APPENDIX.

A. NOTES AND ADDENDA.

I.—STATICS OF STRUCTURES.

RANKINE's treatise on *Applied Mechanics* appeared in 1858. The sixth edition is quoted in the following notes by the letters A.M.

PAGE 2. "The word STRESS has been adopted as a general term to comprehend various forces which are exerted between contiguous bodies or parts of bodies, and which are distributed over the surface of contact" (A.M., p. 68). It appears from this that RANKINE's use of the word is confined to internal forces, but by some writers it is employed for all forces whether external or internal. Ties and struts are, however, defined as in the text (A.M., p. 132).

PAGE 3. The total load on the platform of a timber bridge carrying an ordinary roadway may be assumed as 250 lbs. per sq. ft., of which 120 represents the weight of a closely packed crowd, and the remainder is the weight of the roadway and platform. The weight of a timber roof (slate or tile) is from 12 to 24 lbs. per sq. ft. The travelling load on railway bridges is commonly estimated at 1 ton per foot-run.

PAGE 15. The diagram of forces for a funicular polygon under a vertical load was (probably) first given by ROBISON in his treatise on *Mechanical Philosophy*, Vol. I. Dr. Robison died in 1805, and this work is a collection of his papers published in 1822.

PAGE 22. In the Saltash bridge the compression member of each girder is a tube of elliptical section 15 feet in breadth, 8 feet in depth. A pair of chains, one on each side, carry the platform.

PAGE 23. A. Of the various methods of constructing a parabola the most convenient is that in which a curve is drawn through the intersections of a set of lines radiating from a point, with a set of equidistant lines drawn parallel to a fixed line: the radiating lines being drawn so as to cut off equal intercepts on another fixed line. It can easily be proved that this curve is the funicular polygon proper to a uniform load without introducing any properties of the parabola.

PAGE 23. B. Let P be the vertical tension of the chain at the point P, then, since $dy/dx=P/H$, where H is constant,

$$\frac{d^2y}{dx^2}=\frac{1}{H}\cdot\frac{dP}{dx}=\frac{w}{H}.$$

This equation is equally true if w vary according to any law, and is therefore the general differential equation of a cord or linear arch under any vertical load. Particular cases are:—

(1) *The Common Catenary.* Here if m be the weight of a unit of length of the cord, ds an element of arc,

$$w=m\,.\,\frac{ds}{dx}=m\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

the equation then becomes, if $H=m\,.\,c$,

$$\frac{d^2y}{dx^2}=\frac{1}{c}\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Divide by the right-hand member, multiply by dy/dx, and integrate, then

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{y}{c}\text{ or }\frac{d^2y}{dx^2}=\frac{y}{c^2}$$

an equation which, by integration and a proper determination of the constants, gives for the form of the curve

$$y=\frac{c}{2}\left(\epsilon^{\frac{x}{c}}-\epsilon^{-\frac{x}{c}}\right).$$

(2) *The Catenary of Uniform Strength.* Here, if T be the tension of the chain at P,

$$T=m\lambda=w\lambda\,\frac{dx}{ds},$$

where λ is the length equivalent to the stress (p. 90),

$$\therefore\frac{d^2y}{dx^2}=\frac{T}{H\lambda}\cdot\frac{ds}{dx}=\frac{1}{\lambda}\cdot\left(\frac{ds}{dx}\right)^2.$$

Integrating by the same process as before we find

$$y=\lambda\log_{\epsilon}\sec\frac{x}{\lambda}$$

as the equation to the curve.

In ordinary cases there is very little difference between the catenary and the parabola, and these curves therefore are not of much interest.

If the form of an arch be not such as corresponds to the distribution of the load on it, a horizontal force will be necessary for equilibrium, and the investigation of the magnitude of this force is a problem of wider application. Let p be the intensity of this force per unit of length of a vertical ordinate, then H is no longer constant, but is given by

$$\frac{dH}{dy} = p, \text{ also } \frac{dP}{dx} = w \text{ and } \frac{dy}{dx} = \frac{P}{H},$$

three equations from which p can be found for any distribution of load and form of arch. This is the general problem of the linear arch. For examples see A. M., p. 199. If p = const., w = const., we obtain the ellipse as the proper form of arch to sustain the pressure of a great depth of earth.

PAGE 33. On reciprocal diagrams of forces in general the reader is referred to a memoir by CLERK MAXWELL in the Transactions of the Royal Society of Edinburgh for 1870.

The notation used in the text was suggested by HENRICI in the course of a discussion on a paper by CROFTON read before the Mathematical Society in 1871. The figure in the text was drawn at the time by the writer to illustrate the method. The notation was afterwards given by Bow in the treatise referred to.

PAGE 37. It is convenient to have a general term for the tendency to separate into parts due to the action of external forces on a structure or part of a structure. The term "straining action" used in the text is taken from Ch. II., Part III., of a treatise on *Shipbuilding* (London, 1866), edited and in great part written by RANKINE. By some writers this tendency to separate would be called "stress," and for a simple thrust or pull there is no objection to doing so (A.M., p. 132). In more complex cases a separate word is preferable, as the conception is very different. (Comp. p. 298.)

PAGE 50. In some of his engines, before the introduction of cast iron, WATT employed a timber beam trussed with iron rods, forming a Warren girder in two divisions with diagonals inclined at about 30° to the horizontal. This is perhaps the earliest example of such a construction. (ROBISON, Vol. II., p. 14.)

PAGE 59. See Plate VIII., p. 441.

PAGE 62. The method here detailed is given by RANKINE in his work on Civil Engineering (p. 242), who ascribes it to LATHAM. If M be the bending moment, F the shearing force, w the load per foot-run, we have the equations

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = w,$$

which are the symbolical expression of the method. They may be used to find by integration the bending moment and shearing force at any section due to a given load, the constants of integration being found by considering that the bending moment is zero at two points, which must be known if the problem is determinate. (See Art. 38, p. 85.)

PAGE 70. See Ch. II., Part III. of the work on *Shipbuilding*, cited above.

PAGE 80. A. The properties of funicular polygons were first thoroughly investigated by CULMANN, who based upon them a complete system of graphical calculation. In the semi-graphical methods employed in this treatise the integral calculus, trigonometry, and even, to a great extent, algebra, are replaced by geometrical constructions, but arithmetic is still used, and certain steps of the various processes are conducted by numerical calculations. For example, in Ch. II., the supporting forces of a loaded beam are found by the ordinary process of taking moments. In the modern purely graphical methods every step is taken graphically, whatever the calculation be. For example, the displacement of a vessel at a given draught, or her stability at a given angle at heel, would be found without the use of arithmetic.

The pressure of other matter, and the amount of illustration required, have prevented the writer from giving any account of these methods in this treatise. At present they can hardly be considered suitable for an elementary work, though, if graphical calculations were introduced into our schools, the case might be different. A full account of them will be found in CHALMERS's treatise referred to in the text (p. 82). Readers of German or Italian will find the principles of "graphical arithmetic" clearly explained in a small compass in a treatise the German translation of which is entitled *Elemente des Graphischen Calculs*, von Dr. L. CREMONA. Leipzig. 1875.

PAGE 80. B. The property of the funicular polygon expressed by the equation $Hy=M$ follows immediately by comparing the equations

$$\frac{d^2y}{dx}=\frac{w}{H}\,;\;\frac{d^2M}{dx^2}=w,$$

of which one gives the form of the polygon for a given load, and the other the bending moment due to the same load.

Another fundamental property is that any two sides of the polygon must meet on the line of action of the load on that part of the polygon which lies between the two sides. When the load is vertical and represented by a curve, as in Fig. 36*a*, p. 68, this is equivalent to saying that any two tangents to the curve of moments must intersect on the vertical through the centre of gravity of the area of the curve of loads between the corresponding ordinates. (See p. 334.)

The funicular polygon, considered as a line of transmission of stress, will be again referred to in the notes to Ch. XVII.

PAGE 81. In Fig. 40 the line r should be perpendicular to the force R.

PAGE 87. A full account of the Forth Bridge will be found in a paper by BAKER, published, since the text was in type, in the Report of the British Association for 1882.

PAGE 88. The theory of *linear* arches is merely an introduction to the theory of arches in general. Arches are of two kinds—(1) the stone or brick arch, (2) the metallic arch. In either case the theorem of the text is of equal importance. In a blockwork arch the linear arch corresponding to the load shows the direction and position of the resultant of the mutual action between the blocks, and must therefore (pp. 341, 343) fall within the middle third of the arch ring. (A.M., p. 258.)

PAGE 91. See CLERK MAXWELL's memoir referred to above (p. 559).

PAGE 96. For the effects of changes of temperature, see p. 566.

PAGE 98. One of the most remarkable suspension bridges which have been constructed is the East River Bridge at New York, opened in May, 1883. The principal opening of this bridge is 1600 feet span, the platform 85 feet wide, and 135 feet above the water. Cables, four in number, each of 145 square inches *net* area, constructed of 19 steel wire ropes, each containing 278 wires. Estimated strength of wire, 170,000 lbs. per square inch.

II.—KINEMATICS OF MACHINES.

PAGE 102. Referring to Figs. 1, 2, Plate II., p. 121, it seems clear that the sector pair *BD*, Fig. 1, differs kinematically much more from the turning pair *BA* than it does from the sliding pair *BD* of Fig. 2. The writer, therefore, would have been disposed to classify the three lower pairs as the "oscillating pair," the "turning pair," and the "screw pair." This, however, would have probably involved more considerable alterations in REULEAUX's nomenclature than would have been justified in a general elementary treatise.

PAGE 103. The three incomplete lower pairs are considered by REULEAUX as higher pairs. The writer here follows GRASHOF (Theoretische Maschinen-Lehre, Band II.).

PAGE 108. Diagrams of velocity are considered generally by CLERK MAXWELL (*Matter and Motion*, p. 28). The application to mechanism is, so far as the writer is aware, new.

PAGE 109. The construction of curves of position and velocity of a piston has, for many years past, formed a regular part of the course of instruction at Greenwich, and formerly at South Kensington.

PAGE 114. In Owen's air compressor two such mechanisms (Fig. 50) are placed face to face with the guide *A* and block *D* common, a steam piston is connected with *d* and the air-pump piston with the corresponding point *d* of the other mechanism. The result is that a uniform pressure of steam compresses the air.

PAGE 118. Stannah's pump has been introduced since the publication of REULEAUX's work. The example there given is a mechanism used in the polishing of specula.

PAGE 121. The double-slider mechanism, with sliding pairs and turning pairs alternating, is common in collections of mechanisms, but is not often found in practice. It is omitted in REULEAUX's enumeration. The example given (Rapson's slide) and Stannah's pump, were pointed out to the writer by Mr. Hearson.

PAGE 175. The propositions relating to centroids have long been known and are, perhaps, stated as clearly by BELANGER in his excellent treatise on kinematics (*Traité de Kinématique*, Paris, 1864) as by REULEAUX himself. In the author's opinion it is the conception of a kinematic chain which constitutes REULEAUX's great contribution to the theory of mechanism. It is virtually a complete reconstruction of the whole theory of machines, while the centroids are only a method of stating results which was already known. The kinematic formulae employed by REULEAUX to indicate the component elements of a mechanism, in the same manner as a chemical formula shows the composition of a substance, may be regarded as indispensable, if it be attempted to proceed with the study of descriptive mechanism.

PAGE 185. The author has ventured on the introduction of the terms "driving pair," "working pair." They are simply the natural adaptation of the well-known phrases "driving point" and "working point" to REULEAUX's theory.

PAGE 189. The term "multiple chains" has also been introduced by the author.

III.—DYNAMICS OF MACHINES.

The impossibility of a perpetual motion and the practical application of the principle of work were well understood by SMEATON and others of our great engineers of the last century. Smeaton's papers, read before the Royal Society in 1759–82, were long regarded as an engineering text-book by his successors. The language in which their ideas are expressed, however, was not regarded as consistent with NEWTON's teaching, and this circumstance perhaps concealed the real importance of the ideas themselves. At any rate, although the term "energy" was proposed by YOUNG, no considerable use was made of them by students of mechanical science until the publication by PONCELET, in 1829, of the *Introduction à la Mécanique Industrielle*, a work which has had a great influence on the study of mechanics. The third edition of this work (Paris, 1870) published after PONCELET's death, will be quoted by the abbreviation *Méc. Ind.* Poncelet's methods were explained, and considerable additions made to the theory of machines, by MOSELEY in his *Mechanical Principles of Engineering* (London, 1843).

PAGE 195. This method was probably employed for the first time by WATT in his expansion diagram. See ROBISON'S *Mechanical Philosophy*, Vol. II. It is given by PONCELET (*Méc. Ind.*, p. 66).

PAGE 198. The terms "statical" stability, "dynamical" stability, in relaion to vessels were introduced by MOSELEY (*Phil. Trans.*, 1850) They have recently been criticised by REYNOLDS, perhaps not without justice, but are too firmly rooted to be displaced.

PAGE 199. "Force is an action between two bodies, either causing or tending to cause, change in their relative rest or motion." (A. M., p. 15.) The distinction between internal work and external work is due to PONCELET (*Méc. Ind.*, p. 30).

PAGE 200. The language in which writers on mechanics have expressed the distinctive character of frictional resistances has been severely criticized by REULEAUX in the notes to his work on the Kinematics of Machines (*Kennedy's translation*, p. 595). The author by no means supposes that he can escape this universal censure, for the difficulty of expressing abstract principles in a form to which no objection can be made is almost insuperable. As, however, CLERK MAXWELL remarks in reference to a different question, the language in which a truth may be expressed is less important than the truth itself. Friction always causes energy to disappear, and is never a source of mechanical energy except indirectly through the agency of thermal energy. In mechanics this is a distinction of fundamental importance and justifies, in the author's opinion, the use of such phrases as "loss of energy."

The extension of the term "reversible" from a machine to the resistances which are overcome by the machine has been ventured on, though with some hesitation. The old term "active" can hardly be considered suitable.

PAGE 201. "Envisagé sous ce point de vue, le principe de la transmission du travail comprend implicitement toutes les lois de l'action réciproque des forces, sous un énoncé qui en facilite infinitement les applications á la Méca. nique industrielle, qu'on pourrait nommer la *Science du travail* des forces. Dès les premier pas des jeunes éléves dans l'étude, cet énonce, en effet, se présente á eux comme une sorte d'axiome évident par lui-même, et donc la demonstration leur semble superflue aussitôt qu'ils ont bien saisi ce qu'on entend par *travail mécanique*, et qu'il leur est clairement démontre que ce travail, reduit en unités d'une certaine espèce, est dans les arts, l'expression vraie de l'activité des forces" (*Méc. Ind.*, p. 3). This passage from PONCELET is quoted to show how clearly it was seen, even before the discovery of the conservation of energy in its complete form, that the principle of work ought to be regarded as fundamental, and not merely as a deduction from certain equations.

PAGE 202. The modifications made here in the old statement of the principle of work, as applied to machines, are necessary consequences of REULEAUX' conception of a kinematic chain.

PAGE 207. The value 33,000 ft.-lbs. per minute was derived by BOULTON and WATT from experiments on the work done by the powerful dray horses employed in London breweries.

PAGE 210. To avoid misapprehension, it may be here stated that in this, as much as in the preceding section, the object is to explain and to verify, from results given in elementary treatises on dynamics, the principle of work: not in any sense to demonstrate it.

PAGE 211. Except in the use of the work "kinetic" instead of "actual," the statement here is in the form given by RANKINE (A. M., p. 500). The author is entirely of Mr. W. R. Browne's opinion that this is the best form, and has always used it himself. The idea of energy being stored in a body in motion perhaps first appears clearly in MOSELEY'S treatise.

PAGE 223. The construction by means of which curves of crank effort are drawn was given by PONCELET, but it does not appear that any such curves were actually drawn until they were given by ARMENGAUD in his treatise on the Steam Engine. In Fig. 97, to save room, the curves are placed half above and half below the base, but otherwise the figure is that of ARMENGAUD; it is far the most convenient form for applications.

PAGE 231. The stress due to centrifugal action on the rim of a wheel is given by a formula (p. 294) which may be written in the simple form $V^2 = g\lambda$, where λ is the length due to the stress (p. 90). A velocity of 80 ft. per second gives a length of only 200 feet, or about one-fifth of the stress cast iron would safely bear in tension. The inequality of distribution produced by inextensible arms tying together opposite points on the rim of the wheel probably increases the maximum stress about 50 per cent.; but the principal reason for the low limit required for safety is the alternate bending backwards and forwards of the arms as energy is alternately stored and restored by the wheel. The speed is occasionally increased to 100 feet per second. The author is indebted to Prof. Unwin for the information that when the wheel is in segments the speed should be limited to 40 feet per second.

PAGE 233. The method here given occurred to the author many years back; but it has (we believe) been published in *Engineering*.

PAGE 233. On the effect of inertia of reciprocating parts in high speed engines the reader is referred to *Ueber Dampfmaschinen mit hoher Kolbengeschwindigkeit*, von J. F. RADINGER. Wien, 1872.

PAGE 244. In the original Brotherhood engine (*Vienna Exhibition*, 1873) the steam was admitted to the central chamber and exhausted at the outer ends of the cylinders: but in recent examples the central chamber communicates with the exhaust.

PAGE 248. The Friction Circle was defined and its use explained by RANKINE in his treatise on *Millwork and Machinery*, p. 428.

PAGE 251. The formulae for the maximum efficiency of a screw and its nut have been shown by FROUDE to be applicable very approximately to the case of an element of the blade of a screw propeller.

PAGES 258-9. Since the text was printed, a *Report of the Committee on Friction* of the Institution of Mechanical Engineers has been published. (See *Engineering*, Nov. 16th, 1883.) From these experiments it appears that the mode of lubrication has a great influence on the results, and that when the lubrication is perfect the moment of friction is nearly constant whatever the pressure, as in the surface friction of a fluid. The friction here, no doubt, is chiefly due to the viscosity of the lubricant. The co-efficient was sometimes as low as 1/1000th.

In the discussion of this report (*Engineering*, Feb. 1st, 1884) the reporter (Mr. Tower) stated that in locomotive axles 300 lbs. per sq. inch was the limiting pressure, whereas in the crank pins the limit was as high as 1000 lbs. per sq. inch, the difference being that the load on the pins is alternating whereas that on the axles is constant.

PAGE 269. In screw engines the value of e, inclusive of propeller shafting and screw, may probably be as much as ·25.

PAGE 278. The distinction between internal and external kinetic energy is pointed out by RANKINE (A.M., p. 508).

PAGE 287. On Governors in general the reader is referred to a paper by CLERK MAXWELL in the *Proceedings of the Royal Society*, No. 100, 1868. A full account of the principles of construction of centrifugal regulators will be found in *Theoretische Maschinenlehre*, Band III. Leipzig, 1879, von Dr. F. GRASHOF.

PAGE 290. The following description of Plate VII. should have been inserted here :—

The arrangement of dynamometer shown in Fig. 2 provides a self-acting adjustment of the tension of the strap to compensate to some extent for variations in the co-efficient of friction due to want of uniformity in the lubrication, for if the friction between the strap and wheel is temporarily increased, then instead of the wheel being arrested, the weight is raised a little, and the tension of the spring and strap relaxed, and *vice versa*. A more complete design of a dynamometer possessing the same useful property is shown in Fig. 4. In this a strap made in two portions, and to which blocks of wood are secured, completely encircles the fly-wheel. At one point the two portions are joined by a right and left-handed screw A, which provides a ready means of tightening or loosening the strap to exactly balance the driving couple. The other ends of the two portions of the strap are jointed at B and C to a lever (which in the actual construction is in duplicate), moveable about a pin at D.

The dynamometer is loaded at E to balance the moment of friction and the driving couple. If through deficiency of lubrication the wheel should tend to carry the strap around with it, in the direction of the arrow, then since DC is greater than DB, this would effect a loosening of the strap and a diminution of the friction, whereas if the friction is momentarily not sufficient to sustain the weight, it will in falling lighten the strap, and thus automatically the total moment of friction will be maintained fairly constant.

Page 292. The utility of balance weights, sufficiently heavy to neutralize completely the horizontal forces, is by no means universally admitted. The vertical forces introduced are very great (Ex. 17, p. 297), and, should they synchronize (p. 383) with the period of vertical oscillation of the engine on its springs, most dangerous results might follow.

Page 294. The stress due to rotation is best expressed by the formula $V^2 = g\lambda$—given above.

IV.—STIFFNESS AND STRENGTH.

Page 303. If the temperature of a bar be raised $t°$, the corresponding change of linear dimensions (strain) is given by the formula

$$e = \frac{t}{K},$$

where K is roughly approximately constant for the same material.

If a change of length be forcibly prevented during the change of temperature, the stress

$$p = \frac{E}{K}t$$

will be produced. The change of temperature corresponding to 1 ton per square inch is K/E. This quantity in degrees Fahrenheit and the value of K are given for various materials in the annexed table :—

Material.	Value of K.	Value of K/E.
Wrought Iron,	147,500	11°.3
Cast Iron, . .	162,000	20°
Copper, . . .	104,500	16°
Brass, . . .	95,400	25°

PAGE 338. If an elastic solid or, more generally, a set of connected pieces of perfectly elastic material, be under the action of any number of forces P_1, P_2, ..., and any number of couples M_1, M_2, ..., in equilibrium, the value of U must be

$$U=\tfrac{1}{2}\Sigma Px+\tfrac{1}{2}\Sigma Mi,$$

where x_1, x_2, ..., are the displacements of the points of application of the forces, and i_1, i_2, ..., the angular displacements of the arms of the couples. For if the forces gradually increase from zero, always remaining distributed in the same way, each part of the load (P) will exert the energy $\frac{1}{2}Px$, since the space moved through (x) must clearly be proportional to P. The same argument applies *mutatis mutandis* to couples. Hence the whole energy exerted must be given by the above formula, and this is always represented by the energy stored up in the system when the parts are perfectly elastic.

Now, imagine the solid immoveably fixed at three or more points, and let one of the forces P_1 be increased by a small quantity δP_1, all the other forces retaining their original magnitudes. The effect of this is that the points of application of all the forces move through certain small spaces (δx), and the arms of all the couples through certain small angles (δi). The total additional work done will be

$$\delta U=\Sigma \mathrm{P}\delta x+\Sigma M\delta i.$$

But, on differentiating the value of U on the supposition that P_1 alone varies, we find

$$2\,.\,\delta U=\Sigma P\delta x+\Sigma M\delta i+x_1\,.\,\delta P_1,$$

and therefore by substitution

$$\delta U=x_1\,.\,\delta P_1.$$

A similar equation is derived by supposing one of the couples to vary, and we obtain the general equations

$$\frac{dU}{dP}=x\,;\ \frac{dU}{dM}=i,$$

that is, the displacements are the partial differential co-efficients of U with respect to the forces.

The forces to be considered are partly weights or other loads of known magnitude, and partly arise from the stress between the bounding surfaces (real or ideal) of the solid and external bodies. The boundary forces must be consistent with statical equilibrium, but subject to this condition are determined by equations found by differentiating the function U. In particular, when the bounding surface is fixed, the partial differential co-efficients of U with respect to the corresponding forces must be zero. The value of U is then in most cases (perhaps always) a minimum, as stated in the text.

It appears then that whenever the elastic potential can be found and expressed in terms of the external and boundary forces acting on the system, the necessary equations for determining the boundary forces and the deflection produced by the external forces can all be found by differentiation of U and by the conditions of statical equilibrium. As an example, take the case of a beam loaded in any way and fixed at the ends. Let the beam be AB (Fig. 28), and let the notation be as on pages 45, 46, then (p. 338)

$$U = \int \frac{M^2}{2EI} dx.$$

Substitute for M by the formula on page 46, and integrate between the limits l and o, we find

$$2EI \,.\, U = \tfrac{1}{3}(M_A{}^2 + M_A M_B + M_B{}^2)l + \int_o^l m^2 dx + \frac{2M_A}{l^2} \,.\, \int_o^l m(l-x)dx$$
$$+ \frac{2M_B}{l^2} \,.\, \int_o^l mxdx.$$

The integrals are most conveniently expressed in terms of, S the area of the curve of moments (m), z the distance of its centre of gravity from A, and, k its radius of gyration about AB. The formula then becomes, dividing by 2,

$$EI \,.\, U = \tfrac{1}{6}(M_A{}^2 + M_A M_B + M_B{}^2)\,l + S\frac{M_A(l-z) + M_B z}{l^2} + \tfrac{3}{2}Sk^2.$$

The potential is thus expressed in terms of the load on the beam and the bending moments at its ends. The latter may have any values we please consistently with statical equilibrium, and the partial differential co-efficients of U with respect to $M_A M_B$ will be the slopes at the ends. In particular, if the ends are fixed horizontally,

$$2M_A + M_B + 6\,\frac{l-z}{l^2} \,.\, S = 0,$$

$$2M_B + M_A + 6\,\frac{z}{l^2} \,.\, S = 0,$$

equations which determine M_A M_B, and express that the function U is then a minimum. In the particular case of a symmetrical load

$$M_A = M_B = -\frac{S}{l}.$$

The value given on page 332 for the particular case of a uniform load will be found to agree with this result.

The potential for a continuous beam may be immediately deduced, by addition of the potentials for each span taken separately, in terms of the bending moments at the points of support. The theorem of three moments (p. 337) for the case of supports on the same level, then follows at once by differentiating with respect to the moment at the middle point of support.

In all cases, differentiation of U with respect to any portion of the external load will give the deflection at the point where that load is applied.

In applying this method care must be taken that the supporting forces, in terms of which the potential is expressed, are independent: if they are not, then the equations of statical equilibrium will be conditions subject to which U will be a minimum. To take a simple example, suppose a perfectly rigid four-legged table standing on four similar elastic supports and loaded in any way, then

$$U = n\,(P_1{}^2 + P_2{}^2 + P_3{}^2 + P_4{}^2),$$

where P_1, P_2, P_3, P_4, are the part of the whole load resting on each leg, and n is some multiplier. Here the forces P are partly determined by three statical equations for equilibrium of the table, and only one additional equation is found by making U a minimum.

This method was explained and applied to a number of examples in some papers by the author which appeared in the *Philosophical Magazine* for 1865; the demonstrations there given, however, were partly erroneous and partly insufficient. The author at that time supposed it to be new, but it had already been given in a memoir by M. E. F. MÉNABRÉA. *Comptes Rendus*, vol. xlvi. (1858), page 1056.

PAGE 345. The lateral disturbance is here supposed small. With a larger disturbance the pillar would return even if the value of W were equal to $2EI/l^2$, and with a greater value would bend over into a position of equilibrium given by the formula

$$W = \left(\frac{\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)^2 . \frac{2EI}{l^2},$$

where θ is the angle subtended by the circular arc into which the pillar is bent.

PAGE 346. Referring to Fig. 133, p. 341, let O be the origin of rectangular co-ordinates, ON axis of x, and y the horizontal ordinate of any point in the curve OA. Then the bending moment at that point is, very approximately,

$$M = W(a + \delta - y).$$

The differential equation of bending is therefore (p. 328)

$$\frac{d^2y}{dx^2} = \frac{W(a + \delta - y)}{EI}.$$

For brevity it is convenient to write

$$W = m^2 . EI,$$

where m for a uniform transverse section is constant. The equation then becomes

$$\frac{d^2y}{dx^2} + m^2 . y = m^2(a + \delta).$$

The integral of this equation is well known to be

$$y = a + \delta + A . \cos mx + B . \sin mx,$$

where A and B are the constants of integration. For the case shown in the figure $y = 0$, $dy/dx = 0$, when $x = 0$ and the equation becomes

$$y = (a + \delta)(1 - \cos mx).$$

If we now put $x = l$ we have $y = \delta$, and therefore

$$a + \delta = \frac{a}{\cos ml}$$

an equation which determines a definite value of δ, the lateral deviation of the summit of the pillar, unless $a = 0$, when we must have $\cos ml = 0$, that is

$$W = \frac{\pi^2}{4} . \frac{EI}{l^2}.$$

This is the equation given in the text; and for the pillar flat or rounded at both ends it applies with the modification there explained.

If, however, the pillar be fixed at one end while the other is constrained to lie in the same vertical, as will be the case when one end is flat and the other rounded, the equation requires modification. We must now suppose a horizontal force P at the top of the pillar to prevent lateral deviation, and the bending moment is then, since $a = 0$, $\delta = 0$,

$$+ M = + P(l - x) - Wy.$$

Making the same substitution as before, we find

$$\frac{d^2y}{dx^2} + m^2 . y = m^2 \frac{P}{W}(l - x),$$

the integral of which equation is

$$y = \frac{P}{W}(l - x) + A . \cos mx + B . \sin mx.$$

To determine the constants we have $y = 0$ when $x = 0$ and when $x = a$; also $dy/dx = 0$ when $x = 0$, therefore

$$A = -\frac{P}{W} . l;\ mB = \frac{P}{W};\ A \cos ml + B . \sin ml = 0,$$

from which we find that P is zero unless

$$\tan ml = + ml,$$

a transcendental equation which, being solved by trial, gives $ml = 4{\cdot}493$, that is

$$W = 2{\cdot}047\, \pi^2 \frac{EI}{l^2}.$$

It is sufficiently approximate to replace 2·047 by 2, as is done in the text, and this value is much more approximate than the value 16/9 employed by Rankine (*Useful Rules and Tables*, p. 220), which was obtained by supposing the summit of the pillar to deviate laterally until it is in the same vertical as the point of contrary flexure.

Pages 347 and 348. When the pillar is absolutely straight and homogeneous and of uniform transverse section, the lateral deflection due to an actual deviation a is given by the formula

$$a + \delta = \frac{a}{\cos ml} = a . \sec \frac{\pi}{2}\sqrt{\frac{p}{p_0}},$$

and the formula on p. 348 for the effect of deviation becomes

$$\left(\frac{f}{p} - 1\right) \cos \frac{\pi}{2}\sqrt{\frac{p}{p_0}} = \frac{qa}{nh}.$$

In any actual example, however, this formula would not be exact any more than that given in the text. Each particular example will have its own formula. The result of all such formulas, however, must be nearly the same for a small deviation. Further, a great proportional change in the deviation, always supposing it small, produces little change in the crushing load, and this probably explains why experiment gives tolerably definite values of the crushing load although its precise amount must depend on accidental circumstances.

PAGE 350. It is worth noticing that Gordon's formula with the theoretical values of the constants may be written

$$\frac{1}{p}=\frac{1}{f}+\frac{1}{p_0}.$$

On substituting this value of p in the deviation formula we get

$$\frac{qa}{nh}=\frac{f}{f+p_0}.$$

PAGE 361. The formulae given in different books for the moment of resistance of a shaft of rectangular section exhibit considerable discrepancies. COULOMB, to whom the formula for a circular section is due, supposed that in every case

$$T=f.\frac{I}{r_1},$$

where I is the polar moment of inertia and r_1 is the outside radius. In a rectangular section of sides a and b this gives

$$T=\tfrac{1}{6}fab\sqrt{a^2+b^2},$$

which for a square section of side h becomes

$$T=\cdot 2357f.h^3.$$

If these results were correct it would appear that a shaft of given sectional area was stronger the more unequal the sides were, a result quite contrary to experience. In a memoir on torsion published in the *Mémoires de l'Institut* for 1856, BARRÉ DE SAINT VENANT investigated the question thoroughly, and showed that the exact formula for a square section is

$$T=1\cdot 6653f.\left(\frac{h}{2}\right)^3=\cdot 2082fh^3,$$

giving about 88 per cent. of the preceding result. On comparing this with the moment of resistance of a circular section of equal area we find the ratio to be ·738, which should have been given in the text in place of the value ·8863. The general result for a rectangular section with sides in any proportion can be expressed only as an infinite series involving a separate calculation for each case. The calculation, however, may be replaced approximately by an empirical formula: that given in the text is not exactly the same as that used by SAINT VENANT, but leads to nearly the same result. The result given in the text for an elliptic section is exact.

RANKINE (A.M., p. 358) gives ·281 fh^3 as the result of SAINT VENANT'S calculations without further explanation. This value is greater than that given by COULOMB'S hypothesis, and is certainly too large.

PAGE 391. A line of stress may be regarded as the geometrical axis of a curved rod which is in tension or compression, as the case may be, under the action of a load perpendicular to itself. The whole solid, therefore, may be conceived as made up of a set of rods, each of which is a rope or linear arch in equilibrium under a transverse load. Each rod transmits stress in the direction of its length. If there be no lateral stress the rods are straight, but otherwise they are curved. In a framework structure loaded at the joints, the bars of the frame may be regarded as lines of stress except at the joints where those lines assume complex forms. The tendency of modern science is to regard all force as being due to the transmission of stress through a medium of some kind, even in such cases as that of gravity, where no medium perceptible to our senses exists. All forces on this conception are represented by a system of lines of stress.

PAGE 398. The theory of elastic solids has been much more fully treated with reference to practical application by GRASHOF, SAINT VENANT, and other continental writers than in any English treatise. The author is chiefly indebted to GRASHOF'S work, *Die Festigkeits Lehre* (Berlin, 1866), a new edition of which has (we believe) recently appeared. An attempt has been made to distinguish clearly between those parts of the subject which are necessarily true either exactly or to a degree of approximation which is capable of being exactly calculated, and those parts which depend on hypotheses more or less probable. The first are placed in the present chapter; the second in the chapter which follows.

PAGE 400. Attempts have been made to prove by theoretical reasoning that, in a perfectly elastic isotropic material, the value of m is necessarily 4, and the demonstration is still considered valid by some authorities. It is, however, more probable that such reasoning simply shows that matter is not constituted in the way supposed in the demonstration. It is difficult to obtain material which is really perfectly isotropic, but all the experimental evidence at present goes to show that m may have various values from 2 to infinity.

PAGE 402. Some other points in the theory of bending may here be noticed :—

(1) The effect of curvature is that a lateral stress p' must exist on the longitudinal layers given by the same equation as is used for thick hollow cylinders under internal fluid pressure (p. 404), viz.,

$$\frac{d}{dr}(p'r)=p.$$

Replacing r by $R+y$, and p by the value given in the text, we find

$$\frac{d}{dy}(p'r)=\frac{Ey}{R},$$

and therefore, by integration,

$$p'r=\frac{Ey^2}{2R}+\text{constant}.$$

Since p' is zero at the outer surface where y is $\pm\frac{1}{2}h$,

$$p'=\frac{E}{8R^2}(4y^2-h^2)=p_1\cdot\frac{4y^2-h^2}{4Rh},$$

where p_1 is the stress due to the bending at the outer surface, and r is replaced by its mean value R. At the neutral surface p^1 is greatest, but even there has only the very small value

$$-p'=p_1\cdot\frac{h}{4R}.$$

This lateral stress is therefore never great enough to have any perceptible influence on the elasticity of the layers.

(2) It has been stated on page 302 for the case of tension, page 315 for the case of bending, and page 360 for the case of torsion, that the stress on any transverse section is the same, however the straining forces are applied to a bar, provided only that their resultant be given in magnitude and position. This may be regarded as a general principle applicable in all cases. Any other distribution of stress produced on a transverse section by friction or other external forces applied directly to it will change with great rapidity on passing to transverse sections not directly exposed to such forces. It is, however, generally necessary to provide additional strength at these exceptional sections.

(3) If the beam be fixed at one end and loaded at the other, there will be shearing as well as bending on all sections. Transverse sections are then no longer plane, but are distorted by the action of the shearing force and the distribution of stress is not given exactly by the formulæ on pages 311, 367. Further, the maximum stress on the parts of the beam does not depend on the bending alone, but also on the shearing, as shown on page 392. It is, however, certain that the approximation is very close, even in short pieces, where the shearing is relatively great. The only exception is in beams of I section, in which the web is very thin, as explained in the passage cited. In practice, however, additional strength is always required in the web on account of the necessary stiffening (p. 366).

The various approximations here referred to have been all examined very thoroughly, especially by Saint Venant, and there is no doubt that the formulae are substantially exact within the limit of elasticity.

Page 406. The lines of stress for a thick hollow cylinder under internal fluid pressure, and also under the action of tangential stress applied as in Ex. VI., p. 394, will be found to be equiangular spirals, the angle of the spiral depending on the proportion between the fluid stress and the tangential stress.

The verification given in the text is necessary because, otherwise, we could not be sure that the assumptions on pages 404, 405 were consistent with one another. This is very well shown by supposing the cylinder to rotate and obtaining a solution of the problem when thus modified, assuming the cylinder to remain cylindrical and employing the equation of verification. It will be found that the solution thus obtained can only be true if the stress on the transverse section varies according to a certain law. If the cylinder is long it appears that this must really be the case except very near the ends. The problem of a swiftly rotating circular saw appears not as yet to have been attempted; it is found by experience that a saw to run at high speed must be hammered so as to be "tight" at the periphery. The same difficulty occurs if the material of the cylinder be not isotropic.

PAGE 407. Article 210 belongs more properly to the next chapter and is referred to below.

It may be useful to add the general formula for the elastic potential in the case of stress in two dimensions. Let a cube (side unity) be subject to the normal stresses p, q, o, on its faces, then, if the strains be e_1, e_2,

$$U = \tfrac{1}{2}pe_1 + \tfrac{1}{2}qe_2$$

Substituting from the formula on page 405

$$U = \frac{1}{2E}\left\{ p^2 + q^2 - \frac{2pq}{m} \right\},$$

this gives the elastic potential per unit of volume. When p and q vary as in the cylinder under internal pressure the volume of each element must be multiplied by this expression and an integration performed. This calculation is simple in the case of the cylinder.

PAGE 411. TRESCA'S experiments are described in detail, with a great variety of interesting illustrations in a series of memoirs which have been separately published (*Mémoires sur l'Ecoulement des Corps Solides*). The example in the text is taken from the second memoir (Paris, 1869). It is to be remarked that the influence of time was not taken into account. It is, however, certain that in the softer metals a much greater amount of work must be done in producing a permanent deformation, when that deformation takes place quickly, than when the slow action of hydraulic pressure is employed. Such bodies behave like fluids of very great viscosity.

PAGE 413. The more refined methods employed in the mechanical laboratories recently fitted up show that, when a bar is stretched a second time, the elasticity is sensibly perfect up to a certain limit, and then becomes sensibly imperfect up to a second limit, where drawing out commences. The first of these two limits may be expected to agree with the result obtained by alternate bending (p. 430). Further, if the testing machine be constructed in such a way as to permit the stretching force to diminish after a maximum value has been reached, drawing out will continue much longer before rupture takes place, as in the case of a stick of hot sealing wax.

PAGE 418. The modulus of elasticity in compression is found to be less than that in tension in cast iron as well as wrought iron in about the same ratio. This circumstance, together with the equality of the moduli for bending and tension, leads us to conjecture that the effect is due to lateral bending which cannot be wholly prevented by the trough.

PAGE 419. Crushing by bending has been considered in a previous chapter; it need only be added that when a tube is very thin as compared with its diameter, its resistance to longitudinal crushing is independent of its length, because crushing takes place by local buckling instead of by lateral flexure of the tube as a whole. The limit of length for which this is true is uncertain. The strength of iron tubes of circular section, in practical cases, has been found to be about 36,000 lbs. per square inch.

PAGE 421. The argument of Art. 217 applies equally to any case where stress is not uniformly distributed. In the hydraulic press cylinder (p. 407) the stress is never reversed, and the increase of strength is probably reliable.

PAGE 426. To test experimentally the truth of the various theories of compound strength the tensile strength of a bar should be determined, the ends of which project from a cylinder under hydraulic pressure of a given amount.

PAGE 431. It is much to be regretted that the resistance of cast iron to alternate bending has not been determined. Such an experiment would settle a point of the greatest importance in strength of materials, namely, whether the tenacity of a material can be increased by lateral connection. In wrought iron and steel, it would appear that it is not, the increase of apparent strength being accounted for by the raising of the elastic limit (p. 421).

PAGE 432. No formula of this kind is anything more than a formula of interpolation supplying the place of missing experiments. The author is led to make this remark by the elaborate manner in which such formulae are discussed by some writers. The study of WOHLER's original memoir cannot be too strongly recommended to those interested in the subject.

V.—HYDRAULICS AND PNEUMATICS.

PAGE 444. The value of w for pure water is greatest at 39°, and is then 62·425, while at 100° it diminishes to 62. At temperatures above 75°, 62 is more approximate than $62\frac{1}{2}$, but on the other hand water is seldom entirely free from solid matter, which increases its density.

PAGE 448. The standard experiments on the co-efficients of velocity and contraction in the case of orifices are those made by WEISBACH, and described by him in his treatise *Die Experimental Hydraulik* (Freiberg, 1855), to which the reader is referred for details. A short pipe projecting inwards is known as Borda's mouthpiece. The theoretical minimum value of the co-efficient of contraction (·5, see p. 476) is closely approached when the pipe is very thin and sharp-edged; otherwise the value is somewhat larger, say about ·55. The only case in which a co-efficient of contraction has been found theoretically is that of a long narrow slit, for which RAYLEIGH has obtained the value ·611.

PAGE 454. The use of the term "head" for the energy per unit of weight of a fluid is not free from inconvenience—the two things not being identical unless the datum level be at the surface of the fluid.

PAGE 455 A. In the flow of rivers it is well known that it is the outer side of a bend, not the inner, which suffers erosion, so that the windings of the river have a constant tendency to increase in extent. The reason of this has been explained by THOMSON to be that the layers of water in contact with the bottom are greatly retarded, and hence have less centrifugal force than the upper layers. The excess pressure at the outer side of the bend is therefore partially unbalanced below, and an inward flow takes place, carrying material with it from the outer side to the inner. This was verified by experiment. (*B.A. Report* for 1876, p. 31.)

PAGE 455 B. The velocity of the water in any one of the ideal pipes is inversely proportional to the sectional area of the pipe. Now the form of the pipes depends solely on the form of the bounding surfaces, and it follows, therefore, that the velocities of all parts of the stream bear a fixed proportion to each other, depending only on the nature of the bounding surfaces. In the language of the theory of mechanism, the fluid forms a closed kinematic chain. The chain is closed by the pressure of the bounding surfaces, and when the velocity exceeds a certain limit the chain opens. Energy can then no longer be transmitted uniformly to all parts of the fluid, and is no longer uniformly distributed. When energy is unequally distributed, eddies are formed.

PAGE 458. That hydraulic resistances of all kinds are independent of the pressure is one of the best established laws of experimental mechanics, but how far this may be true at very high pressures is, of course, uncertain. In some books it is stated as confidently as if it were an observed fact that the friction of the skin of a vessel near the keel is greater than that near the surface on account of increased pressure, but there is no foundation for this assertion.

To the three normal laws given in the text may, in all probability, be added a fourth :—

(4) Friction is proportional to the density of the fluid.

The grounds for this statement will be seen on reference to pages 473, 547, 549. It amounts to saying that friction is a kind of eddy resistance. If we assume this, the laws of friction are expressed by an equation similar to that employed for eddy resistance (p. 472),

$$R = f'w \, . \, S \, . \, \frac{V^2}{2g}.$$

The co-efficient of friction, f', is then distinguished from the friction per sq. ft., given in the table on p. 459, and is the same as the co-efficient for pipes.

The explanation in the text of the diminution of friction in long surfaces is that given by FROUDE in his reports on surface friction, and also by BOURGOIS in his treatise referred to farther on.

PAGE 462. The value of C, the co-efficient in the formula for the discharge of a pipe, is connected with $4f$ by the equation

$$C = \frac{4\cdot736}{\sqrt{4f}}.$$

This gives $C = 27\cdot3$ for $4f = \cdot03$, and not 30 as stated in the text. The larger value, corresponding to $4f = \cdot025$, may be used for clean iron pipes not less than 4 inches diameter.

The standard experiments on the friction of pipes are due to DARCY, and are expressed approximately by the formula given in the table on page 470.

PAGE 464. The formation of eddies by the meeting of different streams and the passage of water past solid bodies is familiar to all observers of the motions of fluids, and is described in the earliest treatises on hydraulics. The way in which they absorb energy has long been understood; thus PONCELET says, "En général, la production des tourbillons est l'un des moyens dont la nature se sert pour éteindre ou, *plutôt dissimuler*, la force vive dans les changements brusques de mouvement des fluides" (*Méc. Ind.*, p. 571). The italics are the author's. The passage is too long to quote at length, but is worth studying throughout. The extent to which eddy motion may prevail throughout the mass of a fluid, often without any clear indication at the surface, was not understood till long afterwards.

The theory of simple systems of eddies has of late attracted much attention, but the extreme intricacy of the internal motions of fluids will probably long defy calculation in such cases as commonly occur in practice.

The particular case mentioned in the text (Fig. 172) is one observed by the author, in which conspicuous eddies were formed, one or two at a time, with great regularity.

PAGE 469. If the motion of water in a pipe or channel be supposed of the undisturbed kind (p. 455) and viscosity be taken into account (p. 457), it is possible to find the discharge due to a given head. In the case of tubes of very small diameter it was shown by POISEUILLE that the flow actually does take place according to this law, and the co-efficient of viscosity was found. The loss of head is then proportional not to the (vel.)^2, but to the simple velocity.

In pipes of ordinary diameters through which water is flowing with ordinary velocities, the loss of head is, however, certainly, approximately as the (vel.)^2, and, moreover, BOUSSINESQ has shown that it is enormously greater than it would be according to the law for undisturbed flow with the co-efficient deduced by POISEUILLE. The inevitable conclusion is, that the loss is mainly due to the formation of eddies. In the case of large rivers it is found by experiment that the velocity diminishes as the bottom is approached according to a law represented by the ordinates of a parabola, a result which is consistent with the law of undisturbed flow. Nevertheless, in this case also, the facts cannot be explained except by supposing that the resistance is due to eddies. With fluids, the viscosity of which is so small, as is the case in water, undisturbed flow only occurs at very low velocities in very small channels.

Although these facts were tolerably well established, it is only very recently that any attempt has been made to discover the connection which must exist between the viscosity of the fluid, its velocity, and the dimensions of the channel in which it flows, in order that the flow may or may not be undisturbed. This has at length been done by REYNOLDS, who has succeeded in connecting by a common law POISEUILLE'S experiment on capillary tubes and DARCY'S experiments on full-sized pipes. For particulars the reader is referred to his paper recently published in the *Philosophical Transactions* (1883, Part III.). It need only here be mentioned that it is shown that the loss of head in a pipe may be expressed by a formula which, when stated in a simplified form sufficient for our present purpose, becomes

$$h' = 4f \cdot \frac{l}{d^{3-n}} \cdot \frac{v^n}{2g},$$

where n is an index depending on the nature and condition of the surface. When the surface is rough $n=2$, and we get the formula already given on page 460; this is the case for an incrusted pipe, but for a clean cast-iron pipe it falls off to 1·9, and in a lead pipe is 1·723. This falling off in the index in smooth surfaces is quite analogous to that already found by FROUDE in his direct experiments on surface friction (p. 459).

PAGE 472. The formula for eddy resistance is given in this form by PONCELET, and the reasoning in the text is essentially that employed by him (*Méc. Ind.*, p. 585). It is well suited to show the real nature of the law of hydraulic resistance (p. 447). All that is supposed in this law is, that the velocities of the particles of fluid bear a fixed proportion to each other depending solely on the form of the bounding surfaces, as is actually the case in undisturbed motion. If the bounding surfaces are of invariable form the law should be accurately verified for a fluid absolutely devoid of viscosity. The causes of irregularity are explained on page 468. A variation of 20 per cent. in the course of the same experiment was actually observed by FROUDE.

PAGE 473. Until the establishment of the Institution of Naval Architects and the labours of FROUDE had given so great an impulse to naval science, the resistance of ships, like other branches of the subject, had been much more thoroughly studied in France than in England. The only treatise specially devoted to it, even now, is that entitled *Mémoire sur la Resistance de l'Eau* par M. BOURGOIS, Paris (no date). The earlier experiments by BEAUFOY and others are all very thoroughly discussed in this work, and the reader is referred to it for the values of k in the formula for eddy resistance. These experiments were chiefly on bodies of unfair form, and this is the principal reason why so little progress was made for a long period in discovering the true principles of the resistance of ships. Experiments on the resistance of a full-sized ship made in France are described by BOURGOIS, and it was known that it was much less than would have been supposed in the absence of such experiments, but it was not till the discovery by FROUDE of the true method of comparing the resistance of a ship with that of her model that any great advance was made in the subject.

The formula employed in the text is given by RANKINE, and, if a single constant only is admitted, this form is perhaps the best. The value of k, however, stated by him (*Useful Rules and Tables*, p. 274) is now known to be 50 per cent. too large, being deduced not from direct experiment but from the power required for propulsion. The limit of speed to which the formula applies is also much less than that stated in the passage cited.

If three constants be admitted the formula employed by BOURGOIS

$$R=aV+bV^2+cV^4$$

will give a much closer approximation and apply up to a much higher limit of speed. No formula, however, is true at all speeds, and the only complete way of representing the resistance of a vessel is by means of a curve derived from experiments on a model. This question is far too large to enter on here.

PAGE 487. In finding λ in cranes and other hoisting machines the weight raised, multiplied by the velocity ratio, between it and the ram must be included in the weight of the ram.

PAGE 496. In Fig. 188, the letters a, b, which denote the valves, have been interchanged.

PAGE 509. The vanes of a turbine-wheel are often not radial at the circumference where the water enters. The simpler case, however, is sufficient to illustrate the principle on which such machines are designed.

PAGE 512. The undisturbed motion of a perfect liquid within fixed boundaries is always *reversible*, that is, if every particle of liquid were imagined to be set in motion with the same velocity in the reverse direction, the motion would continue undisturbed. But if water be set in motion from rest this will generally not be the case. If, for example, we imagine a pipe connected with a tank by a mouthpiece in the form of the *vena contracta*, then, when water flows out of the tank, it will issue in a continuous stream with small loss of head; but if the motion be reversed most of the energy of motion of the water in the pipe will be wasted in the internal motions soon after entering the tank. The loss is not unavoidable as will be seen on reference to the case of a trumpet-shaped pipe (Fig. 168, p. 454), but may be rendered small by enlarging the pipe very gradually.

PAGE 513. The spiral passage should not be too large and should be connected with the ascending main by a carefully formed gradual enlargement.

PAGE 514. This calculation is partly the same as that given by UNWIN, (*Min. Proc. Inst. C. E.*, vol. 32).

PAGE 535 A. The author has for some time past been in the habit of distinguishing between the mechanical *equivalent* and the mechanical *value* of heat.

The term "entropy" (CLERK MAXWELL'S *Theory of Heat*, 1st edition, p. 186) is used in a wider sense, and is employed by different writers with different meanings.

PAGE 535 B. In the author's treatise on the steam engine (chap. viii.) attention has been called to the distinction between the mechanical losses and the thermal losses in heat engines, and the terms "mechanical" efficiency and "thermal" efficiency are here introduced to render the distinction more clear.

PAGE 536. In certain pumping engines it is said that the consumption of steam is as little as 14 lbs. per I.H.P. per hour. This, though exclusive of the steam supplied to the jackets, corresponds to a somewhat greater efficiency than that given in the text.

For a detailed discussion of the losses in steam engines the reader is referred to the author's work just cited.

PAGE 538. The value of n in the adiabatic expansion of the permanent gases is commonly given as 1·408, and this is probably more exact than 1·4. The difference, however, is not important, and the simpler value is therefore adopted in the text.

PAGE 541. If the fluid be supposed at rest, and elevation be taken into account, we obtain

$$K_p T + z = \text{Const.}$$

or, as it may also be written,

$$3{\cdot}5\, PV + z = \text{Const.}$$

This gives the distribution of pressure and temperature of the atmosphere for "convective equilibrium" (CLERK MAXWELL'S *Theory of Heat*, 1st edition, p. 301). Energy is then uniformly distributed.

PAGE 545. The explanation here given (Fig. 199) on the whole seems the most natural, but it very probably may not be complete.

PAGE 547. This formula for the flow of air in a long pipe is believed to have been first given by UNWIN (*Min. Proc. Inst. C. E.*, vol. xliii.). It is a question of considerable practical interest. By comparison with experiment it has been shown that the co-efficient is given by a formula of the same form (DARCY'S), as in the flow of water through pipes, an important verification of theoretical principles. The equation for the case where the temperature varies can be obtained without difficulty, but has not as yet been practically applied.