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CHAPTER XIII.

DEFLECTION AND SLOPE OF BEAMS.

163. *Deflection due to the Maximum Bending Moment.*—It is not only necessary that a beam should be strong enough to support the load to which it is subjected, it is also necessary that its changes of form should not be too great, or in other words, that it should be sufficiently stiff, and we next proceed to determine under what conditions this will be the case.

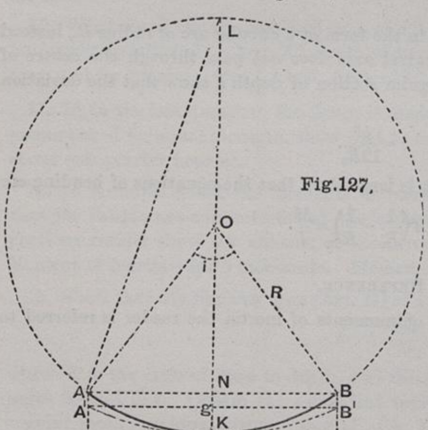


Fig.127.

The question is simplest when the beam is bent into an arc of a circle, we have then

$$\frac{p}{y}=\frac{M}{I}=\frac{E}{R}=\text{constant.}$$

Two cases may be especially mentioned--

(1) Depth uniform. We then have p constant, that the beam is of uniform strength. (See Case 1 of Art. 161.)

(2) Sectional area uniform. We then have, since

$$M=\frac{E}{R}I=n\,.\,\frac{E}{R}\,.\,Ah^2,$$

the depth of the beam varying as the square root of the bending moment, as in Case 3 of the same article.

Let l be the length of the beam, i the angle its two ends make with one another, then since i is also the angle subtended by the beam at the centre

$$i = \frac{l}{R} = \frac{Ml}{EI}.$$

If the beam be supported at the ends i is twice the angle which the ends make with the horizontal, an angle called the Slope at the ends. Let AB be the beam (Fig. 127), O the centre of the circle into which it is bent, KL the diameter of the circle through K the middle point of the beam. Then KN is the deflection which is given by a known proposition of Euclid

$$KN \,.\, NL = AN^2.$$

Hence remembering that the diameter of the circle is very large * we have, if δ be the deflection,

$$\delta = \frac{l^2}{8R} = \frac{Ml^2}{8EI}.$$

This formula gives the deflection in any case where the curvature is uniform.

When the transverse section is uniform the curvature varies. Unless the bending moment be likewise uniform, the deflection curve is not then a circle AKB, but for the same maximum bending moment a flatter curve $A'KB'$. Thus the deflection is less than that calculated by the above formula, which may be described as the "deflection due to the maximum moment." The actual deflection may conveniently be expressed as a fraction of that due to the maximum moment. It is possible to construct the deflection curve graphically by observing that the curvature at every point is proportional to the bending moment. We have then only to strike a succession of arcs with radii inversely proportional to the ordinates of the curve of bending moment. It is however more convenient to proceed by an analytical method.† The fraction is least when the beam is least curved, which is evidently the case when it is loaded in the middle, and we shall show presently that it is then two-thirds, while, when uniformly loaded, it is five-sixths.

* For clearness it is made small in the figure.

† Readers who have no knowledge of the Calculus may pass over the next four articles.

164. *General Equation of Deflection Curve.*—It was shown above that

$$i = \frac{M}{EI} \cdot l.$$

If the bending moment vary, then we must replace l by an element of the length ds and i by the corresponding element of the angle; we shall then have an equation

$$\frac{di}{ds} = \frac{M}{EI},$$

which by integration will furnish i. It will generally be convenient to reckon i from a horizontal tangent and it then means the slope of the beam at the point considered. To perform the integration it is in most cases necessary to suppose the slope of the beam small, as it actually is in most important cases in practice, and we may then replace ds the element of arc by dx, the corresponding element of a horizontal tangent AN (Fig. 128) taken as axis of x, whence

$$\frac{di}{dx} = \frac{M}{EI} \text{ approximately,}$$

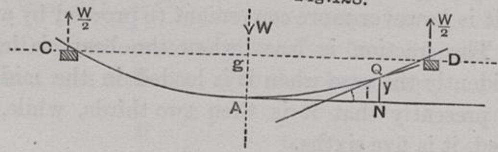
an equation which can generally be integrated because M is usually a function of x.

The deviation y of any point Q of the beam from the straight line AN can now be found since $dy/dx = i$, from which we further obtain the fundamental equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI},$$

which applies to all cases where the bending of the beam is occasioned by a transverse load. We shall first give some elementary examples of the determination of the deflection and slope of a beam and then consider the question more generally.

Fig.128.



165. *Elementary Cases of Deflection and Slope.*—*Case I.* Suppose a beam supported at the ends and loaded in the middle.

In Fig. 128 CD is the beam resting on supports at C, D, and loaded in the middle with a weight W. Take the centre A as origin and the horizontal tangent at A as axis of x, then if l be the whole length

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}.$$

$$\therefore i = \frac{dy}{dx} = \frac{\frac{W}{2}\left(\frac{l}{2}x - \frac{1}{2}x^2\right)}{EI}$$

is the slope of the beam at Q, no constant being required since i is zero when $x = 0$.

If $x = l/2$ we get the slope at the ends of the beam

$$i_1 = \frac{Wl^2}{16EI}.$$

Integrating a second time

$$y = \frac{\frac{W}{2}\left(\frac{1}{4}lx^2 - \frac{1}{6}x^3\right)}{EI}.$$

As before no constant is required because $y = 0$ when $n = 0$.

If now we put $x = l/2$ we get the elevation of D above AN or, what is the same thing, the depression of A below the level of the supports. This is called the Deflection of the beam; if we denote it by δ,

$$\delta = \frac{\frac{W}{2}\left(\frac{1}{16}l^3 - \frac{1}{48}l^3\right)}{EI} = \frac{Wl}{48EI}{}^3,$$

a result which we may also write

$$\delta = \frac{2}{3} \cdot \frac{M_0 l^2}{8EI} = \frac{2}{3} \cdot \delta_0,$$

where M_0 is the maximum moment and δ_0 the deflection due to it.

Case II. Let the beam be supported at the ends and loaded uniformly with w pounds per foot run. It will be sufficient to give the results, which are obtained in precisely the same way, remembering that the bending moment is now $\frac{1}{2}w(a^2 - x^2)$ where a is the half span. We have

$$i_1 = \frac{wa^3}{3EI} = \frac{Wl^2}{24EI}; \quad \delta = \frac{5}{24} \cdot \frac{wa^4}{EI} = \frac{5}{384} \cdot \frac{Wl^3}{EI}.$$

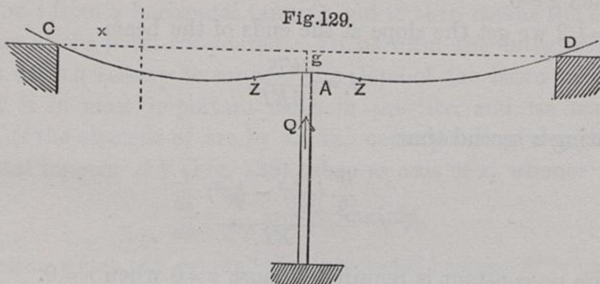
The value of δ may be expressed as in the previous case in terms of the deflection due to the maximum moment. We have $\delta = \frac{5}{6} . \delta_0$.

166. *Beam propped in the Middle.*—When a beam is acted on by several loads the deflection and slope due to the whole is the sum of those due to each load taken separately. An important example is

Case III. Beam supported at the ends and propped in the middle, uniformly loaded. (Fig. 129.)

Here the deflection of the beam is the difference between the downward deflection due to the uniform load and the upward deflec-

Fig.129.



tion due to the thrust Q of the prop. Hence we write down at once for the deflection at the centre,

$$\delta = \frac{5}{384} \cdot \frac{Wl}{EI} - \frac{Ql^3}{48EI},$$

an equation which may be used to determine the load carried by the prop when its length is given, and conversely.

First suppose the centre of the beam propped at the same level as the supports, then $\delta = 0$, and

$$Q = \frac{5 \times 48W}{384} = \tfrac{5}{8}W,$$

so that the prop in this case carries five-eighths of the weight of the beam, the supports C, D only carrying three-eighths. Each supporting force is $\frac{3}{16}wl$, l being as before the whole length of the beam; hence the bending moment at a point distant x from C is given by the formula

$$M = \tfrac{3}{16}wlx - \tfrac{1}{2}wx^2 = \tfrac{1}{2}wx(\tfrac{3}{8}l - x),$$

from which it appears that the beam is bent downwards until a point Z is reached, such that

$$CZ = \tfrac{3}{8}l = \tfrac{3}{4}AC.$$

Here the bending moment is zero, that is, Z is a "point of contrary flexure" or "virtual joint." (Compare Art. 38.)

Beyond Z the beam is bent upwards, and at the centre A we get, by putting $x=\frac{1}{2}l$,

$$-M_0=\tfrac{1}{32}wl^2.$$

The case here discussed is also that of a beam one end of which is fixed horizontally and the other supported at exactly the same level.

Let us next inquire what will be the effect of supposing the centre of the beam propped somewhat out of the horizontal line through the supports at the ends. Let us suppose δ to be $1/n^{\text{th}}$ the deflection of the beam when the prop is removed, then

$$\frac{1}{n}\cdot\frac{5}{384}\cdot\frac{Wl^3}{EI}=\frac{5}{384}\cdot\frac{Wl^3}{EI}-\frac{Ql^3}{48EI},$$

that is

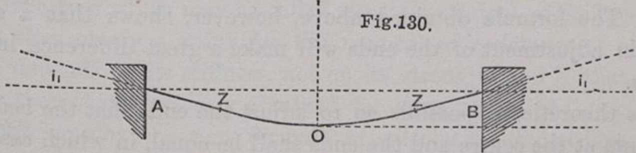
$$Q=\tfrac{5}{8}W\left(1-\frac{1}{n}\right),$$

a formula which gives the load on the prop. If, for example, $n=5$, $Q=\frac{1}{2}W$, or if $n=-5$, $Q=\frac{3}{4}W$; thus if the centre of the beam be out of level, by as much as one-fifth the deflection when the prop is wholly removed, the load on the prop will vary between $\frac{1}{2}W$ and $\frac{3}{4}W$, a result which shows the care necessary in adjustment to obtain a definite result.

167. *Beam fixed at the Ends.—Case IV.* Uniformly loaded beam, with ends fixed at a given slope.

In Fig. 130 AB is a uniformly loaded beam, with the ends A, B fixed not horizontally but for greater generality at a slope i. Here

Fig.130.



the central part of the beam will be bent downwards and the end parts upwards; at Z, Z there will be virtual joints; let $OZ=r$, then taking O as origin the bending moment at any point between O and Z is

$$M=\tfrac{1}{2}w(r^2-x^2),$$

a formula which will also hold for points beyond Z, as can be seen from Art. 38, or proved independently. We have then

$$\frac{d^2y}{dx} = \frac{\frac{1}{2}w(r^2 - x^2)}{EI};$$

$$i = \frac{\frac{1}{2}w(r^2x - \frac{1}{3}x^3)}{EI}.$$

No constant is required, because i is zero at O. Let a be the half span OA, or OB, then putting $x = a$, we get for the slope at the ends

$$i_1 = \frac{\frac{1}{2}w(r^2a - \frac{1}{3}a^3)}{EI},$$

a formula from which r can be determined if i_1 be given. If $r = a$, we get the case where the ends are free; let the slope then be i_0, we have

$$i_0 = \frac{wa^3}{3EI} \text{ as before (p. 329).}$$

Now, assume the actual slope to be $1/n^{\text{th}}$ of this, we get

$$\frac{1}{n} \cdot \frac{wa^3}{3EI} = \frac{\frac{1}{2}w(r^2a - \frac{1}{3}a^3)}{EI};$$

that is,

$$r^2 = \tfrac{1}{3}a^2\left(1 + \frac{2}{n}\right).$$

If the ends are fixed exactly horizontal, then

$$r^2 = \tfrac{1}{3}a^2,$$

and by substitution we find for the bending moment at the centre and the ends

$$M_0 = \tfrac{1}{6}wa^2;\ M_A = M_B = \tfrac{1}{3}wa^2.$$

If the ends were free, the bending moment at the centre would have been $\frac{1}{2}wa^2$, so that the beam will be strengthened in the proportion 3 : 2. The formula obtained above, however, shows that a small error in adjustment of the ends will make a great difference in the results.

It is theoretically possible so to adjust the ends that the bending moments at the centre and the ends shall be equal, in which case the beam will be strongest. For this we have only to put

$$\tfrac{1}{2}wr^2 = \tfrac{1}{2}w(a^2 - r^2),$$

that is,

$$r^2 = \tfrac{1}{2}a^2,$$

whence by substitution we get

$$n = 4;$$

that is, the ends should be fixed at one fourth the slope which they have when free, and the strength of the beam will then be doubled.

By proceeding to a second integration the deflection of the beam can be found. In particular when the ends of the beam are horizontal it can be shown that the deflection is only one fifth of its value when the ends are free.

The graphical representation of the bending moments in Cases III., IV., is easily effected, as in Fig. 42, p. 86.

168. *Stiffness of a Beam.*—The stiffness of a beam is measured by the ratio of the deflection to the span. In practice, the deflection is limited to 1 or 2 inches per 100 feet of span when under the working load; that is, the ratio in question is $\frac{1}{600}$th to $\frac{1}{1200}$th. It appears from what has been said that if M_0 be the maximum moment the deflection is given by

$$\delta = k \cdot \frac{M_0 l^2}{8EI},$$

where k is a fraction, varying from two-thirds to unity, depending on the way in which the beam is loaded. Hence the greatest moment which the beam will bear consistently with its being sufficiently stiff is

$$M_0 = \frac{8E\delta}{kl} \cdot \frac{I}{l}.$$

If we express I as usual in terms of the sectional area and depth, we get

$$M_0 = s\frac{n}{k}A\frac{h^2}{l},$$

where s is a co-efficient depending on the material and on the admissible deflection which may be called the "Co-efficient of Stiffness."

We thus obtain a value for the moment of resistance of a beam which depends on its stiffness, not on its strength, and if that value be less than that previously obtained for strength (p. 314), we must evidently employ the new formula in calculating dimensions. On comparing the two, we find that they will give the same result if

$$\frac{sh}{kl} = \frac{f}{q}; \text{ or } \frac{h}{l} = \frac{fk}{qs};$$

that is to say, for a certain definite ratio of depth to span, and if there is no other reason for fixing on this ratio, it will be best to choose the value thus determined. The two formulæ then give the same result. In large girders a greater depth is generally desirable, then the strength formula must be used; while in small beams it may often be convenient or necessary to have a smaller depth, and then the stiffness formula must be employed.

169. *General Graphical Method.*—The foregoing simple examples of the determination of the deflection and slope of a beam are perhaps those of most practical use, but, by the aid of graphical processes, there is no difficulty in generalizing the results which are of considerable theoretical interest. We can, however, afford space only for a hasty sketch.

The general equations given in Art. 164 show that the angle (i) between two tangents to the deflection curve of a beam is proportional to the area of the curve of bending moments intercepted between two ordinates at the points considered. Starting from the lowest point of the deflection curve, let us now imagine a curve drawn, the ordinate of which represents that area reckoned from the starting point, then that curve will represent the slope of the beam at every point, and may therefore properly be called the "Curve of Slope." But referring again to the general equations we see that the ordinate of the deflection curve reckoned upwards from the horizontal tangent at the lowest point, is connected with the slope in the same way as the slope with the bending moment, and is consequently proportional to the area of the curve of slope. Thus it appears, on reference to Chapter III., that the curves of Deflection, Slope, and Bending Moment are related to each other in the same way as the curves of Bending Moment, Shearing Force, and Load. The five curves, in fact, form a continuous series each derived from the next succeeding by a process of graphical integration.

We now see that any property connecting together the second three quantities must also be true for the first three. For example, we know, from the properties of the funicular polygon, that two tangents in the curve of moments intersect in a point vertically below the centre of gravity of the area of the corresponding curve of loads (see Arts. 31, 35). It must therefore be true that two tangents to the deflection curve intersect vertically below the centre of gravity of the corresponding area of the curve of moments, a useful property, which can be proved directly without much difficulty.

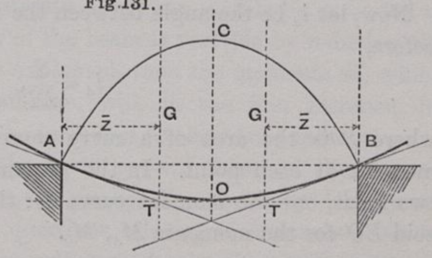
The deflection curve of a beam may therefore be constructed in the same way that the funicular polygon is constructed in Art. 35, the perpendicular distance (H) of the pole from the load line in the diagram of forces being made equal to EI. To do this we have only to divide the moment curve into convenient vertical strips and regard each as representing a weight. Set down these ideal weights as a vertical line and choose a pole at a distance from the line equal to EI, measured (on account of the largeness of E) on a scale less in a given ratio. Now, construct the polygon and draw its closing line, the intercept multiplied by the scale ratio is the deflection of the beam. A parallel to the closing line in the diagram of forces gives the slopes at the extremities of the beam which correspond to the supporting forces of the loaded beam in the original case.

We have hitherto supposed the beam to be of uniform stiffness throughout; if not, let the quantity EI, which is now variable, be E_0I_0 at some datum section. Reduce the ordinates of the curve of moments in the proportion E_0I_0 to EI, then the reduced curve is to be employed in the way just described for the original curve.

170. *Examples of Graphical Method. Theorem of Three Moments.*—Let us now take some examples.

Case I.—Symmetrically loaded beam, of flexibility also symmetrical about the centre. Let ABC (Fig. 131) be the curve of moments, reduced if necessary, AOB the deflection curve; both curves, of course, will be symmetrical about the centre vertical, then from what has been said, tangents at A, B to the deflection curve intersect the tangent at O in points T vertically below the centres of gravity of the two equal areas ACO, BCO. Hence if S be the area of the whole curve of moments, $\bar{z}$ the horizontal distance of either point T from the nearer end,

Fig.131.

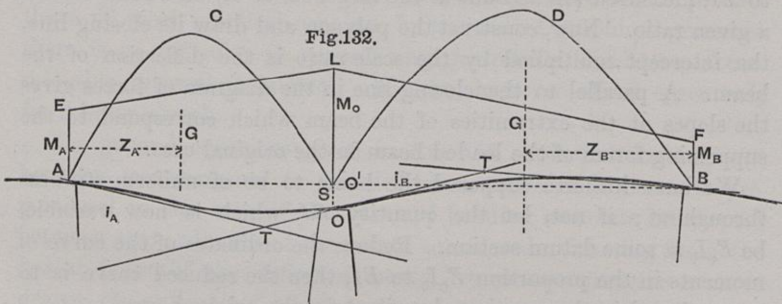


$$i_0 = \frac{S}{2EI}\,;\ \delta = \bar{z}\,.\,i_0 = \frac{S\,.\,\bar{z}}{2EI}$$

must be the slope of the ends of the beam and its deflection.

Case II. Beam continuous over several, spans loaded in any way. (Fig. 132.) Let ACO', BDO' be the moment curves due to the load on two spans AO', BO' of a beam AOB, continuous over three supports A, O, B, of which the centre O is somewhat below the level of A, B. Being continuous, there will be bending moments at A, O, B, which are represented in the diagram by AE, $O'L$, BF. Joining EL, FL, the actual bending moment at each point of the beam will be

Fig.132.



represented by the intercept between the line ELF and the curves of moments due to the load and corresponding supporting forces. (See Art. 38.) The curve AOB is the deflection curve, AT, BT are the tangents at A, B and TOT is the tangent at O, intersecting AT, BT in the points T.

Now, let i_A be the angle between the tangents at O and A, then, as before,

$$i_A = \frac{S}{EI},$$

where S is the area of a curve representing the actual bending moment at each point. In the present case S is the difference of two areas, one the moment curve for the load, the other the trapezoid EO' for the moments M_A, M_0.

$$\therefore S = A - \frac{M_A + M_0}{2} \cdot l_A,$$

where A is the area of the moment curve ACO' and l_A is the span AO'. Let the horizontal distance from A of the common centre of gravity of the two curves be x; then, as before, x is also the horizontal distance of T from A, and

$$y_A = \frac{Sx}{EI}, \text{ as before}$$

To find x, let z_A be the horizontal distance of the centre of gravity of ACB from A, then

$$Sx = Az_A - M_A l_A \cdot \frac{l_A}{2} - \frac{M_0 - M_A}{2} \cdot l_A \cdot \tfrac{2}{3} l_A\,;$$

$$= Az_A - \tfrac{1}{6} M_A \cdot l_A^2 - \tfrac{1}{3} M_0 \cdot l_A^2.$$

We have thus found y_A the distance of A from the tangent through O; and y_B, the corresponding distance of B, is written down by change of letters.

Assuming now the depression of O, the centre of the beam, below the level of the two other supports to be δ, it appears from the geometry of the diagram that

$$\frac{y_A - \delta}{l_A} = \frac{-y_B + \delta}{l_B}\,;$$

or

$$\frac{y_A}{l_A} + \frac{y_B}{l_B} = +\delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right);$$

hence dividing the values of y_A, y_B by l_A, l_B respectively, and adding

$$A\frac{z_A}{l_A} + B \cdot \frac{z_B}{l_B} - \tfrac{1}{3} M_0(l_A + l_B) - \tfrac{1}{6} M_A l_A - \tfrac{1}{6} M_B l_B = \delta\left(\frac{1}{l_A} + \frac{1}{l_B}\right) EI.$$

This equation connects the bending moments at three points of support of a continuous beam, the centre support being below the end supports by the small quantity δ. It can readily be extended to the case where the flexibility of the beam is variable by reducing the moment curves as previously explained, then the moments M, which are the results of the calculation, will, in the first instance, be reduced, and can afterwards be increased to their true values.

The above equation is the most general form of the famous Theorem of Three Moments, originally discovered by Clapeyron, which is always employed in questions relating to continuous beams —a somewhat large subject, on which we have not space to enter.

171. *Resilience of a Bent Beam.*—The work done in bending a beam by a uniform bending moment M is evidently $\frac{1}{2}Mi$, where i is the angle which the two ends of the beam make with each other, as in Art. 163; hence by substitution for i we find for the work U,

$$U = \frac{M^2}{2EI} \cdot l\,;$$

and if the bending moment vary,

$$U = \int \frac{M^2}{2EI} \cdot dx.$$

An important case is when the beam is of uniform strength, then we have

$$p = \frac{My}{I} = \text{constant} = \frac{M_0 y_0}{I_0},$$

where the suffix 0 refers to a datum section. Then

$$U = \frac{M_0^2}{2EI_0} \int \frac{I}{I_0} \cdot \frac{y_0^2}{y^2} \cdot dx.$$

Assuming now the section (A), though varying, to remain of the same type,

$$\frac{I}{I_0} = \frac{Ay^2}{A_0 y_0^2}.$$

If, therefore, we call V the volume of the beam,

$$U = \frac{M_0^2}{2EI_0} \cdot \frac{V}{A_0} = \frac{p^2}{2E} \cdot \frac{I_0}{A_0 y_0^2} V.$$

With the notation of Art. 155 this gives

$$U = \frac{p^2}{E} \cdot \frac{n}{2q^2} \cdot V,$$

For the resilience we have only to change p into f, the proof strength. It thus appears that in beams of uniform strength with transverse sections of the same type the resilience is proportional to the volume, and less than that of a stretched or compressed bar, as might have been foreseen from general considerations. The ratio of reduction is $q^2 : n$, being $3 : 1$ in rectangular sections, $4 : 1$ in elliptic sections. When the beam is not of uniform strength the ratio of reduction must be greater for the same type of section. The reduction is of course least in I sections of uniform strength.

The function U is of great importance in the theory of continuous beams and other similar structures, the relative yielding of the several parts of the structure being always such that this function is less than it would be for any other distribution of stress and strain. It may be called the Elastic Potential, and when known all the equations necessary to determine the distribution of stress may be found by simple differentiation. (See Appendix.)

EXAMPLES.

1. If l be the length of an iron rod in feet, d its diameter in inches, just to carry its own weight with a deflection of 1 inch per 100 feet of span, show that

$$l = \sqrt{233d^2}.$$

Compare this result with that of Ex. 14, p. 324, and state what formula is to be used when both stiffness and strength are required.

2. Find the ratio of depth to span in a beam of rectangular section loaded in the middle, assuming stress $= 8{,}000$, $E = 28{,}000{,}000$, deflection $= \dfrac{\text{span}}{1200}$. *Ans.* $\dfrac{1}{17 \cdot 5}$.

3. A beam is supported at the ends and loaded at a point distant a, b from the supports with a weight W, show that the depression of the weight below the points of support is $\dfrac{Wa^2b^2}{3EI(a+b)}$.

4. In the last question deduce the work done in bending the beam, and verify the result by direct calculation. (See Art. 20.)

5. A dam is supported by a row of uprights which take the whole horizontal pressure of the water. The uprights may be regarded as fixed at their base at the bottom of the water, while their upper ends at the water level are retained in the vertical by suitable struts sloping at 45°, the intermediate part remaining unsupported. Find the bending moment at any point of the upright, and show that the thrust on the struts is about two sevenths the horizontal pressure of the water.

6. A timber balk 20 feet long of square section supports 160 square feet of a floor, find the dimensions that the deflection of the floor, when loaded with 60 lbs. per square foot, may not exceed $\frac{1}{2}$ inch.

7. A shaft carries a load equal to m times its weight (1) distributed uniformly, (2) concentrated in the middle. Considering it as a beam fixed at the ends, find the distance apart of bearings for a stiffness of $\dfrac{1}{1200^{\text{th}}}$. *Ans.* If l be the distance apart in feet, d diameter in inches, then for a wrought iron or steel shaft

$$(1)\ l = 10 \cdot 5 \sqrt[3]{\frac{d^2}{m+1}}; \qquad (2)\ l = 8 \cdot 3 \sqrt[3]{\frac{d^2}{m+\frac{1}{2}}}.$$

8. A beam originally curved, as in Ex. 21, p. 325, is fixed at one end and loaded in any way. If i be the change of slope at any point and X, Y the displacements parallel to axes of x, y of the point consequent on any load, prove that

$$\frac{di}{ds} = \frac{M}{EI}; \quad \frac{dX}{dy} = -i; \quad \frac{dY}{dx} = i.$$

Apply these formulæ to find the straining actions at any point of one of the rings of a chain of circular links.