J. Geophys. 42, 137 – 145, 1976

Journal of Geophysics

Propagation of Love-Type Waves in Heterogeneous Elastic Layers

Arabinda Biswas

Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, India

Abstract. The propagation of Love type waves in isotropic non-homogeneous stratum of finite depth has been studied for two different cases: (i) the layer is imbedded by two isotropic homogeneous elastic half spaces; (ii) the layer is in welded contact with another heterogeneous layer of different properties. The existence of such waves has been proved by obtaining the solutions of frequency equations numerically.

Key words: Wave propagation – Love wave – Nonhomogeneous medium.

1. Introduction

Stoneley (1924) investigated the propagation of generalized type of Love waves in the presence of a homogeneous layer imbedded between two half spaces. He proved the existence of such type of waves if the wave length be not very large or the thickness of the sandwiched layer not too thin. He discussed the possibility of Love type wave propagation when the velocity of the distortional wave in the upper medium is less or greater than that in the lower medium. Datta (1963) considered the imbedded layer to be heterogenous to study the propagation of this type of wave for several variations in the modulus of rigidity and density. Sinha (1966) studied a similar problem of propagation of waves in a layer lying between two elastic half-spaces by varying the rigidity and velocity exponentially with depth.

The propagation of elastic waves in two layers in welded contact having two free surfaces was investigated by Jones (1964). He considered the layers to be homogeneous. But Paul (1966) discussed the propagation of SH-waves by taking the layers in welded contact to be heterogeneous for two simple cases. In one case he assumed the rigidity and density to be linear while in the other exponential.

In this paper we consider the problems of propagation of Love type waves in heterogeneous layers for both the cases:

(i) The heterogeneous elastic layer is sandwiched between homogeneous elastic half spaces.

(ii) The medium of propagation of waves consists of two heterogeneous elastic layers in welded contact and having different properties.

The variations in modulus of rigidity μ and density ρ in one of the layers for both the case are taken as

$$\mu=\mu_0(1-\sin\delta z), \qquad \rho=\rho_0(1-\sin\delta z) \tag{1a}$$

where μ_0, ρ_0 are the constant rigidity and density respectively and δ is a constant. In case (ii) the modulus of rigidity and the density in one of the adjacent layer vary exponentially with depth and those are expressed as

$$\mu=\mu_0\exp(\gamma z), \qquad \rho=\rho_0\exp(\gamma z). \tag{1b}$$

2. Basic Equations

We assume (u, v, w) to be the displacement components in general at any point (x, y, z) in cartesian coordinate system and the conditions for SH-waves advancing parallel to x-axis as $u=w=\partial/\partial y=0$. The equations of motion, in the absence of body forces, for the propagation of such type of waves will then reduce to a single equation of the form

$$\rho\frac{\partial^2 v}{\partial t^2}=\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yz}}{\partial z}, \tag{2}$$

where

$$p_{xy}=\mu\frac{\partial v}{\partial x}, \qquad p_{yz}=\mu\frac{\partial v}{\partial z}. \tag{3}$$

If the rigidity modulus μ and the density ρ are the functions of z only, the Equation (2) will then, after putting the values of p_{xy} and p_{yz} from Equation (3) into it, take the form

$$\rho\frac{\partial^2 v}{\partial t^2}=\mu\nabla^2 v+\frac{d\mu}{dz}\frac{\partial v}{\partial z} \tag{4}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}.$$

The above Equation (4) is the standard differential equation to be solved for any problem dealing with the type of heterogeneity considered in this case.

We consider the waves to be plane harmonic, so that the displacement v may be assumed as

$$v=V(z)\exp\{ik(x-ct)\} \tag{5}$$

where k is the wave number and c is the phase velocity of the wave propagated. This assumption transforms the Equation (4) into the form

$$\frac{d^2V}{dz^2}+\frac{1}{\mu}\frac{d\mu}{dz}\frac{dV}{dz}+k^2\left(\frac{c^2}{\beta^2}-1\right)V=0 \tag{6}$$

where $\beta=\sqrt{\mu/\rho}$ is the distorsional wave velocity.

In order to normalize the Equation (6) we substitute $V(z)=X(z)\,\mu^{-1/2}$ and therefore, obtain

$$\frac{d^2X}{dz^2}+\left[\frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2-\frac{1}{2\mu}\frac{d^2\mu}{dz^2}+k^2\left(\frac{c^2}{\beta^2}-1\right)\right]X=0. \tag{7}$$

The expressions for μ and ρ in Equation (1a) reduce the Equation (7) into the form

$$\frac{d^2X}{dz^2}+l^2X=0 \tag{8}$$

where

$$l=\left\{\frac{\delta^2}{4}+k^2\left(\frac{c^2}{\beta^2}-1\right)\right\}^{1/2} \tag{9}$$

$$\beta=\sqrt{\mu/\rho}=\sqrt{\mu_0/\rho_0}=\text{constant}. \tag{10}$$

The solution of the Equation (8), in case of layered media can be written as

$$X_1=E_1\cos l_1 z+F_1\sin l_1 z \tag{11}$$

E_1, F_1 being constant, and instead of X and l the notations X_1 and l_1 are used for the layered media.

Now from Equations (5) and (11) and the assumption $V(z)=\mu^{-1/2}X(z)$ we obtain

$$v=\frac{1}{(1-\sin\delta z)^{1/2}}\left[B_1\cos l_1 z+B_2\sin l_1 z\right]\exp\{ik(x-ct)\} \tag{12}$$

where $B_1=\mu_0^{-1/2}E_1$, $B_2=\mu_0^{-1/2}F_2$ are constant.

The Equation (7), on application of the Equation (1b) can be reduced to the form

$$\frac{d^2X}{dz^2}+m^2X=0 \tag{13}$$

where

$$m=\left\{k^2\left(\frac{c^2}{\beta^2}-1\right)-\frac{\gamma^2}{4}\right\}^{1/2} \tag{14}$$

$$\beta=\sqrt{\mu/\rho}=\sqrt{\mu_0/\rho_0}=\text{constant}. \tag{15}$$

Now we write the solution of the Equation (13) as

$$X=E_2\cos mz+F_2\sin mz \tag{16}$$

where E_2, F_2 are constant. The displacement component v in this case can then be written as

$$v=\exp\left(-\frac{\gamma}{2}z\right)\left[C_1\cos mz+C_2\sin mz\right]\exp\{ik(x-ct)\} \tag{17}$$

with $C_1=\mu_0^{-1/2}E_2$, $C_2=\mu_0^{-1/2}F_2$ as integration constants.

The Equation (4) for a homogeneous medium reduces to the differential equation

$$\rho \frac{\partial^2 v}{\partial t^2} = \mu \nabla^2 v \tag{18}$$

and the displacement component v, being the solution of the Equation (18) can be written as

$$v = [D_1 e^{nz} + D_2 e^{-nz}] \exp\{ik(x-ct)\} \tag{19}$$

where $n = k(1 - c^2/\beta^2)^{1/2}$ and D_1, D_2 are constant.

In the case of infinitely extended medium the displacement component can be expressed in more simplified form

$$v = D \exp\{\pm nz + ik(x-ct)\} \tag{20}$$

where the sign is to be so chosen that the displacement vanishes at infinity.

3. Problem I

Propagation of Waves in a Heterogeneous Layer Sandwiched between Homogeneous Half-Spaces

Solution of the Problem. The aim of this problem is to study the propagation of Love type waves in non-homogeneous stratum of finite depth imbedded between isotropic elastic half-spaces. It has been investigated that Love type wave propagation is possible for $c < \beta_1 < \beta_3$, where c is the phase velocity of the wave propagated, and β_1, β_3 are the distorsional wave velocities in the upper and the lower medium respectively.

The two homogeneous half-spaces $H \leqq z \leqq \infty$ and $-\infty \leqq z \leqq 0$ have constant modulus of rigidity and density. The rigidity modulus and the density in the upper medium are denoted by μ_1, ρ_1 and those in the lower medium by μ_3 and ρ_3 respectively. The rigidity modulus μ_2 and the density ρ_2 of the heterogeneous layer $0 \leqq z \leqq H$ are considered to be functions of z only, and then assumed as in the Equation (1a)

$$\mu_2 = \mu_0(1 - \sin\delta z), \qquad \rho_2 = \rho_0(1 - \sin\delta z) \tag{21}$$

so that the velocity $\beta_2 = \sqrt{\mu_2/\rho_2} = \sqrt{\mu_0/\rho_0}$ is constant.

For a plane wave travelling in the direction of x increasing, the displacements v_1, v_2 and v_3 in the upper medium, layer and lower medium respectively are expressed from the Equations (20) and (12), as

$$v_1 = D_1 \exp\{-s_1 z + ik(x-ct)\} \tag{22}$$

$$v_2 = \frac{1}{(1-\sin\delta z)^{1/2}} [B_1 \cos s_2 z + B_2 \sin s_2 z] \exp\{ik(x-ct)\} \tag{23}$$

$$v_3 = D_2 \exp\{s_3 z + ik(x-ct)\} \tag{24}$$

where

$$s_1 = k(1 - c^2/\beta_1^2)^{1/2} \tag{25}$$

$$s_2 = \left\{k^2(c^2/\beta_2^2 - 1) + \frac{\delta^2}{4}\right\}^{1/2} \tag{26}$$

$$s_3 = k(1 - c^2/\beta_3^2)^{1/2} \tag{27}$$

$\beta_1 = \sqrt{\mu_1/\rho_1} = \text{constant}$; $\beta_2 = \sqrt{\mu_2/\rho_2} = \sqrt{\mu_0/\rho_0} = \text{constant}$;
$\beta_3 = \sqrt{\mu_3/\rho_3} = \text{constant}$, and $c < \beta_1 < \beta_3$.

Boundary Conditions

The boundary conditions are

$$\left.\begin{array}{ll} \text{(I)} & v_1 = v_2 \\ \text{(II)} & p_{yz} \text{ in the upper medium} = p_{yz} \text{ in the layer,} \\ & \text{or,} \quad \mu_1 \dfrac{\partial v_1}{\partial z} = \mu_2 \dfrac{\partial v_2}{\partial z} \end{array}\right\} \quad \text{at} \quad z = H;$$

$$\left.\begin{array}{ll} \text{(III)} & v_2 = v_3 \\ \text{(IV)} & p_{yz} \text{ in the layer} = p_{yz} \text{ in the lower medium,} \\ & \text{or,} \quad \mu_2 \dfrac{\partial v_2}{\partial z} = \mu_3 \dfrac{\partial v_3}{\partial z} \end{array}\right\} \quad \text{at} \quad z = 0.$$

The above boundary conditions along with the Equations (22), (23) and (24) give the following equations

$$\begin{aligned}
&\text{(i)} \quad \frac{1}{(1 - \sin \delta H)^{1/2}} [B_1 \cos s_2 H + B_2 \sin s_2 H] - e^{-s_1 H} D_1 = 0 \\
&\text{(ii)} \quad \frac{\mu_0}{(1 - \sin \delta H)^{1/2}} [\{\delta \cos \delta H \cos s_2 H - 2 s_2 \sin s_2 H (1 - \sin \delta H)\} B_1 \\
&\qquad + \{\delta \cos \delta H \sin s_2 H + 2 s_2 \cos s_2 H (1 - \sin \delta H)\} B_2] \\
&\qquad + 2 \mu_1 s_1 e^{-s_1 H} D_1 = 0 \\
&\text{(iii)} \quad B_1 - D_2 = 0 \\
&\text{(iv)} \quad \mu_0 \delta B_1 + 2 \mu_0 s_2 B_2 - 2 \mu_3 s_3 D_2 = 0.
\end{aligned} \tag{28}$$

Frequency Equation

Eliminating the unknown constants B_1, B_2, D_1 and D_2 from the Equations (28), the frequency equation is obtained in the form

$$\begin{aligned}
&[(\mu_0 \delta - 2 \mu_3 s_3)(2 \mu_1 s_1 + \mu_0 \delta \cos \delta H) + 4 \mu_0^2 s_2^2 (1 - \sin \delta H)] \tan s_2 H \\
&\quad + 2 \mu_0 s_2 (\mu_0 \delta - 2 \mu_3 s_3)(1 - \sin \delta H) - 2 \mu_0 s_2 (2 \mu_1 s_1 + \mu_0 \delta \cos \delta H) = 0.
\end{aligned} \tag{29}$$

We assume $\mu_0=\mu_3$ and use the following dimensionless parameters

$$\begin{aligned} &a=\delta H, \quad \xi=\mu_1/\mu_3, \quad s=kH, \\ &b_1=\beta_2/\beta_1, \quad b_3=\beta_2/\beta_3, \quad \eta=c/\beta_2 \end{aligned} \tag{30}$$

so that the frequency Equation (29) takes the form

$$\begin{aligned} &[(a-2s_3H)(2\xi s_1H+a\cos a)+4(s_2H)^2(1-\sin a)]\tan s_2H \\ &\quad +2s_2H(a-2s_3H)(1-\sin a)-2s_2H(2\xi s_1H+a\cos a)=0 \end{aligned} \tag{31}$$

and the Equations (25), (26) and (27) also take the forms

$$s_1H=s(1-b_1^2\eta^2)^{1/2} \tag{32}$$

$$s_2H=\left\{s^2(\eta^2-1)+\frac{a^2}{4}\right\}^{1/2} \tag{33}$$

$$s_3H=s(1-b_3^2\eta^2)^{1/2}. \tag{34}$$

Numerical Solutions

The frequency Equation (29) or (31) may be solved for two different cases

(i) $c<\beta_2$ and (ii) $c>\beta_2$.

We consider the Equation (31) to obtain its numerical solutions for the case $\beta_2<c<\beta_1<\beta_3$ and for the following numerical values of the non-dimensional parameters in Equation (30)

$$a=0.3, \quad \xi=0.5, \quad b_1^2=0.5, \quad b_3^2=0.25.$$

The roots in the form c^2/β_2^2 of the frequency Equation (31) for two different values of s are obtained numerically as

(i) $c^2/\beta_2^2=1.4$, when $s(=kH)=2.36693$,

(ii) $c^2/\beta_2^2=1.2$, when $s(=kH)=4.22820$

and this proves that Love type waves may propagate under the assumed condition $\beta_2<c<\beta_1<\beta_3$.

Similarly the existence of this type of wave may be proved for the condition $c<\beta_2$.

4. Problem II

Propagation of SH-Wave in a Two-Layered Heterogeneous Medium

Solution of the Problem. In this problem we have studied the propagation of *SH*-wave in a two-layered heterogeneous medium in which the modulus of rigidity and the density are μ_1 and ρ_1 for the first layer, and those for the second layer are μ_2 and ρ_2 respectively. The variations in the rigidity and density of the layers $H \geqq z \geqq 0$ and $-H \leqq z \leqq 0$ are assumed as

$$\text{(i)} \quad \mu_1 = \mu_{10}(1 - \sin \delta z), \quad \rho_1 = \rho_{10}(1 - \sin \delta z) \tag{35}$$

$$\text{(ii)} \quad \mu_2 = \mu_{20} \exp(\gamma z), \quad \rho_2 = \rho_{20} \exp(\gamma z) \tag{36}$$

so that the distortional wave velocities

$$\beta_1 = \sqrt{\mu_1/\rho_1} = \sqrt{\mu_{10}/\rho_{10}} = \text{constant},$$
$$\beta_2 = \sqrt{\mu_2/\rho_2} = \sqrt{\mu_{20}/\rho_{20}} = \text{constant}.$$

The existence of SH-wave in such a medium has been proved by obtaining the real roots of the frequency equation when the variations are small in each case.

The displacement components v_1 and v_2, in the first and second layers respectively, with the help of the Equations (12) and (17), are expressed as

$$v_1 = \frac{1}{(1 - \sin \delta z)^{1/2}} [E_1 \cos l_1 z + E_2 \sin l_1 z] \exp\{ik(x - ct)\} \tag{37}$$

$$v_2 = \exp\left(-\frac{\gamma}{2} z\right) [F_1 \cos l_2 z + F_2 \sin l_2 z] \exp\{ik(x - ct)\} \tag{38}$$

where

$$l_1 = \left\{k^2 \left(\frac{c^2}{\beta_1^2} - 1\right) + \frac{\delta^2}{4}\right\}^{1/2}$$
$$l_2 = \left\{k^2 \left(\frac{c^2}{\beta_2^2} - 1\right) - \frac{\gamma^2}{4}\right\}^{1/2}.$$

Boundary Conditions

The boundary conditions are that the components of stress across the bounding surfaces vanish at the free surfaces $z = H$ and $z = -H$, and the components of displacement and stress across the interface $z = 0$ are continuous. Applying these conditions we obtain

(I) p_{yz} in the first layer $= 0$ or, $\mu_1 \dfrac{\partial v_1}{\partial z} = 0$, at $z = H$;

(II) $v_1 = v_2$

(III) p_{yz} in the first layer $= p_{yz}$ in the second layer

or, $\mu_1 \dfrac{\partial v_1}{\partial z} = \mu_2 \dfrac{\partial v_2}{\partial z}$ — (II) and (III) at $z = 0$;

(IV) p_{yz} in the second layer $= 0$ or, $\mu_2 \dfrac{\partial v_2}{\partial z} = 0$, at $z = -H$.

The above relations along with Equations (35), (36), (37) and (38) give the following equations

$$\begin{aligned}
&\text{(i)}\quad \{\delta \cos \delta H \cos l_1 H - 2 l_1 \sin l_1 H(1-\sin \delta H)\} E_1 \\
&\qquad + \{\delta \cos \delta H \sin l_1 H + 2 l_1 \cos l_1 H(1-\sin \delta H)\} E_2 = 0 \\
&\text{(ii)}\quad E_1 - F_1 = 0 \\
&\text{(iii)}\quad \mu_{10}\, \delta E_1 + 2\mu_{10}\, l_1 E_2 + \mu_{20}\, \gamma F_1 - 2\mu_{20}\, l_2 F_2 = 0 \\
&\text{(iv)}\quad (\gamma \cos l_2 H - 2 l_2 \sin l_2 H) F_1 - (\gamma \sin l_2 H + 2 l_2 \cos l_2 H) F_2 = 0.
\end{aligned} \tag{39}$$

Frequency Equation

The frequency equation is obtained by eliminating the unknown constants E_1, E_2, F_1 and F_2 from the Equations (39) as

$$\begin{aligned}
&2 l_1 [(1-\sin \delta H)\{\mu_{10}\, \delta\gamma + \mu_{20}(4 l_1^2 + \gamma^2)\} - \mu_{10}\, \delta\gamma \cos \delta H] \cot l_1 H \\
&\quad + 2\mu_{10}\, l_2 [4 l_1^2 (1-\sin \delta H) + \delta^2 \cos \delta H] \cot l_2 H \\
&\quad + 4\mu_{10}\, \delta l_1 l_2 [(1-\sin \delta H) - \cos \delta H] \cot l_1 H \cot l_2 H \\
&\quad + [\mu_{10}\, \gamma \{4 l_1^2 (1-\sin \delta H) + \delta^2 \cos \delta H\} \\
&\quad + \mu_{20}\, \delta \cos \delta H (4 l_2^2 + \gamma^2)] = 0.
\end{aligned} \tag{40}$$

We assume δ and γ so small that the terms containing upto first order in δ and γ are to be retained in the Equation (40) and this assumption reduces the frequency equation to the form

$$\mu_{10}\, l_1 \cot l_2 H + \mu_{20}\, l_2 \cot l_1 H + \frac{\mu_{10}\, \gamma l_1^2 + \mu_{20}\, \delta l_2^2}{2 l_1 l_2 (1-\delta H)} = 0 \tag{41}$$

where l_1 and l_2 in their reduced forms are

$$l_1 = k\left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2}, \quad l_2 = k\left(\frac{c^2}{\beta_2^2} - 1\right)^{1/2}.$$

We consider $\beta_1 < c < \beta_2$ and the following dimensionless parameters

$$\begin{aligned}
&\eta_1 = c/\beta_1, \quad \eta_2 = c/\beta_2, \quad a_1 = \delta H, \quad a_2 = \gamma H, \\
&s = kH, \quad b = \beta_1/\beta_2 = \eta_2/\eta_1, \quad a = a_2/a_1 = \gamma/\delta.
\end{aligned} \tag{42}$$

For the condition $\beta_1 < c < \beta_2$, $l_1 = k(c^2/\beta_1^2 - 1)^{1/2}$ is real but $l_2 = k(c^2/\beta_2^2 - 1)^{1/2}$ is imaginary. We introduce the dimensionless parameters from Equation (42) into the Equation (41) and use the condition $\mu_{10} = \mu_{20}$ to obtain the frequency equation in the form

$$\begin{aligned}
&s(\eta_1^2 - 1)^{1/2} \coth \{s(1 - b^2 \eta_1^2)^{1/2}\} - s(1 - b^2 \eta_1^2)^{1/2} \cot \{s(\eta_1^2 - 1)^{1/2}\} \\
&\quad - \frac{a_1 \{(1+a) - (b_2 + a)\eta_1^2\}}{2(1-a_1)(\eta_1^2 - 1)^{1/2}(1 - b^2 \eta_1^2)^{1/2}} = 0.
\end{aligned} \tag{43}$$

Numerical Solutions

In order to solve the frequency Equation (43) numerically we assume $a_1 = 0.2$; $b^2 = 0.25$, 0.50; and $a = 1, 2$; and the roots of the Equation (43) in the form of η_1 or η_2 are obtained. The numerical values of η_1^2 and η_2^2 for different values of s are given in tabular forms (Tables 1–4) and it is observed that the phase velocity occurs at a lower $s(=kH)$ with increasing γ.

Table 1. Roots of the frequency equation in the form of η_1^2 and η_2^2 when $b^2 = \beta_1^2/\beta_2^2 = 0.25$, $a_1 = \delta H = 0.2$ and $\gamma = \delta$

$s(=kH)$	2.52751	1.27953	0.82235
$\eta_1^2(=c^2/\beta_1^2)$	1.20	1.40	1.50
$\eta_2^2(=c^2/\beta_2^2)$	0.30	0.35	0.375

Table 2. Roots of the frequency equation in the form of η_1^2 and η_2^2 when $b^2 = \beta_1^2/\beta_2^2 = 0.25$, $a_1 = \delta H = 0.2$ and $\gamma = 2\delta$

$s(=kH)$	2.46950	1.15970	0.61275
$\eta_1^2(=c^2/\beta_1^2)$	1.20	1.40	1.50
$\eta_2^2(=c^2/\beta_2^2)$	0.30	0.35	0.375

Table 3. Roots of the frequency equation in the form of η_1^2 and η_2^2 when $b^2 = \beta_1^2/\beta_2^2 = 0.50$, $a_1 = \delta H = 0.20$ and $\gamma = \delta$

$s(=kH)$	3.77493	2.07280	0.89681
$\eta_1^2(=c^2/\beta_1^2)$	1.10	1.20	1.30
$\eta_2^2(=c^2/\beta_2^2)$	0.55	0.60	0.65

Table 4. Roots of the frequency equation in the form of η_1^2 and η_2^2 when $b^2 = \beta_1^2/\beta_2^2 = 0.50$, $a_1 = \delta H = 0.20$ and $\gamma = 2\delta$

$s(=kH)$	3.713125	1.95320	0.51758
$\eta_1^2(=c^2/\beta_1^2)$	1.10	1.20	1.30
$\eta_2^2(=c^2/\beta_2^2)$	0.55	0.60	0.65

References

Datta, S.: Love waves in a non-homogeneous internal stratum lying between two semi-infinite isotropic media. Geophysics **XXVIII** (2), 156, 1963

Datta, S.: On the propagation of Love waves in a non-homogeneous internal stratum of finite depth lying between two semi-infinite isotropic media. Geofis. pura e appl. **55**, 31, 1963

Jones, J. P.: Wave propagation in two-layered medium. J. Appl. Mech. **31** (2), 213, 1964

Paul, M. K.: Propagation of SH-wave in a two layered heterogeneous medium. Ind. J. Mech. Maths. **4**, 26, 1966

Sinha, N. K.: Propagation of Love waves in a non-homogeneous stratum of finite depth sandwiched between two semi-infinite isotropic media. Pure Appl. Geophys. **65**, 37, 1966

Stoneley, R.: Elastic waves at the surface of separation of two solids (transverse waves in an internal stratum). Proc. Roy. Soc. (London) A **106**, 424, 1924

Received December 9, 1975