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Embeddings and Proper Holomorphic Maps of Strictly Pseudoconvex Domains into Polydiscs and Balls

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1. Introduction

In this paper we continue the investigation, started in [10], of the problem of embedding pseudoconvex domains as closed complex submanifolds of bounded domains. We shall be studying the existence of proper holomorphic maps from strictly pseudoconvex domains $\Omega \Subset \mathbb{C}^n$ into either the polydisc Δ_m or the ball $\mathbb{B}_m$ in $\mathbb{C}^m$.

In the equidimensional case, $m=n$, such maps are particularly simple. In fact if Ω is strictly pseudoconvex with C^∞ boundary, then any proper holomorphic map to $\mathbb{B}_n$ is necessarily biholomorphic (see [3]) and proper maps to Δ_n simply do not exist (Theorem 15.2.4 of [11]). Some results about proper maps from balls to higher dimensional balls are also known. In fact, proper holomorphic maps from $\mathbb{B}_n$ to $\mathbb{B}_{n+1}$ which are C^2 up to the boundary are necessarily trivial (equivalent to the inclusion) when $n \geqq 3$ and equivalent to one of four polynomial maps when $n=2$. This was proved by Webster in [12] ($n \geqq 3$) and by Faran in [4] ($n=2$) with C^3 boundary conditions. The condition was lowered to C^2 by Cima and Suffridge ([1]), who also proved similar results for higher codimensions.

The content of this paper is that there are an enormous amount of proper holomorphic maps from strictly pseudoconvex domains to both polydiscs and balls, when the codimension is sufficiently high, and in the case of balls as targets they can be made continuous up to the boundary as well. Thus the boundary smoothness condition mentioned above may very well be crucial.

In particular, we obtain a version of the Fornaess embedding theorem ([5]), which says that every strictly pseudoconvex domain is a closed complex submanifold of a strictly convex domain, when the target is a ball, but with continuity up to the boundary only. Lempert ([7]) had previously shown that it is possible to embed strictly pseudoconvex domains as closed complex submanifolds of the infinite dimensional ball.

2. Some Remarks

The case when $\Omega = \mathbb{B}_n$ and the target is a polydisc was solved in [10]. It is based on the construction of inner functions of Hakim and Sibony ([6]) and the author ([8]). We shall use the notations $S = \partial\mathbb{B}_n$ and for $z \in S$, $B(z, r) = \{w \in S;\ \delta(z, w) < r\}$ where δ is the metric $\delta(z, w) = \frac{|z-w|}{\sqrt{2}}$. When $z, w \in S$, then $1 - \operatorname{Re}\langle z, w\rangle = \delta^2(z, w)$, hence $|\exp(1 - \langle z, w\rangle)| = \exp - \delta^2(z, w)$. The construction is based on the following covering lemma:

For any α there exists an integer $M(= M(n, \alpha))$ such that for any $r > 0$ there exist M families $\mathscr{F}_1, \ldots, \mathscr{F}_M$ of balls with radii αr, $\mathscr{F}_i = \{B(z_{ij}, \alpha r);\ j = 1, \ldots, N_i\}$, such that each family is *disjointed*, i.e. $B(z_{ij}, \alpha r) \cap B(z_{ik}, \alpha r) = \emptyset$ when $j \neq k$ and such that the balls of radii r cover S, i.e.

$$S = \bigcup_{i,j} B(z_{ij}, r).$$

Of course M will increase rapidly with α. In [10] $\alpha = 2$ was used to prove that there exist proper holomorphic maps from $\mathbb{B}_n$ to Δ_{4M}, but any $\alpha > \sqrt{2}$ would have done.

We shall formulate all our results for strictly pseudoconvex domains, but only prove them in the case of the ball. The generalization to strictly pseudoconvex domains is done in exactly the same way as [9] generalizes the construction of inner functions from [8]. It should be noted, though, that in strictly pseudoconvex domains we do not have holomorphic support functions whose real part is exactly the square of the distance, as $1 - \langle z, w\rangle$ for the ball, but only bounded above and below by multiples of the square of the distance. This means that for our arguments to work, we have to choose a larger value of α. Hence the dimension of the target space depends in a dramatic way on the geometry of the boundary with our present methods. We would also like to mention that in our construction of proper continuous maps from balls to balls we were not able to choose α below $\sqrt{6}$. Whether there is a real connection between the boundary regularity and the codimension, remains to be seen.

Although the case of the polydisc was solved in [10], we would like to mention some things that follow from the construction. The proof is essentially just to generalize [10] in the same way that [9] generalized [8] and we shall only briefly sketch the necessary changes in the proofs. We shall use the following notations for two domains Ω, Ω': $H(\Omega, \Omega')$ are the holomorphic functions from Ω to Ω', $A(\Omega, \Omega')$ and $HP(\Omega, \Omega')$ are the functions in $H(\Omega, \Omega')$ which are continuous up to the boundary or proper and $AP(\Omega, \Omega') = A(\Omega, \Omega') \cap HP(\Omega, \Omega')$.

3. The Polydisc as a Target

For a function $f: \Omega \to \mathbb{C}^m$, $f = (f_1, \ldots f_m)$ define $M_f(z) = \max_i |f_i(z)|$. If f is holomorphic then M_f is plurisubharmonic.

Theorem 1. *Let $\Omega \Subset \mathbb{C}^n$ be strictly pseudoconvex with C^2 boundary. Then, for all sufficiently large m the following holds: If $f \in A(\Omega, \mathbb{C}^m)$, $\phi > 0$ is continuous on $\partial\Omega$ such that $M_f(z) < \phi(z)$ on $\partial\Omega$, $K \subset \Omega$ is compact and $\varepsilon > 0$ then there exists $g \in H(\Omega, \mathbb{C}^m)$ such that $M_{f+g}(\xi) \to \phi(z)$ uniformly as $\xi \to z \in \partial\Omega$ and $M_g(\xi) \leqq \varepsilon$ when $\xi \in K$. g can also be chosen to vanish to any prescribed order at an interior point.*

Corollary 1. *Ω is a closed complex submanifold of a polydisc.*

Corollary 2. *If Ω has C^3 boundary, then $HP(\Omega, \Delta_m)$ is dense in $H(\Omega, \Delta_m)$ in the topology of uniform convergence on compacts.*

Proof. Assume $\Omega = \mathbb{B}_n$. It is enough to prove the theorem for one integer m. It then follows in general by putting $0 = g_{m+1} = \dots$. We will prove it for $m = 4M (= 4M(n, 2))$ as in [10]. The proof is to replace the number 1 in the sequence of statements in [10] by the function $\phi(z)$. It is only in the last step that a small difficulty arises. When $M_f(z) < \phi(z)$ we define the weight of f at z by:

$$W_f(z) = \sum_i -\ln(\phi(z) - |f_i(z)|).$$

We have then the equivalent of Lemma 1 in [10]:

(1) There exist constants ε_0, C such that: If $a < 1$, $\varepsilon \leqq \varepsilon_0$, $\varepsilon \leqq (1-a)^6$, $\varepsilon' > 0$ and $R < 1$ and $f = (f_1, \dots, f_{2M})$ and $\phi \leqq 1$ are continuous on S with $U_1 = \{z \in S; M_f(z) > a\phi(z)\}$ then there exist entire functions $g = (g_1, \dots, g_{2M})$ such that

(a) $M_{f+g}(z) \leqq \max\{M_f(z), \phi(z)\} + 3\varepsilon$.
(b) $M_g(z) \leqq \varepsilon'$ when $|z| \leqq R$.
(c) $U_2 = \{z \in S; M_{f+g}(z) > a\phi(z) - 5\varepsilon\} \supset U_1$.
(d) If $z \in S \sim U_2$ then $W_{f+g}(z) \geqq W_f(z) + C\varepsilon^{2/3}$.

It is straightforward to check this (compare also Lemma 3.2 of [9] with Lemma 1 of [8]). The equivalent of Lemma 2 in [10] is:

(2) If $f = (f_1, \dots, f_{2M})$ and ϕ are continuous on S with $M_f \leqq \phi$ and $\varepsilon > 0$, $R < 1$, then there exist entire functions $g = (g_1, \dots, g_{2M})$ such that $\phi(z) - \varepsilon \leqq M_{f+g}(z) \leqq \phi(z) + \varepsilon$ for all $z \in S$ and $M_g(z) \leqq \varepsilon$ when $|z| \leqq R$.

Finally, we can also prove the theorem in essentially the same way as in [10]. So let $f \in A(\mathbb{B}_n, \mathbb{C}^{4M})$, $M_f(z) < \phi(z)$ for all $z \in S$. If $f = (f_1, \dots, f_{4M})$, split f in two parts $k = (f_1, \dots, f_{2M})$ and $k' = (f_{2M+1}, \dots, f_{4M})$, so $M_k(z)$, $M_{k'}(z) < \phi(z)$. Choose sequences $\{a_n\}$ converging to 1 and $\{\varepsilon_n\}$ converging to zero such that $M_k(z)$, $M_{k'}(z) < a_1 \phi(z)$ and $a_n \phi(z) + \varepsilon_n \leqq a_{n+1} \phi(z)$ for all $z \in S$. We then apply (2) to k to get g_1, then to k' to get g'_1, then again to $k + g_1$ etc, letting R increase so fast that we cut off an adjustment so far out that we know the previous adjustment is good there, as in [10]. Thus, if we let $h_n = \sum_{i=1}^n g_i$ and $h'_n = \sum_{i=1}^n g'_i$ then

$$a_n \phi(z) - \varepsilon_n \leqq M_{k+h_n}(z), \quad M_{k'+h'_n}(z) \leqq a_n \phi(z) + \varepsilon_n \quad \text{for all } z \in S \tag{$*$}$$

and it follows as in [10] that $h = \lim h_n$ and $h' = \lim h'_n$ exist and that for any $\varepsilon > 0$ there is some $\delta > 0$ such that, if we let $g = (h, h')$ then $M_{f+g}(\xi) > \phi(z) - \varepsilon$ when $|\xi - z| < \delta$. It now remains to notice that since M_{f+g} is subharmonic, the inequality ($*$) implies that $M_{f+g}(\xi) \leqq \tilde{\phi}(\xi)$ where $\tilde{\phi}$ is the harmonic extension of ϕ to $\overline{\mathbb{B}}_n$.

The property $M_g(\xi)\leqq\varepsilon$ when $\xi\in K$ follows by choosing ε_n very small. Also in (1), the ε' occuring in (b) can be chosen arbitrarily small. This will imply, via the Cauchy estimates, that any prescribed number of derivatives at a given point can be chosen as small as we please, hence we may subtract the first terms in the power series expansion of g without practically disturbing g at all. Hence the function g in (1) can be chosen to vanish to any prescribed order at a given point, which gives the final statement of the theorem.

To prove Corollary 2, notice that by a theorem of Cole and Range ([2]), $A(\Omega, \Delta_m)$ is dense in $H(\Omega, \Delta_m)$. To approximate $f\in A(\Omega, \Delta_m)$ by $h\in HP(\Omega, \Delta_m)$, just apply the theorem to $(1-\varepsilon)f$ and choose $\phi=1$ to obtain g. Then $h=g+(1-\varepsilon)f$ is proper and $M_{f-h}(\xi)\leqq 2\varepsilon$ when $\xi\in K$.

4. The Ball as a Target

We shall now construct proper holomorphic maps from strictly pseudoconvex domains to balls. Once again we shall give the construction in its simplest form, i.e. from balls to balls and then indicate how it can be generalized to give Theorem 3. In the discussion that follows $S=\partial\mathbb{B}_n$ and $N=M(n, 3)$, so for any $r>0$ we have N disjointed families $\mathscr{F}_1, \ldots, \mathscr{F}_N$ of balls with radii $3r$ such that the totality of the smaller balls of radii r cover S. The idea is to push the modulus on S of a map from $\overline{\mathbb{B}}_n$ to $\mathbb{B}_{2N}$ towards 1, always pushing values in the complex tangential direction of the target.

Lemma 1. *There exist constants δ_0, $C, D>0$ such that: If $f\colon S\to\mathbb{C}^{2N}$ is continuous, $b\leqq|f(z)|\leqq 1$ for all $z\in S$, $0<\varepsilon\leqq\delta_0$, $\varepsilon\leqq(1-b)^{3/4}$, $\varepsilon'>0$ and $R<1$, then there exists an entire function $g\colon\mathbb{C}^n\to\mathbb{C}^{2N}$ such that for all $z\in S$:*

1. $|f(z)+g(z)|\leqq 1+C\varepsilon(1-b)^{1/2}$.
2. $|f(z)+g(z)|\geqq b+D\varepsilon^{1/4}(1-b)$.
3. $|g(z)|\leqq C(1-b)^{1/2}$.
4. $|g(\xi)|\leqq\varepsilon'$ *when* $|\xi|\leqq R$.

Proof. Let $e_1, \ldots, e_{2N}$ be the standard basis of $\mathbb{C}^{2N}$, and for each $w\in\mathbb{B}_{2N}$, $w\neq 0$, we define T_w to be the complex tangent space at w of the ball through w with center at the origin, i.e. $T_w=\{w+n;\ \langle w, n\rangle=0\}$. T_w is a $2N-1$ dimensional complex hyperplane which intersects $\mathbb{B}_{2N}$ in a ball of radius r with $r^2=1-|w|^2$. We now define N unit length noncontinuous vector fields $n_1, \ldots, n_N$ in $\mathbb{B}_{2N}$. If $w=(w_1, \ldots, w_{2N})$, then $n_i(w)$ is defined as follows: If $w_{2i-1}=w_{2i}=0$, then let $n_i(w)=e_{2i-1}$. Otherwise, the direction of $n_i(w)$ is defined to be that of $\bar{w}_{2i}e_{2i-1}-\bar{w}_{2i-1}e_{2i}$. The crucial properties of these vector fields are that they are orthonormal, no matter at which point each one is evaluated, and for each $w\neq 0$, $n_i(w)$ lie in the direction of the complex tangent space at w, i.e. $\langle w, n_i(w)\rangle=0$ for all i, so

$$w+\sum_{i=1}^{N}\lambda_i n_i(w)\in\mathbb{B}_{2N}\quad\text{whenever } \lambda\in\mathbb{C}^N \text{ and } |\lambda|^2<1-|w|^2.$$

Let us first assume that $b>0$. It will be clear that the constants are independent of b. Then, since T_w varies continuously outside the origin, the following statement, which is formulated to suit our future purposes, holds:

(a) There exists $\delta>0$ such that: If I is a set of indices, $I\subset\{1,\dots,N\}$ and $w, w_i (i\in I)$ are points in $\mathbb{B}_{2N}$ with $|w-w_i|\leqq\delta$ and $|w|, |w_i|\geqq b$ then there exist orthonormal vectors $n_i\in T_w$, $i\in I$, such that $|n_i-n_i(w_i)|<\varepsilon$.

Now assume $r>0$ is so small that $|f(z)-f(z')|\leqq\min\{\delta,\varepsilon^2(1-b)\}$ whenever $\delta(z,z')<3r$ and let $\mathscr{F}_1,\dots,\mathscr{F}_N$ be the corresponding disjointed families of balls with radius $3r$, $\mathscr{F}_i=\{B(z_{ij},3r);\, j=1,\dots,N_i\}$. Now define $g_i\colon \mathbb{C}^n\to\mathbb{C}^{2N}$ by

$$g_i(z)=\sum_{j=1}^{N_i}\left(\frac{1-|f(z_{ij})|^2}{N}\right)^{1/2}\exp(-m(1-\langle z,z_{ij}\rangle))\,n_i(f(z_{ij}))$$

g_i has nonzero entries only in the $2i-1$ and $2i$ position. It then follows, as in [10], that there is a constant C_1 such that if ε is sufficiently small and

(b) $mr^2=\frac{1}{9}\log\left(\frac{C_1}{\varepsilon}\right)$ then $|g_i(z)|\leqq\varepsilon(1-b)^{1/2}$ if $z\in S$ is not in any ball in $\mathscr{F}_i$.

Also, for any $z\in S$, the same estimate holds if we only sum over the balls in $\mathscr{F}_i$ not containing z.

Now let $g(z)=\sum_{i=1}^{N}g_i(z)$ and for $z\in S$ let $w=f(z)$ and $I(z)=\{i;\, z\in B(z_{ij(i)},3r)$ for some $j(i)\}$. Also let $w_i=f(z_{ij(i)})$ and n_i the corresponding vectors in the direction of the complex tangent space as given by (a). Then

$$\begin{aligned}
&\left|[f(z)+g(z)]-\left[w+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp-m(1-\langle z,z_{ij(i)}\rangle)\,n_i\right]\right|\\
&\leqq\left|g(z)-\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp-m(1-\langle z,z_{ij(i)}\rangle)\,n_i(w_i)\right|+2N^{1/2}\varepsilon(1-b)^{1/2}\\
&\leqq\left|g(z)-\sum_{i\in I(z)}\left(\frac{1-|w_i|^2}{N}\right)^{1/2}\exp-m(1-\langle z,z_{ij(i)}\rangle)\,n_i(w_i)\right|\\
&\quad+\frac{1}{N^{1/2}}\sum_{i\in I(z)}|(1-|w_i|^2)^{1/2}-(1-|w|^2)^{1/2}|+2N^{1/2}\varepsilon(1-b)^{1/2}\\
&\leqq N\varepsilon(1-b)^{1/2}+\frac{1}{N^{1/2}}\sum_{i\in I(z)}|(1-|w_i|^2)^{1/2}-(1-|w|^2)^{1/2}|+2N^{1/2}\varepsilon(1-b)^{1/2}.
\end{aligned}$$

Now, if $|w|\geqq1-2\varepsilon^2(1-b)$, then $|w_i|\geqq1-3\varepsilon^2(1-b)$ so $1-|w_i|^2\leqq6\varepsilon^2(1-b)$ hence

$$\sum_{i\in I(z)}|(1-|w_i|^2)^{1/2}-(1-|w|^2)^{1/2}|\leqq\sqrt{6}N\varepsilon(1-b)^{1/2}.$$

If $|w|\leqq1-2\varepsilon^2(1-b)$, then $|w_i|\leqq1-\varepsilon^2(1-b)$ so $1-|w_i|^2\geqq\varepsilon^2(1-b)$, hence by the mean value theorem there is some s between $1-|w|^2$ and $1-|w_i|^2$ such that

$$(1-|w_i|^2)^{1/2}-(1-|w|^2)^{1/2}=\frac{||w_i|^2-|w|^2|}{2\sqrt{s}}\leqq\frac{|w_i-w|}{\sqrt{s}}\leqq\frac{\varepsilon^2(1-b)}{\varepsilon(1-b)^{1/2}}=\varepsilon(1-b)^{1/2}.$$

Thus we have proved that

$$\left|[f(z)+g(z)]-\left[w+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp -m(1-\langle z, z_{ij(i)}\rangle)\, n_i\right]\right|\leqq 6N\varepsilon(1-b)^{1/2}.$$

This inequality proves 1, 2 and 3. It proves 1 because the expression in the second bracket is in $\mathbb{B}_{2N}$. It proves 3 since

$$|g(z)|\leqq \sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}+6N\varepsilon(1-b)^{1/2}\leqq(2N^{1/2}+6N\varepsilon)(1-b)^{1/2}.$$

Finally, since the vectors w, n_i are orthogonal and there is an index $i\in I(z)$ such that $\delta(z, z_{ij(i)})<r$, (b) gives

$$\begin{aligned}
&\left|w+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp -m(1-\langle z, z_{ij(i)}\rangle)\, n_i\right|^2\\
&\quad=|w|^2+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)\exp -2m\,\delta^2(z, z_{ij(i)})\geqq|w|^2+\left(\frac{1-|w|^2}{N}\right)\exp(-2mr^2)\\
&\quad=|w|^2+\left(\frac{1-|w|^2}{N}\right)\left(\frac{\varepsilon}{C_1}\right)^{2/9}\geqq b^2+\left(\frac{1-b^2}{N}\right)\left(\frac{\varepsilon}{C_1}\right)^{2/9}\geqq b^2+\frac{1}{NC_1^{2/9}}\,\varepsilon^{2/9}(1-b)
\end{aligned}$$

hence since $\varepsilon\leqq(1-b)^{3/4}$

$$\begin{aligned}
|f(z)+g(z)|&\geqq\left|w+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp -m(1-\langle z, z_{ij(i)}\rangle)\, n_i\right|-6N\varepsilon(1-b)^{1/2}\\
&\geqq b+\frac{1}{2NC_1^{2/9}}\,\varepsilon^{2/9}(1-b)-6N\varepsilon^{1/3}(1-b)\geqq b+D\varepsilon^{1/4}(1-b)
\end{aligned}$$

whenever ε is sufficiently small.

In case $b=0$ we just choose δ so small that (a) holds whenever $|w-w_i|<\delta$ and $|w|$, $|w_i|\geqq\varepsilon^2$ and then choose r and g as before. If $|w|\geqq 2\varepsilon^2$ then $|w_i|\geqq\varepsilon^2$ and our argument still holds. If $|w|^2\leqq 2\varepsilon^2$ the inequality

$$\left|[f(z)+g(z)]-\left[w+\sum_{i\in I(z)}\left(\frac{1-|w|^2}{N}\right)^{1/2}\exp -m(1-\langle z, z_{ij(i)}\rangle)\, n_i(w_i)\right]\right|\leqq 2N\varepsilon$$

holds and implies 1, 2 and 3 since the vectors $n_i(w_i)$ still are orthonormal. 4 is standard as in [8]. It follows by choosing m sufficiently large.

Remarks. 1. We shall apply Lemma 1 inductively with the hypothesis $b\leqq|f(z)|\leqq a$ for some $a<1$, in which case we can put a instead of 1 in the conclusion. The numbers a will approach 1 more rapidly than b and we may as well use the following formulation, which is better suited for our purposes (the number D occuring is one half of the D in Lemma 1):

There exist constants δ_0, C, $D>0$ such that: If $f\colon S\to\mathbb{C}^{2N}$ is continuous, $b\leqq|f(z)|\leqq a<1$ for all $z\in S$, $(1-a)<(1-b)/2$, $0<\varepsilon\leqq\delta_0$, $\varepsilon\leqq 1/2\ (1-b)^{3/4}$, $\varepsilon'>0$ and $R<1$ then there exists an entire function $g\colon\mathbb{C}^n\to\mathbb{C}^{2N}$ such that for all $z\in S$:

1. $|f(z)+g(z)| \leqq a + C\varepsilon(1-b)^{1/2}$.
2. $|f(z)+g(z)| \geqq b + D\varepsilon^{1/4}(1-b)$.
3. $|g(z)| \leqq C(1-b)^{1/2}$.
4. $|g(\xi)| \leqq \varepsilon'$ when $|\xi| \leqq R$.

We shall refer to f, a, b, ε as data for the lemma, but supress the role of ε' and R, which are not so important. In fact R will be kept fixed and ε' will just decrease rapidly to zero.

2. Since the ε' appearing in 4 is independent of ε, we can choose g in Lemma 1 vanishing to any prescribed order at a given interior point (see the end of the proof of Theorem 1).

3. In case the target dimension of f is larger than $2N$, i.e. $2N+k$ for some integer k, then Lemma 1 is still true if we just put $g_{2N+1} = \ldots = g_{2N+k} = 0$. The norm of $f(z_{ij})$ occuring in the definition of g_i must *not* be interpreted to mean the norm of the $2N$ first components of f only.

4. Finally, we can replace the number 1 in Lemma 1 by any positive continuous function ϕ on S, the precise statement being:

If $\phi > 0$ is continuous on S, then there exist constants δ_0, $C, D > 0$ such that: If $f: S \to \mathbb{C}^{2N}$ is continuous, $b\phi(z) \leqq |f(z)| \leqq \phi(z)$ for all $z \in S$, $0 < \varepsilon \leqq \delta_0$, $\varepsilon \leqq (1-b)^{3/4}$, $\varepsilon' > 0$ and $R < 1$, then there exists an entire function $g: \mathbb{C}^n \to \mathbb{C}^{2N}$ such that for all $z \in S$:

1. $|f(z)+g(z)| \leqq (1 + C\varepsilon(1-b)^{1/2})\,\phi(z)$.
2. $|f(z)+g(z)| \geqq (b + D\varepsilon^{1/4}(1-b))\,\phi(z)$.
3. $|g(z)| \leqq C(1-b)^{1/2}$.
4. $|g(\xi)| \leqq \varepsilon'$ when $|\xi| \leqq R$.

And, of course, Remark 1, 2 and 3 also apply to Remark 4.

Theorem 2. *Let $f: S \to \mathbb{B}_{2N}$ be continuous and $\varepsilon > 0$, $R < 1$. Then there exists a continuous function $h: \overline{\mathbb{B}}_n \to \mathbb{C}^{2N}$ which is holomorphic in $\mathbb{B}_n$ such that $|f(z)+h(z)| = 1$ for all $z \in S$ and $|h(\xi)| \leqq \varepsilon$ when $|\xi| \leqq R$.*

Proof. All ε_n will be chosen less than δ_0 and we do not mention this anymore. Let $b_0 = 0$ and $k \leqq 1/3$ such that $a_0 = 1 - k \geqq |f(z)|$ for all $z \in S$ and assume $\varepsilon_0 < 1/2$. Furthermore, suppose we have sequences $\{a_n\}$, $n \geqq 0$, increasing strictly to 1 and $\{\varepsilon_n\}$, $n \geqq 0$, decreasing strictly to zero such that, if we let $b_n = 1 - \prod_{i=0}^{n-1} (1 - D\varepsilon_i^{1/4})$ for $n \geqq 1$, then

(i) $a_n + C\varepsilon_n(1-b_n)^{1/2} \leqq a_{n+1}$.
(ii) $(1-a_n) \leqq 1/2(1-b_n)$.
(iii) $\varepsilon_n \leqq 1/2(1-b_n)^{3/4}$.
(iv) $\Sigma(1-b_n)^{1/2} < \infty$.

We may then apply Remark 1 to the data $f, a_0, b_0, \varepsilon_0$ to obtain g_1 such that for all $z \in S$:

1. $|f(z)+g_1(z)| \leqq a_0 + C\varepsilon_0(1-b_0)^{1/2} \leqq a_1$.
2. $|f(z)+g_1(z)| \geqq b_0 + D\varepsilon_0^{1/4}(1-b_0) = D\varepsilon_0^{1/4} = b_1$.
3. $|g_1(z)| \leqq C(1-b_0)^{1/2}$.
4. $|g_1(\xi)| \leqq \varepsilon_1'$ when $|\xi| \leqq$ R.

Then by 1, 2, (ii) and (iii) we may apply Remark 1 to the data $f+g_1$, a_1, b_1, ε_1 to find g_2. Suppose we have inductively found functions $g_1, \dots, g_n$ such that, if we let $h_n = \sum_{i=1}^{n} g_i$, then for all $z \in S$:

1. $|f(z)+h_n(z)| \leqq a_{n-1} + C\varepsilon_{n-1}(1-b_{n-1})^{1/2} \leqq a_n$.
2. $|f(z)+h_n(z)| \geqq b_{n-1} + D\varepsilon_{n-1}^{1/4}(1-b_{n-1}) = 1-(1-b_{n-1})(1-D\varepsilon_{n-1}^{1/4}) = b_n$.
3. $|g_n(z)| \leqq C(1-b_{n-1})^{1/2}$.
4. $|g_n(\xi)| \leqq \varepsilon'_n$ when $|\xi| \leqq R$.

Then by 1, 2, (ii) and (iii) we can apply Remark 1 to the data $f+h_n$, a_n, b_n, ε_n to produce g_{n+1} and properties 1 to 4 follow immediately. The by 3 and (iv) $h(z) = \lim h_n(z) = \sum_{i=1}^{\infty} g_i(z)$ converges uniformly on $\overline{\mathbb{B}}_n$ so h is continuous on $\overline{\mathbb{B}}_n$ and holomorphic in $\mathbb{B}_n$. By 1, $|f(z)+g(z)| \leqq 1$ on S and by (iv), $\lim b_n = 1$, so by 2, $|f(z)+g(z)| \geqq 1$ on S. If we now choose ε'_n such that $\Sigma\varepsilon'_n < \varepsilon$, we are finished.

It remains to find a_n and ε_n such that properties (i) to (iv) hold. We choose for $n \geqq 0$:

$$D\varepsilon_n^{1/4} = \frac{3}{n+K+3}, \qquad a_n = 1-k\left(\frac{K}{n+K}\right)^4.$$

We claim that if K is a large integer, then all the properties hold. Clearly $\varepsilon_n \leqq \delta_0$ for all n when K is large. Now,

$$1-b_n = \prod_{i=0}^{n-1}\left(1-\frac{3}{i+K+3}\right) = \prod_{i=0}^{n-1}\left(\frac{i+K}{i+K+3}\right) = \frac{K(K+1)(K+2)}{(n+K)(n+K+1)(n+K+2)}$$

which gives (iv). Also

$$\frac{2\varepsilon_n}{(1-b_n)^{3/4}} = \frac{2\cdot 3^4[(n+K)(n+K+1)(n+K+2)]^{3/4}}{D^4(n+K+3)^4[K(K+1)(K+2)]^{3/4}} \leqq \frac{2\cdot 3^4}{D^4(n+K+3)^{7/4}K^{9/4}} < 1$$

for all n when K is large, which proves (iii) and

$$\begin{aligned}\frac{2(1-a_n)}{(1-b_n)} &= \frac{2kK^3(n+K+1)(n+K+2)}{(n+K)^3(K+1)(K+2)} \\ &\leqq 2k\frac{(n+K+1)(n+K+2)}{(n+K)^2} \leqq 2k\left(1+\frac{3}{K}+\frac{2}{K^2}\right) < 1\end{aligned}$$

for all n when K is large since $k \leqq 1/3$ which proves (ii) and finally, using the equality $x^4-y^4 = (x^2+y^2)(x+y)(x-y)$ we get

$$\begin{aligned}a_{n+1}-a_n &= kK^4\left[\frac{1}{(n+K)^4} - \frac{1}{(n+K+1)^4}\right] \\ &= \frac{kK^4}{(n+K)^4(n+K+1)^4}[(n+K+1)^4-(n+K)^4] \\ &\geqq \frac{kK^4\, 2(n+K)^2\, 2(n+K)}{(n+K)^4(n+K+1)^4} = \frac{4kK^4}{(n+K)(n+K+1)^4}\end{aligned}$$

hence

$$\frac{C\varepsilon_n(1-b_n)^{1/2}}{a_{n+1}-a_n} \leqq \frac{3^4 C(n+K)(n+K+1)^4[K(K+1)(K+2)]^{1/2}}{4D^4 k(n+K+3)^4 K^4[(n+K)(n+K+1)(n+K+2)]^{1/2}} \leqq \frac{3^4 C(K+1)}{4D^4 kK^4}$$

for all n which proves (i).

Theorem 3. *Let $\Omega \Subset \mathbb{C}^n$ be strictly pseudoconvex with C^2 boundary. Then for all sufficiently large m the following holds: If $f: \partial\Omega \to \mathbb{C}^m$ is continuous and ϕ is continuous on $\partial\Omega$ with $|f(z)| < \phi(z)$ for all z, $K \subset \Omega$ is compact and $\varepsilon > 0$ then there exists $g \in A(\Omega, \mathbb{C}^m)$ such that $|f(z)+g(z)| = \phi(z)$ for all $z \in \partial\Omega$ and $|g(\xi)| \leqq \varepsilon$ when $\xi \in K$. g can also be chosen to vanish to any prescribed order at an interior point.*

Corollary 3. *Ω can be embedded as a closed complex submanifold of a ball. The embedding can be made continuous up to the boundary.*

Corollary 4. *$AP(\mathbb{B}_n, \mathbb{B}_m)$ is dense in $H(\mathbb{B}_n, \mathbb{B}_m)$ when m is sufficiently large.*

Proof. Suppose $\Omega = \mathbb{B}_n$. By Remark 3 we can assume $m = 2N$, as in Theorem 2. Using Remark 4, the proof of Theorem 2 works without any changes. The final statement in the theorem follows from Remark 2.

To prove Corollary 3, we may assume $\bar{\Omega} \subset \mathbb{B}_n$. Now apply Theorem 3 to $f = 0$ and $\phi(z) = (1-|z|^2)^{1/2}$ to find $g \in A(\Omega, \mathbb{C}^{2N})$ such that $|g(z)| = \phi(z)$ on $\partial\Omega$. Now the map $z \to (z, g(z))$, which is in $AP(\Omega, \mathbb{B}_{n+2N})$ is the required embedding.

Corollary 4 follows as Corollary 2, since dilation shows that $A(\mathbb{B}_n, \mathbb{B}_m)$ is dense in $H(\mathbb{B}_n, \mathbb{B}_m)$. We can not prove this corollary for strictly pseudoconvex domains, since the necessary version of the Cole/Range theorem has not been studied.

After this paper was completed I have been informed that M. Hakim/N. Sibony and F. Forstnerič, both using [10], have independently proved that strictly pseudoconvex domains embed as closed complex submanifolds of balls. They also both give examples of embeddings of balls in higher dimensional balls which do not extend continuously up to the boundary. This follows by applying our Theorem 3 to such one dimensional examples

References

1. Cima, J.A., Suffridge, T.J.: A reflection principle with applications to proper holomorphic mappings. Math. Ann. 265, 489–500 (1983)
2. Cole, B., Range, R.M.: A-measures on complex manifolds and some applications. J. Funct. Anal. **11**, 393–400 (1972)
3. Diederich, K., Fornaess, J.E.: Proper holomorphic images of strictly pseudoconvex domains. Math. Ann. **259**, 279–286 (1982)
4. Faran, J.J.: Maps from the two-ball to the three-ball and maps taking lines to plane curves. Preprint
5. Fornaess, J.E.: Embedding strictly pseudoconvex domains in convex domains. Am. J. Math. **98**, 529–569 (1976)
6. Hakim, M., Sibony, N.: Fonctions holomorphes bornees sur la boule unite de $\mathbb{C}^n$. Invent. Math. **67**, 213–222 (1982)

7. Lempert, L.: Imbedding strictly pseudoconvex domains into a ball. Am. J. Math. **104**, 901–904 (1982)
8. Løw, E.: A construction of inner functions on the unit ball in $\mathbb{C}^p$. Invent. Math. **67**, 223–229 (1982)
9. Løw, E.: Inner functions and boundary values in $H^\infty(\Omega)$ and $A(\Omega)$ in smoothly bounded pseudoconvex domains. Math. Zeitschr. **185**, 191–210 (1984)
10. Løw, E.: The ball in $\mathbb{C}^n$ is a closed complex submanifold of a polydisc. Preprint
11. Rudin, W.: Function theory in the unit ball of $\mathbb{C}^n$. Berlin-Heidelberg-New York: Springer 1980
12. Webster, S.M.: On mapping an n-ball into an $(n+1)$-ball in complex space. Pac. J. Math. **81**, 267–272 (1979)

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