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Wave Front Sets and Singular Supports of Convolutions

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1. Introduction

If S and T are distributions with compact support, the convolution theorem of Lions-Titchmarsh [7] states that

$$\operatorname{co}\operatorname{supp}(T*S)=\operatorname{co}\operatorname{supp}T+\operatorname{co}\operatorname{supp}S \tag{1.1}$$

where $\operatorname{co}K$ means the convex hull of the set K in $\mathbb{R}^n$. This theorem is used if one wants to prove that the convolution equation $T*u=f$, $f\in\mathscr{E}(\Omega_1)$ has a solution $u\in\mathscr{E}(\Omega_2)$ where Ω_1, Ω_2 are open sets in $\mathbb{R}^n$ with $\Omega_2\supset\Omega_1+\operatorname{supp}T$, [5–7, 9]. If one looks for solutions $u\in\mathscr{D}'(\Omega_2)$ when $f\in\mathscr{D}'(\Omega_1)$, a similar theorem for singular supports is needed.

$$\operatorname{co}\operatorname{sing}\operatorname{supp}(T*S)=\operatorname{co}\operatorname{sing}\operatorname{supp}T+\operatorname{co}\operatorname{sing}\operatorname{supp}S\,. \tag{1.2}$$

But unfortunately this is only true under additional conditions, e.g. if $\operatorname{supp}T$ (or $\operatorname{supp}S$) consists of a finite number of points [7], or if T is the characteristic function of a compact convex polyhedron [2, 3]. But if T is the characteristic function of a sphere there is a distribution $S\in\mathscr{E}'(\mathbb{R}^n)$ such that (1.2) does not hold [3]. C. A. Berenstein and M. A. Dostal conjectured in [3] that this is always the case for characteristic functions of convex, compact sets with smooth boundary and proved it in the meantime in [4] by carefully estimating the Fourier transform of T. We will prove this conjecture even for non convex bounded sets with smooth boundary. This is an immediate consequence of Theorem 3.2 which gives a necessary condition on T for the validity of (1.2) for all $S\in\mathscr{E}'(\mathbb{R}^n)$.

The proof is based on the following formula for the wave front set of the convolution $T*S$

$$WF(T*S)\subset\{(x+y,\xi)\,;\ (x,\xi)\in WF(T),\ (y,\xi)\in WF(S)\}\,. \tag{1.3}$$

This formula is a special case of Theorem 2.5.14 in Hörmander [9], but we will give in this case a simple direct proof. In Section 2 we state the definitions and simple properties of wave front sets which we need. For the proofs we refer to [9], where the notion of wave front set was introduced in analogy to M. Sato's notion of singular spectrum [12, 13]. Section 3 contains the proof of (1.3) and in Section 4 we show by a counterexample that the condition in Theorem 3.2 is only necessary but not sufficient and therefore (1.3) is in general a proper inclusion. The construction of the counterexample uses ideas of Ehrenpreis [5] and Hörmander [8]. For the consequences concerning convolution equations we refer to [8]. Finally we want to mention that in [6] a connection is made between our Theorem 3.2 and Hörmander's characterization of singular supports by supporting functions [8].

Some of the results of this paper were announced in [1].

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2. Wave Front Sets

As usual we note $|\xi|=\left(\sum_{k=1}^{n}\xi_k^2\right)^{1/2}$ the euclidean norm and $x\xi=\sum_{k=1}^{n}x_k\xi_k$ the scalar product of vectors x, ξ in $\mathbb{R}^n$. For a multiindex α we set $|\alpha|=\sum_{k=1}^{n}\alpha_k$ and $D_\xi^\alpha=\left(-i\frac{\partial}{\partial\xi}\right)$. A pseudo-differential operator of order m is an operator of the form

$$(A\varphi)(x)=\left(\frac{1}{2\pi}\right)^n\int e^{ix\xi}a(x,\xi)\hat{\varphi}(\xi)d\xi$$

where $\hat{\varphi}(\xi)=\int e^{-ix\xi}\varphi(x)dx$ is the Fourier transform of $\varphi\in\mathscr{D}(\mathbb{R}^n)$ and the symbol $a(x,\xi)\in\mathscr{C}^\infty(\mathbb{R}^n\times\dot{\mathbb{R}}^n)$ satisfies inequalities

$$|D_\xi^\alpha D_x^\beta a(x,\xi)|\leqq C_{\alpha,\beta,K}(1+|\xi|)^{m-\alpha}\quad\text{where}\quad(x,\xi)K\times\dot{\mathbb{R}}^n$$

K compact in $\mathbb{R}^n$, for some constant $C_{\alpha,\beta,K}$. A can be extended to a continuous operator $A:\mathscr{E}'(\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^n)$. If A can be extended to an operator $A:\mathscr{D}'(\mathbb{R}^n)\to\mathscr{D}'(\mathbb{R}^n)$ we call A a properly supported operator. For more detailed information we refer to [9] and [11].

The characteristic set $\gamma(A)$ of the pseudo-differential operator A of order 0 is defined by

$$\gamma(A)=\{(x,\xi)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\});\ \liminf_{t\to\infty}|a(x,t\xi|=0\}\,.$$

Definition 2.1. For a distribution $T\in\mathscr{D}'(\mathbb{R}^n)$ the wave front set $WF(T)$ is defined by

$$WF(T)=\{\gamma(A),\ AT\in\mathscr{C}^\infty(\mathbb{R}^n)\}$$

where A runs over properly supported pseudo-differential operators of order 0.

Since the $a(x,t\xi)$ as functions of (x,ξ) are equicontinuous, $\gamma(A)$ and $WF(T)$ is closed and the definition implies immediately that $WF(T)$ is a "cone" in $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$. If π is the projection of $\mathbb{R}^n\times\mathbb{R}^n$ on the first factor we have $\pi(WF(T))\subset\operatorname{sing\,supp}T$ and one can even show ([9], Thm. 2.5.3).

Proposition 2.2. $\pi(WF(T))=\operatorname{sing\,supp}T$.

It is clear that for a $\mathscr{C}^\infty$-function φ the fibres of $WF(T)$ and $WF(\varphi T)$ coincide over every point x where $\varphi(x)\neq 0$. So the fact that $(x,\xi)\in WF(T)$ is a local property of the distribution T and we can restrict ourselves to distributions with compact support. For distributions $T \in \mathscr{E}'(\mathbb{R}^n)$ the wave front set can be characterized by the Fourier transform $\hat{T}$ as follows ([9], 2.5.5).

Proposition 2.3. *$(x, \xi) \notin WF(T)$ if and only if there is a function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ with $\varphi(x) \neq 0$ such that for any integer N there exists a constant C_N with*

$$|\widehat{\varphi T}(\eta)| \leqq C_N(1+|\eta|)^{-N} \tag{2.1}$$

for any η in a conic neighborhood of ξ.

We now determine the wave front set in two cases we need in Section 3.

Examples. 1) Let Ω be an open set in $\mathbb{R}^n$ with smooth ($\mathscr{C}^\infty$) boundary and let T be the characteristic function of Ω, $T = \chi_\Omega$ with $\chi_\Omega = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \phantom{\text{if }} x \notin \Omega \end{cases}$. The singular support of T is $\partial\Omega$ and we have only to determine the fibre of $WF(T)$ over each $x \in \partial\Omega$. We claim that $WF(T) = \{(x, \xi);\ x \in \partial\Omega, \xi = \pm tn,\ t > 0\}$, where n is the normal to the boundary. (This is related to Proposition 2.1.3, Chapter III of [13].) For the proof we suppose $x = 0 \in \partial\Omega$. In a neighborhood U of 0 let $\partial\Omega$ be given by the equation $x_n = h(x_1, \ldots, x_{n-1})$, $h \in \mathscr{C}^\infty$, and let A_j be the differential operator of order 1 $A_j = \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_n} \frac{\partial h}{\partial x_j}$ with the symbol $i\left(\xi_j + \xi_n \frac{\partial h}{\partial x_j}\right)$. A_j is the derivation along the curve $x_n = h(x_1, \ldots, x_{n-1})$, $x_k = \text{const.}$ for $k \neq j$ and we see that $A_j T = 0$. If we multiply A_j by the elliptic pseudo-differential operator of order -1 with the symbol $(1+|\xi|^2)^{-1/2}$ for $|\xi| \geqq 1$ we do not change the characteristic sets. We have $\gamma(A_j) = \left\{(x, \xi);\ \xi_j = -\xi_n \frac{\partial h}{\partial x_j}\right\}$ so the fibre of $WF(T)$ over the point $x = 0$ is generated by the vector $n = \left(-\frac{\partial h}{\partial x_1}, \ldots, -\frac{\partial h}{\partial x_{n-1}}, 1\right)$ the normal at $\partial\Omega$.

By reason of symmetry n and $-n$ are in the fibre of $WF(T)$. By differentiating T with respect to the normal n, we get the distribution $\delta_{\partial\Omega}$ defined by $\langle \delta_{\partial\Omega}, \varphi \rangle = \int_{\partial\Omega} \varphi(x)\, d\omega$. The properties of wave front sets show that

$$WF(T) = WF(\delta_{\partial\Omega}).$$

2) Let T be the distribution defined by

$$\langle T, \varphi \rangle = \lim_{y_1 \to 0+} \int \frac{\varphi(x_1, \ldots, x_n)}{x_1 + iy_1}\, dx_1 \ldots dx_n, \qquad \mathscr{D}(\mathbb{R}^n).$$

We claim that $WF(T) = \{(x, \xi);\ x_1 = 0, \xi = t(1, 0, \ldots, 0) t > 0\}$. In fact $\operatorname{sing\,supp} T = \{x;\ x_1 = 0\}$ and the Fourier transform of T is $\hat{T} = Y(\xi_1) \otimes \delta(x_2, \ldots, x_n)$, where $Y(\xi_1)$ is the Heaviside function $Y(\xi_1) = \begin{cases} 0 & \text{for } \xi_1 < 0 \\ 1 & \text{for } \xi_1 > 0 \end{cases}$. So for any $\varphi \in \mathscr{D}(\mathbb{R}^n)$ we have

$$\widehat{\varphi T}(\xi) = (\hat{\varphi} * \hat{T})(\xi) = \int_0^{\xi_1} \hat{\varphi}(t, \xi_2, \ldots, \xi_n)\, dt$$

and $\widehat{\varphi T}(\xi)$ is rapidly decreasing for any direction $\xi \neq (1, 0, \ldots, 0)$. Since $\operatorname{sing\,supp} T \neq \emptyset$ by Proposition 2.3 $WF(T)$ must be given by the formula above.

3. Convolutions and Wave Front Sets

With these preparations we will give now the announced result on the wave front set of the convolution $T*S$ of two distributions with compact support.

Proposition 3.1. *Let $T, S \in \mathscr{E}'(\mathbb{R}^n)$ be distributions with compact support, $T*S$ their convolution. Then we have*

$$WF(T*S) \subset \{(x+y, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}); \\ (x, \xi) \in WF(T), (y, \xi) \in WF(S)\}$$

Proof. (This is a consequence of Theorem 2.5.14 [9], but a direct proof is simpler in this case.) Let z be a point of $\operatorname{sing\,supp}(T*S)$, and suppose that for any x, y with $x+y = z$ we have $(x, \xi) \notin WF(T)$ or $(y, \xi) \notin WF(S)$. We construct a finite partition of unity $\{\varphi_j\}$, $\varphi_j \in \mathscr{D}(\mathbb{R}^n)$ with $\sum_j \varphi_j(x) \equiv 1$ in a neighborhood of $\operatorname{supp} T$ and for each j a finite partition of unity $\{\psi_{j,k}\}$, $\psi_{jk} \in \mathscr{D}(\mathbb{R}^n)$ with $\sum_k \psi_{j,k}(x) \equiv 1$ in a neighborhood of $\operatorname{supp} S$, such that $z \in \operatorname{supp} \varphi_j + \operatorname{supp} \psi_{j,1}$ but $z \notin \operatorname{supp} \varphi_j + \operatorname{supp} \psi_{j,k}$, for all $k \neq 1$. We define $T_j = \varphi_j T$ and $S_{j,k} = \psi_{j,k} S$, so we have $T*S = \sum_{j,k} T_j * S_{j,k}$. Moreover, by Proposition 2.3 we can suppose that the partitions are so fine that $\hat{T}_j(\eta)$ or $\hat{S}_{j,1}(\eta)$ are rapidly decreasing for η in a conic neighborhood of ξ. It follows that $\hat{T}_j(\eta) \hat{S}_{j,1}(\eta)$ is rapidly decreasing for η in a conic neighborhood of ξ and every j and so by Proposition 2.3 we have $(z, \xi) \notin WF(T_j * S_{j,1})$. Since $z \notin \operatorname{supp}(T_j * S_{j,k})$, $k \neq 1$ we have $(z, \xi) \notin WF(T*S)$.

From this proposition we can deduce now a necessary condition for the validity of formula (1.2) of the introduction.

Theorem 3.2. *Let $T \in \mathscr{E}'(\mathbb{R}^n)$ be a distribution with compact support such that the equation*

$$\operatorname{co\,sing\,supp}(T*S) = \operatorname{co\,sing\,supp} T + \operatorname{co\,sing\,supp} S \tag{1.2}$$

holds for any distribution $S \in \mathscr{E}'(\mathbb{R}^n)$, then the fibre of $WF(T)$ over every extreme point x of co sing supp T *is all of* $\mathbb{R}^n \setminus \{0\}$.

Proof. Let S_1 be the distribution defined in Section 2, Example 2 and put $S = \psi S_1$, where ψ is a function in $\mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} \psi \subset \{x; |x| \leqq \alpha\}$ for some constant α and $\psi(x) > 0$ on $|x| < \alpha$. We have

$$WF(S) = \{(x, \xi);\ x_1 = 0,\ |x| \leqq \alpha,\ \xi = t \cdot (1, 0, \ldots, 0),\ t > 0\}.$$

Now let x be an extreme point of sing supp T and suppose that the fibre of $WF(T)$ over x is not all of $\mathbb{R}^n \setminus \{0\}$. By a linear change of variables we can suppose $(x, \xi) \notin WF(T)$ for $\xi = t(1, 0, \ldots, 0)$, $t > 0$. Now there is a point y, $|y| = \alpha$, such that $x + y$ is an extreme point of co sing supp T + co sing supp S but there is no η such that $(x, \eta) \in WF(T)$ and $(y, \eta) \in WF(S)$ and by Proposition 3.1 it follows that

$(x+y,\eta)\notin WF(T*S)$ for all $\eta\in\mathbb{R}^n\setminus\{0\}$, and $x+y\notin\operatorname{sing\,supp}(T*S)$. But since $x+y$ is an extreme point of $\operatorname{co\,sing\,supp} S+\operatorname{co\,sing\,supp} T$, $x+y\notin\operatorname{sing\,supp}(T*S)$ implies $x+y\notin\operatorname{co\,sing\,supp}(T*S)$.

An immediate consequence is

Corollary 3.3. *Let T be the characteristic function of an open, bounded set Ω in $\mathbb{R}^n$ with smooth boundary, then there is a distribution $S\in\mathscr{E}'(\mathbb{R}^n)$ such that* (1.2) *does not hold.*

Proof. $WF(T)=\{(x,\xi),\ x\in\partial\Omega,\ \xi=t\cdot n,\ t\in\mathbb{R}\}$, n the normal to $\partial\Omega$ at x by Example 1 in Section 2. So the condition in Theorem 3.2 is not fulfilled.

Remark. The distribution S for which (1.2) does not hold is explicitely constructed in the proof of Theorem 3.2.

By a similar argument we deduce from Proposition 3.1

Corollary 3.4. *Let $T, S\in\mathscr{E}'(\mathbb{R}^n)$ be distributions with compact support such that* (1.2) *holds, then for every extreme point $z\in\operatorname{co\,sing\,supp}(T*S)$, there are extreme points $x\in\operatorname{co\,sing\,supp} T$ and $y\in\operatorname{co\,sing\,supp} S$ and $\xi\in\mathbb{R}^n\setminus\{0\}$ such that $(x,\xi)\in WF(T)$ and $(y,\xi)\in WF(S)$.*

4. A Counterexample

In this section we will construct an example which shows that the necessary condition of Theorem 3.2 is not sufficient. So the inclusion in (3.1) is in general a proper inclusion. For this construction we need results of Ehrenpreis [5] and Hörmander [8].

Proposition 4.1 ([5], Theorem 2.6). *Let $T\in\mathscr{E}'(\mathbb{R}^n)$ be a distribution with compact support. If for every $S\in\mathscr{E}'(\mathbb{R}^n)$ with $T*S\in\mathscr{D}(\mathbb{R}^n)$ we have $S\in\mathscr{D}(\mathbb{R}^n)$ then $\hat{T}$ is slowly decreasing in the following sense:*

$$\sup\{|\hat{T}(\eta)|;\ \eta\in\mathbb{R}^n,\ |\eta-\xi|<A\log(2+|\xi|)\}\geqq(A+|\xi|)^{-A} \tag{4.1}$$

for $\xi\in\mathbb{R}^n$ and some constant A.

For the proof we refer to [5] or [7].

Proposition 4.2 ([8], Theorem 5.2). *Let $\{\xi_j\}$ be a sequence in $\mathbb{R}^n$ with $|\xi_j|\to\infty$, E a subset of $\mathbb{R}^n$ such that $d(\xi_j,E)/\log|\xi_j|\to\infty$ for $j\to\infty$.*

Then there exists a continuous function f with compact support such that $\operatorname{sing\,supp} f=\{0\}$ *and*

$$|\xi_j|\hat{f}(\xi_j)\quad \text{does not converge to } 0 \text{ when} \quad j\to\infty, \tag{4.2}$$

$$P_{N,m}(f)=\sup\{|\hat{f}(\zeta)|\,|\xi|^N;\ \xi\in E,\ \zeta\in\mathbb{C}^n,\ |\zeta-\xi|\leqq m\log|\xi|\}<\infty. \tag{4.3}$$

(4.3) *implies that $\hat{f}$ is not slowly decreasing, so by Proposition 4.1 there is an $S\in\mathscr{E}'(\mathbb{R}^n)$, $S\notin\mathscr{D}(\mathbb{R}^n)$ such that $S*f\in\mathscr{D}(\mathbb{R}^n)$, i.e.* $\operatorname{sing\,supp}(S*f)=\emptyset$.

For the proof Hörmander considers the space $\mathscr{F}$ of all continuous functions with support contained in $\{x;|x|\leqq 1\}$ such that $f\in\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})$ and such that the semi-norms defined by (4.3) are finite. These semi-norms together with $\sup|f|$ and $\sup|D^\alpha f|$ where K runs over all compact sets in $\mathbb{R}^n \setminus \{0\}$ induce a locally convex topology on $\mathscr{F}$ for which $\mathscr{F}$ is a Fréchet space. Hörmander then shows that there is an $f \in \mathscr{F}$ for which $|\xi_j|\,|\hat{f}(\xi_j)|$ is not bounded, so (4.2) does not hold. By choosing ξ_j and E we prove now:

Proposition 4.3. *There is a distribution $T \in \mathscr{E}'(\mathbb{R}^n)$ with $WF(T) = \{0, \mathbb{R}^n \setminus \{0\}\}$ and which is not slowly decreasing. By Proposition 4.1 this means that there is a distribution $S \in \mathscr{E}'(\mathbb{R}^n)$ with $S \notin \mathscr{D}(\mathbb{R}^n)$ but $T * S \in \mathscr{D}(\mathbb{R}^n)$.*

Proof. Let $\{\eta_k\}$ be a sequence of points, $|\eta_k| = 1$, which is dense in the unit sphere of $\mathbb{R}^n$. If $t_j = e^{2j}$, $E = \{\eta \in \mathbb{R}^n, |\eta| = e^{2j+1}, j = 1, 2, \ldots\}$ and $\xi_j^{(k)} = t_j \eta_k$ we have $d(\xi_j^{(k)}, E)/\log|\xi_j^{(k)}| \to \infty$ for $j \to \infty$ and each k.

We set $\mathscr{F}_k = \{f \in \mathscr{F}\,;\, |\xi_j^{(k)}|\,|\hat{f}(\xi_j^{(k)})| < \infty\}$, $\mathscr{F}$ the space considered by Hörmander, and introduce a locally convex topology on $\mathscr{F}_k$ by the semi-norms of $\mathscr{F}$ together with the semi-norm $\sup|\xi_j^{(k)}|\,|\hat{f}(\xi_j^{(k)})|$. $\mathscr{F}_k$ is a Fréchet-space and continuously embedded in $\mathscr{F}$, but by Proposition 4.2 $\mathscr{F}_k \neq \mathscr{F}$. By the open mapping theorem $\mathscr{F}_k$ is either closed in $\mathscr{F}$ or of first category in $\bar{\mathscr{F}}_k$. In either case $\mathscr{F}_k$ is of first category in $\mathscr{F}$ and so $\bigcup_{k=1}^{\infty} \mathscr{F}_k$ as countable union is of first category in $\mathscr{F}$. Hence there is an $f \in \mathscr{F}$ with $f \notin \bigcup \mathscr{F}_k$, i.e. $|\xi_j^{(k)}|\,|\hat{f}(\xi_j^{(k)})|$ not bounded for $j \to \infty$ and each k. So $\hat{f}(\xi)$ is not rapidly decreasing in any direction η_k and since wave front sets are closed, we have $WF(f) = \{0, \mathbb{R}^n \setminus \{0\}\}$. On the other hand (4.3) implies that $\hat{f}$ is not slowly decreasing.

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