CHAPTER XVII.

STRESS, STRAIN, AND ELASTICITY.

SECTION I.—STRESS.

199. *Ellipse of Stress.*—Stress consists, as we have said (Art. 147), in a mutual action between two parts, into which we imagine a body divided by an ideal section. If the section be plane, and if the stress be uniform, the intensity and direction of the stress at each point of the section are the same at all points of a given section, and, for a given point, depend only on the position of the plane. In a fluid the intensity is the same for all planes, and the direction is normal to the plane. In simple tension and compression the direction of the stress is the same for all planes, but its intensity varies, becoming zero for planes parallel to the stress. In shearing the intensity is the same for all planes perpendicular to a third given plane, but the direction varies: on one pair of planes it is normal, on another tangential.

We now proceed to consider stress more generally, and we shall first examine the effect of combining together a pair of simple longitudinal stresses, the directions of which are at right angles and the intensities of which are given. Let the plane of the paper be parallel to the directions of the stresses, and let us consider a piece of material of thickness unity. If the stress be uniform, the size and shape of the piece are immaterial. Let us then imagine a rectangular block $ABCD$ (Fig. 149) with sides perpendicular to the stresses p_1, p_2. On the faces AB, CD a stress, of intensity p_1, and of total amount $p_1 \,.\, AB$ will act; while on BC and AD there will be a stress of intensity p_2, and of total amount $p_2 \,.\, BC$. Divide now the rectangle by a diagonal plane AC; there will be a stress on that plane, which it is our object to determine in direction and magnitude. Let θ be the angle which the normal to the plane makes with the direction of p_1; by determining rightly the ratio of the sides of the rectangle this angle may be made what we please. Proceeding as in Art. 81, we find for the normal stress

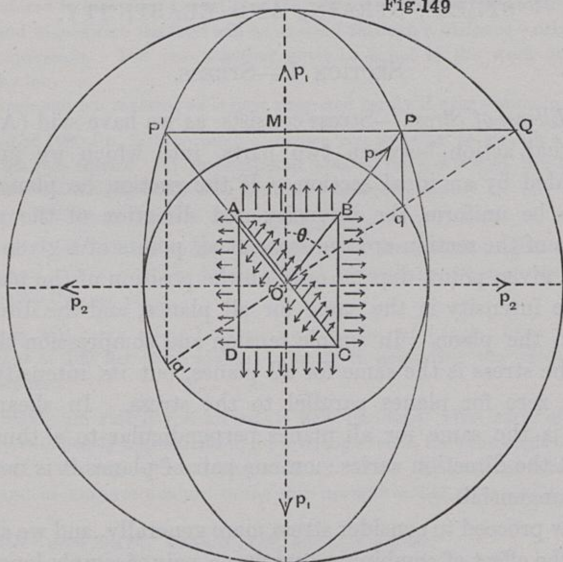
$$p_n = p_1 \cdot \cos^2\theta + p_2 \cdot \sin^2\theta,$$

and for the tangential stress

$$p_t = (p_1 - p_2) \sin\theta \cdot \cos\theta.$$

The resultant stress might be found in direction and magnitude by

Fig.149



compounding these results, but it is better to proceed by a graphical construction. On the perpendicular set off OQ to represent p_1 and Oq to represent p_2; also draw the ordinate QM and qP parallel to p_1 to meet it in P. Then

$$OM = OQ \cdot \cos\theta = p_1 \cdot \frac{AB}{AC};$$

$$PM = Oq \cdot \sin\theta = p_2 \cdot \frac{BC}{AC}.$$

Whence it follows that the intensity of the stress on AC due to p_1 is represented by OM, and that due to p_2 by PM. If then we join OP we shall obtain the resultant stress on AC in direction and magnitude. It is easily seen that P lies on an ellipse of which p_1, p_2 are the semi-axes. This ellipse is called the Ellipse of Stress.

If the pair of stresses p_1, p_2 have opposite signs, then $Oq = p_2$ must be set off on the opposite side of O, and OP' the radius vector of the ellipse lies on the other side of OM, but in other respects the construction is unaltered. When p_1, p_2 are equal the ellipse becomes a circle; if they have the same sign the stress is the same in all directions in magnitude and direction like fluid pressure; if they have opposite signs, as in the chapter on Torsion, the intensity is the same, but the angle of inclination $P'OQ$, called the "obliquity" of stress, is variable, being always equal to QON.

200. *Principal Stresses. Axes of Stress.*—We now propose to show that any state of stress in two dimensions (Art. 204) may always be reduced to a pair of simple stresses such as we have just considered.

For, drawing the same figure as in the last article, let us inquire the effect of replacing p_1, p_2 by other stresses of any magnitude and in any directions. Whatever they be, they evidently must have given tangential and normal components, of which, reasoning as in the last chapter, we know that the tangential must be equal and opposite.

Let the equal tangential components be p_t, and the normal components p_n and p'_n. Consider the equilibrium of the triangular portion ABC (Fig. 150), and let us determine under what conditions it is possible that the stress on AC should be a normal stress only, without any tangential component. Resolve parallel to BC; then, if p be that normal stress,

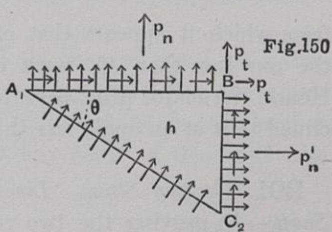


Fig.150

$$p \,.\, AC \,.\, \cos\theta = p_t \,.\, BC + p_n \,.\, AB\,;$$

or

$$p - p_n = p_t \,.\, \tan\theta.$$

Similarly resolving parallel to AB,

$$p - p'_n = p_t \,.\, \cot\theta,$$

whence, subtracting one equation from the other,

$$p_n - p'_n = p_t \,.\, (\cot\theta - \tan\theta) = 2p_t \,.\, \cot 2\theta\,;$$

or

$$\tan 2\theta = \frac{2p_t}{p_n - p'_n}.$$

This equation always gives two values of θ at right angles, showing that two planes at right angles can always be found on which the stress is wholly normal. The magnitude of the stress on these planes is found by multiplying the equations together, when we get the quadratic

$$(p - p_n)(p - p'_n) = p_t^2,$$

the roots of which, p_1, p_2, are the stresses required. Having determined p_1, p_2, the ellipse of stress can now be constructed by the method of the last article.

Every state of stress in two dimensions then can always be represented by an ellipse, the semi-axes of which are called Principal Stresses, and their directions the Axes of Stress.

The particular case in which p'_n is zero is one of constant occurrence in practical applications. If q be the shearing stress, the equations may then be written

$$p_n \tan 2\theta = 2q \quad (1)\,; \qquad p(p - p_n) = q^2 \qquad (2).$$

Of the roots of the quadratic the greater has the same sign as that of p_n, and the other the opposite. Also, we find by dividing the two equations for p by one another,

$$\tan^2\theta = \frac{p - p_n}{p} = \frac{q^2}{p^2},$$

from which it appears that of the two values of θ furnished by (1) the one less than 45° must correspond to the greater value of p. Hence, the major principal stress is of the same kind as p_n, and inclined to it at an angle less than 45°.

201. *Varying Stress. Lines of Stress. Bending and Twisting of a Shaft.*—In proving the two very important propositions just given, we have assumed (1) that the stress was uniform, throughout the region including the portion of matter we have been considering; (2) that gravity or any other force acting not on the bounding surface, but on each particle of the interior, may be neglected. It is however to be observed that by taking the portion of matter

small enough, both these suppositions may be made, in general, as nearly true as we please: the first, because any change of stress must be continuous, and therefore becomes smaller the less the distance between the points we consider; the second, because any internal force is proportional to the volume, while any force on the boundary of a piece of material is proportional to the surface of the piece. Now the volume of a body varies as the cube, and the surface as the square of its linear dimensions, and it follows that the internal force vanishes in comparison with the stress on the boundary when the dimensions diminish indefinitely. Hence these propositions are still true as respects the state of stress at any given point of a body, even though the stress be variable, and notwithstanding the action of gravity. When however we consider the variation of stress from point to point, gravity must be considered. Thus, for example, in the case of a fluid the action of gravity does not prevent the pressure from being the same in all directions, but it does cause the pressure to vary from point to point.

When the stress varies from point to point, both the intensity and the direction may vary; thus, for example, in a twisted shaft the intensity of the stress at any point varies as the distance from the axis, and the direction of the stress varies according to the position of the point, the principal stresses making an angle of 45° with the axis of the cylinder. The axes of stress in this case always touch certain lines which give, at each point they pass through, the direction of the stress at that point. These lines are called Lines of Stress; in a simple distorting stress, or, in other cases where the principal stresses are of opposite signs, one is a Line of Thrust, the other a Line of Tension.

In a twisted shaft of elastic material the lines of stress are spirals traced on a cylinder passing through the point considered, the spirals being inclined at 45° to the axis. If the shaft be bent as well as twisted, the maximum normal stress at any point of the transverse section is given by the equation

$$p_n = \frac{M}{\frac{1}{4}\pi r^3} \qquad \text{(Art. 155)},$$

where M is the bending moment and r the radius. The shearing stress at the external surface due to a twisting moment T is given by

$$q = \frac{T}{\frac{1}{2}\pi r^3} \qquad \text{(Art. 184)}.$$

Combining these two together we get, by solving the quadratic for the principal stresses,

$$p = \frac{M \pm \sqrt{M^2 + \tau^2}}{\frac{1}{2}\pi r^3},$$

which gives the principal stresses at that point of the shaft where the stress is greatest. The maximum stress is the same as would be given by a simple twisting moment equal to $M + \sqrt{M^2 + T^2}$, which is sometimes called the simple equivalent twisting moment. The minor principal stress ought, however, also to be considered in calculations respecting strength, as will be seen hereafter.

The lines of stress here are spirals of variable pitch angle.

202. *Straining Actions on the Web of an I Beam.*—Let us now return to the case of an I beam with a thin web, in which the web resists nearly the whole of the shearing force F, and the flanges nearly the whole of the bending moment M. The intensity of the shearing stress q is approximately

$$q = \frac{F}{ht},$$

where h is the depth and t the thickness. The intensity of the normal stress at a point distant y from the neutral axis is

$$p_n = \frac{M}{I} \cdot y.$$

The principal stresses and axes of stress are given by the equations

$$p(p - p_n) = q^2 \text{ ; } \tan 2\theta = \frac{2q}{p_n}.$$

From this it appears that, even when the web is very thin so that it carries a very small fraction of the total bending moment, it cannot be treated as resisting shearing alone, and if it is so treated will be the most severely strained part of the beam. Let us, for example, suppose the flanges to be subject to a stress of 4 tons per sq. inch at a given section, and the web to a shearing stress also of 4 tons per sq. inch : then at points in the web near the flanges, say, for example, at a distance from the centre, of three fourths the half depth of the beam, the normal stress will be 3 tons per sq. incn. Putting these values in the formula, we get the quadratic equation

$$p(p - 3) = 16 \text{ ;}$$

whence

$$p = 5{\cdot}77, \text{ or } - 2{\cdot}77,$$

a result which shows that the web is much more severely strained than the flanges. The lines of stress are found from the equation for θ. By a graphical method it is possible to draw the lines of stress approximately. As to this the reader is referred to a treatise by Mr. Chalmers, cited on page 82.

203. *Remarks on Stress in General.*—We have hitherto been considering only the stress on planes at right angles to a certain primary plane, to which we have supposed the stress on every plane to be parallel. In most practical questions relating to strength of materials this is sufficient, since, though stress frequently exists on the primary plane, it is usually normal and of relatively small intensity. Thus, for example, in a steam boiler there is stress on the internal and external surface of the boiler due to the pressure of the steam and the atmosphere; but it is of small amount compared to the stress on planes perpendicular to the surface. We therefore content ourselves with a statement without demonstration of corresponding propositions in three dimensions.

(1) Any state of stress at a point within a solid may always be reduced to three simple stresses on planes at right angles.

(2) The resultant stress on any plane due to the action of three simple stresses at right angles to each other is always represented in direction and magnitude by the radius vector of an ellipsoid.

The first of these propositions may be regarded as the last step in a process of analysis, by which we reduce all external forces acting on a structure of any kind: *first*, into a set of forces acting on each piece of the structure; and *second*, into forces acting on each of the small elements of which we may imagine that piece composed; and *lastly*, into three forces at right angles acting upon the element, of which one in practical cases is usually small. All questions in Strength of Materials, then, ultimately resolve themselves into a consideration of the effects of forces so applied.

One method of conceiving the effect of three such forces is to imagine each separated into two parts, one of which is the same for all, being the mean value of the three; while the other is compressive for one and tensile for the two others, or *vice versa*. In isotropic matter (Art. 207) the first set produces change of volume only, and may be called the "volume-stress," or, as no other stress can exist in fluid bodies at rest, a "fluid" stress. The second is a distorting stress, consisting of three simple distorting stresses tending to produce distortion in the three principal planes.

EXAMPLES.

1. A tube, 12 inches mean diameter and $\frac{1}{2}$ inch thick, is acted on by a thrust of 20 tons and a twisting moment of 25 foot-tons. Find the principal stresses and lines of stress.

Taking a small rectangular piece with one side in the transverse section, we find one face acted on by a normal stress of 1·06 tons per square inch due to the thrust, and a tangential stress of 2·66 tons due to the twisting. Substituting these values for p_m, p_i, and observing that the stress on the other face is wholly tangential, we find from the quadratic

Major principal stress = 3·24 (thrust);
Minor principal stress = 2·18 (tension).

Lines of stress are spirals, the lines of tension inclined at $50\frac{1}{2}°$ to the axis, and the lines of thrust at $39\frac{1}{2}°$.

2. A rivet is under the action of a shearing stress of 4 tons per square inch, and a tensile stress, due to the contraction of the rivet in its hole, of 3 tons per square inch. Find the principal stresses.

Ans. Major principal stress = 5·8 tons (tension).
Minor principal stress = 2·77 tons (thrust).

3. The thrust of a screw is 20 tons; the shaft is subject to a twisting moment of 100 foot-tons, and, in addition, to a bending moment of 25 foot-tons, due to the weight of the shaft and its inertia when the vessel pitches. Find the maximum stress and compare it with what it would have been if the twisting moment had acted alone. Shaft 14 inches diameter.

Ans. Major principal stress = 2·9, Ratio = 1·32.
Minor principal stress = 1·6.

4. A half-inch bolt, of dimensions given in Ex. 6, page 271, is screwed up to a tension of 1 ton per square inch of the gross sectional area. Assuming a co-efficient of friction of ·16, find the true maximum stress on the bolt while being screwed up. *Ans.* Principal stresses = 1·95 and ·3 tons.

5. It has been proposed to construct cylindrical boilers with seams placed diagonally instead of longitudinally and transversely. What is the object of this arrangement, and what is the theoretical gain of strength? *Ans.* Increase of strength = $26\frac{1}{2}$ per cent.

6. A thick hollow cylinder is under the action of tangential stress, applied uniformly all over its internal surface in directions perpendicular to its axis, the cylinder being prevented from turning by a similar stress, applied at the external surface. Find the principal stresses and lines of stress. *Ans.* The principal stresses are equal and opposite, forming a simple distorting stress, of intensity varying inversely as the square of the distance from the centre. Lines of stress equiangular spirals of angle 45°.

7. In Ex. 9, page 372, suppose the beam so loaded that the maximum stress due to bending is 4 tons per square inch, and the total shearing force divided by the sectional area of the web also 4 tons per square inch: find the principal stresses at points immediately below the flanges. *Ans.* Principal stresses $4\frac{1}{2}$ and 1·9 tons per square inch.

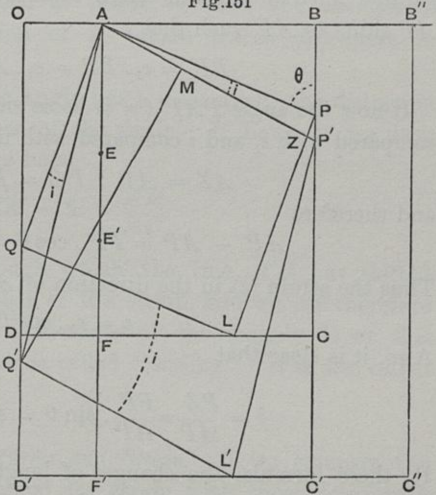
8. In any state of stress at a point in a body show that the sum of the normal stresses on three planes at right angles is the same however the planes be drawn.

Section II.—Strain.

204. *Simple Longitudinal Strain. Two Strains at Right Angles.*—We now go on to consider the changes of form and size which are produced by the action of stress. Such changes, it has already been said, are called Strains.

In uniform strain every set of particles lying in a straight line must still lie in a straight line, and two lines originally parallel must still be parallel. The lengths of all parallel lines are altered in a given ratio $1 + e : 1$, where e is a quantity, in practical cases very small, which measures the strain in the direction of the line considered. Two sets of parallel lines, however, will not in general remain at the same inclination to each other, nor will their lengths alter in the same ratio. Thus the sides of a cube remain plane, and opposite sides are parallel, but the parallelopiped is not generally rectangular, and its sides are not equal.

Fig.151



The simplest kind of strain is a simple longitudinal strain in which all lines parallel to a fixed plane in the body are unaltered in length, while all lines perpendicular to that plane remain so: that is to say, a simple change of length, the breadth, and thickness remaining unaltered.

Fig. 151 shows an extensible band $OBCD$, in which OB is fixed, while CD moves to $C'D'$, the breadth being in the first instance unaltered, and the length altered so that

$$CC' = e_1 . BC.$$

If any line AEF be traced in the band parallel to BC, the points EF will shift to $E'F'$ positions in the same line, such that

$$EE' = e_1 . AE : FF' = e_1 . AF.$$
$$E'F' = (1 + e_1)EF;$$

for since the strain is uniform the change of length of all parts of the band is the same. If, however, we draw a line QL inclined at an angle θ to BC, that line will shift to $Q'L'$, a position such that QL has not increased in so great a ratio, and is not inclined to BC at the same angle as before. We are about to determine the actual change of length and angular position of QL by finding that of a parallel AP drawn through A. It has been already remarked that parallel lines in uniform strain must suffer the same strain. Now AP shifts to AP' such that

$$PP' = e_1 . BP = e_1 . AP . \cos\theta.$$

If now the angle PAP' ($= i$) be so small that i^2 may be neglected compared with i, and i compared with unity,

$$AZ = AP : P'Z = PP' . \cos\theta;$$

and therefore

$$AP' - AP = PP' . \cos\theta = e_1 . AP . \cos^2\theta.$$

Thus the strain (e) in the direction of AP is

$$e = e_1 . \cos^2\theta.$$

Also, it is clear that

$$i = \frac{PZ}{AP} = \frac{PP'}{AP} . \sin\theta = e_1 . \sin\theta . \cos\theta.$$

By these formulæ the changes of length and angular position of all lines in the band are determined.

Next draw a line AQ perpendicular and equal to AP, and let AQ' be the position into which it moves in consequence of the strain; we find for e', the extension of AQ,

$$e' = e_1 . \sin^2\theta;$$

while the angle QAQ' is

$$i' = e_1 . \sin\theta . \cos\theta = i.$$

Imagine now the square AQL completed; this square, in con sequence of the strain, will have its sides altered in length by the quantities e, e', and will have suffered a distortion given by

$$2i = 2e_1 . \sin\theta . \cos\theta.$$

In this way the effect of a simple longitudinal strain is completely determined, for we can calculate the changes taking place in any portion of the band we please.

Next suppose the band to suffer a second simple longitudinal strain e_2 in the direction of the breadth, and observe that since the strains are very small, the effect of e_1, e_2 taken together must be the sum of those due to each taken separately; then we find for the change of length and position of any line AP,

$$e = e_1 . \cos^2\theta + e_2 . \sin^2\theta;$$
$$i = (e_1 - e_2)\sin\theta . \cos\theta,$$

results which may be applied as before to show the changes of dimension and the distortion of a square traced anywhere in the band.

We have here regarded the angle i as a measure of the distortion a square suffers in consequence of the strain. If, however, we drop $Q'M$ perpendicular to AP', we have

$$AQ'M = 2i = \frac{AM}{AQ'}.$$

Now AM is the space through which the line $A'Q'$ has shifted parallel to itself in consequence of the strain, and we see therefore that the angle i also gives a measure of the magnitude of this shifting. By some writers this is called "sliding." It is also called "shearing strain."

205. *Comparison between Stress and Strain.*—If we compare the equations we have just obtained for strain with those previously obtained in Art. 199 for stress, we find them identical; and hence it appears that, so long at least as the strains are very small, all propositions respecting stress must also be true, *mutatis mutandis*, with respect to strain. Thus, for example, a simple distortion must be equivalent to a longitudinal extension accompanied by an equal longitudinal contraction; and, again, every state of strain can be reduced to three simple longitudinal strains at right angles to each other, and represented by an ellipsoid of strain. The simple strains are called Principal Strains, and their directions Axes of Strain. Strain, like stress, generally varies from point to point of the body: but the relations here proved still hold good at each point, and we have Lines of Strain just as we previously had Lines of Stress.

SECTION III.—CONNECTION BETWEEN STRESS AND STRAIN.

206. *Equations connecting Stress and Strain in Isotropic Matter.*—So far we have merely been stating certain conditions which stress must satisfy in order that each element of a body may be in equilibrium, and certain other conditions which strain must satisfy if the body is continuous. We now connect the two by considering the way in which stress produces strain, which differs according to the nature of the material.

We first consider perfectly elastic material (see Art. 147), and suppose that material to have the same elastic properties in all directions, in which case it is said to be isotropic. Metallic bodies are often not isotropic, as will be seen hereafter (Ch. XVIII.). Suppose a rectangular bar under the action of a simple longitudinal stress p_1, then there results (Art. 148) a longitudinal strain e_1 given by

$$p_1 = Ee_1,$$

where E is the corresponding modulus of elasticity. Accompanying the longitudinal extension we find a contraction of breadth that is a lateral strain of opposite sign of magnitude $1/m^{\text{th}}$ the longitudinal strain, where m is a coefficient. The contraction in thickness will be equal, because the material is supposed isotropic. Hence the effect of the simple longitudinal stress p_1 is to produce three simple longitudinal strains at right angles,

$$e_1 = \frac{p_1}{E};\ e_2 = -\frac{p_1}{mE};\ e_3 = -\frac{p_1}{mE}.$$

Next remove p_1, and in its place suppose a simple stress p_2 applied in the direction of the breadth of the bar; we have by similar reasoning the three strains

$$e_1 = -\frac{p_2}{mE};\ e_2 = \frac{p_2}{E};\ e_3 = -\frac{p_2}{mE}.$$

And similarly removing p_2 and replacing it by p_3 acting in the direction of the thickness,

$$e_1 = -\frac{p_3}{mE}; \; e_2 = -\frac{p_3}{mE}; \; e = \frac{p_3}{E}.$$

These three sets of equations give the strains due to p_1, p_2, p_3, each acting alone; and we now conclude that if all three act together we must necessarily have

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE},$$

with two other symmetrical equations.

Hence it appears that the effect of three principal stresses, and consequently of any state of stress whatever on isotropic matter, is to produce a strain, the axes of which coincide with the axes of stress, and in which the principal strains are connected with the principal stresses by the equations just written down.*

207. *Elasticity of Form and Volume.*—The value of the constant m may be found directly by experiment, though with some difficulty, on account of the smallness of the lateral contraction which it measures; but it may also be found indirectly, by connecting it with the co-efficient employed in the last chapter to measure the elasticity of torsion. For if we subtract the second of the three equations just obtained from the first, we get

$$e_1 - e_2 = (p_1 - p_2)\frac{m+1}{mE},$$

or

$$p_1 - p_2 = \frac{m}{m+1} \cdot E(e_1 - e_2).$$

Now referring to Arts. 31, 33 we find

$$p_t = (p_1 - p_2)\sin\theta \cdot \cos\theta,$$
$$2i = 2(e_1 - e_2)\sin\theta \cdot \cos\theta,$$

where p_t is the tangential stress on a pair of planes inclined at angle θ to the axes, and $2i$ is the distortion of a square inclined at that angle to the axes of strain. Since now the axes of strain coincide with the axes of stress, we must have

$$\frac{p_t}{2i} = \frac{p_1 - p_2}{2(e_1 - e_2)} = \frac{1}{2}\frac{m}{m+1} \cdot E,$$

* The form in which these equations are given is due to Grashof. For practical application it is more convenient than any other.

an equation which, compared with Art. 183, shows that the co-efficient of rigidity C must be

$$C=\frac{1}{2}\frac{m}{m+1}\cdot E.$$

Experiment shows that in metallic bodies C is generally somewhat less than $\frac{2}{5}E$, whence it follows that m lies between 3 and 4. In the ordinary materials of construction the comparison cannot generally be made with exactness, because such bodies are rarely exactly isotropic. The value of m for iron is about $3\frac{1}{2}$.

Again, if we add together the three fundamental equations, we find

$$E(e_1+e_2+e_3)=\left(1-\frac{2}{m}\right)(p_1+p_2+p_3).$$

Now the volume of a cube, the side of which is unity, becomes when strained $(1+e_1)(1+e_2)(1+e_3)$, and therefore the volume strain is $e_1+e_2+e_3$ when the strains are very small. Hence, if we separate the stress into a fluid stress N and a distorting stress (Art. 204), we have

$$N=\frac{m}{3(m-2)}\cdot E\times\text{Volume Strain},$$

and the co-efficient

$$D=\frac{m}{3(m-2)}E$$

measures the elasticity of volume. The two constants C and D, which measure elasticity of distinctly different kinds, may be regarded as the fundamental elastic constants of an isotropic body. The ordinary Young's modulus E involves both kinds of elasticity.

208. *Modulus of Elasticity under various circumstances. Elasticity of Flexion.*—When the sides of a bar are free the ratio of the longitudinal stress to the longitudinal strain is the ordinary modulus of elasticity E; but the equations above given show that, when the sides of the bar are subject to stress, the modulus will have a different value. For example, let the bar be forcibly prevented from contracting, either in breadth or thickness, by the application of a suitable lateral tension, $p_2(=p_3)$, then e_2, e_3 are both zero, and

$$Ee_1=p_1-\frac{2p_2}{m};\ 0=p_2-\frac{p_1+p_2}{m},$$

whence we obtain for the magnitude of the necessary lateral stress

$$p_2 = \frac{p_1}{m-1},$$

and for the corresponding extension of the bar

$$Ee_1 = \frac{m^2 - m - 2}{m^2 - m} \cdot p_1.$$

Hence the modulus of elasticity is now

$$A = \frac{m(m-1)}{(m+1)(m-2)} \cdot E.$$

This constant A is what Rankine called the direct elasticity of the substance: it is of course always greater than E. For $m=4$, $A=\frac{6}{5}E$; for $m=3$, $A=\frac{3}{2}E$.

If the bar be free to contract in thickness, but not in breadth, we have p_3 and e_2 zero, and the equations become

$$Ee_1 = p_1 - \frac{p_2}{m}; \qquad 0 = p_2 - \frac{p_1}{m}; \qquad Ee_3 = 0 - \frac{p_1 + p_2}{m},$$

whence we find

$$Ee_1 = p_1 \cdot \frac{m^2 - 1}{m^2},$$

so that the value of the modulus of elasticity is $\frac{m^2}{m^2-1}E$. In a similar way if p_2, p_3 have any given values the modulus can be found.

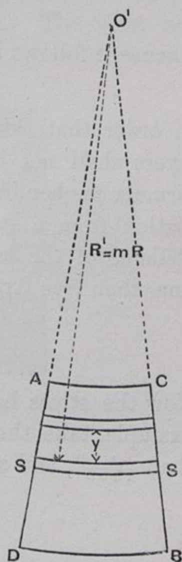


Fig.152

It will now be convenient to examine an important point already referred to in the theory of simple bending, that is to say the assumption (Art. 153) that the modulus of elasticity E was the same as in the case of simple tension, notwithstanding the lateral connection of the elementary bars, into which we imagined the whole beam split up. If these elementary bars were prevented from contracting freely, as they would do if separated from each other, the modulus could not be the same. In fact, however, there is nothing in their lateral connection which prevents them from doing so. Figure 152 shows, on a very exaggerated scale, the form assumed by a transverse section $ACBD$ originally rectangular, cutting a series of longitudinal sections originally parallel to the plane of bending in the straight lines shown. Assuming the upper side stretched as in Fig. 122, page 309, these lines all radiate from a centre O' above the beam, which bends transversely, while the originally straight horizontal layers are cut in arcs of circles struck from the same centre. The upper side of the beam contracts and the lower side expands, and reasoning exactly in the same way as we did when we derived the principal formula,

$$p=\frac{Ey}{R}, \qquad \text{(Art. 153)}$$

we find a corresponding formula for the transverse curvature,

$$p=m\frac{Ey}{R'},$$

whence it follows immediately that

$$R'=mR.$$

In order that this transverse curvature of the originally horizontal layers shall not be inconsistent with the reasoning by which the formula for bending is obtained, all that is necessary is that the deviation from a straight line shall be small as compared with the distance of the layer from the neutral axis. Let x be that deviation, then (see Art. 163) if b be the breadth,

$$x=\frac{b^2}{8R^1}=\frac{b^2}{8mR}=\frac{b^2 . p}{8mEy}.$$

Now the stress being within the elastic limit p/E is very small, for example, take the case of wrought iron, for which p/E is not more than $\frac{1}{1200}$th, and suppose $m=4$,

$$x=\frac{b^2}{38{,}400 . y_1},$$

where y_1 is the greatest value of y, say $\frac{1}{2}h$, where h is the depth, thus

$$x=\frac{b^2}{19{,}200h}.$$

It is obvious that x must be always very small compared with y, except very near the neutral axis, and unless b be very large compared with h. When then a beam is bent *within* the limit of elasticity, the lateral connection of the parts cannot have any sensible influence on its resistance to bending, unless its breadth be great as compared with its depth. The case of a broad thin plate has not been hitherto dealt with theoretically. Beyond the limit of elasticity the lateral connection of the parts may greatly increase the resistance to bending, but this is a matter for subsequent consideration.

209. *Thick Hollow Cylinder under Internal Pressure.*—The equations connecting stress and strain in combination with suitable equations expressing the continuity of the body and the equilibrium of each of its elements are theoretically sufficient to determine the distribution of stress within an elastic body exposed to given forces, and in particular to determine the parts of the body exposed to the greatest stress, and the magnitude of such stress. The most im-

Fig. 153a

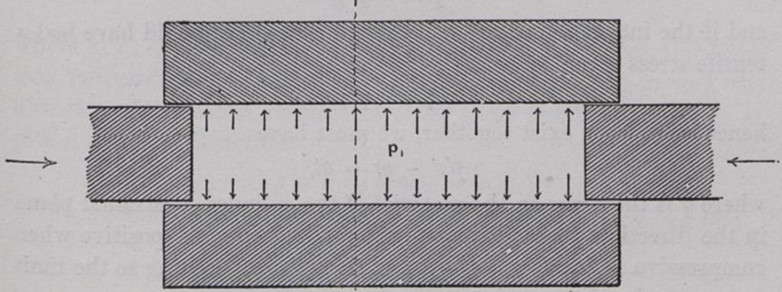
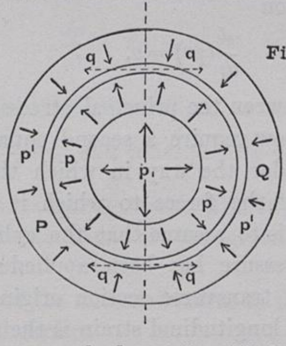


Fig. 153b



portant cases hitherto worked out, in addition to those considered in preceding chapters, are the torsion of non-circular prisms and the action of internal fluid pressure on thick hollow cylinders and spheres. For M. St. Venant's investigations on torsion we must refer to Art. 188, page 360, and the authorities there cited. We shall only consider the comparatively simple case of a homogenous cylinder.

Fig. 153*a* shows a longitudinal section of a hollow cylinder open at the ends, which are flat: the cylinder contains fluid which is acted on by two plungers forced in by external pressure so as to produce an internal fluid pressure p_1. Fig. 153*b* shows the same cylinder in transverse section: imagine a cylindrical layer of thickness t, this thin cylinder will be acted on within and without by stress which symmetry shows must be normal; let these stresses be p and p', and the internal and external radii of the thin cylinder be r and r'. Now, if p' the external pressure had existed alone, a compressive stress q would have been produced on the material of the cylinder given by the equation (see Art. 150)

$$p'r' = qt;$$

and if the internal pressure had existed alone, we should have had a tensile stress given by

$$pr = qt;$$

hence when both exist together, we must have

$$p'r' - pr = qt,$$

where q is the stress on the material of the cylinder on a radial plane in the direction perpendicular to the radius reckoned positive when compressive. Clearly $t = r' - r$, and therefore proceeding to the limit we may write the equation

$$\frac{d}{dr}(pr) = q,$$

which is one relation between the principal stresses p, q at any point of the cylinder. We now require a second equation, to get which it is necessary to consider the way in which the cylinder yields under the application of the forces to which it is exposed. The simplest way to do this is to assume that the cylinder remains still a cylinder after the pressure has been applied: if so, it at once follows that points in a transverse section originally remain so, or, in other words, that the longitudinal strain is the same at all points. It is not to be supposed that there is anything arbitrary about this assumption: no other, apparently, can be made if the ends of the cylinder are free, the pressure on the internal surface exactly uniform, and the cylinder be homogenous and free from initial strain. For when this is the case, there is no reason why the cylinder should be in a different condition in one part of its length than in another. If the ends are not free, or if the pressure is greater in the centre, the middle of the cylinder will bulge, but not otherwise.

It is also clear that the total pressure on a transverse section must be zero because the ends are free, and hence it is natural to suppose that it is also zero at every point of the transverse section, an assumption which we shall presently verify.

The equations connecting stress and strain therefore become

$$Ee_1 = p - \frac{q}{m};$$

$$Ee_2 = q - \frac{p}{m};$$

$$Ee_3 = 0 - \frac{p+q}{m},$$

where e_1, e_2, e_3 are the strains in, the direction of the radius, the direction perpendicular to the radius in the transverse section, and, the direction of the length, respectively. Of these the last is constant, as just stated, and therefore

$$p + q = \text{const.} = 2c_1$$

is the second equation connecting p, q. Substituting for q, we find

$$\frac{d}{dr}(pr) + p = 2c;$$

or

$$\frac{dp}{dr} + 2p = 2c_1.$$

Multiply by r and integrate, then

$$p = \frac{c_2}{r^2} + c_1, \text{ and consequently } q = c_1 - \frac{c_2}{r^2},$$

where c_2 is a constant of integration. The two constants c_1, c_2 are now determined by consideration of the given pressures within and without the cylinder.

If n be the ratio of the external radius to the internal radius R, we have at the internal surface

$$\left.\begin{matrix} p = p_1 \\ r = R \end{matrix}\right\} \quad \therefore p = c_1 + \frac{c_2}{R^2};$$

and at the external surface

$$\left.\begin{matrix} p = 0 \\ r = nR \end{matrix}\right\} \quad \therefore 0 = c_1 + \frac{c_2}{n^2R^2};$$

from which two equations we get

$$c_1 = -\frac{p_1}{n^2-1}, \text{ and } c_2 = \frac{p_1 n^2}{n^2-1} \cdot R^2.$$

Substituting these values in the equation for q,

$$q = -\frac{p_1}{n^2-1}\left\{1 + n^2 \cdot \frac{R^2}{r^2}\right\};$$

the negative sign in this formula indicates that the stress is tensile, as we might have anticipated. The formula shows that the stress decreases from $\frac{n^2+1}{n^2-1} \cdot p_1$ at the internal surface to $\frac{2p_1}{n^2-1}$ at the external surface. The mean stress is obtained from the equation (Art. 150.)

$$q_0(nR - R) = p_1 R;$$

hence the maximum stress is greater than the mean in the ratio $n^2 + 1 : n + 1$, and it is clear that it can never be less than p_1.

Verification of Preceding Solution.—The radial strain (e_1) and the hoop strain (e_2) are given by the above equations in terms of the stress. Now these changes of dimension are not independent, but are connected by a certain geometrical relation which it is necessary to examine in order to see whether it is satisfied by the values we have found.

Returning to the diagram, suppose the internal radius of the ring BQ to increase from r to s, and the external radius from r' to s'; then

$$2\pi s = 2\pi r(1 + e_2),$$

$$2\pi s' = 2\pi r'\left(1 + e_2 + t\frac{de_2}{dr}\right),$$

$$\therefore \quad s' - s = (r' - r)(1 + e_2) + r't\frac{de_2}{dr};$$

or since the thickness of the ring changes from t to $(1 + e_1)t$,

$$1 + e_1 = 1 + e_2 + r'\frac{de_2}{dr},$$

$$e_1 = \frac{d}{dr}(e_2 r).$$

This relation must always hold good, in order that the rings after strain may fit one another, and should therefore be satisfied by our results. On trial it will be found that it is satisfied, and we conclude that the solution we have obtained satisfies all the conditions of the problem, and is therefore the true and only solution, subject to the conditions already explained. For further remarks on this question, see Appendix.

210. *Strengthening of Cylinder by Rings. Effect of great Pressures.*—The stress within a thick hollow cylinder under internal fluid pressure may be equalized, and the cylinder thus strengthened by constructing it in rings, each shrunk on the next preceding in order of diameter. For a cylinder so constructed will be in tension at the outer surface and compression at the inner surface before the pressure is applied, and therefore after the pressure has been applied will be subjected to less tension at the inner and more tension at the outer surface than if it had been originally free from strain. It is theoretically possible to determine the diameters of the successive rings so that the pressure shall be uniform throughout. The principle is important, and frequently employed in the construction of heavy guns.

When the limit of elasticity is overpassed the formula fails, and the distribution of stress becomes different. If the pressure be imagined gradually to increase until the innermost layer of the cylinder begins to stretch beyond the limit, more of the pressure is transmitted into the interior of the cylinder, so that the stress becomes partially equalized. If the pressure increases still further, the tension of the innermost layer is little altered, and in soft materials longitudinal flow of the metal commences under the direct action of the fluid pressure. The internal diameter of the cylinder then increases perceptibly and permanently. This is well known to happen in the cylinders employed in the manufacture of lead piping, which are exposed to the severe pressure necessary to produce flow in the lead. The cylinder is not weakened but strengthened, having adapted itself to sustain the pressure. Cast-iron hydraulic press cylinders are often worked at the great pressure of 3 tons per sq. inch, a fact which may perhaps be explained by a similar equalization.

EXAMPLES.

1. When the sides of a bar are forcibly prevented from contracting, show that the necessary lateral stress is given by

$$p_2 = Be,$$

where $B = \dfrac{mE}{m^2 - m - 2}$. This constant B is what Rankine called the "lateral" elasticity of the substance.

2. With the notation of the preceding question and of Art. 106, prove that

$$C = \frac{A - B}{2}.$$

3. In a certain quality of steel $E = 30{,}000{,}000$; $C = 11{,}500{,}000$: find the elasticity of volume and the values of A and B, assuming the material to be isotropic. *Ans.* $m = 3\frac{2}{7}$; $D = 25{,}400{,}000$.

4. The cylinder of an hydraulic accumulator is 9 inches diameter. What thickness of metal would be required for a pressure of 700 lbs. per square inch, the maximum tensile stress being limited to 2,100 lbs. per square inch? Also, find the tensile stress on the metal of the cylinder at the outer surface. *Ans.* Thickness = 1·84″; Stress = 1,400 lbs. per square inch.

5. If the cylinder in the last question were of wrought iron, proof resistance to simple tension 21,000 lbs. per square inch, at what pressure would the limit of elasticity be overpassed? $m = 3{\cdot}5$. (See Art. 223, page 428.) *Ans.* 6400.

6. Find the law of variation of the stress within a thick hollow sphere under internal fluid pressure. By a process exactly like that for the case of the cylinder (page 404) it is found that the equation of equilibrium is

$$\frac{d}{dr}(pr^2) = 2qr.$$

The equation of continuity is the same as that for a cylinder (Art. 209), and the equations connecting stress and strain are now

$$Ee_1 = p - \frac{2q}{m};$$

$$Ee_2 = q - \frac{p+q}{m}.$$

We can now by elimination of q, reduction, and integration obtain

$$p = c_1 + \frac{c_2}{r^3};$$

$$q = c_1 - \frac{c_2}{2r^3};$$

the constants being found as in the cylinder.

7. The cylinder of an hydraulic press is 8 inches internal and 16 inches external diameter. If the pressure be 3 tons per sq. inch find the principal stresses at the internal and external circumference.

Ans. At inner circumference $\begin{cases}\text{Major Stress} = 5 \text{ (Tension)}.\\ \text{Minor Stress} = 3 \text{ (Thrust)}.\end{cases}$

At outer ,, $\begin{cases}\text{Major Stress} = 2 \text{ (Tension)}.\\ \text{Minor Stress} = 0.\end{cases}$

8. In the last question find the "equivalent simple tensile stress" (p. 428), assuming $m = 3{\cdot}5$. *Ans.* 5·86 and 2 tons.