Note on the Hayford-Bowie tables for calculating g

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It is shown that the resultant attraction of distant topography and compensation is very sensitive to small changes in the assumptions made in making the calculation. It is shown that the outer zones in the Hayford-Bowie tables require corrections up to 200% to allow for the variation of gravity with depth.

1. Introduction. In 1912 Hayford and Bowie*) described a method of calculating the contribution to the force of gravity at a point on the earth's surface made by the attraction of the topography of the whole globe, and also by the compensating defects of density which underlie the topography according to the hypothesis of isostasy. They consider the earth divided into zones by circles described about the point at which the attraction is required. The average height of each of these zones is estimated, and the vertical component of the force due to the matter in the zone above sea level and to the compensating defect of density underlying it, is obtained from tables which are given in the publication referred to. In constructing these tables several approximations have been made, the chief of which are the neglect of the distortion of the geoid and of the vertical variation of gravity. These two errors are opposite in sign and of the same order of magnitude, the resulting effect being small**). It does not seem to be generally recognised that although the absolute error is small, the second of these approximations completely falsifies the contributions from the distant zones, and it is the purpose of this paper to point this out.

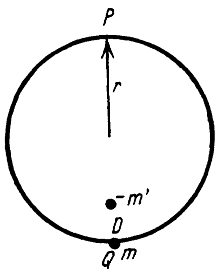

Fig. 1

2. Special Case. The reason for this large proportional effect on the distant zones may most easily be seen by considering a simple special case. Let P (Fig. 1) be a point at sea level at which the attraction of a mass m at Q, and of its compensation is required. Let Q be the antipodes of P and also at sea level. Let the compensation of m be represented by a mass $-m'$ at a depth D. Let r be the radius of the earth. Then the attraction required F is:

$$F = \frac{km}{4r^2} - \frac{km'}{(2r-D)^2} \quad \ldots \ldots \ldots \ldots \quad (1)$$

where k is the gravitational constant, which, since $D \ll r$ is,

$$F = \frac{km}{4r^2}\left[1 - \frac{m'}{m}\left(1 + \frac{D}{r}\right)\right]$$

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*) Coast and Geodetic Survey. Special Pub[n] No. 10, Washington 1912.

**) Jung, Zeitschr. f. Geophys. 8, 40 (1932).

if $m = m'$ this becomes

$$-\frac{km}{4r^2}\cdot\frac{D}{r} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Thus the force at P is upwards and is very small compared with the force due to the topography alone. This is the assumption made in computing the Hayford-Bowie tables.

For m and m' to be in equilibrium, that is, for there to be no nett force on the pair, it is necessary for m' to be less than m, as it is nearer the centre of the earth, where g is greater. For equilibrium:

$$\frac{km}{r^2} = \frac{km'}{(r-D)^2} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Substituting in (2)

$$F = \frac{km}{4r^2}\left[1 - \frac{(r-D)^2}{r^2}\left(1+\frac{D}{r}\right)\right],$$

$$F = \frac{km}{4r^2}\cdot\frac{D}{r} \quad \ldots\ldots\ldots\ldots \quad (4)$$

The attraction is now equal to that in (2) but opposite in sign; that is, it has been changed by 200%.

The attraction of the mass and its compensation are opposite and nearly equal, thus, a small change in the compensating mass has produced a large proportional change in the result. Thus if the outer zones are to be considered at all the effect of increase of gravity with depth should be taken into account, for neglect of it leads to entirely wrong estimates of the contribution from these zones.

3. General Case. The condition for complete isostasy on the Pratt theory is that the attraction of the earth on the topography should be equal and opposite to the attraction on the mass deficiency in the compensation. If δ and h be the density and height of the topography, and δ' the density deficiency in the compensation extending to a depth h_1, then,

$$\int_0^h g\,\delta\,dh = \int_0^{h_1} g\,\delta'\,dx$$

but at a depth x,

$$g = g_0\,\frac{r^2}{(r-x)^2}$$

$$\therefore \delta\left[\frac{1}{r-h} - \frac{1}{r}\right] = \delta'\left[\frac{1}{r-h_1} - \frac{1}{r}\right],$$

and neglecting higher powers of h/r, h_1/r and h/h_1

$$\delta' = \delta\,\frac{h}{h_1}\left(1-\frac{h_1}{r}\right).$$

In constructing the Hayford-Bowie tables it is assumed that $\delta' = \delta h/h_1$ the attraction of the compensation is therefore overestimated by the factor $1 - h_1/r$, that is by 1.8%. For the inner zones the tables give the attraction of the compensation separately and the error can thus be estimated. For the outer zones only the combined attraction of the topography and compensation is given, so we cannot correct directly.

For the outer zones the attraction F_T of the topography is proportional to its average height, and to its density, and to the area of the zone; thus for the zone lying between angular distances θ_1 and θ_2 from the point at which the attraction is required:

$$F_T = kh\,\delta\,2\pi r^2 (\cos\theta_1 - \cos\theta_2)\,\overline{E}_T$$

where $\overline{E}_T$ is the mean value of E_T (a function of θ tabulated by Hayford and Bowie) between θ_1 and θ_2. The attraction of the compensation is:

$$F_c = h_1 \delta'\,2\pi r^2 (\cos\theta_1 - \cos\theta_2)\,\overline{E}_c$$

where $\overline{E}_c$ is also tabulated. Substituting for δ'

$$F_c = h\,\delta\,2\pi r^2 (\cos\theta_1 - \cos\theta_2)\,E_c\left(1 - \frac{h_1}{r}\right).$$

Thus the resultant attraction is

$$F = h\,\delta\,2\pi r^2 (\cos\theta_1 - \cos\theta_2)\left[E_T - E_c\left(1 - \frac{h_1}{r}\right)\right],$$

the value used in constructing the tables is

$$F' = h\,\delta\,2\pi r^2 (\cos\theta_1 - \cos\theta_2)\,[\overline{E}_T - \overline{E}_c].$$

Thus

$$F = F' \cdot \frac{\overline{E}_T - \overline{E}_c\left(1 - \frac{h_1}{r}\right)}{\overline{E}_T - \overline{E}_c} = F'\left(1 + \frac{h}{r}\,\frac{\overline{E}_c}{\overline{E}_T - \overline{E}_c}\right).$$

This can be calculated to a sufficient approximation by plotting $\frac{E_c \sin\theta}{E_T - E_c}$ from Hayford and Bowie's tables, finding the value of θ at which the area between θ_1 and θ is equal to that between θ and θ_2, and taking the value of $\frac{E_c}{E_T - E_c}$ for this value of θ. The factor $\sin\theta$ allows for the lesser weight which must be assigned to those parts of a zone nearest the poles in taking the average, on account of the lesser area per unit θ.

The result is given in Table 1, where $h_1 = 113.7$ km, $r = 6370$ km has been used. The first column gives the number of the zone, the second the radius of the outer edge of the zone, and the third the factor by which the resultant attractions in Hayford and Bowie's tables should be multiplied.

Table 1

Zone	Outer Radius		Correcting Factor	Zone	Outer Radius		Correcting Factor
1	180°	00′	− 0.99	10	10°	44′	+ 0.96
2	150	56	− 0.83	11	7	52	+ 0.97
3	105	48	− 0.30	12	5	47	+ 0.98
4	72	13	+ 0.16	13	4	19	+ 0.98
5	51	04	+ 0.50	14	3	03	+ 0.98
6	35	58	+ 0.71	15	2	34	+ 0,98
7	26	41	+ 0.82	16	2	12	+ 0.98
8	20	41	+ 0.89	17	1	55	+ 0.98
9	14	09	+ 0.94	18	1	41	+ 0.98

4. Application to Special Cases. Table 2 shows the effect of the corrections on the calculations for two American and one European station*). The table gives the attraction due to the topography and compensation calculated according to the Hayford-Bowie tables, and according to the tables corrected by the use of (1) and (2) above. The change produced is also shown. All quantities are in units of 10^{-4} cm/sec^2. Since the antipodes of U. S. A. and of Europe are in deep oceans the effect of the correction is to decrease the calculated value for the attraction of the distant zones.

Table 2

Zone	Key West			Pikes Peak			Zermatt		
	Un-corr^d	Corr^d	Change	Un-corr^d	Corr^d	Change	Un-corr^d	Corr^d	Change
1	+ 1	− 1	− 2	+ 1	− 1	− 2	+ 1	− 1	− 2
2	+ 2	− 2	− 4	+ 3	− 2	− 5	+ 5	− 4	− 9
3	+ 6	− 2	− 8	+ 5	− 2	− 7	+ 4	− 1	− 5
4	+ 8	+ 1	− 7	+ 8	+ 1	− 7	+ 3	0	− 3
5	+ 10	+ 2	− 8	+ 9	+ 4	− 5	+ 1	0	− 1
6	+ 6	+ 4	− 2	+ 9	+ 6	− 3	+ 4	+ 3	− 1
7	+ 5	+ 4	− 1	+ 7	+ 6	− 1	+ 5	+ 4	− 1
8	+ 15	+ 13	− 2	+ 9	+ 8	− 1	+ 8	+ 7	− 1
9	+ 15	+ 14	− 1	+ 0	0	0	+ 3	+ 3	0
10	+ 25	+ 24	− 1	− 17	− 16	+ 1	− 1	− 1	0
11	+ 42	+ 41	− 1	− 30	− 29	+ 1	+ 3	+ 3	0
12	+ 38	+ 37	− 1	− 48	− 47	+ 1	− 6	− 6	0
13	+ 38	+ 37	− 1	− 83	− 81	+ 2	− 11	− 11	0
14	+ 8	+ 8	0	− 59	− 58	+ 1	− 11	− 11	0
15	+ 5	+ 5	0	− 64	− 63	+ 1	− 17	− 17	0
16	+ 3	+ 3	0	− 68	− 67	+ 1	− 23	− 23	0
17	+ 4	+ 4	0	− 68	− 67	+ 1	− 31	− 30	+ 1
18	+ 8	+ 8	0	− 68	− 67	+ 1	− 32	− 31	+ 1
Total for dist. zones			− 39			− 21			− 21
Zones A—O (near zones)			− 2			+ 34			+ 25
Total			− 41			+ 13			+ 4

*) Bowie: U. S. Coast and Geod. Survey. Special Pub^n No. 40, Washington, p. 20—48.

5. Conclusion. It has been shown that the inclusion of the attraction of distant topography and its compensation in the Hayford-Bowie tables is largely illusory, since the attraction for all zones beyond 35^0 is between 50 and 200% different from what it would be if calculated for perfect hydrostatic equilibrium.

In calculating attractions in future there seem to be three possible courses:

1. To continue to use the Hayford and Bowie tables uncorrected.
2. To use the tables and correct by (1) and (2).
3. Not to consider the distant zones at all.

A small change in the assumptions would change the corrections for the outer zones by a large factor, for instance if compensation were only 98% complete, a further change as big as that discussed above would be produced; or if the strain in the crust is assumed to be a minimum the correction vanishes altogether*). It thus seems impossible to predict the attraction of these zones, even approximately, without making very detailed assumptions. There is therefore little to be gained by using the corrected tables. In principle it would probably be best to omit zones 1—5 (that is all zones beyond 35^0) altogether. On the other hand a large number of stations have been reduced using the existing tables, and if it is decided to omit the outer zones in future, it would be disirable to remove their effect from these stations also. The work involved in doing this is not great. The decision whether it is worth while must be left to those directly concerned.

In using results reduced by the tables to discuss such questions as the ellipticity of the equator or the difference between the form of the northern and the southern hemisphere, care must be taken that systematic error is not introduced.

Kann die Laplacesche Differentialgleichung für das Schwerkraftpotential auch innerhalb der Erdkruste als erfüllt angesehen werden?

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Es könnte müßig erscheinen, diese Frage aufzuwerfen, da es ja bekannt ist, daß dies nicht der Fall ist. Wenn ich trotzdem darauf zu sprechen komme, so geschieht es, weil in einigen in diesem Jahre erschienenen Abhandlungen die Behauptung ausgesprochen und zu begründen versucht wird, die Anwendung der Laplaceschen Differentialgleichung sei als Näherung auch innerhalb der Erdkruste gestattet, und aus dieser Behauptung zum Teil auch weitere Folgerungen gezogen werden.

*) Jeffreys: M. N. R. A. S. Geophys. Suppl. Vol. 3, p. 30 (1932).