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Formal and rigid geometry

I. Rigid spaces

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When J. Tate introduced rigid spaces in [T], he was influenced by Grothendieck's idea of associating a generic fibre to a formal scheme satisfying certain finiteness conditions. After Tate's notes [T] had been communicated, considerable efforts were made, for example by the school of Grauert and Remmert, to develop rigid geometry in terms of analytic methods, in analogy to complex analysis. On the other hand, it was Raynaud [R] who suggested to view rigid spaces entirely within the framework of formal schemes. Both approaches have their advantages. Most notably, the approach through formal geometry allows the application of powerful methods from algebraic geometry, thus leading to rigorous solutions of various problems which, from a strictly analytic point of view, can only be dealt with in an ad-hoc-manner.

Apart from the colloquium talk [R], the approach to rigid geometry via formal schemes is not well-documented. It is our intention to elaborate the ideas of Raynaud in order to pave the way for accessing some interesting applications. The present paper is of introductory nature. Its purpose is to motivate and explain Raynaud's definition of the category of rigid spaces as a localization of the appropriate category of formal schemes by admissible formal blowing-ups. This way it is possible to define rigid spaces over complete noetherian rings, not just over complete valuation rings.

In Sects. 1 and 2 the basic results and constructions concerning admissible formal schemes are explained, among them the technique of admissible formal blowing-up. Then, in Sect. 3, we define rig-points of admissible formal schemes, which later on are interpreted as points of rigid spaces. In some sense, they provide the link between an admissible formal scheme and its associated rigid space. After having gathered these technical prerequisites, we show in Sect. 4 how to interpret classical rigid spaces in terms of formal schemes, in the way it was indicated by Raynaud in [R]. Finally, in Sect. 5 the approach is extended to Raynaud's relative rigid spaces over a global noetherian base.

One can start now and generalize classical rigid geometry to the relative case. The point of departure for any activity in this field is the basic result of Raynaud asserting that a flat rigid morphism admits a flat formal model. This theorem, which allows a foundation of rigid spaces different from the one given by Tate, will be the subject of the second part [FII] of this paper.

1 Admissible formal schemes

Let R be a ring with a *finitely generated* ideal $\mathfrak{I} \subset R$ such that R is *complete* and *separated* with respect to the $\mathfrak{I}$-adic topology. Then $\mathfrak{I}$ is contained in the Jacobson radical of R, in particular, each maximal ideal of R is open. Writing $\mathfrak{I} = (g_1, \ldots, g_r)$, we define the ideal of $\mathfrak{I}$-torsion in R as

$$(\mathfrak{I}\text{-torsion})_R = \{r \in R;\ \mathfrak{I}^n r = 0 \text{ for some } n \in \mathbb{N}\};$$

it equals the kernel of the canonical homomorphism

$$R \to \prod_{i=1}^{r} R[g_i^{-1}]\,.$$

We will generally assume that R has no $\mathfrak{I}$-torsion. The latter is equivalent to the fact that the open subscheme of Spec R, which is complementary to the closed subscheme $V(\mathfrak{I})$, is schematically dense in Spec R; for the notion of schematic closure see [EGA $\mathrm{I}_{\mathrm{new}}$, 6.10]. Furthermore, if R has no $\mathfrak{I}$-torsion, there is no associated prime ideal $\mathfrak{p} \in \operatorname{Ass} R$ which belongs to the closed subscheme $V(\mathfrak{I})$ of Spec R. Conversely, the latter property implies that R has no $\mathfrak{I}$-torsion if R is noetherian, cf. [EGA IV_2, 5.10.2]; for the notion of associated prime ideals see [AC, Chap. IV, Sect. 1].

Using rings of the above type as base rings, we will restrict ourselves to the following two cases:

(I) *The classical rigid case.* R is a valuation ring of Krull dimension 1, complete (and separated) with respect to the $\mathfrak{I}$-adic topology generated by a principal ideal $\mathfrak{I} = (\pi) \subset R$ where π is some non-zero element of the maximal ideal of R. The situation is induced from a complete non-trivial height 1 valuation on the field of fractions $Q(R)$.

(II) *The noetherian case.* R is a noetherian ring which is complete and separated with respect to the $\mathfrak{I}$-adic topology given by some ideal $\mathfrak{I} \subset R$. Included as a particular case is the case of (I) where R is a complete *discrete* valuation ring and where $\mathfrak{I}$ coincides with the maximal ideal $\mathfrak{m} \subset R$.

If $\xi = (\xi_1, \ldots, \xi_n)$ is a system of variables, we denote by $R\langle\xi\rangle = R\langle\xi_1, \ldots, \xi_n\rangle$ the R-algebra of strictly convergent power series (in the terminology of [BGR, 1.4]) or restricted formal power series in ξ (in the terminology of [AC, Chap. III, Sect. 4, no. 2]); it consists of all formal power series $\sum c_\nu \xi^\nu \in R[[\xi]]$ whose coefficients c_ν satisfy $\lim c_\nu = 0$. The algebra $R\langle\xi\rangle$ is noetherian if R is noetherian; use [AC, Chap. III, Sect. 2, no. 10, Corollary 5]. Furthermore, we claim that, in the above cases (I) and (II), $R\langle\xi\rangle$ is flat over R. In the classical rigid case, this follows from the fact that $R\langle\xi\rangle$ has no R-torsion, whereas in the noetherian case it is a consequence of the Bourbaki criterion of flatness; cf. [AC, Chap. III, Sect. 5, no. 2, Theorem 1]. In other cases, where R is not noetherian, it is difficult to decide whether or not $R\langle\xi\rangle$ is flat over R. In addition to the flatness we will need to know that $R\langle\xi\rangle$ is a coherent ring; see 1.3. Since the class of rings R satisfying these two properties for any set of variables ξ does not seem to be easily accessible, we have chosen to restrict ourselves to the two cases listed above.

An R-algebra A is called of *topologically finite type* (tf type) if it is isomorphic to a quotient $R\langle\xi\rangle/\mathfrak{a}$, where ξ is a finite set of variables and where $\mathfrak{a} \subset R\langle\xi\rangle$ is an ideal. If, in addition, $\mathfrak{a}$ is finitely generated, we call A of *topologically finite presentation* (tf presentation). For an R-algebra A of tf type, the fact that $\mathfrak{a}$ is finitely generated is independent of the choice of the isomorphism $A \simeq R\langle\xi\rangle/\mathfrak{a}$, as is easily seen. An R-algebra of tf presentation is called an *admissible R-algebra* if it has no $\mathfrak{I}$-torsion. Of course, in the noetherian case, there is no difference between R-algebras of tf type and tf presentation. Furthermore, we will show in the classical rigid case, see 1.1(c) below, that any R-algebra of tf type without $\mathfrak{I}$-torsion is of tf presentation and, thus, admissible. For example, $R\langle\xi\rangle$ is admissible, and $A\langle\xi\rangle$ is admissible if A is admissible. Also note that, in the classical rigid case, an R-algebra of tf type or presentation is admissible if and only if it has no R-torsion, i.e., if and only if it is flat over R. Morphisms between R-algebras of the above type are defined as R-algebra homomorphisms in the usual sense. Such morphisms are automatically continuous with respect to $\mathfrak{I}$-adic topologies.

Proposition 1.1. *Let $A = R\langle\xi\rangle/\mathfrak{a}$ be an R-algebra of* tf *type, and consider the $\mathfrak{I}$-adic topology on A.*

(a) *A is complete.*

(b) *If $\mathfrak{a}$ is finitely generated, $\mathfrak{a}$ is closed in $R\langle\xi\rangle$. In particular, any R-algebra of* tf *presentation is complete and separated with respect to the $\mathfrak{I}$-adic topology.*

(c) *If A has no $\mathfrak{I}$-torsion, $\mathfrak{a}$ is finitely generated. In particular, any R-algebra of* tf *type with no $\mathfrak{I}$-torsion is admissible.*

(d) *In the noetherian case, A is noetherian.*

Proof. Assertion (a) is trivial. Furthermore, in the noetherian case, assertion (b) is easily verified using the Lemma of Artin-Rees, whereas (c) and (d) are clear. To settle assertions (b) and (c) in the classical rigid case, we make use of Lemma 1.2 below. Since $R\langle\xi\rangle$ *is* $\mathfrak{I}$-adically separated, assertion 1.1(b) follows from 1.2(a) and 1.2(b). Furthermore, assertion 1.1(c) follows from 1.2(c), since A having no $\mathfrak{I}$-torsion implies that the ideal $\mathfrak{a} \subset R\langle\xi\rangle$ is saturated. □

Lemma 1.2. *In the classical rigid case, let A be an R-algebra of* tf *type, let F be a finite free A-module, and let $M \subset F$ be a submodule.*

(a) *The $\mathfrak{I}$-adic topology on F restricts to the $\mathfrak{I}$-adic topology on M.*

(b) *If M is finitely generated, it is complete. If, in addition, M or A are separated, M is closed in F.*

(c) *If M is saturated in the sense that an element $f \in F$ belongs to M as soon as $rf \in M$ for some non-zero $r \in R$, then M is finitely generated.*

Proof. Since assertion (c) cannot be obtained using easy ad-hoc arguments, we choose to refer to the existing literature. Assuming $A = R\langle\xi\rangle$ for some finite set of variables ξ, and tensoring over R with the field of fractions $K = Q(R)$, the assertion follows from [B, Satz 2.1], or [BGR, 5.2.7/7]. In [B], probably giving the most elementary access to the problem, the argumentation is based on suitable orthonormal bases of $K\langle\xi\rangle$, whereas Weierstraß division is used as a method in [BGR]. Alternatively, tensoring the situation with $R/\mathfrak{I}$, one can apply [RG, 3.4.6], to the A-module F/M which is flat over R, thus obtaining a proof in terms of flatness and in the spirit of the flattening techniques to be presented in [FII].

Assertion (b) is trivial. So it only remains to verify assertion (a). Let $\pi \in R$ be a generator of the ideal $\mathfrak{I} \subset R$ and consider the saturation

$$M_{\text{sat}} = \{f \in F;\ \pi^n f \in M \text{ for some } n \geqq 1\}$$

of M in F. By the definition of M_{sat}, the $\mathfrak{I}$-adic topology of F restricts to the $\mathfrak{I}$-adic topology of M_{sat}. Since M_{sat} is finitely generated, as we have seen in (c), there is an integer $n \in \mathbb{N}$ satisfying $\pi^n M_{\text{sat}} \subset M$. Consequently, just as for M_{sat}, the $\mathfrak{I}$-adic topology of F restricts to the $\mathfrak{I}$-adic topology of M.

Dealing with modules over admissible R-algebras and thinking of the non-noetherian case, it is convenient to use the notion of *coherent modules*; see [EGA I_{new}, Chap. 0, Sect. 5.3, and Chap. I, 1.4.3], for a version in terms of sheaves. Recall that a finitely generated A-module M is called coherent if all its submodules of finite type are of finite presentation. In particular, M itself is then of finite presentation. We mention the basic fact, [EGA I_{new}, Chap. 0, 5.3.2], that all three modules of a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

are coherent as soon as two of them have this property. From this it follows that sums and intersections of coherent modules are coherent, that image, kernel, and cokernel of homomorphisms between coherent modules are coherent and that $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ are coherent if M and N have this property. The ring A itself is called coherent if it is coherent as an A-module.

Proposition 1.3. *Let A be an R-algebra of* tf *presentation. Then A is a coherent ring; in particular, each A-module of finite presentation is coherent.*

Proof. We have only to look at the classical rigid case. Assuming first that A is admissible, consider a finitely generated ideal $\mathfrak{a} \subset A$ and a presentation

$$0 \to M \to F \to \mathfrak{a} \to 0$$

with a free A-module F of finite type. Since A is admissible, A and $\mathfrak{a}$ have no $\mathfrak{I}$-torsion. So M is saturated in F, and thus, finitely generated by 1.2(c). It follows that A is a coherent ring. In the general case, A is a quotient of the coherent ring $R\langle \xi \rangle$ by a finitely generated and, thus, coherent ideal. But then A is coherent. □

For later use we add an assertion on the coherence of annulators and torsion modules.

Lemma 1.4. *Let A be an R-algebra of* tf *presentation. Let $\mathfrak{a} \subset A$ be a coherent ideal, and let M be a coherent A-module. Then:*

(a) $\mathrm{Ann}_A(M) = \{a \in A;\ aM = 0\}$ *is a coherent ideal in A.*

(b) $\mathrm{Ann}_M(\mathfrak{a}) = \{m \in M;\ \mathfrak{a}m = 0\}$ *is a coherent submodule of M.*

(c) $(\mathfrak{I}\text{-torsion})_M = \{m \in M;\ \mathfrak{I}^n m = 0$ *for some* $n \in \mathbb{N}\}$, *the A-module of $\mathfrak{I}$-torsion in M, is coherent.*

Proof. Assertions (a) and (b) follow from the fact that A is a coherent ring. Namely, $\mathrm{Ann}_A(M)$ is the kernel of the homomorphism $A \to \mathrm{Hom}_A(M, M)$ mapping an element $a \in A$ onto the multiplication by a on M. There is a similar homomorphism $M \to \mathrm{Hom}_A(\mathfrak{a}, M)$ to settle assertion (b). Finally, to verify assertion (c), we have only to show that the A-module N of $\mathfrak{I}$-torsion in M is finitely generated. The latter is clear in the noetherian case. In the classical rigid case, N is saturated by its definition. Reducing to the case where M is a finite free A-module, the assertion follows from 1.2(c). □

In order to deal with formal schemes, we have to interpret R-algebras of the type considered above as projective limits. Writing $R_\lambda = R/\mathfrak{I}^{\lambda+1}$ for $\lambda \in \mathbb{N}$, we have $A = \varprojlim A \otimes_R R_\lambda$ for any R-algebra A which is $\mathfrak{I}$-adically complete and separated.

Lemma 1.5. *Let A be an R-algebra which is complete and separated with respect to the $\mathfrak{I}$-adic topology for some ideal $\mathfrak{I} \subset R$. Then A is of* tf *type if and only if $A \otimes_R R_\lambda$ is an R_λ-algebra of finite type for each $\lambda \in \mathbb{N}$. The latter is the case if and only if $A \otimes_R R_0$ is an R_0-algebra of finite type.*

Similar assertions are valid for finitely generated ideals in A and, more generally, for finitely generated A-modules.

Proof. (See also [AC, Chap. III, Sect. 2, no. 11, Proposition 14]). If A is of tf type, there is a surjective R-homomorphism $\varphi : R\langle\xi\rangle \to A$ for a finite set of variables ξ. Tensoring with R_λ over R yields a surjection $\varphi_\lambda : R_\lambda[\xi] \to A \otimes_R R_\lambda$ so that each R_λ-algebra $A \otimes_R R_\lambda$ is of finite type.

Conversely, assume that $A_0 = A \otimes_R R_0$ is an R_0-algebra of finite type. Then, for a finite set of variables ξ, there is an epimorphism $\varphi_0 : R_0[\xi] \to A_0$, and we can choose a lifting $\varphi : R\langle\xi\rangle \to A$. The surjectivity of φ_0 implies

$$A = \operatorname{im} \varphi + \mathfrak{I}A ,$$

and iteration yields

$$A = \operatorname{im} \varphi + \mathfrak{I} \operatorname{im} \varphi + \ \ldots \ + \mathfrak{I}^\lambda \operatorname{im} \varphi + \mathfrak{I}^{\lambda+1} A$$

for all $\lambda \in \mathbb{N}$. Using a limit argument, we see that φ is surjective and, hence, that A is of tf type. The assertion on A-modules is dealt with similarly. $\square$

Lemma 1.6. *Let $\varphi : A \to B$ be a morphism of R-algebras, where A is of* tf *presentation, and where B is of* tf *presentation or, more generally, of* tf *type and separated. Furthermore, let M be a coherent B-module.*

Then M is flat over A if and only if $M \otimes_R R_\lambda$ is flat over $A \otimes_R R_\lambda$ for all $\lambda \in \mathbb{N}$. The same assertion is true replacing "flat" by "faithfully flat".

In the noetherian case, the assumptions on A and B can be relaxed; $\mathfrak{I}$-adic completeness of B is enough.

Proof. We claim that the flatness assertion is a consequence of the proof of the Bourbaki criterion [AC, Chap. III, Sect. 5, no. 2, Theorem 1]. The only-if-part of the assertion is trivial. In order to justify the if-part, we need to know the following two facts for each finitely generated ideal $\mathfrak{a} \subset A$:

(1) the $\mathfrak{I}$-adic topology on A restricts to the $\mathfrak{I}$-adic topology on $\mathfrak{a}$.

(2) $\mathfrak{a} \otimes_A M$ is separated with respect to the $\mathfrak{I}$-adic topology.

In the noetherian case, (1) follows from the lemma of Artin-Rees, whereas in order to justify (2), we can consider $\mathfrak{a} \otimes_A M$ as a finite B-module. Since $\mathfrak{I}B$ is contained in the Jacobson radical of B, the B-module $\mathfrak{a} \otimes_A M$ is $\mathfrak{I}B$-adically separated due to Krull's intersection theorem.

Next, let us consider the classical rigid case. Condition (1) is settled by 1.2(a). Furthermore, since A is a coherent ring by 1.3, the ideal $\mathfrak{a}$ is of finite presentation. Thus, to justify condition (2), it is enough to show that each B-module M of finite presentation is $\mathfrak{I}$-adically separated. Consider a finite presentation of M

$$0 \to L \to F \to M \to 0$$

where F and L are finite B-modules and where F is free. Then, by 1.1(b) and 1.2(b), L is closed in F with respect to the $\mathfrak{I}$-adic topology of F and, hence, M is separated. This settles the assertion of 1.6 as far as flatness is concerned.

If M is a faithfully flat A-module, $M_\lambda = M \otimes_R R_\lambda$ is faithfully flat over $A_\lambda = A \otimes_R R_\lambda$ for each λ by [AC, Chap. I, Sect. 3, no. 3, Proposition 5]. Conversely, assume that M_λ is faithfully flat over A_λ for all λ. Then M is a flat A-module, as we have seen. Furthermore, consider a finitely generated A-module N such that $N \otimes_R M = 0$. Tensoring with R_λ over R yields $N_\lambda \otimes_{A_\lambda} M_\lambda = 0$ and, since M_λ is a faithfully flat A_λ-module, $N_\lambda = 0$ for all λ. Using this for $\lambda = 0$, we get $N = \mathfrak{I}N$ and, hence, $N = 0$, since $\mathfrak{I}A$ is contained in the Jacobson radical of A. This shows that M is a faithfully flat A-module. □

For the notion of formal schemes we refer to [EGA I, Sect. 10]. An affine formal R-scheme is called *of* tf *presentation* (resp. *admissible*) if it is of type $X = \operatorname{Spf} A$ for some R-algebra A of tf presentation (resp. for some admissible R-algebra A; the class of admissible R-algebras in our sense must not be confused with the class of admissible rings as considered in [EGA I, Sect. 10]). We may write

$$X = \varinjlim X \otimes_R R_\lambda$$

and identify X with the direct system (X_λ) consisting of the ordinary schemes

$$X_\lambda = X \otimes_R R_\lambda = \operatorname{Spec} A \otimes_R R_\lambda = \operatorname{Spec} A/\mathfrak{I}^{\lambda+1}A .$$

For elements $f \in A$, the associated basic open subscheme $X_f \subset X$ equals by definition the affine formal R-scheme $\operatorname{Spf} A\langle f^{-1}\rangle$ where $A\langle f^{-1}\rangle = \varprojlim (A/\mathfrak{I}^{\lambda+1}A)[f^{-1}]$ is the complete localization of A by f. An easy argument on projective limits shows $A\langle f^{-1}\rangle = A\langle\zeta\rangle/(1 - f\zeta)$ and, furthermore, that $A\langle f^{-1}\rangle$ is the $\mathfrak{I}$-adic completion of the ordinary localization $A[f^{-1}]$. We need to know that the property of $X = \operatorname{Spf} A$ to be of tf presentation or to be admissible is local on X.

Proposition 1.7. *Let A be an R-algebra which is complete and separated with respect to the $\mathfrak{I}$-adic topology, and let* $(\operatorname{Spf} B_i)_{i\in I}$ *be an affine open covering of* $\operatorname{Spf} A$. *Then the following are equivalent*:

(a) *A is an R-algebra of* tf *presentation* (*resp. an admissible R-algebra*).

(b) *B_i is an R-algebra of* tf *presentation* (*resp. an admissible R-algebra*) *for each i.*

Furthermore, if one of these conditions is satisfied, the canonical maps $A \to B_i$ are flat and the injection $A \hookrightarrow \prod B_i$ is faithfully flat.

Proof. We may assume that i varies over a finite index set. Using 1.5, A is of tf type if and only if the B_i are. Furthermore, the B_i are of tf presentation if A is. In order to show the converse, assume that all B_i are of tf presentation. Since A is of tf type, there is an exact sequence

$$0 \to \mathfrak{a} \to R\langle\xi\rangle \to A \to 0$$

with a finite set of variables ξ. Dividing by a power $\mathfrak{I}^{\lambda+1}$ of the ideal of definition $\mathfrak{I} \subset R$, yields the exact sequence

$$0 \to \mathfrak{a}/\mathfrak{a} \cap \mathfrak{I}^{\lambda+1}R\langle\xi\rangle \to R_\lambda[\xi] \to A_\lambda \to 0 ,$$

and the assumption on the B_i implies that each A_λ is locally of finite presentation over R_λ. But then, using [EGA $\mathrm{I_{new}}$, 6.2.9], A_λ is of finite presentation over R_λ and, thus, $\mathfrak{a}/\mathfrak{a} \cap \mathfrak{I}^{\lambda+1}R\langle\xi\rangle$ is a finitely generated ideal in $R_\lambda[\xi]$. Since the $\mathfrak{I}$-adic topology on $R\langle\xi\rangle$ restricts to the $\mathfrak{J}$-adic topology on $\mathfrak{a}$, see 1.2(a), it follows that $\mathfrak{a}/\mathfrak{J}\mathfrak{a}$ is a finitely generated $R_0[\xi]$-module. Hence, using a limit process as in the proof of 1.5, we see that $\mathfrak{a}$ is a finitely generated ideal in $R\langle\xi\rangle$. So, A is of tf presentation, and the equivalence of (a) and (b) is clear for R-algebras of tf presentation. In order to extend the equivalence to admissible R-algebras, it is enough to point out that, if (a) or (b) are given, the canonical map $A \hookrightarrow \prod B_i$ is faithfully flat by 1.6. □

As a consequence of 1.7, the notion of formal R-schemes of tf (topologically finite) presentation (resp. of admissible formal R-schemes) can be globalized: a formal R-scheme X is called *locally of tf presentation* (resp. *admissible*) if, locally, X is R-isomorphic to the affine formal scheme Spf A of an R-algebra A which is of tf presentation (resp. admissible). If X is locally of tf presentation and quasi-compact, we say X is of tf presentation.

Corollary 1.8. *Let $X = \varinjlim X_\lambda$ be a formal R-scheme which is locally of* tf *presentation, and consider a point $x \in X_0$. Then:*

(a) *For each formal open subscheme* Spf $A \subset X$ *containing x, the canonical morphism $A \to \mathcal{O}_{X,x}$ is flat. If A_x is the localization of A at x, the morphism $A_x \to \mathcal{O}_{X,x}$ is faithfully flat.*

(b) *If x is a specialization of a point $y \in X_0$, the canonical morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,y}$ is flat.*

Proof. Assertion (a) follows from the flatness of maps of type $A \to A\langle f^{-1}\rangle$, since a direct limit of flat A-modules is flat again. To verify assertion (b), consider a finitely generated ideal $\mathfrak{a}_x \subset \mathcal{O}_{X,x}$. We have to show that the canonical morphism $\sigma : \mathfrak{a}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,y} \to \mathcal{O}_{X,y}$ is injective. Choosing an open affine formal subscheme Spf $A \subset X$ such that $\mathfrak{a}_x$ is induced from a finitely generated ideal $\mathfrak{a} \subset A$, the map σ is a direct limit of morphisms of type $\mathfrak{a} \otimes_A A\langle f^{-1}\rangle \to A\langle f^{-1}\rangle$. The latter are injective, since each $A\langle f^{-1}\rangle$ is flat over A. □

We will allow to replace R or, better, the affine formal base scheme Spf R, by a more general formal scheme S. Similarly as before, working in the admissible context, we consider the following two cases:

(I′) *The classical rigid case.* S is an arbitrary admissible formal scheme over a valuation ring R as considered above. So the topology of S is generated by the ideal $\mathcal{I} = \pi\mathcal{O}_S$, where π is a non-zero element of the maximal ideal of R.

(II′) *The noetherian case.* S is a noetherian formal scheme such that the topology of its structure sheaf $\mathcal{O}_S$ is generated by a coherent ideal $\mathcal{I}$. Of course, we have to require that S (or better, its structure sheaf), has no $\mathcal{I}$-torsion.

In the following we will consider formal schemes X over a base S as above which are locally of tf presentation; in particular, we require that $\mathcal{I}\mathcal{O}_X$ is an ideal of definition of X. Such a formal scheme X is called *admissible* if it has no $\mathcal{I}$-torsion in the sense that the structure sheaf $\mathcal{O}_X$ has no $\mathcal{I}$-torsion. Frequently we will start out from an admissible formal S-scheme X and, performing a certain construction on it, end up with a formal S-scheme X' which is just of locally tf presentation. Then, in order to return to an admissible formal S-scheme, we replace X by the subscheme defined by the ideal of $\mathcal{I}$-torsion in $\mathcal{O}_X$; it is admissible again as is easily seen using 1.4(c). We will fix S as well as the ideal $\mathcal{I}$ in the following and call $\mathcal{I}$ "the" *ideal of definition* of S or X. Similarly as before, reduction modulo $\mathcal{I}^{\lambda+1}$ yields an ordinary scheme X_λ over S_λ. We will call the S_0-scheme X_0 the *special fibre* of X. For each λ, the topological space underlying X_λ, which coincides with the topological space underlying X, is locally noetherian, even in the classical rigid case. Namely, in the latter case the reduced scheme $(X_0)_{\mathrm{red}}$ is locally of finite type over the residue field of the valuation ring R over which we are working. In particular, all formal S-schemes of locally tf presentation are quasi-separated. Formal S-schemes of locally tf presentation and their morphisms form a category which admits fibred products. The same is true for admissible formal S-schemes. However, if the fibred product of two admissible formal S-schemes X and Y over a third one Z is to be constructed in the category of admissible formal S-schemes, one constructs the fibred product in terms of formal schemes $X \times_Z Y$ and, as indicated above, kills its $\mathscr{I}$-torsion. Note that the $\mathscr{I}$-torsion is trivial if X or Y are flat over Z.

Many notions and definitions of algebraic geometry which apply to the scheme case carry over to the case of formal S-schemes of locally tf presentation or to admissible formal S-schemes; one just applies them to each S_λ-scheme X_λ. As an example, we have already discussed the notion of flatness; cf. 1.6. Also we point out that the notion of a coherent module $\mathscr{M}$ over a formal S-scheme X as above is defined as usual. The flatness assertion in 1.7 and the fact that a module M over a ring A is of finite presentation if and only if it has this property after faithfully flat base change of A, see [AC, Chap. I, Sect. 3, no. 6, Proposition 11], imply that an $\mathcal{O}_X$-module $\mathscr{M}$ is coherent if and only if on any affine open formal subscheme $\operatorname{Spf} A \subset X$ it corresponds to an A-module which is coherent in the sense discussed before. The structure sheaf $\mathcal{O}_X$ itself is coherent since algebras of tf presentation are coherent by 1.3.

If $\operatorname{Spf} A$ is an affine open formal subscheme of X, we can identify points of the special fibre X_0 with *open* prime ideals in the corresponding algebra A. Arbitrary non-open prime ideals of A cannot be viewed in a reasonable way as "points" of X, since there can be prime ideals in a localization $A\langle f^{-1}\rangle$ without being induced from prime ideals of A. Nevertheless, in certain considerations on affine formal open parts $\operatorname{Spf} A \subset X$, where we need to consider non-closed prime ideals of A, we will talk about the *complement of the special fibre* X_0. Also we point out that the closed points of this complement may be viewed as points of the so-called associated rigid space X_{rig} of X, to be introduced in Sect. 4.

2 Admissible formal blowing-up

In order to discuss the relationship between admissible formal schemes and rigid spaces we need to introduce the notion of *admissible formal blowing-up*. Consider an admissible formal S-scheme X with ideal of definition $\mathscr{I} \subset \mathcal{O}_X$ and a coherent *open* ideal $\mathscr{A} \subset \mathcal{O}_X$. Then

$$X' = \varinjlim_{\lambda} \quad \operatorname{Proj} \bigoplus_{n=0}^{\infty} (\mathscr{A}^n \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathscr{I}^{\lambda+1})$$

is a formal S-scheme, and the projection $\varphi: X' \to X$ is called an *admissible formal blowing-up*, more precisely, the formal blowing-up of $\mathscr{A}$ on X, or of the formal subscheme $Y \subset X$, corresponding to $\mathscr{A}$, on X. As usual, the subscheme $Y \subset X$ is referred to as the *center* of the blowing-up. The construction of blowing-ups $X' \to X$ is local on X, in fact, it commutes with flat base change. Furthermore, it does not leave the category of admissible formal S-schemes as we prove below. If

$\mathscr{A}_1, \ldots, \mathscr{A}_r$ are finitely many coherent open ideals on X, there will always be an admissible blowing-up $X' \to X$ such that, on X', the ideals $\mathscr{A}_i$ become invertible and locally ordered by inclusion. So, up to admissible blowing-up, the situation is locally as good as over a complete valuation ring of height one.

Proposition 2.1. *Let X be an admissible formal S-scheme and let $\varphi: X' \to X$ be an admissible formal blowing-up of some coherent open ideal $\mathscr{A} \subset \mathcal{O}_X$.*

(a) *Let $U = \operatorname{Spf} A \subset X$ be an affine open formal subscheme, and let $\mathfrak{a} \subset A$ be the ideal corresponding to $\mathscr{A} \subset \mathcal{O}_X$. Then $\varphi^{-1}(U) \to U$ can be interpreted as the $\mathfrak{I}$-adic completion of the ordinary scheme-theoretic blowing-up of $\mathfrak{a}$ on* Spec A.

(b) *X' is an admissible formal S-scheme on which the ideal $\mathscr{A}\mathcal{O}_{X'}$ is invertible.*

(c) (*Universal property*) *If $\psi: Z \to X$ is a morphism of formal S-schemes (Z is not necessarily admissible) such that $\mathscr{A}\mathcal{O}_Z$ is invertible on Z, then there is a unique S-morphism $\psi': Z \to X'$ such that $\psi = \varphi \circ \psi'$.*

Proof. Assertion (a) is part of the definition of admissible blowing-up. The remaining assertions are easy to verify. Assuming X and S to be affine, say $X = \operatorname{Spf} A$, $S = \operatorname{Spf} R$, we can consider the corresponding scheme-theoretic blowing-up $\tilde{\varphi}: \tilde{X}' \to \operatorname{Spec} A$ and use the fact that assertions (b) and (c) are known for $\tilde{\varphi}$ in place of φ; assertion (b) in the sense that $\tilde{X}'$ is an A-scheme of finite type, on which $\mathscr{A}\mathcal{O}_{\tilde{X}'}$ is invertible. In particular, as X has no $\mathfrak{I}$-torsion, the same is true for $\tilde{X}'$, where $\mathfrak{I}$ is the ideal of definition of A. We claim that these results carry over from $\tilde{X}'$ to the $\mathfrak{I}$-adic completion X'. This is clear for assertion (c) and, in the noetherian case, also for assertion (b), since the $\mathfrak{I}$-adic completion $\hat{C}$ of any noetherian R-algebra C is flat over C due to [AC, Chap. III, Sect. 5, no. 4, Corollary to Proposition 3].

In the classical rigid case, we have to use a slightly different argumentation for proving assertion (b). Consider an affine open covering $\mathfrak{U}' = (\operatorname{Spec} A_i')$ of $\tilde{X}'$ as well as the associated covering $(\operatorname{Spf} \hat{A}_i')$ of X' obtained from $\mathfrak{U}'$ via $\mathfrak{I}$-adic completion. Then each $\hat{A}_i'$ is of tf type over R and is separated by definition. Furthermore, $\tilde{X}'$ is flat over R since it has no $\mathfrak{I}$-torsion due to the fact that $\mathscr{A}$ is an open ideal. So using 1.6, the flatness of A_i' over R implies that $\hat{A}_i'$ is flat over R. Thereby we see that X' is admissible again. To verify the second part of assertion (b), we mention that the ideal $\mathscr{A}\mathcal{O}_{X'}$ is principal and, since it is open, contains a power of the generator π of $\mathfrak{I}$. However, since X' has no π-torsion, $\mathscr{A}\mathcal{O}_{X'}$ must be invertible. □

We want to add an explicit description of admissible formal blowing-ups.

Lemma 2.2. *Let $X = \operatorname{Spf} A$ be an affine admissible formal R-scheme, where R and the ideal $\mathfrak{I} = (g_1, \ldots, g_r) \subset R$ are as in* (I) *or* (II). *Let $\varphi: X' \to X$ be an admissible formal blowing-up of a coherent open ideal $\mathfrak{a} = (f_0, \ldots, f_n) \subset A$ and let $\tilde{\varphi}: \tilde{X}' \to \tilde{X}$ be the scheme-theoretic blowing-up of $\mathfrak{a}$ on $\tilde{X} = \operatorname{Spec} A$ so that X' is the $\mathfrak{I}$-adic completion of $\tilde{X}'$. Consider the affine open covering $(\operatorname{Spec} A_i')_{i=0,\ldots,n}$ of $\tilde{X}'$, where $\operatorname{Spec} A_i'$ is the maximal open subscheme of $\tilde{X}'$ where f_i generates the invertible ideal $\mathfrak{a}\mathcal{O}_{\tilde{X}'}$; i.e.,*

$$A_i' = A_i''/(f_i\text{-torsion}),$$

$$A_i'' = A\left[\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}\right] = A\left[\frac{\xi_0}{\xi_i}, \ldots, \frac{\xi_n}{\xi_i}\right] \Big/ \left(f_i \frac{\xi_j}{\xi_i} - f_j\right).$$

Then the $\mathfrak{J}$-adic completions of the algebras A_i' and A_i'', which are described as

$$\hat{A}_i' = \hat{A}_i''/(f_i\text{-torsion})\,,$$

$$\hat{A}_i'' = A\left\langle\frac{f_0}{f_i},\ldots,\frac{f_n}{f_i}\right\rangle = A\left\langle\frac{\xi_0}{\xi_i},\ldots,\frac{\xi_n}{\xi_i}\right\rangle\Big/\left(f_i\frac{\xi_j}{\xi_i}-f_j\right),$$

give rise to the affine open covering $(\mathrm{Spf}\,\hat{A}_i')_{i=0,\ldots,n}$ *of X', where* $\mathrm{Spf}\,\hat{A}_i'$ *is the maximal open subscheme in X' where f_i generates the invertible ideal $\mathfrak{a}\mathcal{O}_{X'}$. Furthermore,* ($f_i$-torsion) *and* ($\mathfrak{J}$-torsion) *coincide on A_i'' and on $\hat{A}_i''$. (The fractions ξ_j/ξ_i serve as indeterminates, except for $j = i$.)*

Proof. It is only to show that the completions of A_i' and A_i'' are as stated. In the noetherian case, we use the fact that $A\langle\xi_j/\xi_i\rangle$ is flat over $A[\xi_j/\xi_i]$. As completion of finite $A[\xi_j/\xi_i]$-modules is done by tensoring with $A\langle\xi_j/\xi_i\rangle$, the descriptions of A_i' and A_i'' carry over to their completions. Furthermore, f_i generates an open ideal in A_i'' as well as in $\hat{A}_i''$. We therefore have

$$(f_i\text{-torsion}) \subset (\mathfrak{J}\text{-torsion})\,.$$

However A_i' has no $\mathfrak{J}$-torsion since A does not have any. (If $U \subset \mathrm{Spec}\,A$ is a schematically dense open subscheme, its inverse image in $\tilde{X}'$ is schematically dense, too.) Therefore the f_i-torsion of A_i'' must coincide with the $\mathfrak{J}$-torsion. The same is true for $\hat{A}_i''$ since $\hat{A}_i'$, obtained from A_i' via flat base change, does not have $\mathfrak{J}$-torsion.

Now we turn to the classical rigid case. Consider a short exact sequence

$$(*) \qquad 0 \to \mathfrak{b} \to B \to C \to 0\,,$$

where B and C are rings and $\mathfrak{b}$ is an ideal in B. Then, by completion with respect to some ideal $\mathfrak{J} \subset B$, we get an exact sequence

$$(**) \qquad 0 \to \hat{\mathfrak{b}} \to \hat{B} \to \hat{C} \to 0\,,$$

where $\hat{B}$ and $\hat{C}$ are the $\mathfrak{J}$-adic completions of B and C and where $\hat{\mathfrak{b}}$ is the completion of $\mathfrak{b}$ under the restriction of the $\mathfrak{J}$-adic topology on B. So $\hat{\mathfrak{b}}$ equals the closure of the ideal generated by $\mathfrak{b}$ in $\hat{B}$. From this fact it follows that $\hat{A}_i''$ satisfies the assertion of the lemma since $(f_i\xi_j/\xi_i - f_j)$ defines a closed ideal in $A\langle\xi_j/\xi_i\rangle$ by 1.2(b). Furthermore, it follows as in the noetherian case that the f_i-torsion and the $\mathfrak{J}$-torsion coincide on A_i''.

Now consider the canonical map $\sigma: A_i'' \to A_i'$ whose kernel is the just mentioned torsion. Via $\mathfrak{J}$-adic completion it gives rise to a surjection $\hat{\sigma}: \hat{A}_i'' \to \hat{A}_i'$ whose kernel is the closure of $\ker\sigma$ in $\hat{A}_i''$. Since the ideal of $\mathfrak{J}$-torsion of $\hat{A}_i''$ is saturated, it is finitely generated by 1.2(c), and it follows that the closure $\ker\hat{\sigma}$ of $\ker\sigma$ (the ideal of $\mathfrak{J}$-torsion of A_i'') is contained in the ideal of $\mathfrak{J}$-torsion of $\hat{A}_i''$. However, $\hat{A}_i' = \hat{A}_i''/\ker\hat{\sigma}$ does not have $\mathfrak{J}$-torsion by 2.1, and so $\ker\hat{\sigma}$ is precisely the $\mathfrak{J}$-torsion of $\hat{A}_i''$. On the other hand, we may view $\ker\hat{\sigma}$ as the closure of the ideal of f_i-torsion in A_i''. So $\ker\hat{\sigma}$ can be thought to be generated by f_i-torsion elements. However, since the f_i-torsion of $\hat{A}_i''$ is contained in the $\mathfrak{J}$-torsion, both must coincide. □

If $\varphi: X' \to X$ is an admissible formal blowing-up with center $Y \subset X$ corresponding to a coherent open ideal $\mathscr{A} \subset \mathcal{O}_X$ and if $\mathscr{M}$ is a coherent $\mathcal{O}_X$-module, the *strict transform* of $\mathscr{M}$ under φ is defined as the $\mathcal{O}_{X'}$-module $\bar{\mathscr{M}}' = \mathscr{M}'/\mathscr{N}$ where

$\mathcal{M}' = (\varphi^*\mathcal{M})$ and where $\mathcal{N} \subset \mathcal{M}'$ is the submodule consisting of all sections whose support is contained in $\varphi^{-1}(Y)$; i.e., $\mathcal{N} = \mathrm{Ann}_{\mathcal{M}'}(\mathcal{A}\mathcal{O}_{X'})$. We claim that $\mathcal{N}$ and, thus, also $\bar{\mathcal{M}}'$ are well-defined coherent $\mathcal{O}_{X'}$-modules. In fact, consider an affine formal open subscheme $\mathrm{Spf}\, A' \subset X'$ and let $\mathfrak{a}' \subset A'$ be the coherent open ideal associated to $\mathcal{A}\mathcal{O}_{X'}$ as well as M' the A'-module corresponding to $\mathcal{M}'$. Then $\mathcal{N}$ corresponds to the submodule $\mathrm{Ann}_{M'}(\mathfrak{a}') \subset M'$ which is coherent by 1.4. Since the formation of $\mathrm{Ann}_{M'}(\mathfrak{a}')$ commutes with flat base change, the annihilator $\mathcal{N} = \mathrm{Ann}_{\mathcal{M}'}(\mathcal{A}\mathcal{O}_{X'})$ is a well-defined coherent submodule of $\mathcal{M}'$. Since $\mathcal{A}$ contains a power of $\mathcal{I}$, we have $\mathrm{Ann}_{\mathcal{M}'}(\mathcal{A}\mathcal{O}_{X'}) \subset (\mathcal{I}\text{-torsion})_{\mathcal{M}'}$ and, hence, $\mathcal{N} = (\mathcal{I}\text{-torsion})_{\mathcal{M}'}$ if the strict transform $\bar{\mathcal{M}}'$ of $\mathcal{M}$ has no $\mathcal{I}$-torsion. For example, the latter is the case if $\bar{\mathcal{M}}'$ is flat over X' or S. Also note that $\bar{\mathcal{M}}'$ coincides with $\mathcal{M}'$ if $\mathcal{M}$ is flat over X.

If $S' \to S$ is an admissible formal blowing-up of some coherent open ideal $\mathcal{A} \subset \mathcal{O}_S$ and if $X \to S$ is an admissible formal S-scheme, we have canonically a commutative diagram

$$\begin{array}{ccc} X & \leftarrow & X' \subsetneq X \times_S S' \\ \downarrow & & \downarrow \\ S & \leftarrow & S' \end{array}$$

where X' is the fibred product of X and S' over S in the category of admissible formal S-schemes, i.e., the subscheme of $X \times_S S'$ given by the ideal of $\mathcal{I}$-torsion. In analogy to the above, we call X' the *strict transform* of X under the blowing-up $S' \to S$.

Remark 2.3. In the above situation, $X' \to X$ is an admissible formal blowing-up, namely the one of the ideal $\mathcal{A}\mathcal{O}_X$ on X.

In fact, the ideal $\mathcal{A}\mathcal{O}_{X'}$ is locally monogenous and, furthermore, invertible since it is open and X' has no $\mathcal{I}$-torsion. So there is a canonical morphism $X' \to X''$ into the blowing-up X'' of $\mathcal{A}\mathcal{O}_X$ on X. On the other hand, X'' lies over X and over S' so that we have a canonical morphism $X'' \to X'$ into the fibred product (in terms of admissible formal schemes) X' of X and S' over S.

We list some properties of admissible formal blowing-ups which are easily deduced from 2.1.

Remark 2.4. (a) Let X be an admissible formal S-scheme and let $\varphi: X' \to X$ be an admissible formal blowing-up of some coherent open ideal $\mathcal{A} \subset \mathcal{O}_X$. Then, for any formal open subscheme $U \subset X$ such that U is disjoint from the closed subscheme $Y \subset X$ given by $\mathcal{A}$, the induced morphism $\varphi^{-1}(U) \to U$ is an isomorphism.

(b) In the category of *admissible* formal S-schemes, admissible formal blowing-ups are compatible with base change.

(c) Admissible formal blowing-ups are proper and surjective on special fibres.

(d) Let $\mathcal{A}, \mathcal{B} \subset \mathcal{O}_X$ be coherent open ideals. If $\varphi: X' \to X$ is the admissible blowing-up of $\mathcal{A}$ in X and $\varphi': X'' \to X'$ the admissible blowing-up of $\mathcal{B}\mathcal{O}_{X'}$ in X', then $\varphi \circ \varphi': X'' \to X$ is the admissible blowing-up of $\mathcal{A}\mathcal{B}$ in X.

Since assertions (a) and (d) are clear and assertion (b) has been justified above, only assertion (c) needs a verification. Consider an admissible blowing-up $\varphi: X' \to X$ of a coherent open ideal $\mathcal{A} \subset \mathcal{O}_X$ on a non-empty formal scheme X. By the definition of admissible blowing-up, $\varphi: X' \to X$ is proper and, thus, closed. So it is enough to show that X' is not empty. We may assume that X is affine, $X = \mathrm{Spf}\, A$.

Let $\mathfrak{a} \subset A$ be the ideal associated to $\mathscr{A}$. Reducing modulo powers of the ideal of definition $\mathfrak{I}$, we have

$$X'_\lambda = \operatorname{Proj} \bigoplus_{n=0}^{\infty} (\mathfrak{a}^n \otimes_A A/\mathfrak{I}^{\lambda+1})$$

since the formation of Proj is compatible with pull-back of the base. We claim that X'_λ is not empty. In fact, if it were, we would have $\mathfrak{a}^n \otimes_A A/\mathfrak{I}^{\lambda+1}A = 0$ for $n \gg 0$; see [EGA II, 2.4.7]. The latter implies $\mathfrak{a}^n = \mathfrak{I}^{\lambda+1}\mathfrak{a}^n$ and, hence, $\mathfrak{a}^n = 0$ for $n \gg 0$. However, $\mathfrak{a}$ contains a power of $\mathfrak{I}$ so that $\mathfrak{I}^n = 0$ for $n \gg 0$. But then, using the fact that A has no $\mathfrak{I}$-torsion, we have $A = 0$, which we had excluded. Alternatively, we could have used the fact that rig-points of X, to be introduced in Sect. 3, correspond one-to-one to the rig-points of X'.

There is a generalization of assertion (d).

Proposition 2.5. *Let X be an admissible formal S-scheme which is quasi-compact. If $\varphi: X' \to X$ and $\psi: X'' \to X'$ are admissible formal blowing-ups, $\varphi \circ \psi: X'' \to X$ is an admissible formal blowing-up.*

Proof. We will use the corresponding fact in the scheme case; cf. [RG, 5.1.4]. Let φ be the blowing-up of the coherent open ideal $\mathscr{A} \subset \mathscr{O}_X$ and let ψ be the blowing-up of the coherent open ideal $\mathscr{B} \subset \mathscr{O}_{X'}$. Assume first that the situation is affine, say $X = \operatorname{Spf} A$ and $S = \operatorname{Spf} R$ with ideal of definition $\mathfrak{I}$. Then $\mathscr{A}$ corresponds to an open ideal $\mathfrak{a} \subset A$, and we can consider the ordinary blowing-up $\varphi': Y' \to Y$ of $\mathfrak{a}$ on $Y = \operatorname{Spec} A$. Furthermore, X' is the $\mathfrak{I}$-adic completion of Y'; let $j: X' \to Y'$ be the associated morphism of ringed spaces and $u: \mathscr{O}_{Y'} \to j_* \mathscr{O}_{X'}$ the corresponding morphism of structure sheaves. Since the ideal $\mathscr{B} \subset \mathscr{O}_{X'}$ is open, there exists an integer n such that $\mathfrak{I}^n \mathscr{O}_{X'} \subset \mathscr{B}$. Then, considering the isomorphism

$$\mathscr{O}_{Y'}/\mathfrak{I}^n \mathscr{O}_{Y'} \to \mathscr{O}_{X'}/\mathfrak{I}^n \mathscr{O}_{X'}$$

induced from u, we see that $\mathscr{B}' = u^{-1}(j_* \mathscr{B})$ is a quasi-coherent open ideal of finite type in $\mathscr{O}_{Y'}$ such that $j^* \mathscr{B}'$ generates $\mathscr{B} \subset \mathscr{O}_{X'}$. Now let $\psi': Y'' \to Y'$ be the ordinary blowing-up of $\mathscr{B}'$ in Y'. Then the composition

$$\varphi \circ \psi: X'' \to X' \to X$$

is obtained from

$$\varphi' \circ \psi': Y'' \to Y' \to Y$$

via $\mathfrak{I}$-adic completion. By [RG, 5.1.4], there is an admissible open ideal $\mathfrak{b} \subset A$ with the property that $\mathfrak{b}\mathscr{O}_{Y'}$ coincides with $\mathscr{A}^m \mathscr{B}'^n \mathscr{O}_{Y'}$ for some integers m, n and such that $\varphi' \circ \psi'$ is the ordinary blowing-up of $\mathfrak{a}\mathfrak{b} \subset A$ on Y. But then, $\varphi \circ \psi$ is the formal blowing-up of $\mathfrak{a}\mathfrak{b}$ in $X = \operatorname{Spf} A$.

In the noetherian case, we may replace $\mathfrak{b}$ by the inverse image $\mathfrak{b}'$ of $\varphi'_*(\mathscr{A}^m \mathscr{B}'^n \mathscr{O}_{Y'})$ with respect to the canonical map $\mathscr{O}_Y \to \varphi'_* \mathscr{O}_{Y'}$; this inverse $\mathfrak{b}'$ is a quasi-coherent ideal of finite type on Y, due to the noetherian hypothesis. Since for any $f \in A$ the blowing-up $Y' \to Y$ is compatible with base change of type $A \to A\langle f^{-1} \rangle$ which is flat by 1.6, the construction is local on X and, thus, can be globalized.

In the classical rigid case we do not know a priori that $\mathfrak{b}'$ as constructed above is of finite type. Nevertheless the construction of $\mathfrak{b}'$ is local on the formal scheme X and, thus, can be globalized to produce a "quasi-coherent" open ideal $\mathscr{D}' \subset \mathscr{O}_X$

such that $\mathscr{D}'\mathcal{O}_{X'} = \mathscr{A}^m\mathscr{B}'^n\mathcal{O}_{X'}$. Reducing modulo a suitable power of $\mathfrak{J}$ and applying [EGA I_{new}, 6.9.9], we can replace $\mathscr{D}'$ by a smaller ideal $\mathscr{D}''$ which is quasi-coherent of finite type and which does the same job. □

We add here a technical lemma which will allow to reduce later considerations on admissible blowing-ups $X' \to X$ to the case where X is affine.

Lemma 2.6. *Let X be an admissible formal S-scheme, which is quasi-compact, and let U_i, $i \in I$, be finitely many open subschemes. For each i, consider a coherent open ideal $\mathscr{A}_i \subset \mathcal{O}_{U_i}$ and the formal blowing-up $\varphi_i: U_i' \to U_i$ of $\mathscr{A}_i$. Then:*

(a) *Each φ_i extends to an admissible formal blowing-up $\psi_i: X_i' \to X$.*

(b) *There is an admissible blowing-up $\psi: X' \to X$ which factors through each ψ_i.*

Proof. Reducing modulo a suitable power of $\mathfrak{J}$ and applying [EGA I_{new}, 6.9.7], we can extend each $\mathscr{A}_i$ to a coherent open ideal $\bar{\mathscr{A}}_i \subset \mathcal{O}_X$. The corresponding formal blowing-up $\psi_i: X_i' \to X$ extends φ_i. Furthermore, the product of the $\bar{\mathscr{A}}_i$ yields a coherent open ideal $\mathscr{A} \subset \mathcal{O}_X$ such that the associated formal blowing-up $\psi: X' \to X$ factors through each ψ_i. □

3 Rig-points of admissible formal schemes

When associating a rigid space X_{rig} to an admissible formal S-scheme X, as we will do in Sects. 4 and 5, the rigid points of X_{rig} (which, in the classical rigid case, are the points of the underlying topological space of X_{rig}) may be interpreted as points of X with values in certain valuation rings. These points will be called rig-*points* of X. In some sense they provide a link between the formal scheme X and the "complement" of its special fibre X_0.

Definition 3.1. *Let X be an admissible formal S-scheme. A* rig-*point (resp. closed* rig-*point) of X is given by a morphism of admissible formal S-schemes $u: T \to X$ with the following properties:*

(a) *u is a locally closed (resp. closed) immersion.*

(b) *T is affine, $T = \operatorname{Spf} B$, and B is a local integral domain of dimension 1. The field of fractions of B is called the residue field of u.*

Rig-points, as defined above, are also referred to as *locally closed* rig-points. In the classical rigid case, locally closed rig-points are closed. The same is true if X_0 is jacobson.

Lemma 3.2. *In the situation of 3.1, $T = \operatorname{Spf} B$ is finite over an open part of S. The integral closure $\bar{B}$ of B in its field of fractions $Q(B)$ is a valuation ring which, in the noetherian case, is finite over B.*

Proof. We may assume that S is affine, say $S = \operatorname{Spf} R$, where, in the classical rigid case, R is a valuation ring. Looking at special fibres, we can view T_0 as a closed point of X_0 and, since the latter is of finite type over S_0, we see that T_0 is finite over an open part of S_0. Then an argument as used in the proof of 1.5 shows that T is finite over an open part of S. Now consider the integral closure $\bar{B}$ of B in $Q(B)$ in the noetherian case. Due to a Theorem of Nagata, $\bar{B}$ is finite over B; cf. [AC, Chap. IX, Sect. 4, no. 2, Theorem 2]. Thus, $\bar{B}$, as a complete semi-local integral domain, must be local. So it is a local noetherian integral domain and, hence, a discrete valuation ring. In the classical rigid case, $Q(B)$ is finite over the field of fractions K of R, and the integral closure of R in $Q(B)$ is a valuation ring; cf. [BGR, 3.2.4/2]. However this integral closure coincides with $\bar{B}$. □

Using the assertion of 3.2, it is easily verified that any morphism $\varphi: Y \to X$ of admissible formal S-schemes gives rise to a canonical map rig-pts$(Y) \to$ rig-pts(X) between associated rig-points. Just as in the ordinary scheme case, residue fields may shrink under this map.

Lemma 3.3. *Let $\varphi: X' \to X$ be an admissible formal blowing-up. Then the associated map between* rig-*points is bijective and respects residue fields.*

Proof. In order to exhibit a map from rig-pts(X) to rig-pts(X') which is an inverse of the canonical map rig-pts$(X') \to$ rig-pts(X), consider a rig-point u: Spf $B \to X$. Let $\mathfrak{a} \subset B$ be the pull-back of the coherent open ideal of the blowing-up $\varphi: X' \to X$. Then $\mathfrak{a}$ is a coherent open ideal in B which becomes invertible over the integral closure $\bar{B}$ of B in $Q(B)$ since $\bar{B}$ is a valuation ring by 3.2. Interpreting $\bar{B}$ as a direct limit of finite extensions of B, we see that $\mathfrak{a}$ is invertible already over a finite extension B' of B; the latter is a local ring. Using the universal property of the blowing-up φ, the morphism

$$\mathrm{Spf}\, B' \to \mathrm{Spf}\, B \to X$$

factors uniquely through a morphism u': Spf $B' \to X'$. Since Spf B' is finite over an open part of S, it is finite over an open part of X' and, hence, replacing B' by a suitable subring containing B, we get a rig-point u': Spf $B' \to X'$. Thereby we obtain a map rig-pts$(X) \to$ rig-pts(X') which, as is easily seen, is an inverse of the canonical map rig-pts$(X') \to$ rig-pts(X). Since $Q(B) = Q(B')$, the maps leave residue fields of rig-points invariant. □

We want to show that the rig-points of X correspond to the closed points of the "complement" of the special fibre of X.

Lemma 3.4. *Let $X = \mathrm{Spf}\, A$ be an affine admissible formal S-scheme, where S is assumed to be affine, say $S = \mathrm{Spf}\, R$. Then points of the following type correspond bijectively to each other*:

(a) *closed* rig-*points of* Spf A,

(b) *prime ideals $\mathfrak{p} \subset A$ with $\mathfrak{I}A \not\subset \mathfrak{p}$ and* $\dim A/\mathfrak{p} = 1$, *where $\mathfrak{I}$ is the ideal of definition of R*,

(c) *closed points of the complement of the special fibre of the ordinary scheme* Spec A.

Proof. It is clear that a point of type (a) induces a point of type (b) and that the latter induces a point of type (c). Thus it remains to show how to obtain a closed rig-point of Spf A from a point of type (c).

First, let us consider the classical rigid case, so we work over a valuation ring R with field of fractions K. Then the generic fibre of Spec A is the scheme Spec A_{rig} with $A_{\mathrm{rig}} = A \otimes_R K$. A closed point of Spec A_{rig} corresponds to a surjective K-morphism $A_{\mathrm{rig}} = A \otimes_R K \to K'$ where K' is a finite extension of K; use [BGR, 6.1.2/3]. The image of A in K' is an admissible R-algebra, which we denote by B. Let R' be the integral closure of R in K'. Since R is a complete valuation ring, the same is true for R', cf. [BGR, 3.2.4/2]. Using a continuity argument, we see

$B \subset R'$ and, thus, that B is integral over R. Therefore B must be a local ring of dimension 1, just as R and R' are. Consequently, the resulting R-morphism $A \to B$ defines a closed rig-point of X in the terminology of 3.1.

In the noetherian case consider a closed point of the complement of the special fibre of the ordinary scheme $\operatorname{Spec} A$, i.e., a surjective morphism $\sigma: A[g^{-1}] \to K$, where $g \in \mathfrak{I}$ and where K is a field. Let B be the image of A in K; it is an admissible R-algebra. In order that σ induces a rig-point of X, we have to show B is a local ring of dimension 1. Since $B[g^{-1}]$ is a field, namely K, we see from the Theorem of Artin-Tate, [EGA 0_{IV}, 16.3.3], that B is a semi-local ring of dimension $\leqq 1$. However, B cannot be a field and, thus, is of dimension precisely 1. As a complete semi-local integral domain, B is local.

Finally, it is easily checked that the above defined maps between closed rig-points and closed points of ordinary schemes are inverse to each other. □

If, for some admissible formal S-scheme X, there is no difference between locally closed and closed rig-points (for example, in the classical rigid case), the assertion of 3.4 states that, locally on affine open pieces of X, the set of rig-points of X "is" the set of closed points of the complement of the special fibre of X. However, for any locally closed rig-point u of X which is not closed, we can only say that there is an affine open part $\operatorname{Spf} A$ of X such that u corresponds to a closed point x of the complement of the special fibre of the ordinary scheme $\operatorname{Spec} A$. In any case, we call such a point x a *generic fibre of* u. Furthermore, we will view the set of rig-points of X as "the" points of the complement of the special fibre of X and use X_{rig} as a provisional notation for this set. Also note that, for each u, we can consider the *special fibre* u_0 of u.

Proposition 3.5. *Let X be an admissible formal S-scheme. For each closed point $x_0 \in X_0$ there exists a* rig-*point u of X with special fibre x_0.*

Proof. We may assume that X is affine, say $X = \operatorname{Spf} A$. Then, in the noetherian case, the assertion follows from arguments on systems of parameters such as [EGA 0 IV, 16.3.4 and 16.3.7]. In the classical rigid case one applies [BGR, 7.1.5/4] to the integral closure of A in $A \otimes K$, where K is the field of fractions of the valuation ring over which we work. □

4 Classical rigid spaces in terms of formal schemes

Working in the classical rigid case (I′) of Sect. 1, we want to explain how to associate a rigid space (in the classical sense of [K, Sect. 0], or [BGR, 9.3.1/4]) to each admissible formal scheme. Later we will extend this procedure to the noetherian case (II′) where we work over a formal base scheme S which might have no interpretation in terms of classical rigid geometry.

Let us start with the classical rigid case (I) of Sect. 1. So R is a complete valuation ring which comes from a complete non-trivial height 1 valuation on the field of fractions K of R, and the ideal $\mathfrak{I} \subset R$ is generated by some element $\pi \in K$ whose absolute value satisfies $0 < |\pi| < 1$. For general facts on rigid spaces over

K we refer to [T] or to [BGR]. Let $A = R\langle\xi\rangle/\mathfrak{a}$ be an admissible R-algebra. Tensoring with K over R, we obtain from A the K-algebra

$$A_{\mathrm{rig}} := A \otimes_R K = K\langle\xi\rangle/\mathfrak{a}K\langle\xi\rangle$$

which is an affinoid K-algebra in the sense of [BGR, 6.1.1]. Furthermore, $A \mapsto A_{\mathrm{rig}}$ constitutes a functor which is compatible with complete localization. Namely, consider an element $f \in A$. Then

$$\begin{aligned} A\langle f^{-1}\rangle \otimes_R K &= A\langle\zeta\rangle/(1 - f\zeta) \otimes_R K \\ &= A_{\mathrm{rig}}\langle\zeta\rangle/(1 - f\zeta) = A_{\mathrm{rig}}\langle f^{-1}\rangle , \end{aligned}$$

using the terminology of [BGR, 6.1.4]. Hence, the rigid map associated to the canonical map from A to the complete localization $A\langle f^{-1}\rangle$ is just the canonical map

$$A_{\mathrm{rig}} \to A_{\mathrm{rig}}\langle f^{-1}\rangle$$

which, on the level of associated affinoid K-spaces corresponds to the inclusion of the Laurent domain $(\mathrm{Sp}\, A_{\mathrm{rig}})\,(f^{-1})$ given by f into $\mathrm{Sp}\, A_{\mathrm{rig}}$; cf. [BGR, 7.2.3/1]. Thereby we see that the functor $A \mapsto A_{\mathrm{rig}}$ yields in a natural way a functor

$$\begin{aligned} \mathrm{rig}\colon (\text{Admissible Formal } R\text{-Schemes}) &\to (\text{Rigid } K\text{-Spaces}) , \\ X &\mapsto X_{\mathrm{rig}} . \end{aligned}$$

As in [R] we call X_{rig} the *generic fibre* of the formal R-scheme X. In fact, locally on any affine open piece $\mathrm{Spf}\, A \subset X$, the points of the associated open subspace $\mathrm{Sp}\, A_{\mathrm{rig}} \subset X_{\mathrm{rig}}$ correspond to the maximal ideals in $A_{\mathrm{rig}} = A \otimes_R K$. So, pointwise, X_{rig} "is" the complement of the special fibre of X in the sense we have defined it in Sect. 3; it coincides with the set of rig-points of X.

If we start with some rigid K-space X_K, there is, of course, the question of finding an admissible formal R-scheme X satisfying $X_{\mathrm{rig}} = X_K$. Furthermore, if X exists, there is the question of unicity for X. We want to state the main result which settles these questions.

Theorem 4.1 (Raynaud [R]). *In the classical rigid case* (I), *the functor*

$$\mathrm{rig}\colon X \mapsto X_{\mathrm{rig}} ,$$

gives rise to an equivalence between

(1) *the category of quasi-compact admissible formal R-schemes, localized by admissible formal blowing-ups, and*

(2) *the category of rigid K-spaces which are quasi-compact and quasi-separated.*

It is easily checked that the functor $X \mapsto X_{\mathrm{rig}}$ can produce only quasi-separated rigid K-spaces, so this finiteness condition is definitely needed when we want to construct a formal model X of a rigid K-space X_K. On the other hand, if X_K is affinoid, the existence of X is quite easy. Writing $X_K = \mathrm{Sp}\, A_K$ with $A_K = K\langle\xi\rangle/\mathfrak{a}$, set $A = R\langle\xi\rangle/\mathfrak{a} \cap R\langle\xi\rangle$ and $X = \mathrm{Spf}\, A$.

Proof of 4.1. To begin, let us briefly recall the notion of localization of a category. Consider a category $\mathfrak{C}$ and a class M of morphisms in $\mathfrak{C}$. A localization of $\mathfrak{C}$ by M is a category $\mathfrak{C}_M$ together with a functor $\mathfrak{C} \to \mathfrak{C}_M$ such that, for any functor $\mathfrak{C} \to \mathfrak{D}$ transforming all morphisms of M into isomorphisms of $\mathfrak{D}$, there is a unique functor $\mathfrak{C}_M \to \mathfrak{D}$ making the triangle

$$\begin{array}{ccc} \mathfrak{C} & \to & \mathfrak{C}_M \\ & \searrow \quad \swarrow & \\ & \mathfrak{D} & \end{array}$$

commutative. Of course, uniqueness and commutativity are meant up to equivalence of functors. Localizations of categories do always exist in a unique way, their notion has been developed in order to introduce derived categories, see [V] or [H].

In order to prove 4.1, we will establish the 5 statements listed below, from which (a), (c), and (e) will yield the assertion.

(a) The functor rig transforms admissible formal blowing-ups into isomorphisms.

(b) Two morphisms $\varphi, \psi: Y \to X$ of admissible formal R-schemes coincide if the associated rigid morphisms $\varphi_{\mathrm{rig}}, \psi_{\mathrm{rig}}$ coincide.

(c) For quasi-compact formal R-schemes X, Y and any rigid morphism $\varphi_K: Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$, there exist an admissible blowing-up $\tau: Y' \to Y$ and a morphism $\varphi: Y' \to X$ of formal R-schemes such that $\varphi_{\mathrm{rig}} = \varphi_K \circ \tau_{\mathrm{rig}}$.

(d) If φ_K as in (c) is an isomorphism of rigid K-spaces, we can choose $\varphi: Y' \to X$ satisfying $\varphi_{\mathrm{rig}} = \varphi_K \circ \tau_{\mathrm{rig}}$ with the additional property that it is an admissible formal blowing-up of X.

(e) For each quasi-compact and quasi-separated rigid K-space X_K, there is a quasi-compact formal R-scheme X satisfying $X_{\mathrm{rig}} \simeq X_K$.

Starting with (a), consider a morphism $\varphi: X' \to X$ of formal R-schemes which is an admissible formal blowing-up of some coherent open ideal $\mathscr{A} \subset \mathcal{O}_X$. In order to verify that the associated morphism $\varphi_{\mathrm{rig}}: X'_{\mathrm{rig}} \to X_{\mathrm{rig}}$ is an isomorphism, we may assume that X is affine, say $X = \mathrm{Spf}\, A$. Let $\mathfrak{a} = (f_0, \ldots, f_n) \subset A$ be the ideal corresponding to $\mathscr{A}$. Using 2.2, the formal R-scheme X' is obtained by gluing affine parts $\mathrm{Spf}\, A'_i$ where

$$A'_i = A\left\langle \frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i} \right\rangle \Big/ (\pi\text{-torsion})\ ;$$

$\mathrm{Spf}\, A'_i$ is the maximal open subscheme of X' where f_i generates the invertible ideal $\mathfrak{a}\mathcal{O}_{X'}$. Then, since $f_0, \ldots, f_n$ generate the unit ideal in A_{rig}, we have

$$A'_{i,\mathrm{rig}} = A_{\mathrm{rig}} \left\langle \frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i} \right\rangle$$

in the terminology of [BGR, 6.1.4]. In fact, the rigid morphism associated to $\mathrm{Spf}\, A'_i \to X$ is just the inclusion of the rational subdomain $X_{\mathrm{rig}}(f_0/f_i, \ldots, f_n/f_i)$ into X_{rig}, where the latter subdomain consists of all $x \in X$ such that $|f_i(x)|$ is maximal among $|f_0(x)|, \ldots, |f_n(x)|$; cf. [BGR, 7.2.3/4]. If i varies from 0 to n, the corresponding rational subdomains form a so-called rational covering of X_{rig}, and we see easily that $\varphi_{\mathrm{rig}}: X'_{\mathrm{rig}} \to X_{\mathrm{rig}}$ is an isomorphism, just as claimed. In particular, the functor rig factors through the category obtained from formal R-schemes by localizing with respect to admissible formal blowing-up. This verifies (a).

Next looking at assertion (b), consider two morphisms $\varphi, \psi: Y \to X$ of admissible formal R-schemes such that φ_{rig} coincides with ψ_{rig}. Then we see from 3.4

that φ and ψ coincide as maps between rig-points and, hence, as maps between the point sets underlying Y and X; consider special fibres of rig-points and use 3.5. Therefore, in order to show $\varphi = \psi$, we can assume that X and Y are affine, say $X = \operatorname{Spf} A$, $Y = \operatorname{Spf} B$. However, since the canonical maps $A \to A_K$ and $B \to B_K$ are injective, due to the fact that X and Y are admissible, it is clear that $\varphi_{\mathrm{rig}} = \psi_{\mathrm{rig}}$ must imply $\varphi = \psi$.

Before we proceed any further with the proof of 4.1, let us state some facts which we will need.

Lemma 4.2. *Let X_K be an affinoid K-space and let $Y_K \subset X_K$ be an open affinoid subspace. Then Y_K is a finite union of rational subdomains of X_K.*

Lemma 4.3. *For any affinoid K-space X_K, the system of rational coverings is cofinal in the system of all coverings which are admissible in the sense of the Grothendieck topology of X_K.*

Lemma 4.4. *Let X be a quasi-compact admissible formal R-scheme and let $\mathfrak{U}_K$ be a finite family of quasi-compact open subspaces of the associated rigid space X_{rig}. Then there is an admissible formal blowing-up $\varphi : X' \to X$ together with a family $\mathfrak{U}'$ of formal open subschemes of X' such that the associated family $\mathfrak{U}'_{\mathrm{rig}}$ coincides with $\mathfrak{U}_K$. If $\mathfrak{U}_K$ covers X_{rig}, the system $\mathfrak{U}'$ covers X'.*

The assertion of 4.2 depends on the notion of open affinoid subspaces of affinoid K-spaces. If we thereby understand open affinoid subdomains of most general type, see [BGR, 7.2.2/2], then 4.2 is a consequence of the Theorem of Gerritzen and Grauert, see [GG] or [BGR, 7.3.5/3]. However, for our purposes it is enough to view the rational subdomains as basic open subsets of affinoid K-spaces. Then we can say that a morphism of affinoid K-spaces $\varphi_K : Y_K \to X_K$ defines Y_K as an open subspace of X_K if there is a finite covering of Y_K by rational subdomains Y'_K such that each restriction $\varphi_K|_{Y'_K}$ defines Y'_K as a rational subdomain in X_K. Proceeding like this, the assertion of 4.2 just reflects this definition. Anyhow, later the Theorem of Gerritzen and Grauert, more precisely, its version concerning open immersions, follows in full generality from the flattening technique, see [FII, 5.4]. The assertion of 4.3 is easily derived from 4.2; cf. [BGR, 8.2.2/2].

To settle 4.4, start with the case where X is affine. Using 4.2, each $U_K \in \mathfrak{U}_K$ is a finite union of rational subdomains in X_{rig}. We may assume that each U_K itself is a rational subdomain in X_{rig}, i.e., that each U_K is of type $X_{\mathrm{rig}}(f_1/f_0, \ldots, f_n/f_0)$ with functions $f_0, \ldots, f_n$ generating the unit ideal in $\mathcal{O}_{X_{\mathrm{rig}}}$. Multiplying the f_i with a suitable constant of K, we may even assume $f_i \in \mathcal{O}_X$ for all i. So we can consider the coherent open ideal $\mathscr{A} \subset \mathcal{O}_X$ generated by all f_i as well as the associated formal blowing-up $X' \to X$. Then X' contains a formal open subscheme U which induces U_K as a subspace of X_{rig}. Working with all $U_K \in \mathfrak{U}_K$, we can blow up the product of the corresponding ideals $\mathscr{A} \subset \mathcal{O}_X$. Thereby we obtain an admissible formal R-scheme X' admitting a system $\mathfrak{U}'$ of open formal subschemes which induces the system $\mathfrak{U}_K$ on X_{rig}. The generalization of this fact to the case where X is not necessarily affine is straightforward, using 2.6. Finally we see with the help of 3.5(c) that $\mathfrak{U}'$ covers X' if $\mathfrak{U}_K$ covers X_{rig}.

We add here another lemma on admissible formal blowing-up which we will need.

Lemma 4.5. *Let A be an admissible R-algebra. Consider A as a subring of the associated affinoid K-algebra A_{rig}, and let $f_1, \ldots, f_n \in A_{\mathrm{rig}}$ be elements satisfying*

$|f_i|_{\sup} \leqq 1$. *Then* $B = A[f_1, \ldots, f_n]$ *is an admissible R-algebra which is finite over A. Furthermore, if* $r \in \mathbb{N}$ *is chosen in such a way that* $\pi^r f_i \in A$ *for all i, the canonical morphism* $\operatorname{Spf} B \to \operatorname{Spf} A$ *can be viewed as the blowing-up of the coherent open ideal generated by* $\pi^r, \pi^r f_1, \ldots, \pi^r f_n$ *on* $\operatorname{Spf} A$.

Proof. Choose a set of variables ξ and an epimorphism $R\langle\xi\rangle \to A$. The associated rigid morphism $K\langle\xi\rangle \to A_{\text{rig}}$ is an epimorphism also, and it follows from [BGR, 6.3.4/1], that B is integral over $R\langle\xi\rangle$ and, hence, over A. Since B is of finite type over A, we see that B is a finite A-module. From this one concludes that B is an admissible R-algebra with associated affinoid K-algebra $B_{\text{rig}} = A_{\text{rig}}$. Now consider the ordinary blowing-up $Y' \to \operatorname{Spec} A$ of the ideal $\mathfrak{a} = (\pi^r, \pi^r f_1, \ldots, \pi^r f_n)$ on $\operatorname{Spec} A$. Since the pull-back of $\mathfrak{a}$ to $\operatorname{Spec} B$ is generated by π^r and, thus, is invertible, there is a canonical factorization

$$\operatorname{Spec} B \to Y' \to \operatorname{Spec} A$$

of the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$. Furthermore, the first of these maps identifies $\operatorname{Spec} B$ with the open part $V' \subset Y'$ where the pull-back of $\mathfrak{a}$ is generated by π^r; use the formulas of 2.2. We claim that $V' = Y'$. To justify this, we have to show that, over each open part $\operatorname{Spec} A_i \subset Y'$ where

$$A_i = A\left[\frac{\pi^r}{\pi^r f_i}, \frac{\pi^r f_1}{\pi^r f_i}, \ldots, \frac{\pi^r f_n}{\pi^r f_i}\right] \Big/ \pi\text{-torsion},$$

the pull-back of $\mathfrak{a}$ is generated by π^r. In fact, A_i contains an "inverse" of f_i, and the extension $A_i \to A_i[f_i]$ is integral. But then, using an integral equation of f_i over A_i and multiplying it with a suitable power of f_i^{-1}, we see $f_i \in A_i$. Thus, the pull-back of $\mathfrak{a}$ to A_i which is generated by $\pi^r f_i$, is also generated by π^r. This shows that the blowing-up $Y' \to \operatorname{Spec} A$ coincides with the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$. It follows then by $\mathfrak{J}$-adic completion, which is trivial in this case, that $\operatorname{Spf} B \to \operatorname{Spf} A$ is the formal blowing-up of $\mathfrak{a}$ on $\operatorname{Spf} A$. This concludes the verification of 4.5. □

Next, resuming the proof of 4.1, let us verify assertion (c) above. Consider two quasi-compact formal R-schemes X, Y and a morphism $\varphi_K: Y_{\text{rig}} \to X_{\text{rig}}$ between associated rigid K-spaces. We have to show that there exist an admissible blowing-up $Y' \to Y$ as well as a morphism of formal R-schemes $\varphi: Y' \to X$ satisfying $\varphi_{\text{rig}} = \varphi_K$. To do this, consider coverings $(X_{i,\text{rig}})$ of X_{rig} and $(Y_{i,\text{rig}})$ of Y_{rig} by finitely many open affinoid K-subspaces such that $\varphi_K(Y_{i,\text{rig}}) \subset X_{i,\text{rig}}$ for all i. Applying 4.4 and refining the coverings $(X_{i,\text{rig}})$ and $(Y_{i,\text{rig}})$ in a suitable way, we can assume that both are represented by affine formal open coverings (X_i) of X and (Y_i) of Y. Now assume that the assertion of (c) is already known in the affine case. Then there exist admissible formal blowing-ups $Y'_i \to Y_i$ and morphisms $\varphi_i: Y'_i \to X_i$ satisfying $\varphi_{i,\text{rig}} = \varphi_{\text{rig}}|_{Y_{i,\text{rig}}}$. Applying 2.6, we can view the Y'_i as open formal subschemes of some blowing-up Y' of Y. Then it follows from (b) above that the φ_i define a morphism $\varphi: Y' \to X$ satisfying $\varphi_{\text{rig}} = \varphi_K$.

It remains to look at the case where X and Y are affine, say $X = \operatorname{Spf} A$ and $Y = \operatorname{Spf} B$. Then the morphism between associated rigid spaces $\varphi_K: Y_{\text{rig}} \to X_{\text{rig}}$ is given by a morphism between the corresponding affinoid K-algebras $\sigma_K: A_{\text{rig}} \to B_{\text{rig}}$. Using 4.5, we can apply an admissible blowing-up to Y and thereby assume that σ_K extends to a morphism $\sigma: A \to B$ of admissible R-algebras. Then the associated morphism $\varphi: Y \to X$ satisfies $\varphi_{\text{rig}} = \varphi_K$.

If, in the preceding situation, $\varphi_K: Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ is an isomorphism, we can find an admissible R-subalgebra C consisting of elements with supremum norm $\leqq 1$ in $A_{\mathrm{rig}} = B_{\mathrm{rig}}$, which contains A and B. Then, by 4.5, Spf C is a common admissible blowing-up of X and of Y. Using 2.6 and 4.4, we can extend this fact to the general case and thereby prove (d).

Finally, it remains to settle the existence of formal models as stated in (e). So, consider a quasi-compact and quasi-separated rigid K-space X_K. Proceeding by induction on the number of open affinoid K-spaces which cover X_K, we assume that X_K is covered by two quasi-compact open subspaces U_K and V_K of X_K, which admit formal models U and V. Set $W_K = U_K \cap V_K$. Since X_K is a quasi-separated, an application of 4.4 shows that, after blowing-up, we may assume that the open immersions $W_K \hookrightarrow U_K$ and $W_K \hookrightarrow V_K$ are represented by open immersions $W' \hookrightarrow U$ and $W'' \hookrightarrow V$ of quasi-compact formal R-Schemes. But then, using assertion (d), we can dominate the formal models W' and W'' by a third formal model W of W_K. Extending the necessary blowing-ups to U and V with the help of 2.6, we may view W as an open part of U and of V. Gluing both along W yields the required formal model X of X_K. This concludes the proof of 4.1. □

From assertion (c) of the above proof we can deduce:

Corollary 4.6. *In the situation of* 4.1 *the functor* $X \mapsto X_{\mathrm{rig}}$ *commutes with fibred products.*

As an application of this fact, it follows that separated rigid K-spaces admit separated formal models; more precisely:

Proposition 4.7. *A morphism of admissible formal R-schemes $X \to Y$ is separated if and only if the associated morphism of rigid spaces $X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$ is separated.*

Proof. We have to show that the diagonal morphism $\Delta: X \to X \times_Y X$ is a closed immersion if and only if the associated rigid morphism, which, by 4.6, is just the diagonal morphism $\Delta_{\mathrm{rig}}: X_{\mathrm{rig}} \to X_{\mathrm{rig}} \times_{Y_{\mathrm{rig}}} X_{\mathrm{rig}}$, is a closed immersion. The only-if-part being trivial, assume that Δ_{rig} is a closed immersion. Then let Z be the schematic closure of $\Delta_{\mathrm{rig}}(X_{\mathrm{rig}})$ in $X \times_Y X$; it coincides with the schematic image of Δ and is an admissible formal R-scheme, use 1.2(c). So it is enough to verify that, in terms of closed points, Z coincides with the image of Δ. To show this, consider a closed point $x_0 \in Z_0$ with projections $x_0', x_0'' \in X_0$. Using 3.5, we can extend x_0 to a rig-point x of Z; let x', x'' be the associated projections on X. Then the generic fibres of x' and x'' must coincide, because the image of Δ_{rig} is closed in $X_{\mathrm{rig}} \times_{Y_{\mathrm{rig}}} X_{\mathrm{rig}}$. Thus, we have $x' = x''$ by 3.4 and, hence, $x_0' = x_0''$. It follows that x_0 belongs to the image of Δ and that the latter coincides with Z. □

Theorem 4.1 provides a means of characterizing classical rigid K-spaces in terms of formal algebraic geometry, at least if we restrict ourselves to rigid K-spaces which are quasi-compact and quasi-separated. If we do not require such finiteness conditions, the assertion of 4.1 is still useful since we can interpret general rigid K-spaces as inductive limits of rigid K-spaces which are quasi-compact and quasi-separated. Anyhow, in most cases where rigid K-spaces occur in nature, they are separated and equipped with a certain class of affinoid coverings leading to formal models.

5 Relative rigid spaces

In classical rigid geometry it is essential that the (absolute) base over which one is working, consists of a field K with a valuation. Therefore, if one wants to extend the notion of rigid spaces to the noetherian cases (II) or (II′) of Sect. 1, it is not possible to imitate the classical construction of Tate. Instead, motivated by the characterization of classical rigid spaces given in Sect. 4, one proceeds via admissible formal schemes. To simplify the terminology, we will require from now on that all admissible formal S-schemes are *quasi-compact*, unless stated otherwise. The base S is as in (I′) (the classical rigid case) or in (II′) (the noetherian case) of Sect. 1.

Denoting the category of admissible formal S-schemes in the above sense by (FSch/S), we can now extend the notion of rigid spaces.

Definition 5.1. *The category* (Rig/S) *of rigid S-spaces is obtained from* (FSch/S) *by localizing with respect to admissible formal blowing-up.*

Due to the definition of the category (Rig/S), there is a canonical functor

$$\mathrm{rig}:(\mathrm{FSch}/S)\to(\mathrm{Rig}/S)\,,$$

which associates a rigid S-space X_{rig} to any formal S-scheme X and a rigid S-morphism φ_{rig} to any morphism φ of formal S-schemes. Note that, for simplicity, we will continue writing S instead of S_{rig} also on the rigid side, although S_{rig} would be more accurate.

We have already explained the notion of a localized category in Sect. 4. However, at this point, we must know how the localization (Rig/S) of (FSch/S) is actually constructed; see [V] or [H] for the corresponding procedure which is used to introduce derived categories. To define (Rig/S), take as objects the objects of (FSch/S). Furthermore, for two objects X, Y of (Rig/S), a morphism $X\to Y$ is given by an equivalence class of diagrams in (FSch/S) of type

$$\begin{array}{ccc} X' & & \\ \downarrow & \searrow & \\ X & & Y\,, \end{array}$$

where $X'\to X$ is an admissible formal blowing-up. Two such diagrams $X\leftarrow X'_i\to Y$, $i=1,2$, are called equivalent if there is a third diagram $X\leftarrow X'_3\to Y$ of this type together with S-morphisms $X'_3\to X'_i$, $i=1,2$, providing factorizations of $X'_3\to X$ and $X'_3\to Y$ through X'_1 and X'_2.

It is not difficult to check directly that the just described relation really is an equivalence relation. On the other hand, it might be more appropriate to interpret the set $\mathrm{Hom}_{(\mathrm{Rig}/S)}(X,Y)$ as the direct limit (in the style of [A, I.1]) of the sets $\mathrm{Hom}_{(\mathrm{FSch}/S)}(X',Y)$ where X' varies over all admissible formal blowing-ups of X. To do this, consider the category $\mathfrak{B}$ of all admissible formal blowing-ups of X; define a morphism $X_1\to X_2$ in $\mathfrak{B}$ between two such objects X_1 and X_2 as an X-morphism $X_2\to X_1$ in (FSch/S). Composition of two arrows $X_1\to X_2$ and $X_2\to X_3$ is done, using the composition of morphisms in (FSch/S). Then $\mathfrak{B}$ satisfies the axioms (L1), (L2), (L3) of [A, I.1]. Namely, (L1) just means that, given a diagram

$$\begin{tikzcd} X'_1 \arrow[dr] & & X'_2 \arrow[dl] \\ & X' \arrow[d] & \\ & X & \end{tikzcd}$$

of X-morphisms in (FSch/S), where X' and the X'_i are admissible formal blowing-ups of X, say of ideals $\mathscr{I}$ and $\mathscr{I}_i$ on X, there is an admissible formal blowing-up X'' of X dominating the X'_i via X'-morphisms. Just define X'' by blowing up the product $\mathscr{I} \cdot \mathscr{I}_1 \cdot \mathscr{I}_2$ on X. Then each of the ideals $\mathscr{I}$ and $\mathscr{I}_i$ becomes invertible on X''. Thus, using the universal property of blowing-ups, see 2.1(c), the existence of the required morphisms follows.

In the same way we can verify axiom (L2). Consider two blowing-ups X_1 and X_2 of X as well as two X-morphisms $X_2 \to X_1$. Then both coincide after dominating X_2 by a blowing-up X_3 of X; just blow up the product of the ideals necessary for X_1 and X_2. Finally, the connectedness axiom (L3) holds since two admissible blowing-ups can be dominated by a third one.

Now, considering the functor $\mathfrak{B} \to$ (Sets) which associates to any admissible formal blowing-up X' of X the set $\mathrm{Hom}_{(\mathrm{FSch}/S)}(X', Y)$, the direct limit

$$\varinjlim_{X' \in \mathfrak{B}} \mathrm{Hom}_{(\mathrm{FSch}/S)}(X', Y)$$

exists, and we can use it to define the set $\mathrm{Hom}_{(\mathrm{Rig}/S)}(X, Y)$. Then the latter corresponds to the above set of equivalence classes of diagrams $X \leftarrow X' \to Y$, where X' varies over the admissible formal blowing-ups of X. Finally, to compose two morphisms $X \to Y$ and $Y \to Z$ in (Rig/S), say given by diagrams

$$X \leftarrow X' \to Y, \quad Y \leftarrow Y' \to Z$$

in (FSch/S), we use a diagram of type

$$\begin{tikzcd} X'' \arrow[d] \arrow[dr] & & \\ X' \arrow[d] \arrow[dr] & Y' \arrow[d] \arrow[dr] & \\ X & Y & Z, \end{tikzcd}$$

where $X'' \to X'$ is the formal blowing-up of the coherent open ideal $\mathscr{I}'$ on X' which is the pull-back of the ideal $\mathscr{I} \subset \mathcal{O}_Y$ corresponding to the formal blowing-up $Y' \to Y$. To end the considerations concerning the construction of (Rig/S), one has, of course, to observe that the resulting category satisfies the universal property of a localized category.

Next we want to introduce the concept of rigid points for rigid S-spaces. Thereby we will see that, even in the general noetherian case, a rigid space in the sense of 5.1 is not too far from a rigid space in the classical sense.

Definition 5.2. *Let X be an admissible formal S-scheme. For any* rig-*point* $u: T \to X$ *(in the notion of* 3.1) *we call the associated morphism* $u_{\mathrm{rig}}: T_{\mathrm{rig}} \to X_{\mathrm{rig}}$ *a rigid point of the associated rigid S-space X_{rig}. The residue field of u (i.e., the field of fractions of the ring $\mathcal{O}_T(T)$) is referred to as the residue field of u_{rig}.*

It follows from 3.3 that the definition of rigid points of a rigid S-space X_{rig} is independent of the choice of the formal model X of X_{rig} and that the residue field of such a point is well-defined. Furthermore, we see from 3.4 that the concept of rigid points in the sense of 5.2 coincides with the classical concept of (closed) points, if classical rigid spaces are considered. In fact, the notion X_{rig}, which was provisionally used in Sect. 3 to denote the set of points of the complement of the special fibre of an admissible formal S-scheme X, just yields the set of rigid points of the rigid S-space X_{rig} which is associated to X in the sense of 5.1.

For any rigid point $u_{\mathrm{rig}}: T_{\mathrm{rig}} \to S_{\mathrm{rig}}$ and any rigid S-space X_{rig}, we can define the pull-back $X_{\mathrm{rig}}(u_{\mathrm{rig}})$ of X_{rig} with respect to u_{rig}. To do this, choose a rig-point $u: \mathrm{Spf}\, B \to S$ representing u_{rig} and let $\bar{B}$ be the integral closure of B in its field of fractions; it is a valuation ring by 3.2. Forming the fibred product of X over S with $\mathrm{Spf}\, \bar{B}$ (in the sense of [EGA I, 10.7]) and killing the $\bar{B}$-torsion, we obtain an admissible formal $\bar{B}$-scheme $X(u)$ whose associated rigid space, which we denote by $X_{\mathrm{rig}}(u_{\mathrm{rig}})$, is a classical rigid space over $Q(B)$. We may visualize X_{rig} as the family of all (classical) rigid spaces $X_{\mathrm{rig}}(u_{\mathrm{rig}})$ with u_{rig} varying over the rigid points of S_{rig}. Also we point out that all notions applicable to classical rigid spaces can now be applied to the fibres over rigid points of the base S_{rig}. In particular, the fibre dimension of rigid S-spaces is defined.

Besides the notion of rigid points, there are several other notions in classical rigid geometry which can be extended to the situation of Definition 5.1. Basically, one can proceed in two ways. The first is to say that a certain object or morphism in (Rig/S) has a certain property (P) if there is a representative in (FSch/S) which satisfies a formal version of (P). For example one proceeds like this with open or closed immersions. That in the classical rigid case open immersions can be characterized in this manner, is part of the proof of 4.1; the case of closed immersions is dealt with similarly. The same procedure works for flatness; to show this is the main objective of the paper [FII].

The second possibility to define properties in (Rig/S) is by applying corresponding properties to complements of special fibres of objects or morphisms in (FSch/S). For example, considering an object $X \in$ (FSch/S), we cover X by open affine pieces $X_i = \mathrm{Spf}\, A_i$ and look at a scheme property (P) which applies to complements of special fibres in the ordinary schemes $\mathrm{Spec}\, A_i$. Of course, in order that (P) defines a reasonable property in (Rig/S), one has to check that the validity of (P) on complements of special fibres is local with respect to the Zariski topology in (FSch/S) and invariant under admissible blowing-up. Frequently, we will say that an object or a morphism in (FSch/S) satisfies rig-(P) if the associated object or morphism in (Rig/S) satisfies (P) in the sense we have just explained.

In the following we want to work out the notion of rig-flatness in (FSch/S) as well as the notion of flatness in (Rig/S). Assuming S affine, say $S = \mathrm{Spf}\, R$, let $B \to A$ be a morphism of admissible R-algebras and consider an A-module M as well as a closed point $x \in \mathrm{Spec}\, A - V(\mathfrak{J})$ of the complement of the special fibre of $\mathrm{Spec}\, A$. We say that M is rig-*flat over B at* x, or at the rig-point of $\mathrm{Spf}\, A$ corresponding to x, if the ordinary localization M_x is a flat B-module. Furthermore, M is said to be rig-*flat* (resp. *faithfully* rig-*flat*) over B if M restricts to a flat (resp. faithfully flat) B-module over $\mathrm{Spec}\, A - V(\mathfrak{J})$. In particular, M is rig-flat over B if and only if it has this property at each closed rig-point of $\mathrm{Spf}\, A$. The notions we have just introduced, can easily be extended to the level of admissible formal schemes. Namely, consider a morphism $\varphi: X \to Y$ in (FSch/S) and a coherent $\mathcal{O}_X$-module $\mathcal{M}$. Then $\mathcal{M}$ is called rig-*flat* over Y at a rig-point $x \in X$ if there are open affine parts $U = \mathrm{Spf}\, A \subset X$ and $V = \mathrm{Spf}\, B \subset Y$ with x closed in U and $\varphi(U) \subset V$, such that, on U, the $\mathcal{O}_X$-module $\mathcal{M}$ corresponds to an A-module which is rig-flat at x over B. Of course, $\mathcal{M}$ is called rig-flat over Y if it is rig-flat over Y at each rig-point of X. Equivalently we can say that, locally on X and for each finitely generated ideal $\mathcal{A} \subset \mathcal{O}_X$, the canonical map $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ has a kernel which is annihilated by a power of the ideal of definition $\mathcal{I} \subset \mathcal{O}_S$; keep in mind that we have restricted ourselves to quasi-compact formal schemes. Furthermore, $\mathcal{M}$ is called *faithfully* rig-*flat* over X, if it is rig-flat and if the pull-back of $\mathcal{M}$ with respect to any rig-point of X is not annihilated by a power of $\mathcal{I}$. Alternatively, we can require that, locally on X, each equation $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} = 0$ with a finitely generated $\mathcal{O}_X$-module $\mathcal{N}$ implies that $\mathcal{N}$ is annihilated by a power of $\mathcal{I}$.

Proposition 5.3. *Let $X \to Y$ be a morphism in* (FSch/S) *and consider a coherent $\mathcal{O}_X$-module $\mathcal{M}$.*

(a) *The notion of* rig-*flatness of $\mathcal{M}$ over Y is local on X and Y; i.e., consider an affine situation, where $S = \mathrm{Spf}\, R$, $X = \mathrm{Spf}\, A$, $Y = \mathrm{Spf}\, B$, and let x be a closed rig-point of X. Then $\mathcal{M}$ is* rig-*flat* (*resp.* rig-*flat at x*) *over Y if and only if the A-module M associated to $\mathcal{M}$ is* rig-*flat* (*resp.* rig-*flat at x*) *over B.*

(b) *The notion of faithful* rig-*flatness of $\mathcal{M}$ over Y is local on Y.*

(c) rig-*flatness,* rig-*flatness at a point, and faithful* rig-*flatness are invariant under admissible blowing-up.*

Due to the properties of rig-points, assertion (b) is clear, once (a) has been proved. Furthermore, assertions (a) and (c) will follow from Lemma 5.5, which we will obtain as a consequence of Lemma 5.4 below.

Lemma 5.4. *Let $X = \mathrm{Spf}\, A$ be an affine formal S-scheme, and consider a formal admissible blowing-up $X' \to \mathrm{Spf}\, A$ of some coherent open ideal $\mathfrak{a} \subset A$. Let $(\mathrm{Spf}\, A_i')_{0,\dots,m}$ be an affine open covering of X'. Then,*

(a) *the morphisms $A \to A_i'$ are* rig-*flat,*

(b) *$A \to \prod_i A_i'$ is faithfully* rig-*flat.*

Lemma 5.5. *In the situation of* 5.4, *consider a morphism $\varphi: \mathrm{Spf}\, A \to \mathrm{Spf}\, B$ of affine formal S-schemes, an admissible formal blowing-up $Y' \to \mathrm{Spf}\, B$ of some coherent open ideal $\mathfrak{b} \subset B$, as well as a coherent A-module M and a closed* rig-*point $x \in \mathrm{Spf}\, A$. Let $(\mathrm{Spf}\, B_j')_{0,\dots,n}$ be an affine open covering of Y'. Then the following are equivalent:*

(a) *M is* rig-*flat* (*resp.* rig-*flat at x*) *over B,*

(b) *$M \otimes_A A_i'$ is* rig-*flat* (*resp.* rig-*flat at x*) *over B for all i.*

(c) *$M \otimes_B B_j'$ is* rig-*flat* (*resp.* rig-*flat at x*) *over B for all j.*

Proof of 5.4. Writing $\mathfrak{a} = (f_0, \dots, f_m)$, we may use 2.2 and thereby assume that $\mathrm{Spf}\, A_i'$ is the affine open part of X' where f_i generates $\mathfrak{a}\mathcal{O}_{X'}$. Furthermore, let $\mathrm{Spec}\, \tilde{A}_i'$ be the affine open part of the scheme-theoretic blowing-up $Y' \to \mathrm{Spec}\, A$ of $\mathfrak{a}$ where f_i generates $\mathfrak{a}\mathcal{O}_{Y'}$. By the definition of admissible blowing-up, A_i' is the $\mathfrak{J}$-adic completion of $\tilde{A}_i'$. But then, considering the noetherian case, (a) is clear since $Y' \to \mathrm{Spec}\, A$ is an isomorphism on complements of special fibres and since $\tilde{A}_i' \to A_i'$ is flat as a completion map. To verify (b), it is enough to mention that the admissible blowing-up $X' \to \mathrm{Spf}\, A$ induces a surjection, in fact, a bijection between associated rig-points, cf. 3.3, and that, for each closed rig-point $x \in \mathrm{Spf}\, A$, the corresponding rig-point of X' is closed in $\mathrm{Spf}\, A_i'$ for some i.

In the classical rigid case, (a) and (b) follow from the fact that, for any open immersion of affinoid K-spaces $\mathrm{Sp}\, B_K \to \mathrm{Sp}\, A_K$, the associated morphism of affinoid K-algebras $A_K \to B_K$ is flat. To justify this fact, consider the induced morphisms $A_{K,x} \to B_{K,x}$ between localizations at points $x \in \operatorname{Sp} B_K$. The latter are flat by [AC, Chap. III, Sect. 5, no. 4, Proposition 4], since the corresponding morphisms between maximal-adic completions are isomorphisms; cf. [BGR, 7.3.2/3]. □

Proof of 5.5. We need only to consider the noetherian case. But then, using [AC, Chap. I, Sect. 3, no. 2, Proposition 4], the equivalences follow from 5.4. □

Although in most cases we will work exclusively on the level of admissible formal S-schemes, we want to point out that rigid S-spaces can be viewed in a natural way as locally ringed spaces. To give some explanations, consider an object $X_{\text{rig}} \in (\text{Rig}/S)$. The points of X_{rig} are, of course, the rigid points (which correspond to the rig-points of any formal S-model X of X_{rig}). The open subsets are induced from formal open subschemes of formal S-models of X_{rig}. To define the structure sheaf $\mathcal{O}_{\text{rig}}$ on X_{rig}, fix a formal S-model X of X_{rig} with ideal of definition $\mathcal{I}$, and consider the presheaf $\mathcal{O}$ which associates to any affine open formal subscheme $\operatorname{Spf} A \subset X$ the ring $\Gamma(\operatorname{Spec}(A) - V(\mathcal{I}), \mathcal{O}_{\operatorname{Spec} A})$ of sections which are defined on the complement of the special fibre of the ordinary scheme $\operatorname{Spec} A$. Using the fact that sections of $\mathcal{O}$ extend to sections of $\mathcal{O}_X$, if multiplied by suitable sections of $\mathcal{I}$, one shows that $\mathcal{O}$ satisfies sheaf properties and, thus, extends to a sheaf which is defined on all open subsets of X. Let us call $\mathcal{O}$ the sheaf of sections on X which are defined on the complement of the special fibre. Now, to define the structure sheaf $\mathcal{O}_{\text{rig}}$ of X_{rig}, consider an open subspace $U_{\text{rig}} \subset X_{\text{rig}}$. By definition, there is a formal model $U \subsetneq X$ of $U_{\text{rig}} \subsetneq X_{\text{rig}}$, which is an open immersion. Then associate with U_{rig} the ring of sections on U which are defined on the complement of the special fibre of U. That the resulting functor $\mathcal{O}_{\text{rig}}$ is well-defined and is a sheaf on X_{rig} (in the sense that it satisfies sheaf properties with respect to *finite* open coverings) follows in the classical rigid case from Tate's acyclicity theorem [BGR, 8.2.1/1], and in the noetherian case using [EGA III_1, 5.1.2]. Actually, we would have to say that $\mathcal{O}_{\text{rig}}$ is a sheaf with respect to the appropriate Grothendieck topology G on X_{rig}, just as in the classical rigid case. However, since all our spaces are quasi-compact and quasi-separated, G is quite simple to describe. The G-open subsets of X_{rig} are the ones which are induced from formal open subschemes of formal S-models of X_{rig}, whereas the G-coverings are those coverings by G-open sets which admit a finite subcover.

Having constructed the structure sheaf $\mathcal{O}_{\text{rig}}$ on a rigid S-space X_{rig}, there is the notion of coherent $\mathcal{O}_{\text{rig}}$-modules; cf. [EGA 0_{I}, 5.3.1]. The latter extends the notion we have in the classical rigid case. Starting with a formal S-model X of X_{rig} and a coherent $\mathcal{O}_X$-module $\mathcal{M}$, we can proceed similarly as above and associate to $\mathcal{M}$ a coherent $\mathcal{O}_{\text{rig}}$-module $\mathcal{M}_{\text{rig}}$. Again we say that $\mathcal{M}$ is a formal $\mathcal{O}_X$-model of $\mathcal{M}_{\text{rig}}$. An alternative way to define coherent $\mathcal{O}_{\text{rig}}$-modules is by localizing the fibred category of coherent modules over the category of admissible formal S-schemes.

Proposition 5.6. *Let $\mathcal{M}_{\text{rig}}$ be a coherent $\mathcal{O}_{\text{rig}}$-module on a rigid S-space X_{rig}. Then $\mathcal{M}_{\text{rig}}$ admits a formal $\mathcal{O}_X$-model $\mathcal{M}$ on any formal S-model X of X_{rig}.*

Proof. Let us start with the noetherian case. Working up to admissible formal blowing-up of X (which is permissible, as we will see at the end of the proof), we want to show that, locally on X_{rig}, there exist formal models of $\mathcal{M}_{\text{rig}}$. To do this, we may assume that the formal model X we are considering is affine and that the ideal of definition of X is principal, say generated by $g \in \mathcal{O}_X$. Furthermore, we can assume that we have an exact sequence

$$\mathcal{O}^m_{\mathrm{rig}} \to \mathcal{O}^n_{\mathrm{rig}} \to \mathcal{M}_{\mathrm{rig}} \to 0 .$$

Multiplying the first morphism by a suitable power of g, we can assume that the sequence extends to an exact sequence

$$\mathcal{O}^m_X \to \mathcal{O}^n_X \to \mathcal{M} \to 0$$

of $\mathcal{O}_X$-modules. Then $\mathcal{M}$ is a formal $\mathcal{O}_X$-model of $\mathcal{M}_{\mathrm{rig}}$ which is coherent.

In the classical rigid case, we can settle the local existence of formal models by a similar argument, this time working locally on X. So assume that X is affine, $X = \mathrm{Spf}\, A$, and let g be a generator of the ideal of definition of A. Then $\mathcal{M}_{\mathrm{rig}}$ is a coherent module on the classical rigid space $\mathrm{Sp}\, A[g^{-1}]$ and, hence, corresponds to an $A[g^{-1}]$-module M_{rig} of finite presentation. Thus, there is an exact sequence of $A[g^{-1}]$-modules

$$F'_{\mathrm{rig}} \to F_{\mathrm{rig}} \to M_{\mathrm{rig}} \to 0$$

with F'_{rig} and F_{rig} finite free. The same argument as the one used above shows that M_{rig} extends to an A-module M of finite presentation, giving rise to a formal model $\mathcal{M}$ of $\mathcal{M}_{\mathrm{rig}}$ on X. The latter is coherent since it is of finite presentation and since $\mathcal{O}_X$ is a sheaf of coherent rings; cf. 1.3.

As a next step we want to show how to produce a global formal $\mathcal{O}_X$-model of $\mathcal{M}_{\mathrm{rig}}$ from local ones. In particular, this will settle the assertion of 5.6 in the classical rigid case.

Lemma 5.7. *For $X \in (\mathrm{FSch}/S)$, consider a finite open covering $(X_i)_{i \in I}$ of X, together with a coherent $\mathcal{O}_{X_i}$-module $\mathcal{M}_i$ on each X_i. Furthermore, assume that, on X_{rig}, we have gluing data for the $\mathcal{M}_{i,\mathrm{rig}}$. Then there exist a coherent $\mathcal{O}_X$-module $\mathcal{M}$ as well as $\mathcal{O}_{X_i}$-morphisms $\mathcal{M}|_{X_i} \to \mathcal{M}_i$ which are* rig-*isomorphisms respecting the gluing data.*

The procedure of *proof* is similar to the one we applied for the construction of formal models of classical rigid spaces. Using the quasi-compactness and quasi-separatedness of X, the basic case to deal with, is the one where $I = \{1, 2\}$ and where X_1 and X_2 are affine; we set $X_{12} = X_1 \cap X_2$. To settle this case, let $\mathcal{M}_{12}$ (resp. $\mathcal{M}_{21}$) be the restriction of $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) to X_{12}. Furthermore, let $g \in \mathcal{O}_S$ be a generator of the ideal of definition $\mathfrak{J}$ of X (for simplicity, we assume that $\mathfrak{J}$ is principal). Then there is a power g^n with the property that $g^n\mathcal{M}_{12}$ may be viewed as a submodule of $\mathcal{M}_{21}$ via an injection $g^n\mathcal{M}_{12} \hookrightarrow \mathcal{M}_{21}$ respecting the gluing data we have on the complement of the special fibre of X. Working modulo powers of $\mathfrak{J}$ and applying [EGA $\mathrm{I}_{\mathrm{new}}$, 6.9.7], it is possible to extend $g^n\mathcal{M}_{12}$ to a coherent submodule $\mathcal{M}'_2$ of $\mathcal{M}_2$. Then $g^n\mathcal{M}_1$ and $\mathcal{M}'_2 + g^n\mathcal{M}_2$ can be glued along $g^n\mathcal{M}_{12}$ over X_{12}, thus providing the desired $\mathcal{O}_X$-module $\mathcal{M}$ and thereby establishing the assertion of 5.7.

Finally, it remains to justify that, in the noetherian case, we are allowed to work up to admissible formal blowing-up of X, as we have done by using a local argument on X_{rig}. So let $\varphi: X' \to X$ be an admissible formal blowing-up, and assume that $\mathcal{M}'$ is a coherent $\mathcal{O}_{X'}$-module which is a formal model of $\mathcal{M}_{\mathrm{rig}}$. Then one can use the algebraization theorem [EGA III_1, 4.1.5], in conjunction with [EGA III_1, 5.1.4], to show that $\varphi_*(\mathcal{M}')$ is a coherent $\mathcal{O}_X$-module such that $\varphi^*\varphi_*(\mathcal{M}')$ is rig-isomorphic to $\mathcal{M}'$. In particular, $\varphi_*(\mathcal{M}')$ is a formal $\mathcal{O}_X$-model of $\mathcal{M}_{\mathrm{rig}}$. This finishes the proof of 5.6. □

We end our general discussion of coherent modules by discussing briefly the notion of flatness for coherent modules over rigid S-spaces. Consider a morphism $\varphi_{\text{rig}}: Y_{\text{rig}} \to X_{\text{rig}}$ in (Rig/S) and a coherent $\mathcal{O}_{\text{rig}}$-module $\mathcal{M}_{\text{rig}}$ on Y_{rig}. We say that $\mathcal{M}_{\text{rig}}$ is *flat* over X_{rig} if there is a formal model $\varphi: Y \to X$ of φ_{rig} with a formal model $\mathcal{M}$ of $\mathcal{M}_{\text{rig}}$ on Y such that $\mathcal{M}$ is rig-flat over X. In fact, the latter means that $\mathcal{M}$ is flat over X at all points of X_{rig}; i.e., at all points of the complement of the special fibre of X. The morphism φ_{rig} itself is called flat if $\mathcal{O}_{Y_{\text{rig}}}$, the canonical Y_{rig}-module induced from structure sheaves of formal models of Y_{rig}, is flat over X_{rig}. As follows from the properties of rig-flatness, the notion of flatness is local on X_{rig} and Y_{rig} as well as independent of the choice of formal models.

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