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A Comparison of Methods for Computing Surface Densities of the Geopotential from Satellite Altimetry

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Abstract. In regard to the expected altimetry data a solution for the earth's gravity field represented by the potential of a simple layer is obtained by means of analyzing 11700 altimeter- and 500 distance-measurements. The unknown density values of the simple layer model are fitted to the given data-covering and the global gravity field is separated into several regional representations. To evaluate the data for a fine structured and global geoid resolution two methods are used and their differences and advantages are examined. First an iterative method uses a global adjustment to determine coarse structured density values. Based on this approximated solution the analysis of all data may be repeated step by step for all regional representations of fine structured density values. On the other hand a detailed resolution for a regional part of the geoid is solved directly. Thus the global solution may be kept step by step by adjusting the regional ones. As no real data is available the numerical tests have been done by means of simulated data of known accuracy.

Key words: Satellite geodesy – Altimetry data – Surface densities of the geopotential – Geoid undulations.

Introduction

The determination of the earth's gravity field from a combination of Doppler data, satellite triangulation and gravity anomalies by means of the simple layer model has been applied by Koch (1968, 1974), and gave excellent results for the gravity field. Hence this method has been proposed to represent the earth's gravity field for the analysis of satellite altimetry data (Benning, 1974).

At present the launch of the earth-satellite GEOS-C is scheduled for the end of 1975. GEOS-C will carry a radar altimeter on board to measure the shortest distance between the satellite and the surface of the ocean. With this new type of data the problem arises to analyze a huge amount of very accurate data collected over the surface of the oceans. If altimetry data will be measured within an accuracy of ± 1 m or ± 2 m this means the possibility to improve existing solutions of the geoid undulations up to this accuracy. Combining this type of data with gravimetry about all $1° \times 1°$-squares of the earth's surface will be covered with data. Here the altimeter measurements are assumed to represent the shortest distance between the satellite and the geoid, that means the surface of the oceans and the geoid coincide. All periodic and time invariant gravitational influences of sun and moon, currents and waves have to be filtered out of the data.

To represent the gravity field the geopotential may be expressed by an expansion into spherical harmonics. But if it is asked for a detailed resolution of the unknowns, representing distances of about hundred kilometers, the spherical harmonics must be computed up to the 180-th degree and order, a task to determine nearly 32400 coefficients which yet raises some unresolved technical problems.

1. Representation of the Gravity Field

The potential W of the earth

$$W = U + T \tag{1}$$

is divided into a known potential U, which is expressed by an expansion into spherical harmonics,

$$U = \frac{kM}{r}\left[1 + \sum_{n=2}^{n_c} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n \bar{P}_{nm}(\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda)\right] + \tfrac{1}{2}\omega^2 r^2 \cos^2\beta \tag{2}$$

and into a potential T, which is unknown and represented by a potential of a simple layer distributed over the surface of the earth. In (2) r, β, λ are spherical coordinates in the usual earth-fixed coordinate system, whose 3-axis is identical with the instantaneous axis of the earth and whose 1-axis points towards the intersection of the meridian of Greenwich with the equator. k is the gravitational constant, M the mass of the earth, a is the mean equatorial radius, $\bar{P}_{nm}$ are the fully normalized associated Legendre functions of degree n and order m, and ω is the angular velocity of the earth. The fully normalized harmonic coefficients $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are taken up to the 15th degree and order from Koch (1974, p. 14). The values of a and kM are

$$kM = 3.986013 \times 10^{14}\ \mathrm{m}^3\ \mathrm{sec}^{-2}, \qquad a = 6378145\ \mathrm{m}.$$

The potential T is evaluated by numerical integration of

$$T = \sum_{i=1}^{p} \chi_i \iint_{\Delta E_i} \frac{dE}{l}, \tag{3}$$

where l is the distance between the fixed point at which T is computed and the moving point on the surface E of the earth, which is divided into p elements ΔE_i. The unknown density values χ_i are assumed to be constant in ΔE_i, which are bordered by meridians and parallels. The kernel of (3) is computed for the midpoint of each ΔE_i and is assumed to be constant over ΔE_i. To determine the coordinates of these midpoints the reference surface $U = U_0$ is introduced. The value U_0 is defined by the potential of a level ellipsoid, whose constants kM, a, C_{20} and ω are taken from (2), (Koch, 1971). Hence r is computed by the inversion of (2) iteratively.

The density values can be converted into normalized spherical harmonics by (Koch, 1974):

$$\bar{C}_{nm}=\bar{C}_{nmu}+\frac{1}{(2n+1)kM}\sum_{i=1}^{p}\chi_i\iint_{\Delta E_i}\left(\frac{r}{a}\right)^n\bar{P}_{nm}\cos m\lambda\, dE$$
$$\bar{S}_{nm}=\bar{S}_{nmu}+\frac{1}{(2n+1)kM}\sum_{i=1}^{p}\chi_i\iint_{\Delta E_i}\left(\frac{r}{a}\right)^n\bar{P}_{nm}\sin m\lambda\, dE, \qquad (4)$$

where $\bar{C}_{nmu}$ and $\bar{S}_{nmu}$ are the original harmonic coefficients which define (2).

2. Computational Procedures

Satellite orbits should be computed in an inertial reference frame. In this analysis orbit computations do not exceed a time interval of 7 days. Thus an inertial system is well approximated by a geocentric coordinate system, whose 3-axis coincides with the instantaneous axis of the earth and whose 1-axis points towards a point east of the true vernal equinox which takes into account precession and nutation in right ascension.

The altimeter measurement h

$$h=|\mathbf{r}-\mathbf{r}_g| \qquad (5)$$

is a function of the satellite position $\mathbf{r}$, which in turn is a function of the initial position and velocity vectors $\mathbf{r}_0$, $\dot{\mathbf{r}}_0$, the density values and – in this analysis negligible – of air drag and radiation pressure. In (5) $\mathbf{r}_g$ fixes the position of the subsatellite point on the surface of the geoid and is a function of the unknowns of the gravity field, the density values χ_i. The vector $(\mathbf{r}-\mathbf{r}_g)$ is the normal to the surface of the geoid.

A distance s between the satellite and a tracking station is given by

$$s=|\mathbf{r}-\mathbf{r}_s|. \qquad (6)$$

The position vectors $\mathbf{r}_s$ of 34 tracking stations are introduced. They denote a selection of the 41 worldwide distributed stations defined in (Koch, 1974, p. 14).

To evaluate the altimetry- and distance-measurements a least squares adjustment based on a differential correction process is applied. The derivation of the corresponding observation equations and normal equations is published in (Benning, 1975). The orbit is computed by numerical integration methods with a 24-second time step using a 12th order Cowell-Störmer integration for positions, a 10th order Adams-Bashforth predictor and a 10th order Adams-Moulton corrector for the velocities (Witte, 1971). Lagrange's interpolation is applied between time steps of the orbit. To avoid errors in the orbit computation resulting from the numerical integration in (3), the preliminary values of χ_i are set equal to zero.

The determination of the subsatellite point on the reference surface $U=U_0$ is solved iteratively by Lagrange's method (Benning, 1974). The neglected deflection from the vertical amounts on the oceans to a maximum of 10 seconds. The relative error in the computed heights raises beyond $2.5\cdot 10^{-9}$ of the satellite's height in meters.

Assuming that all areas of the earth's surface are covered by a huge amount of data this means for a detailed resolution of the geopotential a division of this

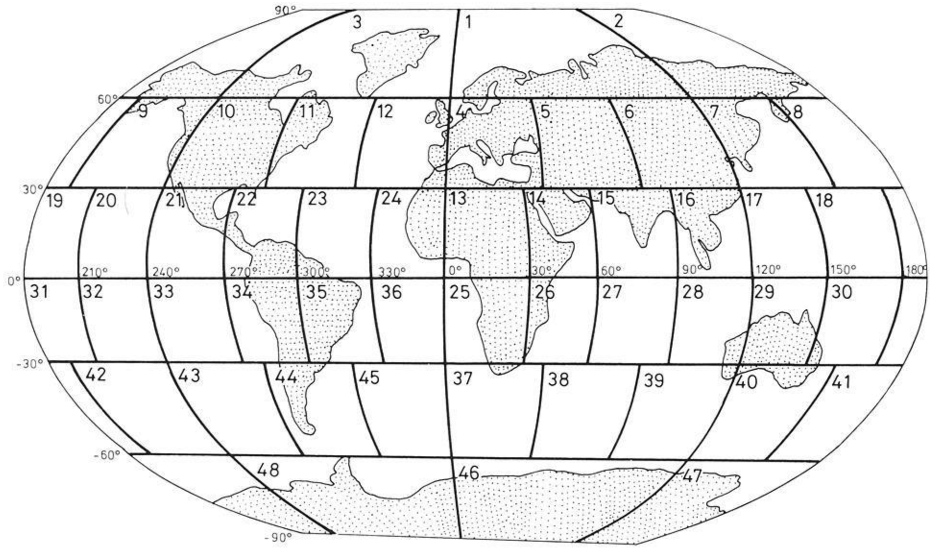

Fig. 1. ($30° \times 30°$) surface elements ΔE_i

surface into small, say $1° \times 1°$-elements which in turn causes a huge global system of about 40000 normal equations. At present this can neither directly nor iteratively be inverted on computers. Thus in this analysis the number of unknowns is limited to 200. If the reference surface $U = U_0$ is computed by spherical harmonics up to the 15th degree and order the density values and geoid undulations will be very small in comparison to those ones which refer to a reference ellipsoid. Hence the correlations depending on the attraction between the unknown density values decrease nearly exponentially with the increasing distance between the corresponding surface elements. They may be neglected beyond the spherical distance of 20° of the connected surface elements, because they are smaller than $|0.04|$ (Benning, 1974).

Thus for a high geoid resolution the unknown parameters are distributed to a limited, at least ($60° \times 60°$)-area of the earth's surface. Fig. 1 shows a division of the earth's surface into 48 elements which approximate the size of a $30° \times 30°$-area at the equator. For a locally restricted but fine structured geoid resolution one of these elements, the central one, is defined as the inner zone and is divided into small elements of nearly equal size, *i.e.* $1° \times 1°$-squares. This inner zone of density values is surrounded by a buffer zone, which according to the maximum number of unknowns is divided into coarse, say $5° \times 5°$-squares. The densities in the remaining surface elements of Fig. 1 are set equal to zero. For the accurate determination of the densities χ_i at least one observation should be measured to the connected surface element ΔE_i as to avoid a singular system of normal equations.

Now the density values of the inner zone are determined together with those of the buffer zone. While the solution for the inner zone gives the final values, the density values for the buffer zone are discarded. Then the inner zone together with the buffer zone is moved to the next surface element of Fig. 1. This process (=method A) is repeated for all areas which are sufficiently covered with data.

It has been tested by Benning (1975) and gave for the analysis of "errorless" altimetry data small discontinuances between two neighbouring inner-zone-solutions.

Another method to evaluate the data, "method B", consists of a global adjustment to determine coarse structured density values χ_i for all, say $15° \times 15°$-elements ΔE_i of the earth's surface. Hence from this solution new spherical harmonics $\bar{C}_{nm}$ and $\bar{S}_{nm}$ can be computed by (4). The results of this transformation give a better flattening of the best fitting ellipsoid, a corrected value U_0 and a corresponding reference surface $U = U_0$ as well as improved orbital elements.

Now the analysis of all data can be repeated with method A as to determine fine structured density values. According to the accuracy of the recomputed geoid undulations the two methods in question have to be compared. Since at present no real data is available, the numerical tests will be done with simulated altimeter- and distance-measurements.

3. Numerical Results

To examine the stability and accuracy of the formulated two methods from simulated GEOS-C orbit (Nasa, 1972) a data set of 11700 altimeter- and 500 distance-measurements was generated. The altitudes are sequentially collected in a time interval of one measurement per 12 seconds whereas the distances s are continuously distributed in time intervals of 3.6 minutes over the whole time of nearly 87 hours of the satellite-borne altimeter. The data refers to an orbit and reference surface computed by an expansion U into spherical harmonics up to the 15th degree and order ($U = U(n_c \leqq 15)$). Neglecting the topographic heights above sea-level the altitudes are also computed for land areas. The distance measurements were included for the orbit recovery. The altimetry data alone does not give well determined orbits, because the geoid undulations are smooth and the orbits of geodetic satellites do not possess great excentricities.

To save computer time the coordinates of the tracking stations were held fixed and the amount of the whole data was limited to 12200. Errorless data was generated, that means for altimetry data shortest distances between the satellite and the reference surface $U_0 = U$ ($n_c \leqq 15$).

Since the inclination of GEOS-C is 115°, the two pole-caps are without ground tracks beyond $\pm 65°$. With respect to a sufficient data-covering the density values of the correspondent pole-elements are set equal to zero. Furthermore, since nearly all $4° \times 4°$-elements of the remaining surface are covered with data, the distribution of fine structured geoid parameters in method A is limited to 11 elements ΔE_i of Fig. 1, that means to 36 ($5° \times 5°$)-squares of the inner zone and to 90 ($10° \times 10°$)-elements of the buffer zone.

When adjusting a ($15° \times 15°$)-resolution for the density values by evaluating all data the rms discrepancy between geoid heights computed at 10°-intervals is ± 34 cm, the maximum deviation -87 cm. The standard deviations for the density values χ_i lie between ± 0.09 mgal and ± 0.02 mgal and the correlation coefficients for χ_i are less than 0.50.

When evaluating the data with method A detailed density values for two neighbouring inner zones – the elements No. 26, No. 27 of Fig. 1 – were computed.

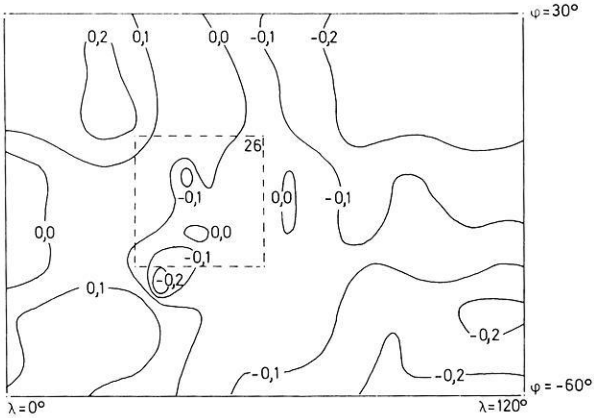

Fig. 2. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 15$), computed for inner zone ΔE_{26} – method A

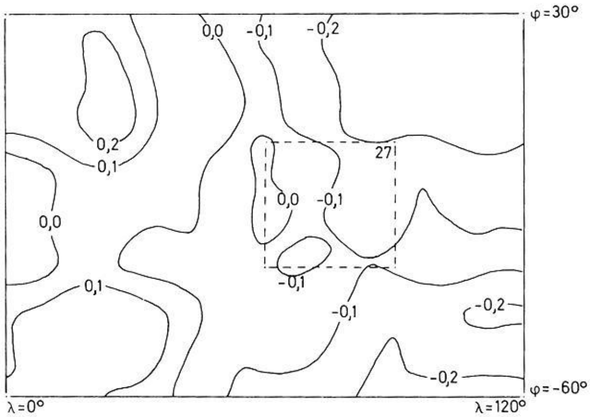

Fig. 3. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 15$), computed for inner zone ΔE_{27} – method A

The iso-contours of the correspondent geoid undulations N are given in Figs. 2 and 3. They show good agreement for all identical areas even for inner zones. In the borders of the two neighbouring inner zones the discontinuances are within the accuracy of ± 15 cm of all computed geoid undulations.

But this result changes if the values N grow.

To proof this the evaluation of the data was done with the identical density-distribution with the examples of Figs. 2 and 3 but with an orbit and reference surface $U_0 = U(n_c \leqq 7)$ which were computed by a spherical expansion U up to the 7th degree and order. Thus the N grow up to $+30$ m and to -21 m respectively.

Nevertheless the geoid heights of identical areas of the buffer zone only differ up to a maximum of ± 2 m to ± 3 m. If we compare the values at the borders of the neighbouring inner zones, they show good conformity within a maximum deviation of ± 1 m till ± 2 m. Thus we may conclude that the analysis of altimetry

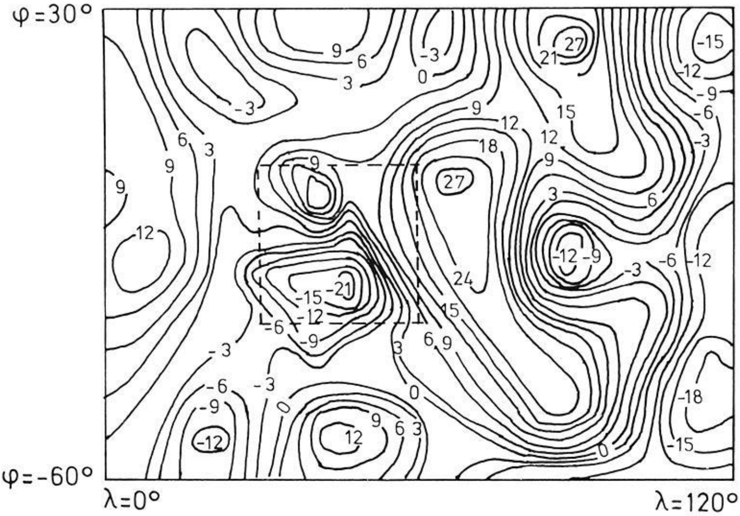

Fig. 4. Geoid Heights (m), referred to $U_0 = U$ $(n_c \leqq 7)$, computed for inner zone ΔE_{26} – method A

data for the solution of fine structured density values and geoid undulations with method A gives an accuracy of the recomputed N of at least 10%.

Then the determination of the geoid heights referred to spherical harmonics up to the same degree was repeated with method B, *i.e.* Fig. 6. Hence a global 15°-resolution of 168 density values was applied. Afterwards using the improved spherical harmonics up to the 15th degree and order the detailed resolution with method A was tried. As to compare the methods in question the inner zone was chosen as in the example of Fig. 4.

The discrepancies between correspondent geoid heights of the inner zones in Fig. 4 and 6 reach nearly ± 3 m, those in the buffer zones grow up to ± 9 m. When constraining the known spherical harmonics up to the 11th degree and order for the computation of the orbit and reference surface $U_0 = U(n_c \leqq 11)$ and when repeating the evaluation of the data with methods A and B according to Figs. 4, 5, and 6, the results of Figs. 7, 8 and 9 were obtained respectively:

The comparison of the determined neighbouring inner zone values of the Figs. 7 and 8 show discrepancies up to and about ± 2 m, whereas the correspondent geoid heights in the buffer zones only differ up to ± 1 m. On the other hand the correspondent results of methods A and B represented in Figs. 7 and 9 differ up to ± 3 m for the buffer zone and up to ± 2 m for the inner zone.

The comparison of these solutions with their directly calculated values in Fig. 10 gives maximum deviations of ± 1 m (method A) and ± 2 m (method B) for the inner zones and ± 2 m resp. ± 4 m for the buffer zones. Thus method A yields for somewhat better results than method B.

This indicates deficiencies in the model being applied, for example in the numerical integration methods to compute T in the subsatellite point (Benning and Fröhlich, 1974), in other simplifications or round off errors. Although the worldwide distribution of the tracking stations is good more distance measure-

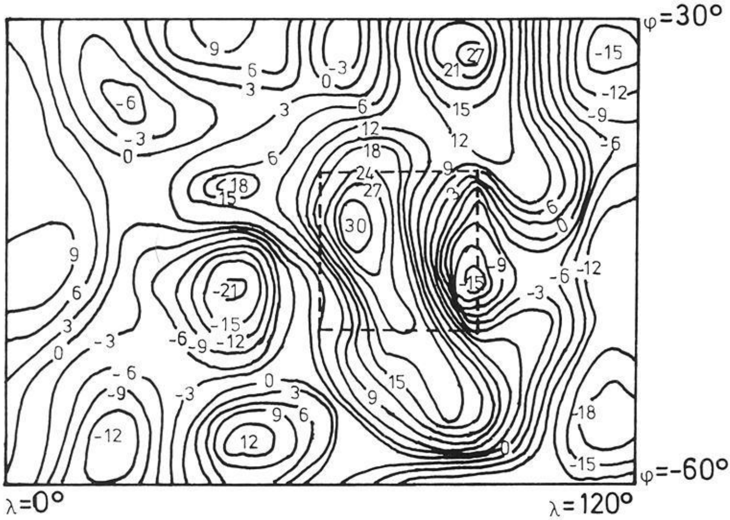

Fig. 5. Geoid Heigths (m), referred to $U_0 = U$ ($n_c \leqq 7$), computed for inner zone ΔE_{27} – method A

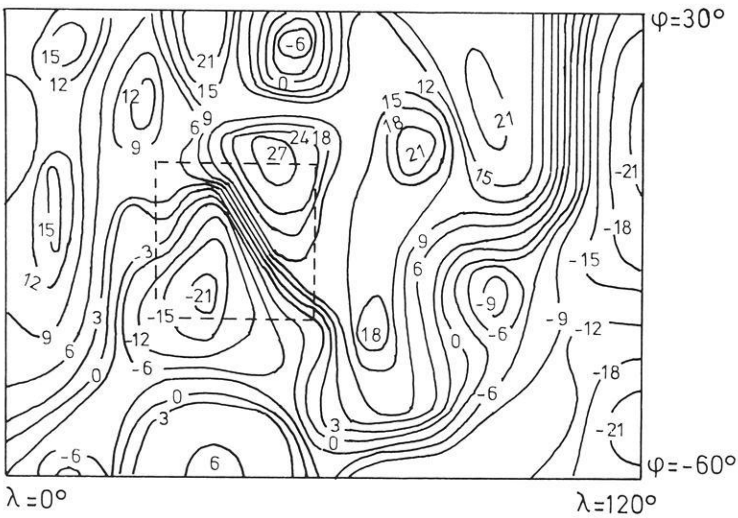

Fig. 6. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 7$), computed for inner zone ΔE_{26} – method B

ments must be included as to improve the estimate and accuracy of the fitted orbit. Data of perhaps 4 or 5 satellites of different inclinations should be available. Furthermore the model of the global force field in method B could be improved by introducing physically relevant information or data for the pole-caps. This would permit a global density-recovery with smaller residuals, an important fact, because the uncertainties of the global density values enter directly the spherical harmonics. Thus they indirectly falsify the recomputed orbit which mainly depends

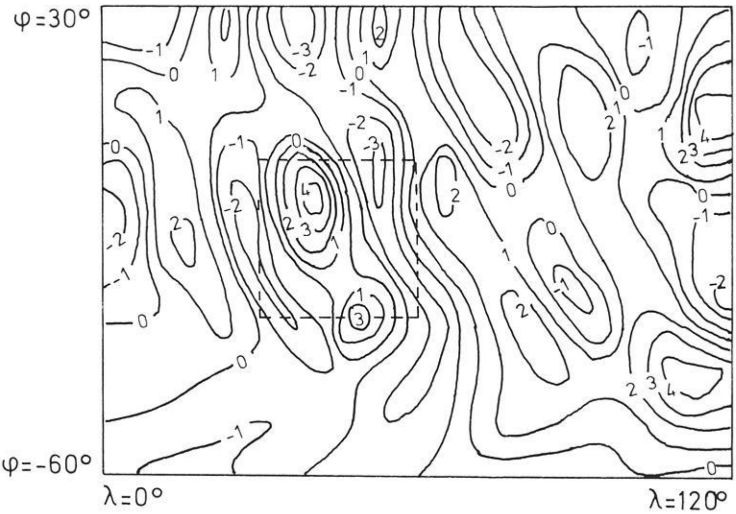

Fig. 7. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 11$), computed for inner zone ΔE_{26} – method A

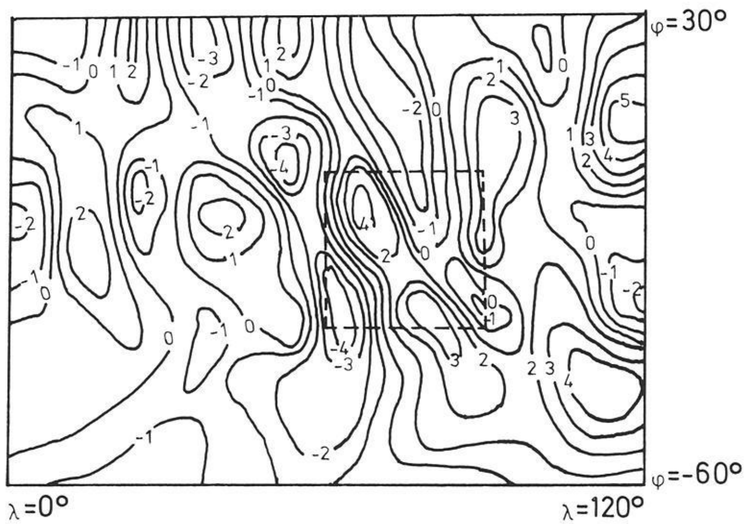

Fig. 8. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 11$), computed for inner zone ΔE_{27} – method A

on the spherical harmonics. The better the orbit the more accurate and the more direct may altimetry data be used for the recovery of the geometry of the oceans.

Concerning to the needed computer time and weighing the obtained accuracies method A, the direct regional resolution of detailed geoid parameters should be preferred.

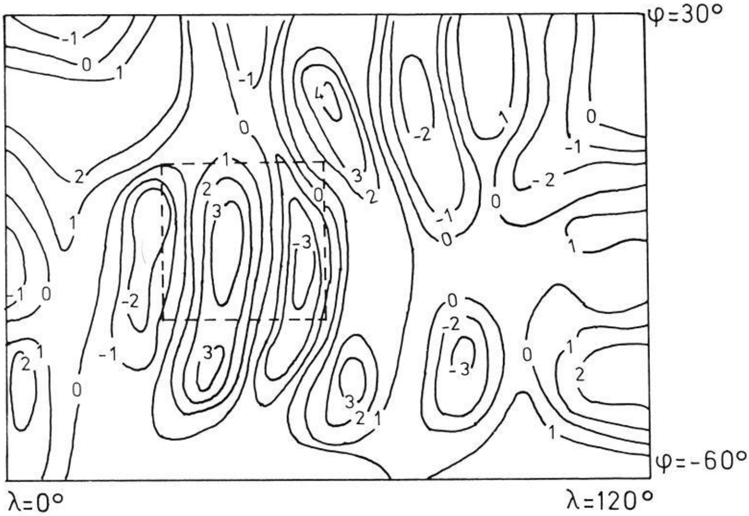

Fig. 9. Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 11$), computed for inner zone ΔE_{26} – method B

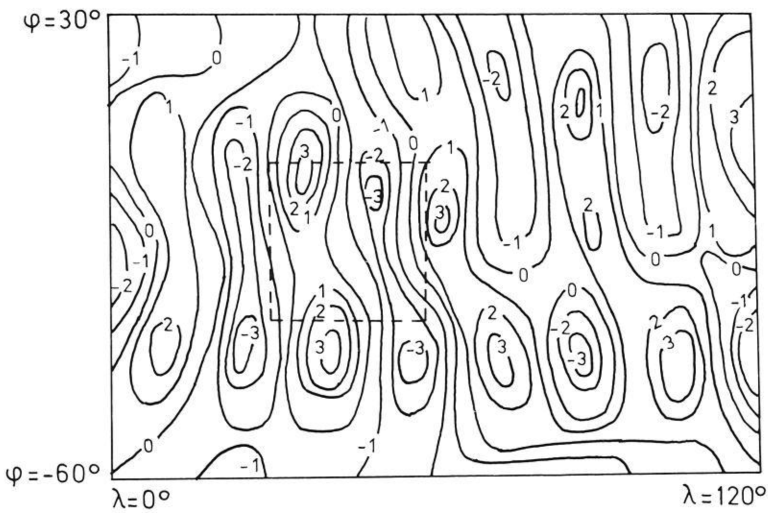

Fig. 10. Directly computed Geoid Heights (m), referred to $U_0 = U$ ($n_c \leqq 11$)

4. Conclusions

Regarding the existing solutions of the gravity field up to spherical harmonics of at least 15th degree and order, the geoid undulations referred to the correspondent reference surface $U = U_0$ will be less than 15 m. Hence, the methods to analyze altimetry data for a detailed knowledge of the earth's gravity field, the methods pointed out here give results of the recomputed geoid undulations with an accuracy of ± 2 m. This result surpasses existing solutions by one order of magnitude.

The feasibility and flexibility of the simple layer model for the geopotential in satellite altimetry is shown by the possibility to limit the unknown parameters to those surface elements which are sufficiently covered with data and which may be fitted to the given data-covering by any mesh size.

As to improve the analytical model for the physical attraction the distribution of the unknown density values may be extended with respect to the technical means (computer storage and -time). This would for example not be possible by means of spherical harmonics, because they denote integrals over the mass distribution of the earth and may not be locally varied nor regionally be limited. Thus the simple layer model is well suited for the analysis of altimetry data with a high resolution and accuracy of the unknowns of the global gravity field. But to strengthen results for the higher order harmonics other worldwide distributed data like gravity anomalies will be introduced into a combined solution.

References

Benning, W.: Auswertung von Altimetermessungen für das Modell der einfachen Schicht des Erdschwerefeldes in der Satellitengeodäsie. Deutsche Geod. Komm. (DGK), München, Series C, No. 200, 1974

Benning, W.: Eine globale Geoidbestimmung aus stückweise regionalen Detaillösungen mit Hilfe von Altimeter- und Entfernungsmessungen in der Satellitengeodäsie. Allg. Verm. Nachr. (AVN) **82**, 12–22, 1975

Benning, W., Fröhlich, H.: Integrationsfehler für das Modell einer einfachen Schicht in der Satellitengeodäsie bei verschwindender Aufpunktshöhe. Mitt. a. d. Inst. f. Theor. Geod. d. Univ. Bonn, No. 29, Bonn, 1974

Koch, K.R.: Alternate representation of the earth's gravitational field for satellite geodesy. Boll. Geofis. Teor. Appl. **10**, 318–325, 1968

Koch, K.R.: Errors of quadrature connected with the simple layer model of the geopotential. NOAA Techn. Report NOS 11, National Ocean Survey, Rockville, Md., 1971

Koch, K.R.: Earth's gravity field and station coordinates from doppler data, satellite triangulation and gravity anomalies. NOAA Techn. Report NOS 62, National Ocean Survey, Rockville, Md., 1974

Nasa: GEOS-C mission. Proposal briefing information. Wallops Island, Virg., 1972

Witte, B.: Berechnungsverfahren für die Bestimmung des Erdschwerepotentials aus Doppler-Beobachtungen mit Hilfe des Modells einer einfachen Schicht. Deutsche Geod. Komm. (DGK), München, Series C, No. 167, 1971

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