay_0^2.$$

The moment of inertia of an area about any axis is, therefore, determined by adding to the moment of inertia of the area about a parallel axis through the centre of gravity the product of the area into the square of the distance between the two axes.

This theorem, together with previously quoted values of I_0, will enable us to determine the following results, which will be useful in application to beams—

Rectangle about its base, ... $I = \frac{1}{3}Ay^2$.

Triangle ,, ,, ... $I = \frac{1}{6}Ay^2$.

Triangle about a parallel to its base through vertex, $I = \frac{1}{2}Ay^2$.

Many other forms will divide up into rectangles or triangles, or both; for example, the moment of inertia of a trapezoid about the neutral axis may be readily determined by taking, for the area above the neutral axis, the I for a rectangle about one end, and triangles about the base. For the area below, a rectangle about one end and triangles about the vertex, and add the results.

158. *Beams of I Section with Equal Flanges.*—The case of a beam of I section is very important.

First, suppose the flanges of equal breadth and thickness, and the web of uniform thickness b', the depth being h', b being the breadth of the flange, and h the whole depth of the beam. The moment of inertia of the section may be taken as the difference of the moments of inertia of two rectangles (see Fig. 124).

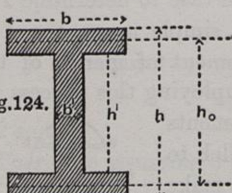
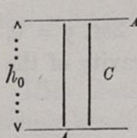


Fig.124.

$$I = b\tfrac{1}{12}h^3 - \tfrac{1}{12}(b - b')h'^3.$$

This is the accurate value of I, and when the flanges are thick this expression for I must be used; but if the flanges are thin compared with the depth, a very close approximation can be obtained with less trouble by supposing each flange to be concentrated in its centre line, and taking for the depth of the beam the distance h_0 to the centre of flanges.

If A = area of each flange and C = area of web,



then $I = A\dfrac{h_0^2}{4} + A\dfrac{h_0^2}{4} + \tfrac{1}{12}Ch_0^2 = \dfrac{h_0^2}{2}\left(A + \dfrac{C}{6}\right)$.

Putting $p = f$ and $y = \frac{1}{2}h_0$, in the formula $\dfrac{p}{y} = \dfrac{M}{I}$,

$$M = \frac{f}{\frac{1}{2}h_0}\frac{h_0^2}{2}\left(A + \frac{C}{6}\right) = fh\left(A + \frac{C}{6}\right).$$

This shows that, area for area, the web has only one-sixth the power resisting bending that the flanges have.

We previously deduced an approximate expression for the strength of an I beam, viz.,

$$M = Hh = fhA \text{ (see Art. 27)},$$

in which the effect of the web in resisting bending was neglected, the whole of the bending action being supposed to be taken by the flanges. The present formula shows the amount of the error involved in that assumption. In using this approximation when h the effective depth is reckoned from centre to centre of the flanges, two errors are made, one in supposing the resistance to bending of the web neglected, and the other, often much greater, in supposing the mean stress on the flange equal to the maximum, hence it is better to take for the effective depth

$$h = \frac{h_0^2}{h'},$$

where h' is the outside depth and h_0 the depth from centre to centre of flanges.

159. *Ratio of Depth to Span in I Beams.*—The formula just obtained for the moment of resistance of a beam of I section shows that the greater the depth of the beam and the thinner the web the stronger will the beam be for the same weight of material, or in other words that the best distribution of material is as far away from the neutral axis as possible. The practical limitation to this is that a certain thickness of web is necessary to hold the flanges together and give sufficient power of resistance to lateral forces and to the direct action of any part of the load which may rest on the upper flange. Hence the weight of web rapidly increases as the depth increases, and a certain ratio of depth to span is best as regards economy of material (see Ex. 17, page 325). This is especially important in large girders in which economy of material is the primary consideration. In smaller beams the proper ratio of depth to span is generally in great measure a question of stiffness, a part of the subject to be considered in Chapter XIII. The moment of resistance of I sections of practical proportions is generally about double that of a rectangular section of equal area. The straining actions on the web will be considered in Ch. XV.

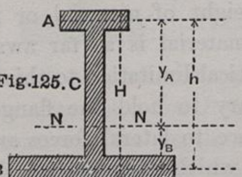
160. *Proportions of I Beams for Equal Strength.*—Materials in general are not equally strong under tension and compression, so that a beam whose section is symmetrical above and below the neutral axis will yield on one side before the material on the other side of the neutral axis has reached its limiting stress. Accordingly we might obtain a more economical distribution of material if we were to take some from the stronger side and put it on the weaker, so that the limiting tensile on one side and the limiting compressive stress on the other may be produced simultaneously. The section of the beam will be different above and below the neutral axis, which will not now be at the centre of depth of the beam, but in such a position that the distances to the top and bottom of the beam are in the proportion of the greatest allowed stresses to one another. The neutral axis in all cases must pass through the centre of gravity of the section.

Let f_A, f_B be the co-efficients of strength under compression and tension respectively, y_A, y_B distances of the most strained layer from neutral axis, then the beam will be strongest when

$$\frac{y_A}{f_A}=\frac{y_B}{f_B}=\frac{y_A+y_B}{f_A+f_B}=\frac{h}{f_A+f_B}.$$

For simplicity of calculation we will consider a beam (Fig. 125) in which the web is of uniform thickness throughout the depth, and so of rectangular section, and each flange also of rectangular section, and determine the relation which should hold between the areas of flanges and web for maximum strength of beam, and the moment of resistance to bending where this condition is satisfied. We will further suppose each flange to be concentrated in its centre line.

Fig.125.



Let A = area of compressed flange, B = area of stretched flange, C = area of web. Since the neutral axis is at the centre of gravity of the section, we obtain, by taking moments about that axis,

$$A \cdot y_A + C\frac{y_A - y_B}{2} = By_B\,;$$

or, substituting the previously given values of y_A and y_B,

$$Af_A + C\frac{f_A - f_B}{2} = Bf_B.$$

Supposing f_A and f_B known, A, B, and C must be such as to satisfy this relation. We have some liberty of choice between these quantities, and frequently find one of the flanges omitted, so producing a beam of T or ⊥ section.

In a cast-iron beam, where the resistance to compression is greater than for tension, the compressed flange A may be omitted. Putting $A = 0$ we get $C = \dfrac{2f_B}{f_A - f_B}B$, and supposing $\dfrac{f_A}{f_B} = 4$; $C = \frac{2}{3}B$, or $B = 1\frac{1}{2}C$. In a wrought-iron beam on the other hand f_A/f_B is about $\frac{3}{5}$, and the stretched flange B is the area to be omitted. Putting $B = 0$, we find $A = \dfrac{f_B - f_A}{2f_A}C = \frac{1}{3}C$.

Otherwise we may assume the depth and thickness of the web to be given (Art. 159), then the equation

$$Af_A + C\,.\frac{f_A - f_B}{2} = Bf_B,$$

furnishes a relation between the areas of the flanges. For example, in cast iron, if we assume $f_A = 4f_B$, we find

$$B = 4A + \tfrac{3}{2}C.$$

Having decided on the proportions between the parts of the section we can now calculate the moments of inertia and resistance. Still considering the flanges concentrated in their centre lines,

$$I = Ay_A^2 + By_B^2 + \tfrac{1}{3}C\,.\frac{y_A}{h}\,.\,y_A^2 + \tfrac{1}{3}C\,.\frac{y_B}{h}\,.\,y_B^2.$$

$$= Ay^2_A + By_B^2 + \tfrac{1}{3}C\,.\frac{y_A^3 + y_B^3}{h},$$

a result which admits of ready calculation. Further

$$\frac{M}{I} = \frac{f_A}{y_A} = \frac{f_B}{y_B} = \frac{f_A + f_B}{h},$$

whence we obtain

$$M = (f_A + f_B)\frac{I}{h}.$$

The calculation just now made is one which has been frequently given in dealing with beams of I section,* but in applying it to actual examples it should be remembered that the results are obtained on the supposition that the flanges are concentrated in their centre lines, and are consequently only approximate when the coefficients f_A, f_B mean the intensities of the stress at those centre lines, *not* at the surface of the beam where the stress is greatest. If, for example, F_A be the maximum stress on the flange A

$$F_A = f_A \cdot \frac{y_A + \frac{1}{2}t_A}{y_A},$$

where t_A is the thickness of the flange. The difference is especially great in the case of the larger flange of cast-iron beams, and the true ratio of maximum compressive and tensile stress is much less than it appears in the preceding article. On the other hand, in extreme cases, such as we are now considering, the stress may not be uniformly distributed along a line parallel to the neutral axis.

Extensive experiments were made on cast-iron beams by Hodgkinson, with the object of determining the best proportions between the flanges, with the result that rupture always took place by tearing asunder of the lower flange, unless it was at least six times the size of the compressed flange. This proportion is rarely adopted in practice, from the difficulties of obtaining a sound casting, and the necessity of having sufficient lateral strength. Nor is it certain that the proportions which are best for resisting the ultimate load are also best in the case of the working load; it is, in fact, probable that a smaller proportion is better even on the score of strength. If we take $f_A = 2\frac{1}{2}f_B$, instead of $4f_B$, we find

$$B = 2\tfrac{1}{2}A + \tfrac{3}{4}C,$$

which agrees more closely with practice. The ratio of maximum compressive and tensile strength is in this case about 2, which, according to some authorities, is the ratio of *elastic* strengths in the two cases.

In wrought-iron beams the areas of the flanges are usually equal, and this is correct if the elastic strength, and not the ultimate strength, is regarded as fixing the proper proportions, and if there be sufficient provision against the yielding of the top flange by lateral flexure. Small-sized beams of this kind are rolled in one piece, while large girders are constructed of iron or steel

* See Rankine's *Civil Engineering*, page 257.

plates and angle irons, rivetted together. Some of the forms they assume are shown in Plate VIII., Ch. XVIII.

In making calculations respecting girders, approximate methods may be used for preliminary tentative calculations, but should be checked by a subsequent accurate determination of the neutral axis and moment of inertia. A previous reduction of the section to an equivalent solid section is required when, as is often the case, all parts of the section do not offer the same elastic resistance to the stress applied to them, either because they are not sufficiently rigidly connected or from the material being different. This is especially the case in determining the resistance to the longitudinal bending of a vessel occasioned by the unequal distribution of weight and buoyancy already considered in Chapter III. On this important question the reader is referred to a treatise on Naval Architecture by Mr. W. H. White. In many cases of built-up girders the shearing action which generally exists has considerable influence, a matter for subsequent consideration (Ch. XV.). The effect of the weight of the girder itself has been considered in Ch. IV. (See also Ex. 13, p. 324, and Ex. 11, p. 372.)

161. *Beams of Uniform Strength.*—A beam of uniform strength is one in which the maximum stress is the same on all sections. For beams of the same transverse section throughout this can only be the case when the bending moment is uniform, but, by properly varying the section, it is possible to satisfy the condition however the bending moment vary. For this purpose we have only to consider the equation

$$M = f \cdot \frac{n}{q} \cdot Ah,$$

which must now be satisfied at all sections. Suppose

$$A = kbh,$$

where k is a numerical factor depending on the type of section, then

$$M = f \cdot \frac{nk}{q} \cdot bh^2.$$

All sections of the beam being supposed of the same type we have only to make Ah or bh^2 vary as M, that is as the ordinates of the curve of bending moments. The principal cases are—

(1) Depth uniform. Here the breadth must vary as the bending moment, whence it is clear that the curve of moments may be taken as representing the half plan of the beam.

(2) Sectional Area uniform. Here the depth must vary as the bending moment, that is, the curve of moments may be taken to represent the elevation or half elevation of the beam.

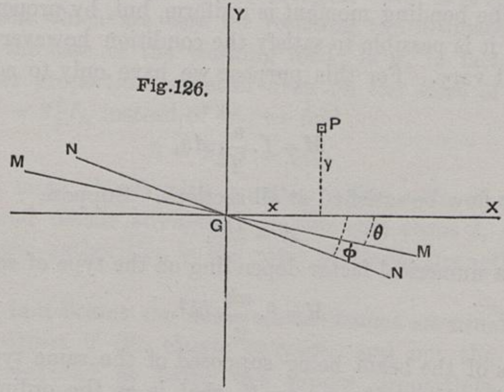
(3) Breadth uniform. Here the elevation or half elevation of the beam must be a curve, the co-ordinates of which are the square roots of the co-ordinates of the curve of moments.

(4) Ratio of breadth to depth constant. Here the half plan and half elevation are each curves, the ordinates of which are the cube roots of the ordinates of the curve of moments.

The first, third, and fourth of these cases are common in practice with some modifications occasioned by the necessity of providing strength at sections of the beam where the bending moment vanishes, as it usually does at one or both ends.

162. *Unsymmetrical Bending.*—It occasionally happens that the plane of the bending moment is not a principal plane of the beam, as for example when a vessel heels over, the plane of longitudinal bending will not coincide with the plane of symmetry of the vessel which is obviously the plane of the masts. The neutral axis does not now coincide with the axis of the bending couple, though in other respects the theory of bending still holds good.

Fig.126.



In Fig. 126 let MM be the axis of the bending moment, M inclined at an angle θ to the principal axis of inertia GX, GY of the plane section. Then the couple M may be resolved into two components $M \cos \theta$ and $M \sin \theta$, each of which will produce stress at any point

P as if the other did not exist. Let p be the stress, x, y the co-ordinates of P referred to the axes GX, GY, the moments of inertia about which are I_1, I_2, then

$$p = \frac{M \cdot \cos\theta \cdot y}{I_1} + \frac{M \cdot \sin\theta \cdot x}{I_2}.$$

The position of the neutral axis NN is found by putting $p = 0$, then the angle ϕ which it makes with GX is given by

$$\tan\phi = -\frac{y}{x} = \frac{I_1}{I_2} \cdot \tan\theta.$$

This equation shows that the neutral axis is parallel to a line joining the centres of the circles into which the beam would be bent by the component couples supposed each to act alone.

The neutral axis being thus determined and laid down on the diagram the points can be found which lie at the greatest distance from that axis. At these points the stress will be greatest, and if X, Y be their co-ordinates, still referred to the axes GX, GY, the moment of resistance will be determined by the equation

$$f = M\left\{\frac{Y \cdot \cos\theta}{I_1} + \frac{X \cdot \sin\theta}{I_2}\right\}.$$

For a different method of expressing the moment of resistance see Rankine's Applied Mechanics, p. 314.

EXAMPLES.

1. A bar of iron 2″ diameter is bent into the arc of a circle 372′ diameter. Find in tons per square inch, 1st, the greatest stress at any point of the transverse section; 2nd, the stress on a line parallel to the neutral axis half an inch from the centre, E being taken = 29,000,000. *Ans.* Maximum stress = 5·8. Stress at ½″ from centre = 2·9.

2. Find the diameter of the smallest circle into which the bar of the last question can be bent; the stress being limited to 4 tons per square inch. *Ans.* Diameter = 540 feet.

3. Find the position of the neutral axis of a trapezoidal section; the top side being 3″, bottom 6″, and depth 8″. Also find the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 3·56 inches from bottom. Ratio of stresses 5 to 4.

4. A cast-iron beam is of I section with top flange 3″ broad and 1″ thick and bottom flange 8″ broad and 2″ thick; the web is trapezoidal in section ½″ thick at top and 1″ at bottom; total outside depth of beam 16″. Find the position of the neutral axis and the ratio of maximum tensile and compressive stresses. *Ans.* Neutral axis 4·81 inches from bottom. Ratio of stresses 3 to 7.

5. A wrought iron beam of rectangular section is 9″ deep, 3″ broad, and 10 feet long. Find how much it will carry loaded in the centre, allowing a co-efficient of 3 tons per square inch. Also deduce the load the same beam will bear when set flatways. *Ans.* When upright load = 4·05 tons. When set flatways load = 1·35 tons.

6. A piece of oak of uniform circular section is 16″ diameter and 12 feet long. It is supported at the two ends and loaded at a point 5 feet from one end. How great may the load be, allowing a stress of $\frac{1}{2}$ ton per square inch? *Ans.* Load may be 5·74 tons.

7. In Example 5 suppose the same weight of metal formed into a beam of I section, each flange being equal to the web; what load will the beam carry? *Ans.* Load may then be 9·45 tons.

8. Find the moment of resistance to bending of the section given in Example 4, the co-efficient for tension being 1 ton per square inch. *Ans.* I = 798 inch units. Moment of resistance to bending = 166·4 inch-tons.

9. Suppose the skin and plate deck of an iron vessel to have the following dimensions at the midship section, measured at the middle of the thickness of the plates. Find the position of the neutral axis and moment of resistance to bending. Breadth 48′ and depth of vertical sides 24′, the bilges being quadrants of 12′ radius. Thickness of plate $\frac{5}{8}$″ all round, and co-efficient of strength 4 tons in compression. *Ans.* Neutral axis 14″ above centre of depth. Moment of resistance to hogging = 40,000 ft.-tons.

10. What should be the sectional area of a ⊤ beam of wrought iron to carry 4 tons uniformly distributed? Span 20′, depth of beam 10″ Co-efficient for compression 3 tons, and for tension 5 tons? *Ans.* Area = 13·7 square inches.

11. If, in the last question, the flange is made equal to the web instead of being proportioned for equal strength, show that to carry the same load the beam must be about one quarter heavier.

12. In Example 8 find the moments of inertia and resistance on the supposition that the flanges are concentrated at the centre lines, and thus by comparison with previous results show the amount of the error involved in the assumption. *Ans.* Moment of inertia = 861·5 inch units. Moment of resistance = 227 inch-tons.

13. Show that the limiting span (Art. 41) of a beam of uniform transverse section is

$$L = \lambda \, . \, \frac{8n}{Nq},$$

where N is the ratio of span to depth, and the rest of the notation is the same as on pages 90 and 314. Obtain the numerical result for a wrought iron beam of rectangular section, taking λ from Table II., Ch. XVIII., and supposing $N = 12$. *Ans.* $L = 336$ ft.; in an ordinary I section the result would be doubled. For the case of large girders see page 372.

14. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight when supported at the ends, show that when the stress allowed is 4 tons per square inch $l = \sqrt{224\,d}$.

15. If I_1, I_2 be the moments of inertia of two plane areas A_1, A_2, about their neutral axis which are supposed parallel at distance apart z, show that the moment of inertia of their sum or difference about their common neutral axis is $I = I_1 \pm I_2 + z^2 \, . \, \dfrac{A_1 A_2}{A_1 \pm A_2}$.

Apply this formula to the trapezoidal section of Question 3. *Ans.* $I = 185$ inch units nearly.

16. Find the moment of resistance to bending of a beam of I section, each flange consisting of a pair of angle irons $3\frac{1}{2}'' \times \frac{1}{2}''$ rivetted to a web ·37″ thick and 16″ deep between them. Assuming it 24 feet span, find the load it would carry in the middle, using a co-efficient of 3 tons per square inch. *Ans.* $M = 288$ inch-tons. $W = 4$ tons.

17. If it be assumed that for constructive reasons the thickness of web of an I beam with equal flanges must be a given fraction of the depth, show that for greatest economy of material the sectional area of the web should be equal to the joint sectional area of the flanges. Prove that in this case $M = \frac{1}{3} f \,.\, Sh$. (See p. 372.)

18. In a cast-iron beam of I section of equal strength for which $f^A = 2\frac{1}{2} f_B$; if it be assumed that for constructive reasons the thickness of the web should be a given fraction of the depth, show that for greatest economy of material the large flange, the web, and the small flange should be in the proportion 25, 20, 4. Prove also that the moment of resistance is given by the same formula as in Question 17 supposing $2/f = 1/f_A + 1/f_B$.

19. A beam of rectangular section of breadth one half the depth is bent by a couple the plane of which is inclined at 45° to the axes of the section. Find the neutral axis, and compare the moment of resistance to bending with that about either axis. *Ans.* Ratio $= 2\sqrt{2}/3$ and $\sqrt{2}/3$.

20. If a beam be originally curved in the form of a circular arc of radius R_0, instead of being straight, show that the neutral axis does not pass through the centre of gravity of the section. In a rectangular section of depth h show that the deviation is, approximately,

$$z = \frac{h^2}{12R_0}.$$

21. In the preceding question if R_0 is large show that the equations of bending are

$$\frac{p}{y} = E\left(\frac{1}{R_0} - \frac{1}{R}\right) = \frac{M}{I}.$$

Reference.

For the graphical determination of moments of inertia the reader is referred to the treatises cited on page 82.