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Extension of Matrix Methods to Structures with Slightly Irregular Stratification*

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Abstract. A perturbation matrix method is proposed for stratifications differing but slightly from the classical ones (parallel planes; coaxial circular cylinders, concentric spheres). It implies the simultaneous evaluation of the effects of a dislocation field and of a body forces field.

The theory is applied to the case of Love waves travelling in a medium consisting of a layer over a half space, the layer having a localized perturbation. The response to an isolated incident surface wave mode depends on the shape of the perturbed region, and can be obtained analytically in the frequency domain to the first order in the perturbation. Inversion to the time domain is done numerically. The perturbed phase velocity depends on the shape of the perturbed region, and can be obtained analytically in the frequency domain to the first order in the perturbation. Inversion to the time domain is done numerically. The perturbed phase velocity depends on the position of the observer, and on the sense of the incident field.

Key words: Matrix methods – Irregular stratification.

1. Introduction

Since the early determinations of local phase velocity by the method of tripartite stations, different authors have found that the measurements over tectonic regions show anomalous oscillations. In fact, it is reasonable to expect that the presence of continental margins, ridges, basins, and other geological features may produce important deviations from the case where the heterogeneity of the medium is only vertical. Alexander (1963), in a very careful study of the crust in California showed examples of this effect. Moreover, the development of global tectonics has reinforced the need to study lateral heterogeneities in order to understand the physics of plate motion and the changes in the deep structures due to the convection process.

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Several theoretical works produced lately are reported by Woodhouse (1974). They consist in the application of either the representation theorems or perturbation methods or the variational techniques to specific problems, and they require the boundary conditions of the perturbed problem to be satisfied approximately. The work done by Jobert (1975, 1976a, b) on the application of the Thomson-Hakell (later T.H.) method to a generally stratified medium, allows us to use the perturbation method in a very general context. Moreover, the procedure leads to the automatic evaluation of a system of secondary sources that replaces the geometrical perturbation.

In this paper we present rapidly the fundamentals of the method, and we illustrate the theory with an application to the case of a Love wave field travelling against a perturbed region.

2. Notation

Summation over repeated literal indices is used unless they are present in both sides of an equation. Latin indices run from 1–3, greek indices from 1–2.

$$\boldsymbol{\delta}_i = \mathrm{col}(\delta_i^1\ \delta_i^2\ \delta_i^3)$$

are the unit vectors,

$\mathbf{I} = \{\delta_{ij}\} = \{\delta_j^i\} = \{\delta^{ij}\}$ is the unit matrix.

$\mathbf{A}^T$ is transposed from $\mathbf{A}$.

$$\partial_i,\ |_i,\ |^i$$

are respectively the partial, covariant, contravariant derivatives with respect to y^i

$$\boldsymbol{\nabla} = \mathrm{col}(\partial_1\ \partial_2\ \partial_3)$$

$$\mathbf{A} = \boldsymbol{\nabla} z \cdot \boldsymbol{\delta}_3^T + \boldsymbol{\delta}_3 \cdot \boldsymbol{\nabla} z^T.$$

$\mathbf{G} = \{G_{ij}\}$ is the covariant metric tensor, with contravariant components G^{ij}

$$\Gamma_{ij}^{\kappa} = \tfrac{1}{2} G^{\kappa m} [\partial_i G_{mj} + \partial_j G_{mi} - \partial_m G_{ij}]$$

are Christoffel's symbols

$\mathbf{u}$ displacement vector with contravariant components u^k
$\mathbf{T} = \{T^{ij}\}$ stress contravariant tensor
$\boldsymbol{\tau}$ is the stress vector acting on a $y^3 = \text{const.}$ surface, $\tau^j = T^{3j}$.
ω angular frequency
ρ density, λ, μ Lamé's parameters functions of y^3 only.
$\kappa = \lambda/(\lambda + 2\mu)$ $\kappa' = \mu/(\lambda + 2\mu)$ $\mathrm{e} = \kappa + \kappa'$
$V_p = \sqrt{(\lambda + 2\mu)/\rho}$ $V_s = \sqrt{\mu/\rho}$ P, S wave velocities
$c(\omega)$ phase velocity
$c_n(\omega)$ phase velocity of the n^{th} mode.

$k=\omega/c$ wave number, $k_{P,S}=\omega/V_{P,S}$
$v^P=\sqrt{k^2-k_P^2}$ $v^S=\sqrt{k^2-k_s^2}$
$\mathbf{C}_P$ primary source
$\mathbf{C}_s$ secondary source

3. Extension of the T.H. Method to Quasi-Plane Stratifications

The method which we shall use here has been already presented in two previous papers (Jobert, 1975, 1976b). We shall follow the derivation made in the first of them.

The physical properties of the medium are supposed constant on each member S of a family of surfaces. These surfaces differ but slightly from parallel planes defining a Cartesian system of coordinates (x^i). The x^3 axis is normal to the stratification. A system of curvilinear coordinates (see Fig. 1) (y^i) is chosen such that y^3 is constant on each S and

$$y^1=x^1 \quad y^2=x^2 \quad y^3+\varepsilon z(y^i)=x^3. \tag{1}$$

Here $\varepsilon=o(1)$, $z=O(y^3)$ and is subjected to other conditions discussed later.

The metric tensor covariant components are deduced from

$$ds^2=dx^i\,dx^i=dy^\alpha\,dy^\alpha+(dy^3+\varepsilon\,dz)^2.$$

It is easy to see that we may write then

$$G_{ij}=\delta_{ij}+\varepsilon A_{ij} \tag{2}$$

with

$$\mathbf{A}=\{A_{ij}\}=\nabla z\cdot\boldsymbol{\delta}_3^T+\boldsymbol{\delta}_3\cdot\nabla z^T. \tag{3}$$

The contravariant components are

$$G^{ij}=\delta^{ij}-\varepsilon A^{ij}$$

where, to terms in ε^2, we may replace A^{ij} by A_{ij}.

From these expressions it is possible to show that the only non zero Christoffel's symbols are given by

$$\Gamma_{ij}^3=\varepsilon\,\partial_{ij}z. \tag{4}$$

3.1. Equation of Motion

The elastic force $\mathbf{e}$ is derived from the stress tensor $\mathbf{T}$ by taking its divergence

$$e^j=T^{ij}|_i=\partial_i T^{ij}+\Gamma_{im}^i\,T^{mj}+\Gamma_{im}^j\,T^{im}.$$

Using (4) we may write

$$e^j=\partial_i T^{ij}+\varepsilon[(\partial_{3m}z)\,T^{mj}+\delta_3^j(\partial_{im}z)\,T^{im}]. \tag{5}$$

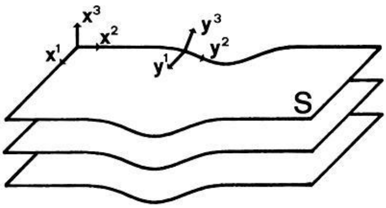


Fig. 1. Quasi-plane layered structure. The properties of the medium are constant on each surface S. The coordinates (x^1, x^2, x^3) refer to the unperturbed, plane layered, medium, and (y^1, y^2, y^3) are the local coordinates on the surfaces S

We shall sort out the components of the stress vector $\boldsymbol{\tau}$ from the other elements $T^{\alpha\beta}$, we may write (5) as:

$$\begin{aligned} e^\alpha &= \partial_3 \tau^\alpha + \partial_\beta T^{\alpha\beta} + \varepsilon b^\alpha \\ e^3 &= \partial_3 \tau^3 + \partial_\beta \tau^\beta + \varepsilon b^3, \end{aligned} \tag{6}$$

with

$$\begin{aligned} b^\alpha &= (\partial_{33} z)\,\tau^\alpha + (\partial_{3\beta} z)\, T^{\alpha\beta} \\ b^3 &= 3(\partial_{3\alpha} z)\,\tau^\alpha + 2(\partial_{33} z)\,\tau^3 + (\partial_{\alpha\beta} z)\, T^{\alpha\beta} \;. \end{aligned} \tag{7}$$

After a Fourier transformation with respect to time, the equation of motion may be written (keeping the same notation for the variables and their transform)

$$e^j + f^j = -\rho\,\omega^2 u^j \tag{8}$$

where f^j are the body force components.

3.2. Hooke's Law

The other equations are given by Hooke's law

$$T^{ij} = \lambda \operatorname{div} \mathbf{u}\, G^{ij} + \mu(u^i|^j + u^j|^i).$$

Using Christoffel's symbols we obtain

$$\begin{aligned} T^{ij} &= \lambda\, G^{ij}(\partial_k u^k + \Gamma^k_{km} u^m) + \mu[G^{im}(\partial_m u^j + \Gamma^j_{mp} u^p) + G^{jm}(\partial_m u^i + \Gamma^i_{mp} u^p)] \\ &= \Lambda^{ij} + \varepsilon\, \Pi^{ij} \end{aligned} \tag{9}$$

with

$$\Lambda^{ij}(\mathbf{u}) = \lambda\, \delta^{ij}\, \partial_k u^k + \mu(\partial_i u^j + \partial_j u^i) \tag{10}$$

$$\begin{aligned} \Pi^{ij}(\mathbf{u}) &= \lambda[\delta^{ij}(\mathbf{u}^T \cdot \partial_3 \nabla z) - A^{ij}\, \partial_k u^k] \\ &+ \mu[\mathbf{u}^T(\partial_i \nabla z \cdot \delta^j_3 + \partial_j \nabla z \cdot \delta^i_3) - A^{im}\, \partial_m u^j - A^{jm}\, \partial_m u^i] \end{aligned} \tag{11}$$

(not tensorial expressions).

We may write in particular

$$\Lambda^{33} = \lambda\, \partial_k u^k + 2\mu\, \partial_3 u^3 = (\lambda + 2\mu)\, \partial_k u^k - 2\mu\, \partial_\gamma u^\gamma$$

so that

$$\Lambda^{\alpha\beta}=\lambda\,\delta^{\alpha\beta}[\Lambda^{33}+2\mu\,\partial_\gamma u^\gamma]/(\lambda+2\mu)+\mu(\partial_\alpha u^\beta+\partial_\beta u^\alpha)$$
$$=\kappa\,\delta^{\alpha\beta}\Lambda^{33}+2\mu\,\kappa\,\delta^{\alpha\beta}\,\partial_\gamma u^\gamma+\mu(\partial_\alpha u^\beta+\partial_\beta u^\alpha). \tag{12}$$

To eliminate $T^{\alpha\beta}$ from (6.1) it is possible to apply the differentiation ∂_β since κ and μ do not depend on y^β. We may also write

$$\tau^j=\Lambda^{3j}+\varepsilon\,\Pi^{3j}. \tag{13}$$

3.3. Primary and Secondary Fields

We shall decompose the total field into a primary field (valid for $\varepsilon=0$) and a secondary field due to the perturbation of the stratification:

$$\begin{aligned}\mathbf{u}&=\mathbf{u}_p+\varepsilon\,\mathbf{u}_s\\ \boldsymbol{\tau}&=\boldsymbol{\tau}_p+\varepsilon\,\boldsymbol{\tau}_s.\end{aligned} \tag{14}$$

Equation (13) becomes

$$\tau_p^j+\varepsilon\,\tau_s^j=\Lambda^{3j}(\mathbf{u}_p+\varepsilon\,\mathbf{u}_s)+\varepsilon\,\Pi^{3j}(\mathbf{u}_P)$$

or

$$\tau_p^j=\Lambda^{3j}(\mathbf{u}_p) \tag{15}$$

$$\tau_s^j=\Lambda^{3j}(\mathbf{u}_s)+\Pi^{3j}(\mathbf{u}_p). \tag{16}$$

Similarly the system (6) (8) becomes $\partial_3(\tau_p^3+\varepsilon\,\tau_s^3)+\partial_\beta(\tau_p^\beta+\varepsilon\,\tau_s^\beta)+\varepsilon\,b_p^3+f^3=-\rho\,\omega^2(u_p^3+\varepsilon\,u_s^3)$

$$\partial_3(\tau_p^\alpha+\varepsilon\,\tau_s^\alpha)+\partial_\beta(\Lambda^{\alpha\beta}(\mathbf{u}_p+\varepsilon\,\mathbf{u}_s)+\varepsilon\,\Pi^{\alpha\beta}(\mathbf{u}_p))+\varepsilon\,b_p^\alpha+f^\alpha=-\rho\,\omega^2(u_p^\alpha+\varepsilon\,u_s^\alpha)$$

or

$$\begin{aligned}&\partial_3\tau_p^3+\partial_\beta\tau_p^3+f^3=-\rho\,\omega^2\,u_p^3\\ &\partial_3\tau_p^\alpha+\partial_\beta\Lambda^{\alpha\beta}(\mathbf{u}_p)+f^\alpha=-\rho\,\omega^2\,u_p^\alpha\end{aligned} \tag{17}$$

$$\begin{aligned}&\partial_3\tau_s^3+\partial_\beta\tau_s^\beta+b_p^3=-\rho\,\omega^2\,u_s^3\\ &\partial_3\tau_s^\alpha+\partial_\beta\Lambda^{\alpha\beta}(\mathbf{u}_s)+\partial_\beta\Pi^{\alpha\beta}(\mathbf{u}_p)+b_p^\alpha=-\rho\,\omega^2\,u_s^\alpha\end{aligned} \tag{18}$$

The system (15) (17) corresponds to the non-perturbed problem. Introducing the T.H. vector $\mathbf{X}=\mathrm{col}(\mathbf{u}\,\boldsymbol{\tau})$ we obtain the first order differential equation

$$\partial_3\mathbf{X}_p=\mathbf{M}\mathbf{X}_p-\mathbf{C}_p \tag{19}$$

where

$$\mathbf{M}=\begin{pmatrix}\mathbf{M}_1 & \mathbf{M}_2\\ \mathbf{M}_3 & \mathbf{M}_1^T\end{pmatrix}$$

$$\mathbf{M}_1=-(\boldsymbol{\delta}_1\boldsymbol{\delta}_3^T+\boldsymbol{\delta}_3\boldsymbol{\delta}_1^T)\,\partial_1-(\boldsymbol{\delta}_2\boldsymbol{\delta}_3^T+\boldsymbol{\delta}_3\boldsymbol{\delta}_2^T)\,\partial_2$$

$$\mathbf{M}_2=\mu^{-1}\,\mathrm{diag}\,(1\;1\;\kappa) \tag{20}$$

$$\mathbf{M}_3 = -\mu[-\boldsymbol{\delta}_1 \boldsymbol{\delta}_1^T (k_s^2 + 4e\,\partial_{11} + \partial_{22}) - \boldsymbol{\delta}_2 \boldsymbol{\delta}_2^T (k_s^2 + 4e\,\partial_{22} + \partial_{11}) + \boldsymbol{\delta}_3 \boldsymbol{\delta}_3^T k_s^2 + (2\kappa+1)(\boldsymbol{\delta}_1 \boldsymbol{\delta}_2^T + \boldsymbol{\delta}_2 \boldsymbol{\delta}_1^T)\,\partial_{12}]$$

$$\mathbf{C}_p = \mathrm{Col}(0\,0\,0\,f^1 f^2 f^3). \tag{21}$$

The system (16) (18) corresponds to the secondary field induced by the interaction of the primary field with the perturbation of the stratification.

We may write it as

$$\partial_3 \mathbf{X}_s = \mathbf{M}\mathbf{X}_s - \mathbf{C}_s \tag{22}$$

where $\mathbf{C}_s$ is given by

$$\mu \mathbf{C}_S = \mathrm{Col}(\Pi^{13}(\mathbf{u}_p), \Pi^{23}(\mathbf{u}_p), \kappa' \Pi^{33}(\mathbf{u}_p), \mu(b^1 + \partial_\alpha \Pi^{1\alpha}), \mu(b^2 + \partial_\alpha \Pi^{2\alpha}), \mu b^3). \tag{23}$$

The first 3 components correspond to a dislocation field, the 3 others to a body force field.

4. The Passage of Love Waves through a Region of Irregular Layering

We are going to show an application of the above theory. For the sake of simplicity we consider a Love wave travelling against a localized irregularity (Fig. 2). The unperturbed medium consists of a layer of thickness H over a half space. The index 1 refers to the layer, the index 2 to the half space. The shape of the irregularity is given by a function $z = z(x^1, x^3)$. We select as primary field an SH field with particle motion along the x^2 axis, and propagating along the x^1 axis in the positive sense. We have, thus, a plane problem, and we may see that only the second and fifth terms in (23) are different from zero. We keep these terms and write (23) in the form:

$$\mathbf{C}_S = \begin{pmatrix} \Pi^{32}(\mathbf{u})/\mu \\ b^2 + \partial_\alpha \Pi^{2\alpha} \end{pmatrix}, \tag{24}$$

where

$$\Pi^{32} = -\mu A_{3m} \partial_m u^2 \qquad \text{(from (11))},$$

$$\partial_1 \Pi^{12} = -\mu \partial_1 (A_{1m} \partial_m u^2) \qquad \text{(from (11))},$$

$$b^2 = (\partial_{33} z)\tau^2 + (\partial_{3\beta} z) T^{2\beta} = \mu[\partial_{33} z \cdot \partial_3 u^2 + \partial_{31} z \cdot \partial_1 u^2] \qquad \text{(from (7))},$$

$$A_{33} = 2\partial_3 z; \qquad A_{31} = \partial_1 z. \qquad \text{(from (3))}.$$

If, moreover, the perturbation is only a function of x^1, we have $b^2 = 0$, $A_{33} = 0$, and then (24) takes the form:

$$\mathbf{C}_S = -\begin{pmatrix} \partial_1 z \cdot \partial_1 u^2 \\ \partial_1 (\partial_1 z \cdot T^{23}) \end{pmatrix}, \tag{25}$$

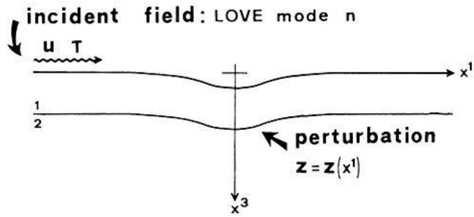


Fig. 2. Geometry of the example: Love wave propagation in a medium consisting of a layer over a half-space. The perturbation $z=z(x^1, x^3)$ is located near the origin of coordinates. The figure shows the case of variation with respect to x^1 only

that represents a continuous distribution of sources, generated by the primary field u^2, T^{23}, over the region where the values of $\partial_1 z$ and $\partial_{11} z$ are different from zero.

From here on we drop the indices in u^2, T^{23}, since they are the only components that are being used.

4.1. The Primary SH Field

We have chosen as primary field an incident Love wave (mode n) travelling in the positive sense along the x^1 axis. Then the field at the surface, in the frequency domain, is written as:

$$\mathbf{X}(0)=\begin{pmatrix} u \\ T \end{pmatrix}=\begin{pmatrix} S(\omega)\, e^{ik_n x^1} \\ 0 \end{pmatrix} \tag{26}$$

where $S(\omega)$ is the spectrum of the displacement at $x^1=0$, and $k_n=k_n(\omega)$ is the wave number. A frequency dependence of the form $\exp(-i\omega t)$ is assumed.

At depth x^3 the field is given by the well known formulae (see for instance, Woodhouse (1974), though the notation is somewhat different):

$$\begin{aligned} \mathbf{X}(x^3) &= \begin{pmatrix} u_n \\ T_n \end{pmatrix} e^{ik_n x^1} = S(\omega) \begin{pmatrix} \cosh \nu^s_{1n} x^3 \\ \mu_1 \nu^s_{1n} \sinh \nu^s_{1n} x^3 \end{pmatrix} e^{ik_n x^1}; \quad (x^3 \leqq H) \\ &= S(\omega) \begin{pmatrix} 1 \\ -\mu_2 \nu^s_{2n} \end{pmatrix} \cosh \nu^s_{1n} H \cdot e^{ik_n x^1 - \nu^s_{2n}(x^3-H)}; \quad (x^3 \geqq H) \end{aligned} \tag{27}$$

where $\nu^s_{in}=\sqrt{k_n^2-k_{\beta_i}^2}$, $i=1, 2$.

The wave number $k_n(\omega)$ satisfies the period equation:

$$\mu_2 \nu^s_{2n} \cdot \cosh \nu^s_{1n} H + \mu_1 \nu^s_{1n} \cdot \sinh \nu^s_{1n} H = 0\,. \tag{28}$$

4.2. The Fourier Transform of the Secondary Field

We Fourier-transform the problem with respect to x^1. The transform of the secondary source (25) is

$$\bar{\mathbf{C}}_s = -\int_{-\infty}^{+\infty} e^{-ik_n x^1} \begin{pmatrix} \partial_1 z \cdot \partial_1 u \\ \partial_1(\partial_1 z \cdot T) \end{pmatrix} dx^1 = (k-k_n)\, \bar{z}(k-k_n) \begin{pmatrix} k_n u_n \\ k\, T_n \end{pmatrix} \tag{29}$$

where we have used the fact that the spectrum of the primary field (27) is proportional to $\delta(k-k_n)$. In fact:

$$\overline{\partial_1\, z\, \partial_1 u} = [i\,k\,\bar{z}(k)] * [i\,k_n\,\bar{u}]$$

$$= \frac{1}{2\pi}[i\,k\,\bar{z}(k)] * [i\,k_n\, 2\pi\, u_n\, \delta(k-k_n)] = -(k-k_n)\,\bar{z}(k-k_n)\,k_n\,u_n.$$

The star indicates convolution.

4.3. The Secondary SH Field

We assume that the field produced by the secondary sources at the surface is:

$$\bar{\mathbf{X}}_S(0) = \begin{pmatrix} \bar{u}_S \\ 0 \end{pmatrix} \tag{30}$$

where $\bar{u}_s$ is to be determined.

We may continue the field at depth x^3 by the use of propagators (Gilbert and Backus, 1966):

$$\bar{\mathbf{X}}_S(x^3) = \mathbf{P}(x^3, 0)\,\bar{\mathbf{X}}_S(0) - \int_0^{x^3} \mathbf{P}(x^3, \zeta)\,\bar{\mathbf{C}}_S(\zeta)\,d\zeta \tag{31}$$

where the propagator for the case of Love waves is (Woodhouse, 1974)

$$\mathbf{P}(\zeta_2, \zeta_1) = \begin{pmatrix} \cosh q & \sinh q/\mu\, v^s \\ \mu\, v^s \sinh q & \cosh q \end{pmatrix}, \quad q = v^s(\zeta_2 - \zeta_1) \tag{32}$$

and permits the passage from $\bar{\mathbf{X}}(\zeta_1)$ to $\bar{\mathbf{X}}(\zeta_2)$.

The radiation condition in the half-space implies that the coefficient of $\exp(v_2^s\, x^3)$ must vanish. We must then have (Appendix I):

$$\begin{aligned} &\bar{u}_S \cdot \Delta_L(k) - (k-k_n)\,\bar{z}(k-k_n)\,S(\omega) \\ &\cdot [\alpha_1\, S^- + \beta_1\, S^+ + \mu_1\, v_1^s(\alpha_1\, C^- - \beta_1\, C^+)/\mu_2\, v_2^s \\ &+ 2\beta_2 \cosh v_{1n}^s\, H \cdot / v_2^+] = 0 \end{aligned} \tag{33}$$

where

$$\Delta_L(k) = \cosh v_1^s\, H + \sinh v_1^s\, H \cdot (\mu_1\, v_1^s/\mu_2\, v_2^s)$$

is the Love determinant;

$$\alpha_1 = k_n + k(v_{1n}^s/v_1^s); \quad \beta_1 = k_n - k(v_{in}^s/v_i^s) \quad (i = 1, 2)$$

$$S^{\pm} = (\sinh v_{1n}^s\, H \pm \sinh v_1^s\, H)/(v_{1n}^s \pm v_1^s)$$

$$C^{\pm} = (\cosh v_{1n}^s\, H - \cosh v_1^s\, H)/(v_{1n}^s \pm v_1^s)$$

$$v_2^+ = v_{2n}^s + v_2^s.$$

We obtain, thus, the displacement at the surface for the secondary field in the form:

$$\bar{u}_S = S(\omega)(k-k_n)\,\bar{z}(k-k_n)\,F(k,k_n)/\Delta_L(k)$$

where

$$F(k,k_n) = \alpha_1 S^- + \beta_1 S^+ + \mu_1 v_1^s(\alpha_1 C^- - \beta_1 C^+)/\mu_2 v_2^s$$
$$+ 2\beta_2 \cosh v_{1n}^s H/v_2^+. \tag{34}$$

4.4. Evaluation by Residues

Inversion to the spatial coordinates is done by the Fourier integral:

$$u_s(x^1, 0; \omega) = \frac{S(\omega)}{2\pi} \int_{-\infty}^{+\infty} (k-k_n)\,\bar{z}(k-k_n)\frac{F(k,k_n)}{\Delta_L(k)} e^{ikx^1}\,dk. \tag{35}$$

The integrand has poles at the zeros of $\Delta_L(k)$, and branch points at k_{β_1}, k_{β_2}. If no additional singularities are introduced by the function $\bar{z}(k)$, it is possible to evaluate u_S by the residue theorem. If $x^1 > 0$, it is necessary to close the contour on the half-plane $\operatorname{Im} k > 0$ to obtain the transmitted field:

$$u_s^+(x^1, 0; \omega) = i\,S(\omega) \sum_{m \neq n} (k_m - k_n)\,\bar{z}(k_m - k_n)\frac{F(k_m, k_n)}{\Delta_L'(k_m)} e^{ik_m x^1}$$
$$+\text{Body waves}. \tag{36}$$

In the case $x^1 < 0$, the contour is closed on the lower half k-plane to evaluate the reflected field:

$$u_s^-(x^1, 0; \omega) = i\,S(\omega) \sum_m (k_m + k_n)\,\bar{z}^*(k_m + k_n)\frac{F(-k_m, k_n)}{\Delta_L'(-k_m)} e^{-ik_m x^1}$$
$$+\text{Body waves}. \tag{37}$$

4.5. Numerical Results

The first order perturbation may be evaluated exactly in the frequency domain by means of expressions (36), (37). The part corresponding to body waves is obtained in the form of branch line integrals. To test the method, we select a long period incident Love fundamental mode, in such a way that the coupling with body waves would be negligible.

In Figure 3 we show the amplitude of the incident field: $|S(\omega)|$, the fundamental mode, and the amplitude of the response function: $(k_m - k_n)\,\bar{z}(k_m - k_n) \cdot F(k_m, k_n)/\Delta_L'(k_m)$, for each higher mode, up to $m = 5$, for the case of a bell-shaped perturbation of exponential type. We have chosen a layer thickness $H = 30$ km, and a perturbation width of the order of 30 km. The velocity of shear waves in

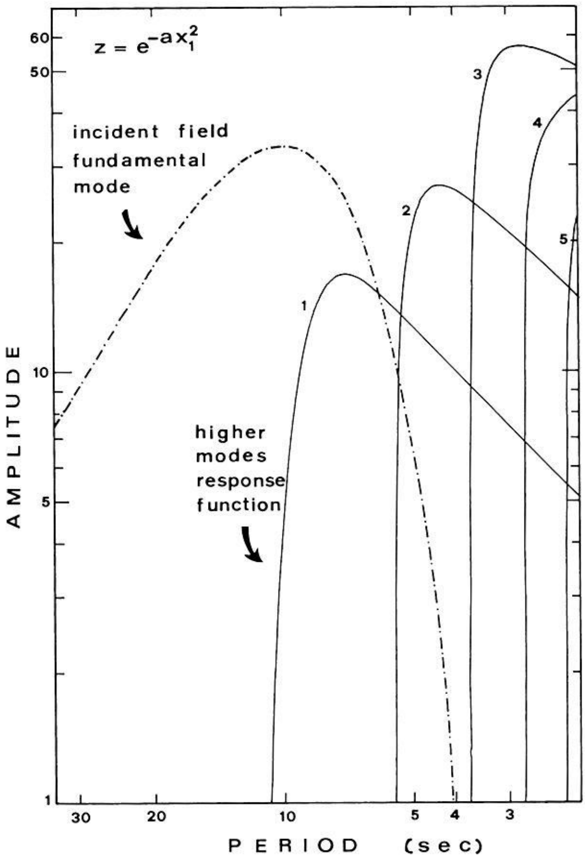


Fig. 3. Case of an exponential perturbation: $z=\exp(-a(x^1)^2)$. The dashed line shows the amplitude spectrum $S(\omega)$ of the incident fundamental Love mode, peak amplitude is about 10 s. The solid lines show the product $|(k_m-k_0)\ Z(k_m-k_0)\ F(k_m,k_0)/\Delta'_L(k_m)|$ for $m=1$ to 5. The parameters are: $H=30$ km, $a=0.003$, $V_{S1}=3.464$ km/s, $V_{S2}/V_{S1}=1.297$, $\mu_2/\mu_1=2.159$, $z_{max}=1$ km

the layer is $V_{s1}=3.464$ km/s; $V_{S2}/V_{S1}=1.297$; $\mu_2/\mu_1=2.159$; the maximum deviation from plane layer geometry is $z_{max}=1$ km. From the figure, it is clear that the product of the incident field and the response function will be significative only for the first higher mode. The contribution of the other higher modes (and of body waves) will become very small due to the weak coupling with the chosen incident field.

Since the exponential function decays very sharply for large x^1, the perturbation is very concentrated in space. To see the effect of a larger spreading, we have chosen an algebraic function $z(x^1)=a^2/(a^2+(x^1)^2)$. The results are shown in Figure 4 for the higher modes $m=1,2$. The elastic properties and the geometry of the above example are employed. It may be seen that the coupling becomes stronger, since the response amplitudes are larger than in the previous case.

Figure 5 shows the inversion of the displacement to time for the incident field, and for the first higher mode at $x^1=0$. The inversion has been performed

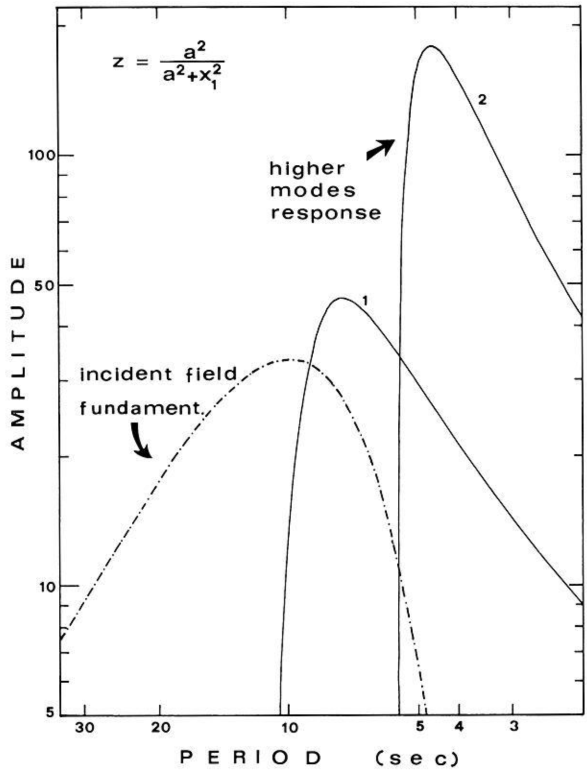


Fig. 4. Case of an algebraic perturbation: $z = a^2/(a^2 + (x^1)^2)$. The parameters are the same as in Figure 3, except $a = 15$. A stronger coupling may be observed in relation to Figure 3

numerically by means of the Fast Fourier Transform, for the displacements of the transmitted field (36). The maximum amplitude ratio is of the order of 5% for the first mode, and it becomes several orders of magnitude smaller for the other modes.

The knowledge of the analytic form of the perturbation in the frequency domain allows us to compute the perturbation to the phase velocity analytically without going to the time domain. In fact, if we take only one of the terms in (36), the one which corresponds to mode m, we may write the local displacement in the form:

$$\bar{u} = A_n \exp(i(\varphi_n + k_n x^1)) + B_m \exp(i(\varphi_m + k_m x^1)) \tag{38}$$

where the amplitude B_m is much smaller than A_n, the amplitude of the incident field; φ_n and φ_m are the phases of the incident and secondary fields at $x^1 = 0$, respectively. The perturbed phase velocity may be shown to be equal to (Appendix II):

$$c(x^1, \omega) \approx c(\omega)/[1 + \alpha(c_n/c_m - 1)\cos \Delta\varphi] \tag{39}$$

where $\alpha = B_m/A_n$, $\Delta\varphi = \varphi_m - \varphi_n + (k_m - k_n)x^1$ and $c(x^1, \omega)$ depends upon the position of the observer.

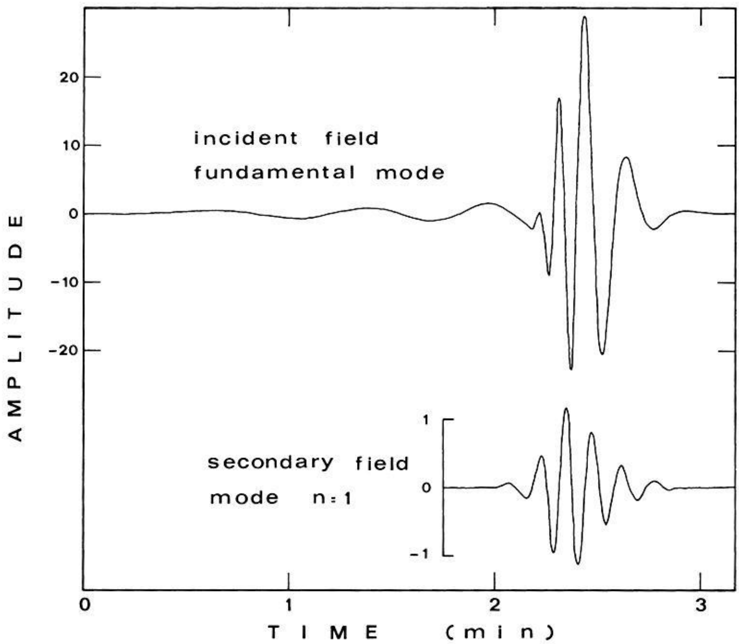
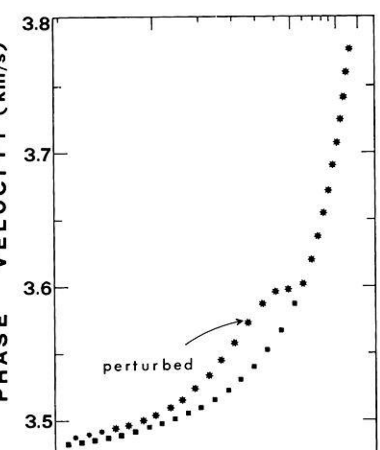


Fig. 5. Theoretical seismogram to show the relative importance of the secondary to the incident field for the case of Figure 4. The maximum amplitude of the first mode is about 5% of the maximum amplitude of the exciting field. The amplitude of other higher modes and of body waves is orders of magnitude smaller

6



7

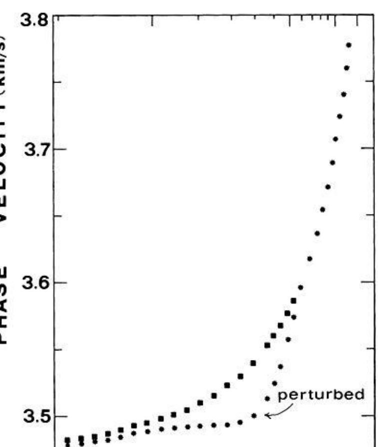


Fig. 6. Phase velocity as observed over a basin with the shape indicated in Figure 4. The observation point is 50 km away from the axis $x=0$, and on the side of the transmitted field. The perturbation disappears at the cut-off frequency

Fig. 7. Same as in Figure 6, but the perturbation is now due to a ridge of the same dimensions

A calculation of the perturbation has been made for the case of a basin (Fig. 6), and for a ridge (Fig. 7). In both cases it is possible to observe the phase velocity oscillation that has been reported by several authors (see for example S. Alexander). It is clear that the mode interference is the cause for such oscillations in much the same way that has been indicated by Thatcher and Brune (1969), and by Boore (1969), for other mode interference phenomena. The perturbation to the normal phase velocity curves disappears at the cut-off frequency of the first higher mode, about 11 s in this case. When x^1 is very large, that is to say, at great distance from the perturbed region, the wave trains separate in the seismogram due to the fact that the velocities are different for different modes, and the phase velocity returns to the normal values.

Another related question refers to the fact that phase velocity measurements at a point close to an irregularity give different values when we change the sense of propagation of the incident field. In one case we measure transmitted waves and we should employ formula (36), in the other case a reflected field is measured and formula (37) should be taken into account. Since the transmitted field doesn't contain a perturbation for the incident mode, $m=n$, and the reflected field does contain such a perturbation, different results should be expected. Our results disagree with those of Herrera (1964) and we believe this due to his incomplete consideration of the effect of the perturbed boundaries.

5. Conclusions

The Thomson-Haskell matrix method may be generalized to media with a stratification that differs slightly from the plane parallel case. The effect of the slope and curvature of the surfaces on the incident field may be represented as a continuous distribution of secondary sources located at the region where the plane parallel geometry is perturbed. The perturbation field is then obtained by the application of the same operator that generates the solution of the unperturbed problem. In other words, the secondary field is expressed as a superposition of eigenvectors of the unperturbed problem.

In the case of Love waves (plane problem), the secondary sources excite transmitted and reflected surface waves and body waves. The transmitted field doesn't modify the incident field to the first order, in the case where the latter consists of a single surface wave mode. The expressions for the first order perturbation may be calculated analytically in the frequency domain. The contribution to each mode depends upon three factors: the spectrum of the incident field, the spectrum of the horizontal variation of the perturbation, and a vertical interference function.

The perturbation of the phase velocity may be calculated analytically also. The known oscillations of the phase velocity are found in this way. The perturbed phase velocity is associated with the interference of the incident field and the excited secondary modes. The measured phase velocity will be dependent on the position of the observer, and on the sense of propagation of the incident field.

Appendix I: Generation of the SH Secondary Field

The Thomson-Haskell vector at the free surface is assumed to be:

$$\bar{\mathbf{X}}(0)=\begin{pmatrix}\bar{u}_S\\ 0\end{pmatrix}.$$

Thence, the Love wave propagator gives at depth H [see (27) and (29)]:

$$\bar{\mathbf{X}}(H)=\mathbf{P}(H,0)\,\bar{\mathbf{X}}(0)-\int_0^H \mathbf{P}(H,\zeta)\,\bar{\mathbf{C}}_S(\zeta)\,d\zeta$$

$$=\begin{bmatrix}\cosh q_1 & \sinh q_1/\mu_1 v_1^s\\ \mu_1 v_1^s \sinh q_1 & \cosh q_1\end{bmatrix}\begin{pmatrix}\bar{u}_S\\ 0\end{pmatrix}$$

$$+K\int_0^H \begin{bmatrix}\cosh q_1^* & \sinh q_1^*/\mu_1 v_1^s\\ \mu_1 v_1^s \sinh q_1^* & \cosh q_1^*\end{bmatrix}\begin{bmatrix}k_n \cosh v_{1n}^s \zeta\\ k\,\mu_1 v_{1n}^s \sinh v_{1n}^s \zeta\end{bmatrix} d\zeta$$

where

$$q_1=v_1^s H,\qquad q_1^*=v_1^s(H-\zeta),\qquad K=-S(\omega)(k-k_n)\,\bar{z}(k-k_n).$$

After the integration, we obtain:

$$\bar{\mathbf{X}}(H)=\bar{u}_s\begin{bmatrix}\cosh q_1\\ \mu_1 v_1^s \sinh q_1\end{bmatrix}+K\begin{bmatrix}\alpha_1 S^-+\beta_1 S^+\\ \mu_1 v_1^s(\alpha_1 C^- -\beta_1 C^+)\end{bmatrix}=\begin{pmatrix}r\\ s\end{pmatrix}$$

where:

$$S^\pm=[\sinh v_{1n}^s H \pm \sinh v_1^s H]/(v_{1n}^s \pm v_1^s),\qquad k_n=\alpha_1+\beta_1$$

$$C^\pm=[\cosh v_{1n}^s H - \cosh v_1^s H]/(v_{1n}^s \pm v_1^s),\qquad k\,v_{1n}^s/v_1^s=\alpha_1-\beta_1.$$

At depth $x^3>H$ we continue the field in the form [see (27) and (29)].

$$\bar{\mathbf{X}}(x^3)=\mathbf{P}(x^3,H)\,\bar{\mathbf{X}}(H)-\int_H^{x^3} \mathbf{P}(x^3,\zeta)\,\bar{\mathbf{C}}_S(\zeta)\,d\zeta$$

$$=\begin{bmatrix}\cosh q_2 & \sinh q_2/\mu_2 v_2^s\\ \mu_2 v_2^s \sinh q_2 & \cosh q_2\end{bmatrix}\begin{bmatrix}r\\ s\end{bmatrix}$$

$$+K\cosh v_{1n}^s H\, e^{v_{2n}^s H}\int_H^{x^3}\begin{bmatrix}\cosh q_2^* & \sinh q_2^*/\mu_2 v_2^s\\ \mu_2 v_2^s \sinh q_2^* & \cosh q_2^*\end{bmatrix}$$

$$\cdot\begin{bmatrix}k_n\\ -k\,\mu_2 v_{2n}^s\end{bmatrix}\exp(-v_{2n}^s\zeta)\,d\zeta$$

where

$$q_2=v_2^s(x^3-H),\qquad q_2^*=v_2^s(x^3-\zeta).$$

After performing the integration, the coefficient of $\exp(\nu_2^s x^3)$ should be null, to satisfy the radiation condition. Therefore:

$$(1/2)\begin{bmatrix} r+s/\mu_2 \nu_2^s \\ s+r\mu_2 \nu_2^s \end{bmatrix} + K\cosh \nu_{1n}^s H \begin{bmatrix} \beta_2/\nu_2^+ \\ \mu_2 \nu_2^s \beta_2/\nu_2^+ \end{bmatrix} = 0$$

where $\beta_2 = (1/2)(k_n - k\,\nu_{2n}^s/\nu_2^s)$, $\nu_2^+ = \nu_{2n}^s + \nu_2^s$.

The first and second rows are proportional, hence it is enough to write:

$$r + s/\mu_2 \nu_2^s + 2K\beta_2 \cosh \nu_{1n}^s H/\nu_2^+ = 0.$$

Replacing r and s by their expressions we find:

$$\begin{aligned} &\bar{u}_S[\cosh q_1 + \mu_1 \nu_1^s \sinh q_1/\mu_2/\nu_2^s] \\ &+K[\alpha_1 S^- + \beta_1 S^+ + \mu_1 \nu_1^s(\alpha_1 C^- - \beta_1 C^+)/\mu_2 \nu_2^s] \\ &+2K\beta_2 \cosh \nu_{1n}^s H/\nu_2^+ = 0 \end{aligned}$$

which leads to (33).

Appendix II

The perturbation of the phase velocity of mode n, when another mode m is present, and interferes, making the separation difficult, may be obtained in the frequency domain, in a way similar to that given by Thatcher and Brune (1969) and Boore (1969). Let the total field be:

$$u = A_n \exp[i(\varphi_n + k_n x^1)] + B_m \exp[i(\varphi_m + k_m x^1)]$$

as in (38). The observer would try to obtain phase velocity by considering the spectrum of the total field in the form:

$$u = A\exp(i\varphi)$$

and computing the wave number by means of:

$$k = d\varphi/dx^1$$

Assuming that $B_m \ll A_m$ we may write to the first order:

$$\varphi = \varphi_n + k_n x^1 + \alpha \sin \Delta\varphi$$

where

$$\alpha = \beta_m/A_n, \qquad \Delta\varphi = \varphi_m - \varphi_n + (k_m - k_n)x^1$$

Then

$$k = d\varphi/dx' = k_n + \alpha(k_m - k_n)\cos \Delta\varphi.$$

Hence

$$c = c_n / [1 + \alpha (c_n / c_m - 1) \cos \Delta \varphi].$$

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