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Propagation of Elastic Waves in Vertically Inhomogeneous Media

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Abstract. A finite-difference scheme is proposed for studying the propagation of waves in vertically inhomogeneous media. The scheme can be applied to either continuous or discontinuous inhomogeneity of the media. When the inhomogeneity is continuous, the scheme approximates the elastic wave equation for vertically inhomogeneous media. The scheme cannot be obtained directly from the elastic wave equation for vertically inhomogeneous media since it contains additional terms of second order in the increments which are deduced from the boundary conditions but not from the wave equation. When the inhomogeneity is discontinuous, the contribution of these additional terms becomes significant, thus ensuring that the boundary conditions on discontinuities are satisfied.

Key words: Elastic wave propagation – Vertically inhomogeneous media.

1. Introduction

The solution, by finite-difference methods, of problems of elastic wave propagation in horizontally homogeneous layered media has received a great deal of attention in recent years (Alterman and Karal, 1968), but much less has been devoted to problems of wave propagation in inhomogeneous media. A finite-difference scheme which approximates the elastic wave equation for vertically inhomogeneous media has been given by Kelly et al. (1976). However, their scheme cannot be applied to horizontally layered media for which the vertical inhomogeneity is discontinuous.

In the present work we propose a finite-difference scheme Equations (2.10), which approximates simultaneously the elastic wave equation for vertically in-

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homogeneous media and the elastic wave equation with the corresponding boundary conditions for horizontally layered media. Thus the proposed finite-difference scheme can be applied to vertically inhomogeneous media with either continuous or discontinuous inhomogeneity.

The response of a vertically inhomogeneous medium can be simulated by a model which consists of many thin homogeneous layers. Theoretical seismograms can then be obtained by the generalized ray theory. Chapman (1974) has reformulated the generalized ray theory to avoid approximating the inhomogeneous model by homogeneous layers. The generalized ray theory of Chapman for inhomogeneous media involves letting the thickness of each layer approach zero and the number of layers approach infinity. Hence in Chapman's theory the vertically inhomogeneous medium is not approximated by reflecting interfaces. However, the theory is still in terms of reflections from the velocity and density gradients.

The considerable success of this theory in the interpretation of seismograms justifies the simulation of an inhomogeneous medium by considering it as the limiting case of a layered medium when the thickness of each layer approaches zero. This limiting process regards every point of a vertically inhomogeneous medium as an interface point of two homogeneous layers.

Adopting the layered approximation principle, according to which every point of a vertically inhomogeneous medium behaves like an interface point between two homogeneous layers, we construct a finite-difference scheme to approximate elastic wave propagation in vertically inhomogeneous media. It is then found that the finite-difference scheme resulting from the layered approximation principle approximates the elastic wave equation for vertically inhomogeneous media.

The proposed finite-difference scheme for vertically inhomogeneous media contains additional terms which can be ignored for continuous vertical inhomogeneity, but whose contribution is considerable when the vertical inhomogeneity is discontinuous. Thus, the additional terms ensure that the boundary conditions are satisfied at discontinuities in the vertical inhomogeneity.

2. Finite-Difference Scheme for Vertically Inhomogeneous Media with either Continuous or Discontinuous Homogeneity

In order to construct a finite-difference scheme for the motion of elastic waves in vertically inhomogeneous media, a grid is imposed on the xz-plane and every grid point regarded as an *interface point* between two horizontal layers. Hence such a finite-difference scheme must approximate, for every grid point, the equation of motion in homogeneous media and the boundary conditions as well. It will be shown that the scheme, which approximates the elastic wave equation and the boundary conditions in horizontally homogeneous layered media, approximates the elastic wave equation in vertically inhomogeneous media.

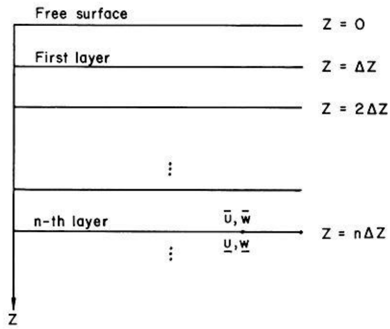

Fig. 1. Geometry of the elastic layered medium. On interfaces, z-derivatives of the displacement vector (U, W) are not continuous and hence a distinction is made between $(\bar{U}, \bar{W})$ and $(\underline{U}, \underline{W})$

Since all the grid points are regarded as *interface points*, the equations of motion (2.1) and the boundary conditions (2.2) must be satisfied at each point of the grid. For the case where the components of the displacement vector referred to cartesian coordinates (x, z) are (U, W), the equations of motion in homogeneous and isotropic solids are:

$$\begin{aligned} &\frac{1}{\beta^2} U_{tt} - U_{zz} - \frac{\lambda+2\mu}{\mu} U_{xx} - \frac{\lambda+\mu}{\mu} W_{xz} = 0 \\ &\frac{1}{\alpha^2} W_{tt} - W_{zz} - \frac{\mu}{\lambda+2\mu} W_{xx} - \frac{\lambda+\mu}{\lambda+2\mu} U_{xz} = 0 \end{aligned} \tag{2.1}$$

and the boundary conditions for horizontal interfaces $z = \text{const.}$ are:

$$\begin{aligned} \mu_n(\bar{U}_z + \bar{W}_x) &= \mu_{n+1}(\underline{U}_z + \underline{W}_x) \\ (\lambda_n + 2\mu_n)\bar{W}_z + \lambda_n \bar{U}_x &= (\lambda_{n+1} + 2\mu_{n+1})\underline{W}_z + \lambda_{n+1}\underline{U}_x \\ \bar{U} &= \underline{U} \\ \bar{W} &= \underline{W}. \end{aligned} \tag{2.2}$$

In Equation (2.1), λ and μ are the elastic parameters $\alpha^2 = (\lambda + 2\mu)/\rho$ and $\beta^2 = \mu/\rho$ where ρ is the mass density. In Equation (2.2), $\bar{U}$, $\bar{W}$ and $\underline{U}$, $\underline{W}$ are the displacements corresponding to the two different layers whose common boundary is $z = \text{const.} = n\Delta z$ and whose elastic parameters are λ_n, μ_n and λ_{n+1}, μ_{n+1}, respectively (Fig. 1).

The functions U and W are continuous and continuously differentiable with respect to x and t. The functions U and W cannot, however, be differentiated with respect to z on the interfaces $z = \text{const.}$ Since every grid point is regarded as an *interface point*, derivatives with respect to z must be eliminated. The elimination of the z-derivatives from the equations of motion (2.1) is achieved by employing the boundary conditions (2.2).

For the grid imposed on the xz-plane, we let $x = m\Delta x$, $z = n\Delta z$ and $t = p\Delta t$, where Δx and Δz are the mesh sizes, taken to be equal, and Δt is the increment in time. The functions U and W and their x-derivatives U_x, U_{xx}, W_x and W_{xx} are well defined on the grid points. The derivatives U_z, U_{zz}, U_{xz}, W_z, W_{zz} and W_{xz} are well defined in the layers between the interfaces, but they are not defined on the interfaces or, equivalently, on the grid points.

For the derivative U_z of U with respect to z in the interior of a layer, let $\bar{U}_z$ and $\underline{U}_z$ be the different limits of U_z on an interface, approaching it from above and from below (Fig. 1). Also, let a similar notation be used for the other z-derivatives of U and W. Since $\bar{U}_z \neq \underline{U}_z$ and $\bar{W}_z \neq \underline{W}_z$ on the interfaces, U_z and W_z cannot be defined on the grid points although $\bar{U}_z$, $\underline{U}_z$, $\bar{W}_z$ and $\underline{W}_z$ are well defined there.

In previous works (e.g. Alterman and Karal, 1968; Kelly et al., 1976), the derivatives $\bar{U}_z, \underline{U}_z, \bar{W}_z$ and $\underline{W}_z$ were approximated by one-sided finite differences which are of first order in the increments, and hence, plane interfaces introduced an error into the schemes. Outside the interfaces the scheme was of second order in the increments since two-sided finite differences were utilized. In the present work, however, we do not use one-sided differences; the derivatives $\bar{U}_z, \underline{U}_z, \bar{W}_z$ and $\underline{W}_z$ which cannot be approximated by two-sided differences, are eliminated before the introduction of the finite differences.

From the equations of motion (2.1) and the boundary conditions (2.2) we shall now derive a system of equations from which the z-derivatives of U and W can be eliminated to obtain a reduced system which can be approximated by two-sided finite differences.

The equations of motion (2.1) must be satisfied on the interfaces for both $\bar{U}_z, \bar{W}_z$ and $\underline{U}_z, \underline{W}_z$. Hence

$$
\begin{aligned}
&\bar{U}_{zz}+\frac{\lambda_n+\mu_n}{\mu_n}\bar{W}_{xz}=\frac{1}{\beta_n^2}U_{tt}-\frac{\lambda_n+2\mu_n}{\mu_n}U_{xx}\\
&\underline{U}_{zz}+\frac{\lambda_{n+1}+\mu_{n+1}}{\mu_{n+1}}\underline{W}_{xz}=\frac{1}{\beta_{n+1}^2}U_{tt}-\frac{\lambda_{n+1}+2\mu_{n+1}}{\mu_{n+1}}U_{xx}\\
&\bar{W}_{zz}+\frac{\lambda_n+\mu_n}{\lambda_n+2\mu_n}\bar{U}_{xz}=\frac{1}{\alpha_n^2}W_{tt}-\frac{\mu_n}{\lambda_n+2\mu_n}W_{xx}\\
&\underline{W}_{zz}+\frac{\lambda_{n+1}+\mu_{n+1}}{\mu_{n+1}+2\mu_{n+1}}\underline{U}_{xz}=\frac{1}{\alpha_{n+1}^2}W_{tt}-\frac{\mu_{n+1}}{\lambda_{n+1}+2\mu_{n+1}}W_{xx}.
\end{aligned}
\tag{2.3}
$$

The first two boundary conditions in Equations (2.2) yield

$$
\begin{aligned}
&\mu_n\bar{U}_z-\mu_{n+1}\underline{U}_z=-(\mu_n-\mu_{n+1})W_x\\
&(\lambda_n+2\mu_n)\bar{W}_z-(\lambda_{n+1}+2\mu_{n+1})\underline{W}_z=-(\lambda_n-\lambda_{n+1})U_x.
\end{aligned}
\tag{2.4}
$$

Equations (5.4) can be differentiated with respect to x (but not with respect to z), to obtain the equations

$$
\begin{aligned}
&\mu_n\bar{U}_{xz}-\mu_{n+1}\underline{U}_{zx}=-(\mu_n-\mu_{n+1})W_{xx}\\
&(\lambda_n+2\mu_n)\bar{W}_{xz}-(\lambda_{n+1}+2\mu_{n+1})\underline{W}_{xz}=-(\lambda_n-\lambda_{n+1})U_{xx}.
\end{aligned}
\tag{2.5}
$$

Some expansions are now required. $\bar{U}_z$ can be expanded in the variable z in the n-th layer while $\underline{U}_z$ can be expanded in the $(n+1)$-th layer, and similarly for the other z-derivatives. Here $\bar{U}_z$ is a function of x, z and t and similarly for the other z-derivatives, whose expansions are given below. For the first and second derivatives with respect to z we have from Taylor expansions

$$\begin{aligned}
-\Delta z \bar{U}_z + \tfrac{1}{2}\Delta z^2 \bar{U}_{zz} &= U(x, z-\Delta z, t) - U(x, z, t) + 0(\Delta z^3) \\
\Delta z \underline{U}_z + \tfrac{1}{2}\Delta z^2 \underline{U}_{zz} &= U(x, z+\Delta z, t) - U(x, z, t) + 0(\Delta z^3) \\
-\Delta z \bar{W}_z + \tfrac{1}{2}\Delta z^2 \bar{W}_{zz} &= W(x, z-\Delta z, t) - W(x, z, t) + 0(\Delta z^3) \\
\Delta z \underline{W}_z + \tfrac{1}{2}\Delta z^2 \underline{W}_{zz} &= W(x, z+\Delta z, t) - W(x, z, t) + 0(\Delta z^3)
\end{aligned} \tag{2.6}$$

while for the mixed derivatives with respect to x and z we have

$$\begin{aligned}
-2\Delta x \Delta z \bar{U}_{xz} &= U(x+\Delta x, z-\Delta z, t) - U(x-\Delta x, z-\Delta z, t) - U(x+\Delta x, z, t) \\
&\quad + U(x-\Delta x, z, t) + 0(\Delta x \Delta z^2) \\
-2\Delta x \Delta z \bar{W}_{xz} &= W(x+\Delta x, z-\Delta z, t) - W(x-\Delta x, z-\Delta z, t) - W(x+\Delta x, z, t) \\
&\quad + W(x-\Delta x, z, t) + 0(\Delta x \Delta z^2)
\end{aligned} \tag{2.7$_1$}$$

and

$$\begin{aligned}
2\Delta x \Delta z \underline{U}_{xz} &= U(x+\Delta x, z+\Delta z, t) - U(x-\Delta x, z+\Delta z, t) - U(x+\Delta x, z, t) \\
&\quad + U(x-\Delta x, z, t) + 0(\Delta x \Delta z^2) \\
2\Delta x \Delta z \underline{W}_{xz} &= W(x+\Delta x, z+\Delta z, t) - W(x-\Delta x, z+\Delta z, t) - W(x+\Delta x, z, t) \\
&\quad + W(x-\Delta x, z, t) + 0(\Delta x \Delta z^2).
\end{aligned} \tag{2.7$_2$}$$

Equations $(2.7)_1$ and $(2.7)_2$ are alternatives and a combination of them will be taken into account for a reason which will be given later.

The equations (2.3)–(2.6) and either $(2.7)_1$ or $(2.7)_2$ provide a system of 14 equations for the 14 unknowns $\bar{U}_z$, $\underline{U}_z$, $\bar{W}_z$, $\underline{W}_z$, $\bar{U}_{zz}$, $\underline{U}_{zz}$, $\bar{W}_{zz}$, $\underline{W}_{zz}$, $\bar{U}_{xz}$, $\underline{U}_{xz}$, $\bar{W}_{xz}$, $\underline{W}_{xz}$, $U(x, z, t+\Delta t)$ and $W(x, z, t+\Delta t)$.

The last two unknowns appear in Equations (2.3) in the centered finite-difference approximation of U_{tt} and W_{tt}.

After cumbersome but straightforward manipulation, one arrives at the 2 Equations (2.8) for the 2 unknowns $U(x, z, t+\Delta t)$ and $W(x, z, t+\Delta t)$:

$$\begin{aligned}
&\Delta z(\mu_{n+1} - \mu_n) W_x + \tfrac{1}{2}\Delta z^2 \{(\rho_n + \rho_{n+1}) U_{tt} - [\lambda_{n+1} + \lambda_n + 2(\mu_n + \mu_{n+1})] U_{xx} - G_k\} \\
&= \mu_{n+1} U(x, z+\Delta z, t) + \mu_n U(x, z-\Delta z, t) - (\mu_n + \mu_{n+1}) U(x, z, t) + 0(\Delta z^3) \\
&\Delta z(\lambda_{n+1} - \lambda_n) U_x + \tfrac{1}{2}\Delta z^2 \{(\rho_n + \rho_{n+1}) W_{tt} - (\mu_n + \mu_{n+1}) W_{xx} - H_k\} \\
&= (\lambda_{n+1} + 2\mu_{n+1}) W(x, z+\Delta z, t) + (\lambda_n + 2\mu_n) W(x, z-\Delta z, t) \\
&\quad - [\lambda_n + \lambda_{n+1} + 2(\mu_n + \mu_{n+1})] W(x, z, t) + 0(\Delta z^3).
\end{aligned} \tag{2.8}$$

Here the index k stands for either 1 or 2. G_k and H_k are the following expressions, obtained from the system of the 14 Equations (2.3)–(2.6) and $(2.7)_k$:

$$\begin{aligned}
G_1 &= \frac{1}{2\Delta x\Delta z}\left[\lambda_{n+1}+\mu_{n+1}+\frac{(\lambda_n+\mu_n)(\lambda_{n+1}+2\mu_{n+1})}{\lambda_n+2\mu_n}\right][W(x+\Delta x, z+\Delta z, t)\\
&\quad -W(x-\Delta x, z+\Delta z, t)-W(x+\Delta x, z, t)+W(x-\Delta x, z, t)]\\
&\quad +\frac{(\lambda_n+\mu_n)(\lambda_{n+1}-\lambda_n)}{\lambda_n+2\mu_n}U_{xx}+0(\Delta z)\\
G_2 &= \frac{1}{2\Delta x\Delta z}\left[\lambda_n+\mu_n+\frac{(\lambda_{n+1}+\mu_{n+1})(\lambda_n+2\mu_n)}{\lambda_{n+1}+2\mu_{n+1}}\right][W(x+\Delta x, z, t)\\
&\quad -W(x-\Delta x, z, t)-W(x+\Delta x, z-\Delta z, t)+W(x-\Delta x, z-\Delta z, t)]\\
&\quad +\frac{(\lambda_{n+1}+\mu_{n+1})(\lambda_n-\lambda_{n+1})}{\lambda_{n+1}+2\mu_{n+1}}U_{xx}+0(\Delta z)\\
H_1 &= \frac{1}{2\Delta x\Delta z}\left[\lambda_{n+1}+\mu_{n+1}+\frac{(\lambda_n+\mu_n)\mu_{n+1}}{\mu_n}\right][U(x+\Delta x, z+\Delta z, t)\\
&\quad -U(x-\Delta x, z+\Delta z, t)-U(x+\Delta x, z, t)+U(x-\Delta x, z, t)]\\
&\quad +\frac{\lambda_n+\mu_n}{\mu_n}(\mu_{n+1}-\mu_n)W_{xx}+0(\Delta z)\\
H_2 &= \frac{1}{2\Delta x\Delta z}\left[(\lambda_n+\mu_n)+\frac{(\lambda_{n+1}+\mu_{n+1})\mu_n}{\mu_{n+1}}\right]\\
&\quad [U(x+\Delta x, z, t)-U(x-\Delta x, z, t)-U(x+\Delta x, z-\Delta z, t)+U(x-\Delta x, z-\Delta z, t)]\\
&\quad +\frac{\lambda_{n+1}+\mu_{n+1}}{\mu_{n+1}}(\mu_n-\mu_{n+1})W_{xx}+0(\Delta z).
\end{aligned} \tag{2.9}$$

For G_k and H_k in Equations (2.8), the values 1 and 2 of k are alternatives and actually, any combination

$$\begin{aligned}
&pG_1+(1-p)G_2, \quad 0\leqq p\leqq 1\\
&qH_1+(1-q)H_2, \quad 0\leqq q\leqq 1
\end{aligned}$$

can replace G_k and H_k in (2.8).

The scheme (2.8) with either $k=1$ or $k=2$, is not consistent with a scheme for homogeneous media. However, it can be shown that if G_k and H_k in Equations (2.8) are replaced by $(G_1+G_2)/2$ and $(H_1+H_2)/2$ respectively, Equations (2.10), then the finite-difference scheme, (2.10), for vertically inhomogeneous media is consistent with the scheme for horizontally homogeneous layered media.

Upon approximating the derivatives in Equations (2.8) by centered finite-differences we obtain the following formulation, (2.10), for grid points (m, n), which is accurate to the second order in the increments. In the following formulation, U and W at the point $x=m\Delta x$, $z=n\Delta z$ and $t=p\Delta t$ are denoted by $U^p_{m,n}$ and $W^p_{m,n}$ and G_k, H_k of Equations (2.8) are replaced by $(G_1+G_2)/2$ and $(H_1+H_2)/2$, respectively.

$$
\begin{aligned}
U^{p+1}_{m,n} = & -U^{p-1}_{m,n} + 2U^{p}_{m,n} + \frac{1}{\bar{\rho}_n}\left(\frac{\Delta t}{\Delta z}\right)^2 [\mu_{n+1} U^{p}_{m,n+1} + \mu_n U^{p}_{m,n-1} - 2\bar{\mu}_n U^{p}_{m,n}] \\
& + \frac{1}{\bar{\rho}_n}\left(\frac{\Delta t}{\Delta x}\right)^2 \Bigg[\bar{\lambda}_n + 2\bar{\mu}_n \\
& + \frac{(\lambda_n+\mu_n)(\lambda_{n+1}-\lambda_n)}{\lambda_n+2\mu_n} + \frac{(\lambda_{n+1}+\mu_{n+1})(\lambda_n-\lambda_{n+1})}{\lambda_{n+1}+2\mu_{n+1}}\Bigg] (U^{p}_{m+1,n} - 2U^{p}_{m,n} + U^{p}_{m-1,n}) \\
& + \frac{1}{2\bar{\rho}_n}\frac{\Delta t^2}{\Delta x \Delta z}\Bigg[(\mu_{n+1}-\mu_n)(W^{p}_{m+1,n} - W^{p}_{m-1,n}) \\
& + \left(\lambda_{n+1}+\mu_{n+1} + \frac{(\lambda_n+\mu_n)(\lambda_{n+1}+2\mu_{n+1})}{\lambda_n+2\mu_n}\right) \\
& \cdot (W^{p}_{m+1,n+1} - W^{p}_{m-1,n+1} - W^{p}_{m+1,n} + W^{p}_{m-1,n}) \\
& + \left(\lambda_n+\mu_n + \frac{(\lambda_{n+1}+\mu_{n+1})(\lambda_n+2\mu_n)}{\lambda_{n+1}+2\mu_{n+1}}\right) \\
& \cdot (W^{p}_{m+1,n} - W^{p}_{m-1,n} - W^{p}_{m+1,n-1} + W^{p}_{m-1,n-1})\Bigg]
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
W^{p+1}_{m,n} = & -W^{p-1}_{m,n} + 2W^{p}_{m,n} + \frac{1}{\bar{\rho}_n}\left(\frac{\Delta t}{\Delta z}\right)^2 [(\lambda_{n+1}+2\mu_{n+1}) W^{p}_{m,n+1} + (\lambda_n+2\mu_n) W^{p}_{m,n-1} \\
& - (\bar{\lambda}_n + 2\bar{\mu}_n) W^{p}_{m,n}] \\
& + \frac{1}{\bar{\rho}_n}\left(\frac{\Delta t}{\Delta x}\right)^2 \left[\bar{\mu}_n + \frac{(\lambda_n+\mu_n)(\mu_{n+1}-\mu_n)}{\mu_n} + \frac{(\lambda_{n+1}+\mu_{n+1})(\mu_n-\mu_{n+1})}{\mu_{n+1}}\right] \\
& \cdot (W^{p}_{m+1,n} - 2W^{p}_{m,n} + W^{p}_{m-1,n}) \\
& + \frac{1}{2\bar{\rho}_n}\frac{\Delta t^2}{\Delta x \Delta z}\Bigg[(\lambda_{n+1}-\lambda_n)(U^{p}_{m+1,n} - U^{p}_{m-1,n}) \\
& + \left(\lambda_{n+1}+\mu_{n+1} + \frac{(\lambda_n+\mu_n)\mu_{n+1}}{\mu_n}\right)(U^{p}_{m+1,n+1} - U^{p}_{m-1,n+1} - U^{p}_{m+1,n} + U^{p}_{m-1,n}) \\
& + \left(\lambda_n+\mu_n + \frac{(\lambda_{n+1}+\mu_{n+1})\mu_n}{\mu_{n+1}}\right)(U^{p}_{m+1,n} - U^{p}_{m-1,n} - U^{p}_{m+1,n-1} + U^{p}_{m-1,n-1})\Bigg]
\end{aligned}
$$

where

$$\bar{\lambda}_n = (\lambda_n + \lambda_{n+1})/2$$

$$\bar{\mu}_n = (\mu_n + \mu_{n+1})/2$$

and

$$\bar{\rho}_n = (\rho_n + \rho_{n+1})/2.$$

When the elastic parameters λ, μ and ρ are continuous functions of the depth z, i.e. $0(\Delta\lambda)$, $0(\Delta\mu)$, $0(\Delta\rho) = 0(\Delta z)$ the finite-difference scheme (2.10) approximates the elastic wave equation for vertically inhomogeneous media. When the elastic parameters are piecewise constant, than (2.10) approximates the elastic wave equation for horizontally homogeneous layered media. Actually, the finite-difference scheme (2.10) approximates the elastic wave equation for vertically inhomogeneous media in which the inhomogeneity may be either continuous or discontinuous.

Hence the scheme (2.10) for vertically inhomogeneous elastic media is superior to the scheme which can be obtained directly from the elastic wave equation given by Kelly et al. (1976).

The scheme (2.10) cannot be used for the free surface (Fig. 1). A second order approximation scheme for a free surface point has already been given by Ilan et al. (1975), and will not be repeated here.

3. Synthetic Seismograms

Synthetic seismograms computed by the finite-difference scheme (2.10) for horizontally layered media are the same as those given by Ilan et al. (1975). Their scheme for layered media and the present one (2.10), when applied to layered media, differ by terms which are of second order and can be ignored. Thus for horizontally layered media, the synthetic seismograms which can be obtained by using the scheme (2.10) are those already reported by Ilan et al. (1975). The scheme (2.10) is more accurate than the previously known schemes for vertically inhomogeneous media since in (2.10) derivatives are approximated by centered differences, in contrast to what was done in previous works (e.g. Alterman and Karal, 1968; Kelly et al., 1976) where one-sided differences were also employed.

In a forthcoming work, synthetic seismograms will be presented for vertically inhomogeneous media with either continuous or discontinuous inhomogeneity, an investigation which should contribute to the understanding and interpretation of wave patterns observed on field seismograms.

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