Math. Ann. 286, 593–612 (1990)

Mathematische Annalen

On free holomorphic $\mathbb{C}$-actions on $\mathbb{C}^n$ and homogeneous Stein manifolds

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Dedicated to Hans Grauert

1. Introduction

Let $\mu: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ denote a free holomorphic group action, i.e. let us assume that μ is a holomorphic map with

$$\mu(s, \mu(t, z)) = \mu(s+t, z) \quad \text{and} \quad \mu(t, z) = z \Leftrightarrow t = 0.$$

This yields an equivalence relation: $x \sim y$ if both x and y are contained in the same $\mathbb{C}$-orbit, i.e. if there exists a $t \in \mathbb{C}$ such that $\mu(t, x) = y$. We want to consider the quotient space $\mathbb{C}^n/\sim$ equipped with the quotient topology and the natural structure as a $\mathbb{C}$-ringed space defined as follows: Let U be an open subset of $\mathbb{C}^n/\sim$ and $\pi: \mathbb{C}^n \to \mathbb{C}^n/\sim$ the projection. Then we define $\mathcal{O}(U) = \mathcal{O}(\pi^{-1}(U))^\mu$ where the latter denotes the invariant functions on $\pi^{-1}(U)$. Our main goal is to understand this quotient space. A natural question is whether this quotient space is always biholomorphic to $\mathbb{C}^{n-1}$.

We will give some examples which show that in general this is not true.

- There is a free algebraic triangular $\mathbb{C}$-action by quadratic transformations on $\mathbb{C}^5$ such that the quotient is diffeomorphic to $\mathbb{C}^4$ but not biholomorphic to $\mathbb{C}^4$. In fact the quotient X is a non-Stein quasi-affine variety: $X = Q \backslash E$ where Q is a smooth four-dimensional quadric and E is a smooth subvariety of codimension two.
- This variety X can furthermore be obtained as a quotient of an *affine-linear* $(\mathbb{C}^2, +)$-action on $\mathbb{C}^6$.
- There is a free algebraic triangular $\mathbb{C}$-action on $\mathbb{C}^4$ with all orbits closed such that the quotient is non-Hausdorff.
- There is a free holomorphic $\mathbb{C}$-action on $\mathbb{C}^5$ such that the quotient is a complex manifold with a one-dimensional compact complex submanifold.
- There is a free holomorphic $\mathbb{C}$-action on $\mathbb{C}^5$ with some non-closed orbits.

These examples show that one needs some additional assumptions to obtain nice quotients. In particular if one assumes that the quotient of $\mathbb{C}^n$ by a $\mathbb{C}$-action is

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already Hausdorff then it follows that the quotient is a complex space [H]. If the action is furthermore algebraic then the quotient is a quasi-affine variety [FM].

For a rather special class of $\mathbb{C}$-actions we have stronger conclusions: If a free holomorphic $\mathbb{C}$-action $\mu: \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\mu(t,z) = z + t\psi(z)$ for some holomorphic map $\psi: \mathbb{C}^n \to \mathbb{C}^n$ then it follows that the quotient is a closed submanifold of $(\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{\frac{n(n-1)}{2}}$.

One can ask whether one can "deform" $\mathbb{C}$-actions. Let $\mu: P \times \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ be a parametrized free group action, i.e. P is a complex space and for all $p \in P$ the evaluation $\mu(p)$ yields a free holomorphic group action. This yields quotient spaces parametrized by elements in P. Now there do exist parametrizations such that all these parametrized quotients are non-isomorphic, i.e. free $\mathbb{C}$-actions on $\mathbb{C}^n$ are not rigid.

Our initial motivation for studying quotients of free $\mathbb{C}$-actions was given by the fact that they are related to the following problem:

If G/H is the quotient of a connected complex Lie groups G by a closed connected Lie subgroup H, under which conditions is the quotient Stein?

Matsushima gave a necessary condition which is also sufficient in the special cases where G is semisimple or solvable. The above mentioned example of a free holomorphic $\mathbb{C}$-action on $\mathbb{C}^5$ with a non-Stein quasi-affine variety as quotient yields an example which shows that this condition is not always sufficient.

To state this condition let us assume that G acts almost effectively on G/H. Then H contains a maximal connected normal simply-connected solvable Lie subgroup U. Let M denote a maximal reductive Lie subgroup of G. Then the condition is the following:

– *For all $g \in G$ the reductive group $M^g = g \cdot M \cdot g^{-1}$ has only discrete intersection with U.*

We would like to thank A. T. Huckleberry for the suggestion to consider this problem. We would also like to thank him, J. P. Demailly and D. M. Snow for inspiring discussions.

2. A non-Stein quotient

We construct an example of a quotient X of $\mathbb{C}^6$ by a unipotent group $U \cong (\mathbb{C}^2, +)$ acting freely by affine-linear transformations. The same quotient X arises from a certain free $\mathbb{C}$-action on $\mathbb{C}^5$ given by quadratic transformations. X is diffeomorphic to $\mathbb{C}^4$, but not biholomorphic to $\mathbb{C}^4$. In fact the quotient space is a quasi-affine variety, which is neither affine nor Stein: $X = Q \setminus E$, where Q is a smooth four-dimensional quadric and E is a two-codimensional closed subvariety of Q.

We define U as the following group of affine-linear transformations on $\mathbb{C}^6$ (for $s, t \in \mathbb{C}$):

$$\phi_{s,t}: \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \\ z_3 + tz_1 + sz_2 \\ z_4 + tz_2 \\ z_5 + t(1+z_1) \\ z_6 + s + \frac{1}{2}t^2(1+z_1) + tz_5 \end{pmatrix}.$$

Obviously the ring of $\phi_{s,0}$-invariant functions is generated by $z_1, z_2, z_3 - z_2 z_6, z_4, z_5$. This yields $\mathbb{C}^6/(\phi_{s,0}) = \mathbb{C}^5$. Since $(\mathbb{C}^2, \dot{+})$ is commutative it is clear that the $\phi_{0,t}$-action on $\mathbb{C}^6$ may be pushed down to this quotient. Now we introduce new coordinates on $\mathbb{C}^5$ by $w_1 = \lambda(z_1+1)$, $w_2 = \lambda z_2$, $w_3 = \lambda z_5$, $w_4 = \lambda z_4$, and $w_5 = z_5 - \frac{1}{2} z_4 z_5 + z_2 z_6$ with $\lambda = \sqrt{-\frac{1}{2}}$. This yields

$$\phi_t: \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} \mapsto \begin{pmatrix} w_1 \\ w_2 \\ w_3 + t w_1 \\ w_4 + t w_2 \\ w_5 + t(1 + D_w) \end{pmatrix}$$

with $D_w = \det \begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix}$. Thus we have realized $\mathbb{C}^6/U$ as a quotient of $\mathbb{C}^5$ by a free algebraic $\mathbb{C}$-action of quadratic transformations.

Observe that the following functions are holomorphic invariants for the U-action:

$$\alpha_1 = w_1, \quad \alpha_3 = w_3(1 + D_w) - w_1 w_5, \quad \alpha_5 = D_w,$$
$$\alpha_2 = w_2, \quad \alpha_4 = w_4(1 + D_w) - w_2 w_5.$$

This yields a holomorphic map $\alpha: \mathbb{C}^5 \to \mathbb{C}^5$. An easy calculation shows that

$$\det \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{pmatrix} = \alpha_5(\alpha_5 + 1).$$

Hence the image of α is contained in the smooth quadric

$$Q = \{ z \in \mathbb{C}^5 \mid z_1 z_4 - z_2 z_3 = z_5(z_5 + 1) \}.$$

Our next step is to show that fibers of α are exactly the $\mathbb{C}$-orbits.

Lemma 1. *Let $x, w \in \mathbb{C}^5$. Assume $\alpha(w) = \alpha(x)$. Then there exists an element $t \in \mathbb{C}$ such that $\phi(t)(w) = x$.*

Proof. Obviously $w_1 = x_1$ and $w_2 = x_2$. First assume that $D_w \neq 0$. Then $D_w = D_x$ $\left[\text{with } D_x = \det \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}\right]$ implies that

$$\begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \zeta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Now let $z = \phi(\zeta)(x)$. Then $(w_1, \ldots, w_4) = (z_1, \ldots, z_4)$. Furthermore from the definition of α_3 and α_4 it follows that

$$w_5 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = z_5 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Since $(w_1, w_2) \neq (0,0)$ this implies $w = z = \phi(\zeta)(x)$.

Secondly assume $D_w = 0$. Now there exists a $t \in \mathbb{C}$ such that $w_5 = z_5$ for $z = \phi(t)(x)$ since $(\phi(t)(x))_5 = x_5 + t(1 + D_w)$. Observe that for $D_w = 0$ we obtain $\alpha(w) = (w_1, w_2, w_3 - w_1 w_5, w_4 - w_2 w_5, 0)$. Thus $\alpha(w) = \alpha(z)$ and $w_5 = z_5$ imply $w = z = \phi(t)(x)$. $\square$

Lemma 2. *The map α has rank 4 everywhere.*

Proof. Since α is invariant under the G-action, it has rank four at most. Conversely observe that

$$D\alpha=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ * & * & 1+D_w-w_2w_3 & w_1w_3 & -w_1 \\ * & * & -w_2w_4 & 1+D_w+w_1w_4 & -w_2 \\ * & * & -w_2 & w_1 & \end{pmatrix}.$$

Distinguishing the cases $(w_1, w_2)=(0,0)$ and $(w_1, w_2)\neq(0,0)$ one can easily verify that the rank of $D\alpha$ is four for all values of (w_1, w_2, w_3, w_4). □

So far we have checked that the holomorphic map $\alpha: \mathbb{C}^5 \to Q$ has rank four everywhere and separates the G-orbits. This implies that the image of α in the four-dimensional variety Q is an open subset and furthermore that the holomorphic functions on the image of α in Q are exactly the G-invariant holomorphic functions on $\mathbb{C}^5$. Hence α yields a natural complex structure on the geometric quotient space $\mathbb{C}^5/G$.

Next we want to show that the quotient space equipped with this complex structure is a quasi-affine variety but not biholomorphic to $\mathbb{C}^4$. This is a consequence of the following lemma.

Lemma 3. *The variety Q defined as above is a smooth four-dimensional affine variety. Let $E=\{z\in Q \mid z_1=z_2=0,\ z_5=-1\}$. Then E is a analytic subvariety of codimension 2 in Q. The image of α is precisely $Q\setminus E$.*

Proof. Observe that $f(z)=z_1z_4-z_2z_3-z_5(1+z_5)$ is a defining function for $Q\subset\mathbb{C}^5$. Note that $df|_z=0$ only for $z=(0,0,0,0,-\frac{1}{2})$ which is not contained in Q. Hence Q is smooth.

We want to show that $z\in Q$ is contained in the image of α iff $z\notin E$.

First let $z\in E$. Assume that $\alpha(w)=z$. Then $\alpha_1=\alpha_2=0$, hence $w_1=w_2=0$. This implies $\alpha_5(w)=0$ contrary to $z\in E$. This is the desired contradiction.

For the case $z_1=z_2=z_5=0$ let $w=z$ and note that $\alpha(w)=z$.

Now assume $D_z=\det\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4\end{pmatrix}=0$ (i.e. $z_5\in\{0,-1\}$) and $(z_1, z_2)\neq(0,0)$. This implies that $(z_3, z_4)=\lambda(z_1, z_2)$ for some $\lambda\in\mathbb{C}$. Then $z=\alpha(z_1, z_2, 0, 0, -\lambda)$ for $z_5=0$. For $z_5=-1$ let $(w_1, w_2)=(z_1, z_2)$ and choose (w_3, w_4) such that $D_w=-1$. Then $z=\alpha(w_1, w_2, w_3, w_4, -\lambda)$.

Finally we consider the case $D_z\neq 0$. Here we let $(w_1, w_2)=(z_1, z_2)$ and choose (w_3, w_4) such that $D_w=z_5$. Now

$$D_z=\det\begin{pmatrix} z_1 & w_3(1+D_w) \\ z_2 & w_4(1+D_w)\end{pmatrix}\neq 0.$$

It follows that

$$\begin{pmatrix} z_3 \\ z_4\end{pmatrix}=\begin{pmatrix} w_3(1+D_w) \\ w_4(1+D_w)\end{pmatrix}+\zeta\begin{pmatrix} z_1 \\ z_2\end{pmatrix}$$

for some $\zeta\in\mathbb{C}$. This implies that $z=\alpha(w_1, w_2, w_3, w_4, -\zeta)$. □

Remark. A non-singular quadric in $\mathbb{C}^5$ is always homotopy-equivalent to S^4. In particular it is not biholomorphic to $\mathbb{C}^4$. Hence $\mathcal{O}(\mathbb{C}^5)^G = \mathcal{O}(Q) \neq \mathcal{O}(\mathbb{C}^4)$ and $\mathcal{P}(\mathbb{C}^5)^G = \mathcal{P}(Q) \neq \mathcal{P}(\mathbb{C}^4)$, where $\mathcal{O}$ and $\mathcal{P}$ denote the holomorphic resp. algebraic functions.

Although the quotient $\mathbb{C}^5/G$ is not biholomorphic to $\mathbb{C}^4$ it is diffeomorphic to $\mathbb{C}^4$:

Lemma 4. *Let $\phi: \mathbb{C} \to \mathbb{C}$ be a C^∞-diffeomorphism such that $\phi(z)(\phi(z)+1) = z$ for all z in some open neighbourhood W of 0. Then the map $\psi: \mathbb{C}^4 \to Q \setminus E$ defined by*

$$\psi(x_1, x_2, t_1, t_2) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \frac{\phi(D)(\phi(D)+1) - D}{|x|^2} \begin{pmatrix} -\bar{x}_2 \\ \bar{x}_1 \end{pmatrix}; \phi(D) \right)$$

for $x = (x_1, x_2) \neq (0,0)$ and

$$\psi(0,0,t_1,t_2) = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}; 0 \right)$$

with $D = \det \begin{pmatrix} x_1 & t_1 \\ x_2 & t_2 \end{pmatrix}$ is a C^∞-diffeomorphism.

It should be noted that this property of our quotient is not a priori obvious. In fact a result of Ramanujam [Ra] implies that there exists a three-dimensional complex affine variety V with an free algebraic $\mathbb{C}$-action such that $V/(\mathbb{C})$ is an affine surface which is *not* homeomorphic to $\mathbb{C}^2$ although V *is* homeomorphic to $\mathbb{C}^3$.

3. $\mathbb{Z}$-actions on $\mathbb{C}^n$

Let us consider the restriction of the above defined $\mathbb{C}$-action to $\mathbb{Z}$. We obtain a $\mathbb{C}^*$-principal bundle over $\mathbb{C}^5/G = Q \setminus E$ as quotient of $\mathbb{C}^5$ by this $\mathbb{Z}$-action. This yields an example for a properly discontinuous $\mathbb{Z}$-action on $\mathbb{C}^n$ such that the quotient is not holomorphically separable.

Lemma 5. *Let $\mathbb{C}$ act on $\mathbb{C}^5$ as described above. Then every holomorphic function on $\mathbb{C}^5$ which is invariant under the respective $\mathbb{Z}$-action is also invariant under the $\mathbb{C}$-action. In particular the quotient $\mathbb{C}^5/(\mathbb{Z})$ is not holomorphically separable.*

Proof. Note that $\mathbb{C}^5/\mathbb{Z}$ is a non-trivial $\mathbb{C}^*$-principal bundle over $\mathbb{C}^5/\mathbb{C} = Q \setminus E$. Hence it suffice to prove that for every non-trivial $\mathbb{C}^*$-principal bundle H over $Q \setminus E$ every holomorphic function on H is constant along the fibers of the projection to $Q \setminus E$.

Let H be a non-trivial $\mathbb{C}^*$-principal bundle over $Q \setminus E$ and f a holomorphic function on H. Let L be the associated line bundle. Let $t_{(i)}$ be local fiber coordinates. Then f has a Laurent development $f(x,t) = \sum_{k \in \mathbb{Z}} a_{(i),k} t_{(i)}^k$. Now the coefficients $a_{(i),k}$ yield sections a_k in the line bundle L^{-k} for each k. Thus we obtain a non-trivial holomorphic section in L^{-k} for some $k \in \mathbb{Z}$ if f is not constant along the fibers. Then L^{-k} is generated by some divisor D on $Q \setminus E$. Since E is of codimension two in Q this divisor can be extended to Q. Thus L^{-k} is the restriction of some line bundle over the affine variety Q. One can check that $H^2(Q, \mathbb{Z}) = \{0\}$, hence $\mathrm{Pic}(Q) = \{e\}$ and this line bundle is trivial. Now observe that $H^2(Q \setminus E, \mathbb{Z}) = \{0\}$.

Hence $H^1(Q\backslash E,\mathcal{O})\cong H^1(Q\backslash E,\mathcal{O}^*)$ and in particular $H^1(Q\backslash E,\mathcal{O}^*)$ is torsion-free. Therefore L^{-k} being trivial implies that L itself must be trivial.

This a $\mathbb{C}^*$-principal bundle over $Q\backslash E$ has holomorphic functions which are not constant along the fibers if and only if it is trivial. □

The first example of a $\mathbb{Z}$-action on $\mathbb{C}^n$ such that the quotient is a complex manifold, but not holomorphically separable, is due to Demailly [D]. He considered the quotient of $\mathbb{C}^3$ by the infinite cyclic group generated by the automorphism $\phi:(z_1, z_2, z_3)\mapsto(z_1+1, z_3-z_2^k, z_2)$ for $k\geqq 2$. The obvious invariant function $\exp(z_1/2\pi i)$ realizes the quotient as a locally holomorphically trivial $\mathbb{C}^2$-bundle over $\mathbb{C}^*$ where all the holomorphic functions are coming from the base.

However, this $\mathbb{Z}$-action on $\mathbb{C}^3$ can *not* be extended to a $\mathbb{C}$-action.

Lemma 6. *There is no holomorphic group action* $\mu:(\mathbb{C},+)\times\mathbb{C}^3\to\mathbb{C}^3$ *such that* $\mu(1;z)=(z_1+1, z_3-z_2^k, z_2)$ *for* $k\geqq 2$.

Proof. Assume the contrary. Since the $\mathbb{Z}$-action induced by $\mu(1)$ is a free, properly discontinuous $\mathbb{Z}$-action it follows that this $\mathbb{C}$-action on $\mathbb{C}^3$ can be pushed down to a fixed point-free $\mathbb{C}^*$-action on the quotient $\mathbb{C}^3/(\mathbb{Z})$. By [D] the quotient is a $\mathbb{C}^2$-bundle over $\mathbb{C}^*$ where the projection is exactly the holomorphic reduction of the total space. Hence the projection is equivariant for all automorphisms. Since any $\mathbb{C}^*$-action on some $\mathbb{C}^m$ has a fixed point it follows that the $\mathbb{C}^*$-action on the base is non-trivial. Therefore every $\mathbb{C}^*$-orbit in the total space is a finite cover over the base $\mathbb{C}^*$. This leads to a contradiction, because now we can choose a fiber $F\cong\mathbb{C}^2$ and a non-constant holomorphic function $f\in\mathcal{O}(F)$, take the mean value over the finite subgroup of $\mathbb{C}^*$ stabilizing F and extend this function along the $\mathbb{C}^*$-orbits to a holomorphic function on the total space which is not constant along the fibers. □

4. A family of the non-Stein quotients

The example given above is not the only case in which the quotient of a free $\mathbb{C}$-action on $\mathbb{C}^n$ is a quasi-affine variety, but not isomorphic to $\mathbb{C}^{n-1}$. We will now described a family of free holomorphic $\mathbb{C}$-actions on $\mathbb{C}^7$ yielding infinitely many non-biholomorphic such quotients.

For a holomorphic function $f\in\mathcal{O}(\mathbb{C}^3)$ with $f(0,0,0)=0$ define a $\mathbb{C}$-action as follows

$$\psi_f(t):\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\\ z_5\\ z_6\\ z_7\end{pmatrix}\mapsto\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4+tz_1\\ z_5+tz_2\\ z_6+tz_3\\ z_7+t(1+f(D_1,D_2,D_3))\end{pmatrix}$$

with $D_1=\det\begin{pmatrix} z_2 & z_5\\ z_3 & z_6\end{pmatrix}$, $D_2=\det\begin{pmatrix} z_3 & z_6\\ z_1 & z_4\end{pmatrix}$, and $D_3=\det\begin{pmatrix} z_1 & z_4\\ z_2 & z_5\end{pmatrix}$. Now we define a number of invariant functions by

$$\begin{aligned} \alpha_1 &= z_1, & \alpha_4 &= D_1, & \alpha_7 &= z_7 z_1 - z_4(1+f(D_1,D_2,D_3)), \\ \alpha_2 &= z_2, & \alpha_5 &= D_2, & \alpha_8 &= z_7 z_2 - z_5(1+f(D_1,D_2,D_3)), \\ \alpha_3 &= z_3, & \alpha_6 &= D_3, & \alpha_9 &= z_7 z_3 - z_6(1+f(D_1,D_2,D_3)). \end{aligned}$$

By explicit calculations similar to those for the above example one can check that the following is true:

Lemma 7. *The holomorphic map $\alpha: \mathbb{C}^7 \to \mathbb{C}^9$ has the following properties*
(i) *The fibers of α are exactly the G-orbits.*
(ii) *The map α has rank 6 everywhere.*
(iii) *The image of α is contained in the affine variety Q_f defined by the equations*

$$\begin{aligned} &\det\begin{pmatrix} x_2 & x_8 \\ x_3 & x_9 \end{pmatrix} = -(1+F)x_4, \qquad x_1x_4 + x_2x_5 + x_3x_6 = 0, \\ &\det\begin{pmatrix} x_3 & x_9 \\ x_1 & x_7 \end{pmatrix} = -(1+F)x_5, \qquad x_7x_4 + x_8x_5 + x_9x_6 = 0, \\ &\det\begin{pmatrix} x_1 & x_7 \\ x_2 & x_8 \end{pmatrix} = -(1+F)x_6 \end{aligned}$$

with $F = f(x_4, x_5, x_6)$.
The variety Q_f is an irreducible variety with

$$\operatorname{Sing}(Q_f) \subset \{x \in \mathbb{C}^9 \mid x_1 = x_2 = x_3 = x_7 = x_8 = x_9 = 0,\ dF = 0,\ F = -1\}.$$

(iv) *The image of α is exactly $Q_f \setminus E_f$ with*

$$E_f = \{x \in \mathbb{C}^9 \mid x_1 = x_2 = x_3 = 0,\ F = -1,\ x_4x_7 + x_5x_8 + x_6x_9 = 0\}.$$

The subvariety E_f of Q_f has pure codimension two and $Q_f \setminus E_f$ is not Stein.

Now let f, g be different holomorphic functions on $\mathbb{C}^3$ vanishing in the origin. Assume that ξ is a biholomorphic map from $Q_f \setminus E_f$ to $Q_g \setminus E_g$. Then it follows from the Riemann extension theorem that ξ can be extended to a biholomorphic map between the normalizations of Q_f and Q_g. This in turn implies that the normalizations of E_f and E_g must be holomorphic to each other. Note that E_f is a vector bundle over the zero set of $f+1$. Hence for suitable chosen functions $f \in \mathcal{O}(\mathbb{C}^3)$ this procedure yields infinitely many non-isomorphic non-Stein quotients.

5. A non-Hausdorff quotient

In this section we will describe an example of a free, algebraic and triangular $\mathbb{C}$-action of quadratic transformations on $\mathbb{C}^4$ such that the quotient space is not Hausdorff. We will discuss the properties of this example in some detail and in particular show that there exists a categorical quotient for the category of affine varieties, but not for the category of arbitrary varieties.

Lemma 8. *Let $\mu:(\mathbb{C},+)\times\mathbb{C}^4\to\mathbb{C}^4$ be the group action given by*

$$x_1\mapsto x_1,\qquad x_3\mapsto x_3+tx_2+\tfrac{1}{2}t^2x_1,$$
$$x_2\mapsto x_2+tx_1,\qquad x_4\mapsto x_4+t(x_2^2-2x_1x_3-1).$$

Then μ defines a free, algebraic group action with closed orbits, but the quotient space $\mathbb{C}^4/G$ is not Hausdorff.[1]

Proof. It is easy to check that this is a free group action (observe that $x_2^2-2x_1x_3$ is an invariant). Since the action is free and algebraic it is clear that the orbits are closed.

Now $p=(0,1,0,0)$ and $q=(0,-1,0,0)$ are two points in $\mathbb{C}^4$ which are not contained in the same orbit, since $\mu(t,p)=(0,1,t,0)$. But for $t=\frac{2}{\varepsilon}$ we obtain

$$\mu(t,(\varepsilon,1,0,0))=(\varepsilon,-1,0,0).$$

Hence it is not possible to find disjoint G-invariant open neighbourhoods for p and q. Thus $\mathbb{C}^4/G$ equipped with the quotient topology is not Hausdorff. □

We will now study this example in more detail.

Lemma 9. *Let $E^+=\{x\in\mathbb{C}^4\mid x_1=0,\ x_2=1\}$, $E^-=\{x\in\mathbb{C}^4\mid x_1=0,\ x_2=-1\}$ and $E=E^+\cup E^-$. Let $p\in E^+$ and $q\in E^-$. Then p and q can not be separated by G-stable open neighbourhoods.*

Proof. Note that for $x\in E$ the group action is given by $x=(0,\pm1,x_3,x_4)\mapsto(0,\pm1,x_3\pm t,x_4)$. Hence we may wlog assume that $p=(0,1,0,p_4)$ and $q=(0,-1,0,p_4+d)$ with $p_4,d\in\mathbb{C}$. Now we can choose arbitrarily small complex numbers ε,δ such that $[\varepsilon:\delta]=[-4:d]$. Then $\mu(t,(\varepsilon,1+\delta,0,p_4))=(\varepsilon,\delta-1,\frac{1}{2}d,p_4+d)$ for $\varepsilon t=-2$, $\delta t=\frac{1}{2}d$. Since $(0,-1,\frac{1}{2}d,p_4+d)$ is contained in the G-orbit through $(0,-1,0,p_4+d)=q$ this completes the proof. □

Corollary. *Every G-invariant continuous function on $\mathbb{C}^4$ is constant on E.*

It is a straight-forward calculation to check that the following functions are invariant polynomials on $\mathbb{C}^4$:

$$\alpha_1=x_1,\qquad \alpha_3=x_1x_4-x_2(x_2^2-2x_1x_3-1),$$
$$\alpha_2=x_2^2-2x_1x_3,\qquad \alpha_4=\frac{1}{\alpha_1}(\alpha_3^2-\alpha_2(1-\alpha_2)^2).$$

This yields an invariant algebraic morphism $\alpha:\mathbb{C}^4\to\mathbb{C}^4$.

Lemma 10. *Let $\alpha:\mathbb{C}^4\to\mathbb{C}^4$ be defined as above. Then α has the following properties:*

(i) *The map α has rank three on $\mathbb{C}^4\setminus E$ and rank two on E.*

(ii) *The set E is a fiber of α. Otherwise the fibers of α coincide with the G-orbits.*

(iii) *The image is contained in the smooth cubic $C=\{w\in\mathbb{C}^4\mid w_1w_4=w_3^2-w_2(1-w_2)^2\}$. Moreover $Image(\alpha)=C\setminus\{w\in\mathbb{C}^4\mid w_1=0,\ w_2^2=1,\ w_3=0,\ w_4\neq0\}$. In particular the image of α is a constructible set, but not a quasi-affine variety.*

[1] D. Snow recently informed us that M. Smith (University of Texas at Austin) has constructed a similar example of an algebraic $\mathbb{C}$-action with non-Hausdorff quotient

(iv) *Every G-invariant holomorphic or algebraic function on $\mathbb{C}^4$ fibers through α, i.e. $\mathcal{O}(\mathbb{C}^4)^G \cong \mathcal{O}(C)$ and $\mathcal{P}(\mathbb{C}^4)^G \cong \mathcal{P}(C)$.*

Proof. The statements (i)–(iii) follow from explicit calculation. Statement (iv) is an immediate consequence of the other statements. □

From these results it follows that for this group action there exists a categorical quotient for the category of affine varieties but not for the category of arbitrary varieties. For a precise statement let us introduce the following notation:

Definition. Let $\mathscr{C}$ denote a category of varieties and let $V \in \mathscr{C}$ be a variety on which a group G acts. Then a variety $W \in \mathscr{C}$ with a G-invariant morphism $\tau: V \to W$ is called categorical quotient if for every variety $Z \in \mathscr{C}$ and every G-invariant morphism $\phi: V \to Z$ there exists a morphism $\psi: W \to Z$ such that $\phi = \psi \circ \tau$.

It follows now immediately from the preceding Lemma that the cubic variety C is the categorical quotient for the category of affine varieties. Now assume that there exists a categorical quotient W for the category of quasi-projective (or quasi-affine or abstract) varieties. Then we have morphisms $\tau: \mathbb{C}^4 \to W$ and $\psi: W \to C$ such that $\alpha = \psi \circ \tau$. Part (ii) of the Lemma and the G-invariantness of τ imply that ψ is injective. It follows that W is actually an open subvariety of C. Note that for every point $p \in C$ which is not contained in the image of α we have a G-invariant morphism $\alpha: \mathbb{C}^4 \to C \setminus \{p\}$. Since $C \setminus \{p\}$ is a quasi-projective variety, the definition implies that $p \notin W$. Thus W must coincide with the image of α in C. But this image is only a constructible set and not an open subvariety. Hence there does not exists a categorical quotient for the category of quasi-projective varieties.

6. Non-closed orbits

If $\mu: (\mathbb{C}, +) \times \mathbb{C}^n \to \mathbb{C}^n$ is a free algebraic action then it follows from the Constructible Set Theorem that all the orbits are constructible sets. Since the closure of a G-orbit is again G-stable, this implies that for a free algebraic group action every orbit is closed.

However, if μ is a free non-algebraic group action non-closed orbits may occur.

Lemma 11. *Let $\mu: (\mathbb{C}, +) \times \mathbb{C}^5 \to \mathbb{C}^5$ denote the group action defined by*

$$\begin{aligned}
x_1 &\mapsto \cos t x_1 + \sin t x_2, \\
x_2 &\mapsto -\sin t x_1 + \cos t x_2, \\
x_3 &\mapsto \cos \sqrt{2t}) x_3 + \sin(\sqrt{2t}) x_4, \\
x_4 &\mapsto -\sin(\sqrt{2t}) x_3 + \cos(\sqrt{2t}) x_4, \\
x_5 &\mapsto x_5 + t(1 - (x_1^2 + x_2^2)(x_3^2 + x_4^2)).
\end{aligned}$$

Then μ defines a free holomorphic $\mathbb{C}$-action on $\mathbb{C}^5$ where certain orbits are not closed.

Proof. Assume that $\mu(t,x)=x$ for $t\neq 0$. Then $1=(x_1^2+x_2^2)(x_3^2+x_4^2)$, hence $(x_1,x_2)\neq(0,0)\neq(x_3,x_4)$. Observe that

$$\det\left(\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}-I_2\right)=2-2\cos t.$$

Hence $\cos t=1$, i.e. $t\in 2\pi\mathbb{Z}$. But in the same way it follows that $\sqrt{2}t\in 2\pi\mathbb{Z}$. Hence $\mu(t,x)=x$ implies $t=0$, i.e. this is a free action.

Let $p=(1,0,1,0,0)$. Then $\mu(t,p)=(\cos t, \sin t, \cos\sqrt{2}t, \sin\sqrt{2}t, p)$. Now assume $\mu(t,p)\in\mathbb{R}^5$. Then $\sin t$, $\cos t\in\mathbb{R}$. Using $e^{it}=\cos t+i\sin t$ and $\sin^2 t+\cos^2 t=1$ we can deduce that $|e^{it}|=1$. Hence $t\in\mathbb{R}$. It follows that $G(p)\cap\mathbb{R}^5=\mu(\mathbb{R},p)\cong\mathbb{R}$ is not compact. On the other hand $\sin^2 t+\cos^2 t=1$ implies that $G(p)\cap\mathbb{R}^5$ is contained in the unit sphere of $\mathbb{R}^5$. Therefore $G(p)\cap\mathbb{R}^5$ cannot be closed. Hence $G(p)$ is a non-closed orbit. □

This example also has other strange properties. In particular there are very few invariant holomorphic functions.

Lemma 12. *Let $G=(\mathbb{C},+)$ act on $\mathbb{C}^5$ as described above. Let $\alpha_1=z_1^2+z_2^2$ and $\alpha_2=z_3^2+z_4^2$. Choose $x,y\in\mathbb{C}^5$ such that $\alpha_i(x)=\alpha_i(y)$ for $i=1,2$ and $\alpha_1(x)\alpha_2(x)\cong 1$. Assume furthermore that x,y are not contained in the same G-orbit.*

Then every G-invariant holomorphic function f on $\mathbb{C}^5$ has the same values at x and y, but there exists a G-invariant C^∞-function g on $\mathbb{C}^5$ such that $g(x)\neq g(y)$.

Proof. Let $E=\{x\in\mathbb{C}^5\,|\,\alpha_1(x)\alpha_2(x)=1\}$ and $\Omega=\mathbb{C}^5\backslash E$. Then every G-orbit in Ω is closed and intersects $H=\{z\in\mathbb{C}^5\,|\,z_5=0\}$ in exactly one point. Hence the G-invariant functions on Ω corresponds to the functions on $H\cap\Omega$. Using C^∞-functions on Ω which vanish identically near $H\cap E$, it is therefore clear that there does exist an invariant C^∞-function g on Ω with $g(x)\neq g(y)$ which extends to an invariant function on $\mathbb{C}^5$.

The following lemma shows that $f(x)=f(y)$ for all $f\in\mathcal{O}(\mathbb{C}^5)^G$. □

Lemma 13. *Under the above assumptions $\mathcal{O}(\mathbb{C}^5)^G=\mathbb{C}[[\alpha_1,\alpha_2]]$.*

Proof. Let Ω, H, E be the same as above. Since every G-orbit in Ω intersects H in exactly one point it is clear that every G-invariant holomorphic function f on Ω can be developed in a power series in $\beta_1,\dots,\beta_4$, where the β_i are holomorphic functions on Ω such that for $z\in\Omega$ the point $(\beta_1(z),\dots,\beta_4(z),0)$ is contained in the G-orbit through z. Thus

$$\beta_1(z)=\cos\left(\frac{z_5}{\alpha_1(z)\alpha_2(z)-1}\right)z_1+\sin\left(\frac{z_5}{\alpha_1(z)\alpha_2(z)-1}\right)z_2$$

etc. Now we consider the differential operator $\Theta=i(\alpha_1(z)\alpha_2(z)-1)\frac{\partial}{\partial z_5}$. This operator stabilizes both $\mathcal{O}(\Omega)^G$ and $\mathcal{O}(\mathbb{C}^5)$ and therefore also the intersection which is exactly $\mathcal{O}(\mathbb{C}^5)^G$. Let $\gamma_1=\beta_1+i\beta_2$, $\gamma_2=\beta_1-i\beta_2$, $\gamma_3=\beta_3+i\beta_4$, $\gamma_4=\beta_3-i\beta_4$. Then we obtain $\Theta\gamma_1=\gamma_1$, $\Theta\gamma_2=-\gamma_2$, $\Theta\gamma_3=\sqrt{2}\gamma_3$, and $\Theta\gamma_4=-\sqrt{2}\gamma_4$.

Let $f \in \mathcal{O}(\mathbb{C}^5)^G$ be an Θ-eigenfunction for the eigenvalue λ. Then $\frac{\partial}{\partial z_5} f = \frac{\lambda}{\alpha_1\alpha_2 - 1} f$. Since $\frac{\partial}{\partial z_5}\alpha_i = 0$ it follows that $\left[\frac{\partial}{\partial z_5}\right]^n f = \left(\frac{\lambda}{\alpha_1\alpha_2 - 1}\right)^n f$. This implies that if $\lambda \neq$), then f must vanish of arbitrarily high order on $E = \{\alpha_1\alpha_0 - 1 = 0\}$. Hence a holomorphic function $f \in \mathcal{O}(\mathbb{C}^5)^G$, $f \not\equiv 0$, can be a Θ-eigenfunction only for the eigenvalue 0.

Now observe that $\gamma_1^k\gamma_2^l\gamma_3^m\gamma_4^n$ is a Θ-eigenfunction for the eigenvalue $(k-l) + \sqrt{2}(m-n)$. This implies that a holomorphic function $f \in \mathcal{O}(\mathbb{C}^5)$ which can be expressed as a power series in the γ_i can in fact already be expressed as a power series in the two functions $\gamma_1\gamma_2$ and $\gamma_3\gamma_4$. This completes the proof, since $\gamma_1\gamma_2 = \alpha_1$ and $\gamma_3\gamma_4 = \alpha_2$. □

7. A quotient containing a torus

For a free *algebraic* $\mathbb{C}$-action on $\mathbb{C}^n$ Fauntleroy and Magid [FM] have proved that if the quotient is Hausdorff then it is already a quasi-affine variety and hence in particular holomorphically separable.

This result does not hold for non-algebraic actions; we will give an example of a free holomorphic $\mathbb{C}$-action on $\mathbb{C}^5$ where the quotient is a complex manifold which contains a one-dimensional complex torus as closed submanifold.

Proposition. *Let μ be the $\mathbb{C}$-action on $\mathbb{C}^5$ defined by*

$$\mu(t): (z_1, \ldots, z_5) \mapsto (z_1 e^{it}, z_2 e^{-it}, z_3 e^t, z_4 e^{-t}, z_5 + t(1 - z_1z_2z_3z_4)).$$

Let X denote the induced quotient space. Then X is Hausdorff and moreover a complex manifold containing a one-dimensional complex torus as closed complex submanifold.

Proof. Let $\phi = z_1z_2z_3z_4$, $U_0 = \{z \in \mathbb{C}^5 \mid \phi \neq 0\}$ and $U_1 = \{z \mid \phi \neq 1\}$. Then ϕ is an invariant function and the U_i are invariant open sets. We will first prove that the restricted quotients $I_i/\sim$ are Hausdorff.

To prove this for U_1 let $H = \{z \in U_1 \mid z_5 = 0\}$. Note that every orbit in U_1 intersects H in exactly one point, because $(1 - z_1z_2z_3z_4) \neq 0$ for all $z \in U_1$. Therefore $H \cong U_i/\sim$ are Hausdorff.

Next consider U_0. Observe that $U_0 = \{z \mid z_1, z_2, z_3, z_4 \in \mathbb{C}^*\}$. Let $z_1 = \theta_1 e^{p_1}$ and $z_2 = \theta_2 e^{p_2}$ with $\theta_i \in S^1$ and $p_i \in \mathbb{R}$. Then we have a diffeomorphism from U_0 to $Y = S^1 \times S^1 \times \mathbb{R}^2 \times (\mathbb{C}^*)^2 \times \mathbb{C}$ given by

$$z \mapsto (\theta_1 e^{-ip_2}, \theta_2 e^{ip_1}, p_1, p_2, z_1z_2, \phi, z_5 + (ip_1 - p_2)(1 - \phi)).$$

By explicit calculations one can now show that the induced $\mathbb{C}$-action on Y is given by

$$t: (\varrho_1, \varrho_2, p_1, p_2, w_1, w_2, x) \mapsto (\varrho_1, \varrho_2, p_1 - \Im\mathrm{m}\, t, p_2 + \Re\mathrm{e}\, t, w_1, w_2, x).$$

Therefore $U_2/\sim$ is diffeomorphic to $S^1 \times S^1 \times (\mathbb{C}^*)^2 \times \mathbb{C}$ and in particular is Hausdorff.

Now let $p, q \in \mathbb{C}^5$ be two points which do not lie in the same orbit. If both of these two points are contained in the same U_i then they have invariant disjoint open neighbourhoods because the quotients $U_i/\sim$ are Hausdorff. But if neither both are contained in U_0 nor both are contained in U_1 then by the definition of the U_i it follows that $\phi(p) \neq \phi(q)$, which implies that p and q have invariant disjoint open neighbourhoods. Therefore p and q can be separated by invariant open neighbourhoods in any case. Thus $\mathbb{C}^5/\sim$ is Hausdorff.

By [H] it now follows that the quotient is a complex manifold.

Finally let $E = \{z \in \mathbb{C}^5 \mid z_1 z_2 = 1,\ z_3 z_4 = 1,\ z_5 = 0\}$. Then E is invariant and biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and it is easy to check that $E/\sim$ is a one-dimensional torus. □

8. On a special class of quotients

From the above examples it follows that without additional assumptions one can not hope for nice quotients. In this section we will introduce a rather strong condition which in particular guarantees that the quotient is a holomorphically separable complex manifold.

Proposition. *Let $\mu : (\mathbb{C}, +) \times \mathbb{C}^n \to \mathbb{C}^n$ define a holomorphic group action on $X = \mathbb{C}^n$. Assume that $\mu(t, z) = z + t\psi(z)$ for some $\psi \in \mathcal{O}^n(\mathbb{C}^n)$. Assume furthermore that this is a free action, i.e. $\psi(z) \neq 0$ for all z.*

Then

(i) *The group action is proper, i.e. the induced map $\phi : G \times X \to X \times X$ given by $\phi(t, x) = (\mu(t, x), x)$ is a proper map.*

(ii) *There is a G-invariant holomorphic map $\alpha : \mathbb{C}^n \to (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{\frac{n(n-1)}{2}}$ with rank $n-1$ everywhere given by the functions ψ_i and $\chi_{ij} = z_i \psi_j - z_j \psi_i$.*

(iii) *The quotient space X/G has a natural complex structure and is biholomorphic to a closed smooth complex-analytic subset of $(\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{\frac{n(n-1)}{2}}$.*

(iv) *If ψ is an algebraic morphism then the quotient space is a quasi-affine variety such that $\mathscr{A}(X/G) = \mathscr{A}(X)^G$.*

(v) *The projection $\pi : X \to X/G$ is a $\mathbb{C}$-principal bundle. If ψ is algebraic then π is locally trivial even with respect to the algebraic Zariski-topology.*

Proof. To show that the action is proper it suffices to prove that for all $R > 0$ the set

$$G_R = \{t \in G \mid \exists x, y \in X : \|x\|, \|y\| \leqq R, \mu(t, x) = y\}$$

is bounded. But $x, y \in X$ with $\|x\|, \|y\| \leqq R$ and $\mu(t, x) = y$ implies that $\|t\psi(x)\| \leqq 2R$. Recall that we assumed that the group action is free. Hence $x \to \|\psi(x)\|$ is a *nowhere vanishing* function on X and has a non-zero minimum $C > 0$ on the compact set $\{x \in X \mid \|x\| \leqq R\}$. Hence $t \in G_R$ implies $|t| \leqq \frac{2R}{C}$, i.e. G_R is bounded. Thus the action is proper and in particular by [H] the quotient space is Hausdorff and moreover a complex manifold.

That the functions χ_{ij} are G-invariant follows immediately from the definition. To see that the functions ψ_i are also invariant, observe that

$$x + (s + t)\psi(x) = x + t\psi(x) + s\psi(\mu(t, x))$$

for all s, $t \in G$, $x \in X$, because μ defines a group action. It follows that $\psi(x) = \psi(\mu(t, x))$ for all $t \in G$.

Next we want to verify that the invariant functions ψ_i and χ_{ij} separate the G-orbits. Let $x, x' \in X$ such that $\psi(x) = \psi(x')$ and $\chi(x) = \chi(x')$ where $\chi: X \to \mathbb{C}^{\frac{n(n-1)}{2}}$ denotes the map given by all functions χ_{ij} with $1 \leqq i < j \leqq n$. Since ψ is nowhere vanishing, we may wlog assume that $\psi_1(x) \neq 0$. Then there exist $y, y' \in X$ such that $y \in G \cdot x$, $y' \in G \cdot x'$ and $y_1 = y'_1 = 0$. Now

$$-y_j \psi_1(y) = \chi_{ij}(y) = \chi_{ij}(y') = -y'_j \psi_1(y')$$

implies that $y_j = y'_j$ for all j. Hence $y = y'$ and consequently $x' \in G \cdot x$.

Thus ψ and χ yield a holomorphic map $\alpha: X \to (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{\frac{n(n-1)}{2}} = Y$ such that the fibers of α are exactly the G-orbits. This gives an injective holomorphic map $\beta: X/G \to Y$. Our next step is to show that α has rank $n-1$ everywhere in order to show that β is locally biholomorphic. Let $x \in X$ and $\mathbf{X} \in T_x(X)$. Assume that $D\alpha(\mathbf{X}) = 0$. Then $d\psi_i(\mathbf{X}) = 0$ and $d\chi_{ij}(\mathbf{X})$ for all i, j. With $\chi_{ij} = z_i \psi_j - z_j \psi_i$ it follows that $\psi_j dz_i(\mathbf{X}) - \psi_i dz_j(\mathbf{X}) = 0$. This implies that $\mathbf{X}$ is a scalar multiple of the tangent vector $\sum_i \psi_i \frac{\partial}{\partial z_i}$. Hence $\mathbf{X}$ is tangent to the G-orbit through x. It follows that α has everywhere rank $n-1$.

Next we show that $\beta: X/G \to Y$ is a proper map. For this purpose it suffices to verify that each point $y \in Y$ has an open neighbourhood $U(y)$ such that $\beta^{-1}(\overline{U(y)})$ is compact. Let $y = (y^1, y^2) \in (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{\frac{n(n-1)}{2}}$ and let $\| \ \|_\infty$ denote the maximum norm. Let $a = \|y^1\|_\infty > 0$ and $C = \|y^2\|_\infty$. We may wlog assume $|y_1^1| = a$. Choose $\varepsilon \in \mathbb{R}$ such that $a > \varepsilon > 0$. Let

$$U_\varepsilon(y) = \{y' \in Y \mid \|y' - y\|_\infty < \varepsilon\}.$$

Assume that $y' \in U_\varepsilon(y) \cap \alpha(X)$. Then $y' = \alpha(x)$ for some $x \in X$. Since $|y_1^1| = a$ and $a > \varepsilon$ it is clear that $\psi_1(x) \neq 0$. Hence there exists an element $x' \in G \cdot x$ such that $x'_1 = 0$. Now $\chi_{1,j}(x') = x'_j \psi_1(x')$. Observe that $\alpha(x') \in U_\varepsilon(y)$ implies that $|\chi_{1,j}(x')| \leqq C + \varepsilon$ and $|\psi_1(x')| \geqq a - \varepsilon$. It follows that $|x'_j| \leqq \frac{C+\varepsilon}{a-\varepsilon}$ for all j. Therefore

$$U_\varepsilon(y) \cap \alpha(X) = U_\varepsilon(y) \cap \alpha(K),$$

where $K = \left\{x \in X \mid \|x\|_\infty \leqq \frac{C+\varepsilon}{a-\varepsilon}\right\}$. This implies that $B = \beta^{-1}(\overline{U_\varepsilon(y)}) \subset X/G$ is a closed subset of X/G which is contained in the image of the compact set K under the projection $\pi: X \to X/G$. Hence B is compact. It follows that $\beta: X/G \to Y$ is a proper holomorphic map. Therefore the image of β is a closed complex-analytic subset of Y. Since β is moreover locally biholomorphic it follows that X/G is biholomorphic to a closed analytic subset of Y.

To show that $\pi: X \to X/G$ is a locally trivial $\mathbb{C}$-principal bundle, it suffices to show that for any $y \in X/G$ there exists an open neighbourhood U and a cross-section $\sigma: U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = id_U$, because then $\phi: U \times \mathbb{C}$ defined by $\phi(y, t) = \mu(t, \sigma(y))$ gives a local trivialization. For the euclidean topology the existence of local cross-sections follows immediately from the above proven fact that the projection has rank $n-1$ everywhere. Now assume that ψ is algebraic. Let $y \in X/G$, $x \in \pi^{-1}(y)$ and H a hyperplane intersecting $G \cdot x$ transversely at x. Observe that $A = \{p \in H \mid \mathrm{rk}(D\pi)_p \leqq n-2\}$ is an algebraic variety in H. Since both $\pi(A)$ and $(X/G) \setminus \pi(H)$ are constructible subsets of X/G, it follows that there exist Zariski-open subsets $U \subset X/G$ and $V = \pi^{-1}(U) \subset H$ such that the restricted projection $\pi|_V : V \to U$ is surjective and has rank $n-1$ everywhere. Hence $\pi|_V : V \to U$ is a finite morphism. Note that $\mu(t, x) = x + t\psi(x)$ implies that every G-orbit in $X = \mathbb{C}^n$ is an affine line. Since V is an open subset of a hyperplane of $\mathbb{C}^n$ it follows that for every G-orbit $G \cdot q$ the assumption $\#(G \cdot q \cap V) < \infty$ implies that $\#(G \cdot q \cap V) \leqq 1$. Thus $\pi|_V : V \to U$ is a bijective morphism. Since $U \subset X/G$ is smooth and in particular normal, we have the desired cross-section $\sigma = \pi^{-1} : U \to V = \pi^{-1}(U)$. □

Remarks. (i) The examples of Sects. 2 and 4 show that this class contains quotient manifolds which are not biholomorphic to $\mathbb{C}^{n-1}$.

(ii) The example of Sect. 5 shows that for group actions of the form $\mu(t, z) = z + t\psi_1(z) + t^2\psi_2(z)$ it is already possible that the quotient is not Hausdorff.

(iii) Statement (v) of the proposition is a special case of a more general fact. If $\mu : \mathbb{C} \times X \to X$ is a free algebraic $\mathbb{C}$-action on an algebraic variety X such that the quotient is a quasi-affine variety then the projection map is always an algebraic principal bundle which is locally trivial in the Zariski-topology [Se].

9. Triangular actions on $\mathbb{C}^3$

Here we want to study a special situation in which one can give precise conditions for the quotient to be Hausdorff.

Let us call a group action $\mu : G \times \mathbb{C}^n \to \mathbb{C}^n$ triangular iff $\mu_i(t, z) = z_i + f_i(t, z_1, \ldots, z_{i-1})$ for all i.

In [S2] Snow has proved that for every free algebraic triangular $\mathbb{C}$-action on $\mathbb{C}^3$ the geometric quotient space is naturally isomorphic to $\mathbb{C}^2$. We would like to mention that there *do* exist free *holomorphic* triangular $\mathbb{C}$-actions on $\mathbb{C}^3$ with all orbits closed such that the geometric quotient space is not Hausdorff.

Lemma 14. *Let $\mu : \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ be a holomorphic group action induced by a holomorphic vector field $\mathbf{X} = f(z_1)\frac{\partial}{\partial z_2} + g(z_1, z_2)\frac{\partial}{\partial z_3}$. Assume that μ is a free action, i.e. $\mathbf{X}(z) \neq 0$ for all z.*

Let $G(z_1, z_2) = \int_0^{z_2} g(z_1, x)\,dx$, $\alpha_1 = z_1$, $\alpha_2 = z_3 f(z_1) - G(z_1, z_2)$. If f vanishes identically, then $\mathcal{O}(\mathbb{C}^3)^G = \mathbb{C}[[z_1, z_2]]$ and consequently $\mathbb{C}^3/G \cong \mathbb{C}^2$. Otherwise the following assertions are fulfilled:

(i) *$\mathcal{O}(\mathbb{C}^3)^G = \mathbb{C}[[\alpha_1, \alpha_2]] \cong \mathcal{O}(\mathbb{C}^2)$.*

(ii) *For every $p_1 \in f^{-1}(0)$ the map $\zeta_{z_1} : \mathbb{C} \to \mathbb{C}$ defined by $\zeta_{z_1} : z_2 \mapsto G(z_1, z_2)$ is locally biholomorphic, i.e. has a nowhere vanishing derivative.*

(iii) *The geometric quotient space $\mathbb{C}^3/G$ is Hausdorff if and only if for all $z_1 \in f^{-1}(0)$ the map ζ_{z_1} is biholomorphic.*

(iv) *If $\mathbb{C}^3/G$ is Hausdorff then α induces an isomorphism $\mathbb{C}^3/G \cong \mathbb{C}^2$.*

Proof. An easy computation show that α_1 and α_2 are invariants. Moreover, the assumption that μ is a free action implies that f, g never vanish simultaneously. Hence

$$d\alpha_1 \wedge d\alpha_2 = f(z_1)dz_1 \wedge dz_2 + g(z_1, z_2)dz_1 \wedge dz_3$$

is nowhere vanishing. Thus $\alpha : \mathbb{C}^3 \to \mathbb{C}^2$ has maximal rank and is an open map. For $p_1 \notin f^{-1}(0)$ we have

$$\alpha^{-1}(p) = \left\{ z \in \mathbb{C}^3 \mid z_1 = p_1, z_3 = \frac{p_2 + G(z_1, z_2)}{f(p_1)} \right\} \cong \mathbb{C}.$$

Hence for $p_1 \notin f^{-1}(0)$ the α-fibers are exactly the G-orbits. For $p_1 \in f^{-1}(0)$ we obtain

$$\alpha^{-1}(p) = \{ z \in \mathbb{C}^3 \mid z_1 = p_1, -\zeta_{z_1}(z_2) = p_2 \}.$$

Now $\zeta_{z_1}(z_2) = G(z_1, z_2)$ implies $d\zeta_{z_1} = g(z_1, z_2)dz_2$. Hence $\zeta_{z_1} : \mathbb{C} \to \mathbb{C}$ is locally biholomorphic for $z_1 \in f^{-1}(0)$. If ζ_{z_1} is biholomorphic for all $z_1 \in f^{-1}(0)$ then all the α-fibers are isomorphic to $\mathbb{C}$ and therefore coincide with the G-orbits. Hence in this case $\mathbb{C}^3/G \cong \mathbb{C}^2$ and the proof is finished.

If ζ_{z_1} is not bijective then it has necessarily an essential singularity at infinity. By the Picard Theorem it follows that, with at most one exception, all of the fibers are infinite. In particular ζ_{z_1} is not injective. Thus we have an α-fiber which is not connected. Let z, z' be two points in different connected components of the same α-fiber. In particular z and z' are not in the same G-orbit. Recall that $\alpha : \mathbb{C}^3 \to \mathbb{C}^2$ is an open map and that for all $p_1 \in f^{-1}(0)$ the fiber $\alpha^{-1}(p)$ consists of a single G-orbit only. It follows that for all open neighbourhoods $U(z)$, $V(z')$ there exists a G-orbit intersecting both $U(z)$ and $V(z')$. Consequently z and z' do not have disjoint G-invariant open neighbourhoods, i.e. $\mathbb{C}^3/G$ is not Hausdorff. Furthermore, it follows that every continuous G-invariant function is constant along the α-fibers. Since α has maximal rank, we obtain $\alpha_* : \mathcal{O}(\mathbb{C}^3)^G \cong \mathcal{O}(\mathbb{C}^2)$. □

Remark. Since a polynomial with nowhere vanishing derivative is already an affine-linear function, we obtain as a corollary that every free algebraic triangular $\mathbb{C}$-action on $\mathbb{C}^3$ has $\mathbb{C}^2$ as quotient.

We now mention an explicit example where the quotient is not Hausdorff. Define

$$\mu(t) : \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 + tz_1 \\ z_3 + e^{z_2} \left[\sum_{k=1}^{\infty} \frac{t^k z_1^{k-1}}{k!} \right] \end{pmatrix}.$$

This is a free action induced by $X = z_1 \frac{\partial}{\partial z_2} + e^{z_2} \frac{\partial}{\partial z_1}$. We obtain $\alpha = (z_1, z_3 z_1 - e^{z_2})$ and $\zeta_0(z_2) = e^{z_2}$. Thus by the lemma the quotient is not Hausdorff. This can also be seen explicitly: For all $z \in \mathbb{C}$, $k \in \mathbb{Z} \setminus \{0\}$ the points $(0, z, 0)$ and $(0, z + 2\pi ik, 0)$ are contained in different orbits, but for all $\varepsilon \neq 0$ the points $(\varepsilon, z, 0)$ and $(\varepsilon, z + 2\pi ik, 0)$ are contained in the same orbit, because $\mu(t, (\varepsilon, z, 0)) = (\varepsilon, z + 2\pi ik, 0)$ for $t = \frac{2\pi ik}{\varepsilon}$. Thus $(0, z, 0)$ and $(0, z + 2\pi ik, 0)$ can not be separated by G-invariant open sets.

10. Parametrized $\mathbb{C}$-actions

Here we consider parametrized group actions, i.e. holomorphic mappings $\mu: P\times G\times X\to X$ where P is a complex manifold and for all $p\in P$ the evaluation map $\mu(p): G\times X\to X$ is a group action. A class of group actions is called *rigid* if for all such parametrized group actions the group actions $\mu(p)$ and $\mu(q)$ are isomorphic for all $p, q\in P$. We will see that free holomorphic $\mathbb{C}$-actions on $\mathbb{C}^n$ are not rigid. First consider the following example. Let $P=\mathbb{C}$, $G=(\mathbb{C}, +)$, $X=\mathbb{C}^5$ and define μ as follows:

$$\mu:(p;t;w_1,\ldots,w_5)\mapsto(w_1,w_2,w_3+ptw_1,w_4+ptw_2,w_5+t(1+p^2D_w))$$

with $D_w=\det\begin{pmatrix} w_1 & w_3\\ w_2 & w_4\end{pmatrix}$. Using the results of Sect. 2 above we see that for $p\neq 0$ the quotient is always the same four-dimensional non-Stein manifold, although for $p=0$ the quotient is simply $\mathbb{C}^4$.

For the construction and discussion of further examples we need some auxiliary results on triangular actions on $\mathbb{C}^3$.

Lemma 15. *For all $f\in\mathcal{O}(\mathbb{C})$, $g\in\mathcal{O}(\mathbb{C}^2)$ the vector field $\mathbf{X}=f(z_1)\frac{\partial}{\partial z_2}+g(z_1,z_2)\frac{\partial}{\partial z_3}$ is globally integrable on $\mathbb{C}^3$, i.e. induced by a group action.*

Proof. The vector field $\mathbf{X}$ is induced by a group action $\mu:\mathbb{C}\times\mathbb{C}^3\to\mathbb{C}^3$ if

$$\mu(t,z)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{X}^k z. \qquad (*)$$

$\left[\text{This follows from } \frac{\partial}{\partial t}\mu(t,z)\Big|_{t=0}=\mathbf{X}(z)\right]$. Thus we have to show that, for the special case of triangular actions on $\mathbb{C}^3$, the power series $(*)$ converges. Let $G(z_1,z_2)=\int_0^{z_2} g(z_1,x)dx$. Now a comparison of the power series $(*)$ with the Taylor series for G yields:

$$\mu\left(t,\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix}\right)=\begin{pmatrix} z_1\\ z_2+tf(z_1)\\ z_3+\dfrac{1}{f(z_1)}[G(z_1,tf(z_1)+z_2)-G(z_1,z_2)]\end{pmatrix}$$

for $f(z_1)\neq 0$ and $\mu(t,z)=(z_1,z_2,z_3+tg(z_1,z_2))$ for $f(z_1)=0$. In particular this power series always converges, i.e. $\mathbf{X}$ is globally integrable. □

For a triangular $\mathbb{C}$-action μ induced by a vector field $\mathbf{X}=f(z_1)\frac{\partial}{\partial z_2}+g(z_1,z_2)\frac{\partial}{\partial z_3}$, we define an exceptional set E. Let $E=\emptyset$, if f vanishes identically. Otherwise define

$$E=\{z\in\mathbb{C}\mid f(z)=0 \text{ and } \zeta_z \text{ is not bijective}\}$$

with $\zeta_z: \mathbb{C} \to \mathbb{C}$ defined by $\zeta_z(x) = G(z, x)$. This definition seems to depend on the coordinate system used. However, the following lemma shows that this is inessential.

Lemma 16. *For $i = 1, 2$ let μ_i be triangular $\mathbb{C}$-actions on $\mathbb{C}^3$ induced by vector fields $X_i = f_i(z_1)\frac{\partial}{\partial z_2} + g_i(z_1, z_2)\frac{\partial}{\partial z_3}$. Let E_i denote the corresponding exceptional sets defined as above. Assume that μ_1 and u_2 are isomorphic, i.e. conjugate in* $\mathrm{Aut}_{\mathcal{O}}(\mathbb{C}^3)$.
Then there exists an automorphism $\xi \in \mathrm{Aut}_{\mathcal{O}}(\mathbb{C})$ such that $E_2 = \xi(E_1)$.

Proof. For $i = 1, 2$ define F_i^0 to be the set of points in $\mathbb{C}^2$ where the quotient is not Hausdorff. To be more precise $x \in F_i^0$ iff there exists a point $y \in \mathbb{C}^3$ such that $f(x) = f(y)$ for all $f \in \mathcal{O}(\mathbb{C}^3)^G$, but $y \notin G \cdot z$ for the respective G-action μ_i. Let F_i denote the closure of F_i^0. Then $F_i = \{z \in \mathbb{C}^3 \mid z_1 \in E_i\}$. Now $\mu_1 \sim \mu_2$ implies that there exists a holomorphic automorphism $\psi: \mathbb{C}^3 \to \mathbb{C}^3$ such that $\psi(F_1) = F_2$. Hence in particular F_1 and F_2 have the same number of connected components, i.e. E_1 and E_2 have the same number of elements. If $\# E_1 \leqq 2$ there obviously exists an automorphism $\xi \in \mathrm{Aut}_{\mathcal{O}}(\mathbb{C})$ such that $\xi(E_1) = E_2$. On the other side for $\# E_1 \geqq 2$ the projection $\pi: \mathbb{C}^3 \backslash F_i \to \mathbb{C} \backslash E_i$ given by $\pi(z) = z_1$ is a canonically defined holomorphic map: $\pi(p) = \pi(q)$ for $p, q \in \Omega_i = \mathbb{C}^3 \backslash F_i$ if and only if $d_{\Omega_i}(p, q) = 0$ where d_{Ω_i} denotes the Kobayashi-pseudodistance of Ω_i. (This follows since $\mathbb{C} \backslash E_i$ is hyperbolic for $\# E_i \geqq 2$ and $d_{\mathbb{C}^2} \equiv 0$). Thus ψ may be pushed down to a biholomorphic map $\pi_* \psi = \xi: \mathbb{C} \to \mathbb{C}$ with $\xi(E_1) = \xi(E_2)$. □

Now we have the necessary tools to prove the following statements on parametrized $\mathbb{C}$-actions on $\mathbb{C}^3$.

Lemma 17. *There are parametrized group actions $P \times G \times \mathbb{C}^3 \to \mathbb{C}^3$ with $\dim_{\mathbb{C}}(P) \geqq 1$ such that for all $p, q \in P$ with $p \neq q$ the group actions $\mu(p)$, $\mu(q)$ are not conjugate in* $\mathrm{Aut}_{\mathcal{O}}(\mathbb{C}^3)$.

Proof. Let P be the unit disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$. Then the parametrized group action $\Delta \times \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ induced by the vector fields

$$X(p) = (2 + i - z_1)(2 + 2i - z_1)(p - z_1)\frac{\partial}{\partial z_2} + e^{z_2}\frac{\partial}{\partial z_3}$$

has the desired properties, since the respective exceptional sets $E_p = \{2 + i, 2 + 2i, p\}$ are all non-isomorphic for $p \in \Delta$. □

Lemma 18. *Let P be a complex manifold and let $h_1, \ldots, h_k$ be holomorphic functions on P. Then there exists a parametrized group action $\mu: P \times G \times \mathbb{C}^3 \to \mathbb{C}^3$ such that the quotient of $\mathbb{C}^3$ by the $\mu(p)$-action is Hausdorff if and only if $h_1(p) = \ldots = h_k(p) = 0$.*

Proof. Take $X(p) = z_1\left(\sum_i h_i(p) z_1^i\right)\frac{\partial}{\partial z_2} + e^{z_2}\frac{\partial}{\partial z_3}$. □

Lemma 19. *Let $\mu: P \times \mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3$ by a parametrized group action induced by a parametrized vector field $X(p) = f(p, z_1)\frac{\partial}{\partial z_2} + g(p, z_1, z_2)\frac{\partial}{\partial z_3}$. Let W denote the set of all $p \in P$ such that the quotient of $\mathbb{C}^3$ by the $\mu(p)$-action is not Hausdorff. Then W is an open subset in P.*

Proof. Let $V=\{(p,w)\in P\times\mathbb{C}\mid f(p,w)=0, f(p,\cdot)\not\equiv 0\}$. Let $U=\pi(V)\subset P$ where π is the natural projection of $P\times\mathbb{C}$ on P. From the Weierstraß Preparation Theorem it follows that W is open in P and that $\pi: V\to W$ is an open map. Consider an element $(p,w)\in V$. Then $p\in W$ iff the holomorphic function $\zeta_{p,w}:\mathbb{C}\to\mathbb{C}$ is not bijective. Let $\zeta_{p,w}=\sum_k a_k(p,w)x^k$. Since $\zeta_{p,w}$ has nowhere vanishing derivative it is clear that $\zeta_{p,w}$ is bijective iff $a_k(p,w)=0$ for all $k\geqq 2$. Thus $p\in W$ implies that for some k we have $a_k(p,w)\neq 0$. It follows that for all (p',w') in some open neighbourhood of (p,w) in V the map $\zeta_{p',w'}$ is not bijective. Since $\pi: V\to U$ is an open map it furthermore follows that for all $p\in W$ there exists an open neighbourhood in P which is already contained in W. Hence W is open. □

Thus, in the special case of triangular actions on $\mathbb{C}^3$, it is an open property for a group action to have a non-Hausdorff quotient. But for more general $\mathbb{C}$-actions on $\mathbb{C}^n$ it might be just the other way around, i.e. it is possible that for a parametrized group action the quotient is simply $\mathbb{C}^{n-1}$ for a dense open subset in the parameter space P, but non-Stein or non-Hausdorff for some small analytic subset in P.

Lemma 20. *Let $\mu_0:\mathbb{C}\times\mathbb{C}^n\to\mathbb{C}^n$ be an arbitrary free group action. Then there exists a parametrized group action $\psi: P\times\mathbb{C}\times\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ with $P\cong\mathbb{C}$ such that the quotient is $\mathbb{C}^n$ for $p\neq 0$ and $\mathbb{C}\times(\mathbb{C}^n/\mu_0)$ for $p=0$.*

Proof. Let $\psi: P\times\mathbb{C}\times\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ be defined by $\psi(p,t,z,w)=(\mu_0(t,z), w+tp)$. □

11. Stein homogeneous manifolds

For a reductive complex Lie group G Matsushima has proved [M1] that a quotient $X=G/H$ of G by a closed Lie subgroup H of G is a Stein manifold if and only if H is reductive. He furthermore considered the more general situation where X is the quotient of an arbitrary connected complex Lie group by a *connected* closed Lie subgroup H. We may always assume that G acts almost effectively on G/H, i.e. that

$$L=\{h\in G\mid g\cdot h\cdot g^{-1}\in H\,\forall g\in G\}$$

is discrete. Under this assumption the Lie group H has a unique maximal normal simply-connected solvable Lie subgroup U so that H/U is reductive [M2]. (If H is a linear-algebraic group then U is just the unipotent radical.) Now $G/U\to G/H$ is a principal bundle with a reductive group as structure group. Hence G/H is Stein if and only if G/U is Stein [MM]. Let M be a maximal connected reductive Lie subgroup of G. If G/U is Stein then the closed M-orbits are obviously Stein. By Richardson [R, Proposition 3.1.] the minimal M-orbits are closed. Since M is reductive, the M-orbit $M/(M\cap U)$ is Stein if and only if its isotropy $M\cap U$ is reductive. But $M\cap U$ is reductive if and only if $M\cap U=\{e\}$ because a simply-connected solvable Lie group can not contain any non-trivial reductive Lie group as Lie subgroup. Thus the assumption that G/U is Stein implies that M acts freely on the minimal M-orbits in G/U. Then it follows that the minimal orbits are already maximal, i.e. all M-orbits are minimal. Thus for G/H being Stein it is necessary that the following condition is fulfilled:

Condition *M. Let G be a complex Lie group, H a connected Lie subgroup which contains no positive-dimensional normal Lie subgroup of G, M a maximal connected reductive Lie subgroup of G and U a maximal connected, simply-connected, solvable, normal Lie subgroup of H.*

Then G and H fulfills Condition M if and only if for all $g \in G$ the intersection of U with $M^g = g \cdot M \cdot g^{-1}$ contains only the neutral element.

The question is now whether this condition is also sufficient to ensure that G/H is a Stein manifold. Matsushima proved that it is indeed sufficient if G is nilpotent [M2]. Snow showed that it is also sufficient if G is solvable [S1]. Furthermore it is sufficient if $\dim_{\mathbb{C}}(G/M) \leqq 3$ [S2].

Moreover G/U is Stein if and only if the double coset space $M\backslash G/U$ is a Hausdorff space and moreover a Stein space (by a result of Holmann [H]) $M\backslash G/U$ always has a natural complex structure if it is Hausdorff with respect to the quotient topology).

The assumption that $M^g \cap U$ is discrete for all $g \in G$ is equivalent to the assumption that M acts freely on G/U. Furthermore under this assumption the quotient space $M\backslash G$ is biholomorphic to some $\mathbb{C}^n$ and $M\backslash G/U$ is the quotient of a free action of the simply-connected solvable Lie group U on $M\backslash G \cong \mathbb{C}^n$. Now U has a composition series where all the factors are isomorphic to $(\mathbb{C}, +)$. Therefore in order to prove that Matsushima's criterion is sufficient in general it would suffice to show that every free holomorphic $\mathbb{C}$-action on $\mathbb{C}^n$ has $\mathbb{C}^{n-1}$ as quotient. But this is not true as we have seen by the examples above. In fact we can use the example in Sect. 2 to show that condition M in general does not ensure that G/H is Stein. For this let $G = \mathrm{Aff}(\mathbb{C}^6)$ and $H = U$ be the two-dimensional unipotent subgroup of G described in Sect. 2. In this case $M = GL_6(\mathbb{C})$ and the condition M is equivalent to the statement that U acts freely on $\mathbb{C}^6$. Hence the condition M is fulfilled. Now we have a $GL_6(\mathbb{C})$-principal bundle $G/U \to (\mathbb{C}^6)/U$ and the fact that $(\mathbb{C}^6/U)$ is not a Stein manifold (see Sect. 2) proves that G/U itself is not Stein. Thus in the condition M is not sufficient to guarantee that G/H is Stein.

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Received August 10, 1988; in revised form February 8, 1989