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On Tracing Seismic Rays with Specified End Points

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Abstract. A method for tracing seismic rays with specified end points is described. It is based on a technique due to Euler in which integrals of variational problems are replaced by sums to be minimized. Thus ray tracing is reduced to solution of a system of algebraic equations. Application of the method is illustrated by tracing a ray between two given points in a medium in which the seismic velocity varies linearly with distance from a plane.

Key words: Seismic Ray Defermination – Numerical Methods

1. *Introduction*

A little used technique due to Euler (Courant and Hilbert, 1953; Elsgolc, 1962) for direct solution of variational problems provides a method of constructing seismic rays. Attention is confined in this initial paper to flat-earth models in which seismic velocity varies only with depth below the free surface in an arbitrary but known way.

An important feature of the present method is that the starting and termination points of the ray to be traced are specified. This is in contrast to other methods of seismic ray tracing in current use (Jackson, 1970) in which the starting point and the direction of the ray at the start are specified. However, as in the other numerical methods of ray tracing, when the seismic velocity varies continuously in the medium, the present method also yields only polygonal approximations to actual rays.

Conceptually, the method adopted in many optics books (Jenkins and White, 1957) to show that Snell's law is a consequence of Fermat's principle may be regarded as a particularly simple application of the present method.

2. *Statement of the Problem*

Let S (Fig. 1) be the source and R the receiver of seismic waves in an isotropic elastic medium in which the wave velocity v(y) varies with the y-coordinate only in a prescribed manner. It is required to construct the refracted ray between S and R.

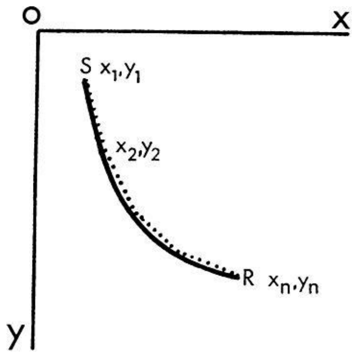

Fig. 1. Actual seismic ray (full line) between source S and receiver R and its polygonal approximation (dotted line) to be determined

2.1. Solution of the Problem by the Euler Technique

By Fermat's principle, the ray SR will be a path of minimum time; i.e., the integral representing the total travel time (t_{SR}) along the ray

$$t_{SR} = \int_S^R ds/v(y) = \int_S^R (1 + x'^2)^{\frac{1}{2}}\, dy/v(y) \tag{1}$$

will be a minimum for small variations of the ray path. Here, $ds = (1 + x'^2)^{\frac{1}{2}}$ dy is the length of an element of arc along the ray and $x' = dx/dy$.

The idea of the Euler technique is that the integrals of variational problems, such as the present, are replaced by sums to be minimized. To this end, the continuous ray of Fig. 1 is replaced by a polygonal curve (dotted line in Fig. 1), along each linear segment of which the wave velocity is assumed constant. Let (x_1, y_1) and (x_n, y_n) be the specified coordinates of S and R. Let $(x_p, y_p; p = 2, 3, \ldots, n-1)$ be the coordinates of the n—2 intervening vertices of the polygonal curve. Let $V_p = (v_{p-1} + v_p)/2$ be the mean velocity along the segment connecting the (p—1)—st and p—th vertices. Then, with $dy_p = y_p - y_{p-1}$ and $x'_p = (x_p - x_{p-1})/(y_p - y_{p-1})$,

$$t_{SR} = \int_S^R (1 + x'^2)^{\frac{1}{2}}\, dy/v(y) \approx \sum_{p=1}^{n} [(y_p - y_{p-1})^2 + (x_p - x_{p-1})^2]^{\frac{1}{2}}\, V_p^{-1} \tag{2}$$

The next step is to fix the y-coordinates $(y_2, y_3, \ldots, y_{n-1})$ of the n—2 vertices. The corresponding x-coordinates $(x_2, x_3, \ldots, x_{n-1})$ of these vertices will render the above sum a minimum for small variations in the values of each x_p individually when the following n—2 equations are satisfied simultaneously.

$$
\begin{aligned}
\frac{\partial t_{\mathrm{SR}}}{\partial \mathrm{x}_p} &= (\mathrm{x}_p - \mathrm{x}_{p-1})\,[(\mathrm{y}_p - \mathrm{y}_{p-1})^2 + (\mathrm{x}_p - \mathrm{x}_{p-1})^2]^{-\frac{1}{2}} \mathrm{V}_p^{-1} \\
&\quad - (\mathrm{x}_{p+1} - \mathrm{x}_p)\,[(\mathrm{y}_{p+1} - \mathrm{y}_p)^2 + (\mathrm{x}_{p+1} - \mathrm{x}_p)^2]^{-\frac{1}{2}}\, \mathrm{V}_{p+1}^{-1} \qquad (3) \\
&= 0; \quad p = 2, 3, \ldots, n-1 .
\end{aligned}
$$

The x_2, x_3, ... x_{n-1} which are solutions to these equations along with the pre-assigned y_2, y_3, ..., y_{n-1} enable the minimum-time polygonal curve approximating the ray SR to be traced.

3. Application of the Method of Ray Tracing

The applicability of the above method of ray tracing was tested in several cases for which the results could also be obtained by other methods. Details of one of these tests are given below in Section 3.2.

3.1 Scheme for Numerical Solution of the Simultaneous Equations

The modified Newton-Raphson iterative scheme for solution of a system of non-linear equations (Stark, 1970) was adopted for the present purpose. The $(k+1)$-st iterated values of the x-coordinates of the polygonal curve are obtained from the k-th iterated values by the relations

$$
(\mathrm{x}_p)_{k+1} = (\mathrm{x}_p)_k - [(\partial t_{\mathrm{SR}}/\partial \mathrm{x}_p)\,(\partial^2 t_{\mathrm{SR}}/\partial \mathrm{x}_p^2)^{-1}]_k\,, p = 2, 3, \ldots, n-1; \quad (4)
$$

or,

$$
\begin{aligned}
(\mathrm{x}_p)_{k+1} = (\mathrm{x}_p)_k &- \{[(\mathrm{x}_p)_k - (\mathrm{x}_{p-1})_{k+1}]\, l_p^{-1}\, \mathrm{V}_p^{-1} \\
&- [(\mathrm{x}_{p+1})_k - (\mathrm{x}_p)_k]\, l_{p+1}^{-1}\, \mathrm{V}_{p+1}^{-1}\} \\
&\div \{(\mathrm{y}_p - \mathrm{y}_{p-1})^2\, l_p^{-3}\, \mathrm{V}_p^{-1} \\
&+ (\mathrm{y}_{p+1} - \mathrm{y}_p)^2 l_{p+1}^{-3}\, \mathrm{V}_{p+1}^{-1}\}, p = 2, 3, \ldots, n-1; \quad (5)
\end{aligned}
$$

where,

$$
l_p^2 = (\mathrm{y}_p - \mathrm{y}_{p-1})^2 + [(\mathrm{x}_p)_k - (\mathrm{x}_{p-1})_{k+1}]^2
$$

and

$$
l_{p+1}^2 = (\mathrm{y}_{p+1} - \mathrm{y}_p)^2 + [(\mathrm{x}_{p+1})_k - (\mathrm{x}_p)_k]^2
$$

Convergence to the required set of values is accelerated by using the latest available values of x-coordinates at each stage of the computation. This is why the subscript indicating the iteration number is $k+1$ for x_{p-1} and k for x_p and x_{p+1} on the right hand sides of the above equations.

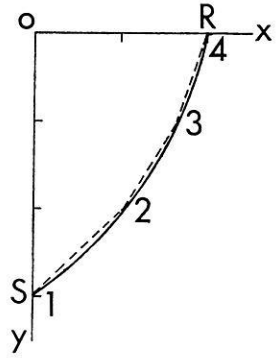

Fig. 2. Comparison of the actual seismic ray (full line) and the minimum-time four-cornered polygonal approximation (dotted line) between source S and receiver R in a medium in which velocity is given by the relation $v(y) = 2 + 2y$. The actual ray is an arc of a circle with centre at —2 · 75, —1)

A Compucorp 425/44 desk calculator was programmed to accept numerical values of $(x_p)_k$, $(x_p)_k - (x_{p-1})_{k+1}$, $(x_{p+1})_k - (x_p)_k$, $y_p - y_{p-1}$, $y_{p+1} - y_p$, V_p, V_{p+1} and yield a new estimate $(x_p)_{k+1}$ using Eq. 5. A straight forward programme of 135 steps required 3 seconds for execution after the 7 input data were entered from the key board. The process was repeated for each p $(= 2, 3, \ldots, n-1)$ to complete one iteration.

3.2 An Example

A medium with the following velocity depth relation was chosen for constructing a seismic ray in one of the tests.

$$v(y) = 2 + 2\,y \qquad (\mathrm{km\ s^{-1}}) \tag{6}$$

The source S (Fig. 2) was assumed to be at a depth of 3 km and the receiver R at the surface at a point removed 2 km from the vertical line through S. Choosing the y-axis to coincide with this line, and x-axis through R, the coordinates of these points are (0,3) and (2,0) respectively. The exact ray path in this case is an arc of a circle with its centre on a line 1 km above the x-axis and parallel to it (Nettleton, 1940). An approximation to it was sought with a polygonal curve of 4 vertices, with the two extreme vertices (1 and 4) coinciding with S and R. The depths of the intervening vertices (2 and 3) were fixed at $y_2 = 2$ km and $y_3 = 1$ km. The mean velocities V_2, V_3, and V_4 are 7, 5, and 3 $\mathrm{kms^{-1}}$ respectively. Starting with $(x_2)_0 = 0.7$ km and $(x_3)_0 = 1.5$ km the successive iterated values shown in Table 1 were obtained.

Table 1

k	$(x_2)_k$ km	$(x_3)_k$ km
0	0.7	1.5
1	0.948	1.641
2	1.042	1.665
3	1.058	1.670
4	1.062	1.671
5	1.062	1.671
	iteration stopped	

Thus the vertices 2 and 3 of the minimum-time four-cornered polygonal curve approximating the ray have coordinates (1.062, 2) and (1.671, 1). For comparison, the exact ray passes through points (1.066, 2) and (1.673, 1).

4. *Conclusion*

A seismic ray tracing method inspired by a little used variational technique due to Euler has been described. Positions of both the end points of the ray to be traced are specified in this method, in contrast to other current methods of seismic ray tracing in which the position of the starting point and the direction of the ray at the start are specified When the seismic velocity of the medium varies with a single space-coordinate only, the computations are simple enough to be carried out on programmable desk-top computers. Thus rapid tracing of rays with specified and points becomes a practicable proposition even for quite complicated seismic velocity variations in situations where large digital computers may not be available.

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