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# Modal Approach to Wave Propagation in Layered Media with Lateral Inhomogeneities

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**Abstract.** A mathematical treatment of wave propagation in a layered elastic structure has been presented. Modal methods have been used to solve the unperturbed (plane layered) problem. Lateral inhomogeneities have been taken into account using a perturbation scheme around the unperturbed state.

**Key words:** Wave propagation in layered media – Lateral inhomogeneities

## 1. Introduction

There has been great interest in geophysics in the problem of wave propagation in layered media. For the purposes of this paper only works involving modal techniques are reviewed. The majority of such works are concerned with wave propagation in plane layered media having isotropic symmetry. Here the reader is referred to the paper by Cochran et al. (1970) which gives a good account of much of the important research in this field. For wave propagation in isotropically symmetric and elastic structures with lateral inhomogeneities reference should be made to Kennett (1972).

The aim of this paper is to study wave propagation in layered and elastic media. No reference will be made to symmetry properties of materials. Consequently, further considerations are necessary when dealing with specific situations. Lateral inhomogeneities are treated using a perturbation scheme. In order to make the notation more compact mathematically, coordinate free methods are used.

## 2. Transformation of the Equations of Motion into an Operator Form

$$\rho \ddot{u} = \mathrm{div}_{x'} T + f \tag{1}$$

and

$$T = C[\mathrm{grad}_{x'} u] \tag{2}$$

express the motion of elastic and inhomogeneous media. In (1) and (2)

$\rho$ is the density field of the material,
$u$ is the displacement vector field,
$\ddot{u}$ is the acceleration vector field,
$x'$ is a point in the 3 space,
$T$ is the stress tensor field,
$f$ is the volume density of the source vector field (assumed to be of bounded support),
$C[A]$ is the fourth order tensor field obtained from the tensor of elasticity $C_0[A]$ using the equation

$$C[A] = C_0[SA].$$

The operator $S$ in turn extracts the symmetric part from the tensor quantity $A$.

Prior to the actual decomposition each point $x'$ will be expressed as

$$x' = (n, x). \tag{3}$$

The first term $n$ is chosen to parametrise the three space in the direction of $n^0$ the unit normal to the surface $S_n$:

$$n(x') = \text{const}. \tag{4}$$

The other member $x$ in turn parametrises the 3 space tangentially to the same surface.

The differential operators $\text{grad}_{x'}$ and $\text{div}_{x'}$ decompose under (3) as follows

$$\text{grad}_{x'}[0] = \frac{d[0]}{dn} \otimes n^0 + \text{grad}_x[0] \tag{5}$$

and

$$\text{div}_{x'}[0] = \frac{d[0]}{dn} \cdot n^0 + \text{div}_x[0]. \tag{6}$$

In (5) and (6):

$\frac{d}{dn}$ is the scalar derivate with respect to $n$, and

$\otimes$ is the notation used for a tensor product.

In case the field quantity [0] is a tensor (linear transformation) the expression $\frac{d[0]}{dn} \cdot n^0$ in (6) will be used without dot.

After these preliminary considerations the substitution of (5) into Equation (2) gives

$$T = C\left[\frac{du}{dn} \otimes n^0\right] + C[\text{grad}_x u].$$

Further, the introduction of the stress vector field

$$t = Tn^0$$

(acting on $S_n$) makes it possible to write the relation

$$t = C\left[\frac{du}{dn}\otimes n^0\right] n^0 + C[\mathrm{grad}_x u]\, n^0. \tag{7}$$

In order to solve (7) for $\frac{du}{dn}$ the auxiliary tensor field $Z$ is introduced by

$$Z\frac{du}{dn} = C\left[\frac{du}{dn}\otimes n^0\right] n^0.$$

This relation together with the positivity of $Z$, a consequence of the properties of the strain energy, are used in (7) to yield the result

$$\frac{du}{dn} = -Z^{-1}\{C[\mathrm{grad}_x u]\, n^0\} + Z^{-1}\, t \tag{8}$$

as the decomposed form of Equation (2).

Preliminary to the decomposition of Equation (1) it is found by (6) that

$$\mathrm{div}_{x'}\, T = \frac{dT}{dn}\, n^0 + \mathrm{div}_x\, T.$$

Under the assumption $\frac{dn^0}{dn} = 0$, the surfaces $\{S_n\}$ form a parallel family, one finds that

$$\frac{dT}{dn}\, n^0 = \frac{dt}{dn}.$$

Therefore,

$$\mathrm{div}_{x'}\, T = \frac{dt}{dn} + \mathrm{div}_x\, T. \tag{9}$$

A manipulation on Equation (1) in connection with Equations (2), (8) and (9) reveals the result

$$\begin{aligned}\frac{dt}{dn} = {} & \rho\ddot{u} + \mathrm{div}_x\{C[Z^{-1}(C[\mathrm{grad}_x u]\, n^0)\otimes n^0]\}\\ & - \mathrm{div}_x\{C[\mathrm{grad}_x u]\} - \mathrm{div}_x\{C[Z^{-1}\, t\otimes n^0]\} - f\end{aligned} \tag{10}$$

as the form of (1) decomposed under (3).

For further treatment Equations (8) and (11) are presented as the system

$$\frac{du}{dn} = A_{11}\, u + A_{12}\, t \tag{11}$$

and

$$\frac{dt}{dn} = A_{21}\, u + A_{22} - f \tag{12}$$

of vector valued, ordinary differential equations acting in certain vector space $V$. $A_{11}, A_{12}, A_{21}$ and $A_{22}$ are linear operators in the same space and are defined by the relations

$$\begin{aligned} A_{11}u &= -Z^{-1}\{C[\mathrm{grad}_x u]\, n^0\} \\ A_{12}t &= Z^{-1}t \\ A_{21}u &= \rho\ddot{u} + \mathrm{div}_x\{C[Z^{-1}(C[\mathrm{grad}_x u]\, n^0)\otimes n^0]\} \\ &\quad -\mathrm{div}_x\{C[\mathrm{grad}_x u]\} \\ A_{22}t &= -\mathrm{div}_x\{C[Z^{-1}t\otimes n^0]\}. \end{aligned}$$

For the purposes of this paper it appears useful to study the single equation

$$\frac{ds}{dn} = As + g \tag{13}$$

instead of the system (11)–(12). The notations

$$\begin{aligned} s &= (u, t) \\ As &= (A_{11}u + A_{12}t, A_{21}u + A_{22}t) \\ g &= (0, -f) \end{aligned}$$

are used in (13). The differential Equation (13) is seen to be defined in the product space $V \times V$.

Representation (13) appears to have many merits over the Equations (1) and (2) in studying the wave propagation in layered structures. For a treatment somewhat similar to the one presented in this paper the reader is referred to Woodhouse (1974).

The operator $A$ is a field quantity in $(n, x)$. $A$ is represented in the form of the sum

$$A(n, x) = A_0(n) + A'(n, x)$$

where $A_0$ is the unperturbed part, takes constant values on each of the surfaces $\{S_n\}$. The other member $A'$ is called the perturbed part of $A$. It presents the deviations, lateral perturbations,

$$A'(n, x) = A(n, x) - A_0(n)$$

of $A$ from the nicely behaving part $A_0$. In order the perturbed part to be a small quantity it is important to fit the surfaces $\{Sn\}$ as closely as possible with the actual layering of the physical materials.

In connection with Equation (13) the usual boundary conditions:

1. The stress vector $t$ vanishes at the surface $n=0$,
2. The displacement vector $u$ vanishes at a certain reference depth $n=n_\infty$,

should be presented in a proper form. To achieve this result the projection operators $P_1$ and $P_2$ are introduced by

$$P_1 s = u \quad \text{and} \quad P_2 s = t.$$

Consequently, the boundary conditions are expressed as follows:

$$P_2 s_0 = 0 \quad \text{and} \quad P_1 s_\infty = 0. \tag{14}$$

In (14) $s_0$ and $s_\infty$ are the evaluations of $s$ at the boundaries $n=0$ and $n=n_\infty$.

## 3. Modal Problem

To obtain the boundary value problem (13) and (14) in a modal form the Laplace transform

$$\mathscr{L}[0](p)=\int_0^{\infty} e^{-pt}[0](t)\,dt$$

and the Fourier transform

$$\mathscr{F}[0](k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ik\cdot x}[0](x)\,dx$$

are introduced. Above, the notation [0] means a field quantity.

The reader should notice that the applicability of the Fourier transform restricts the family of surfaces $\{S_n\}$ to a system of planes. In consequence, with the condition $\frac{dn^0}{dn}=0$, it is found that the surfaces $\{S_n\}$ should form a parallel family of planes.

An application of the Laplace and Fourier transforms to the boundary value problem (13) and (14) results in

$$\frac{d\mathscr{F}\mathscr{L}s}{dn}=\mathscr{F}\mathscr{L}(A_0 s)+\mathscr{F}\mathscr{L}(A' s)+\mathscr{F}\mathscr{L}g,$$

$$P_2(\mathscr{F}\mathscr{L}s_0)=0 \quad \text{and} \quad P_1(\mathscr{F}\mathscr{L}s_\infty)=0.$$

Finally, under the properties:

1. $\mathscr{F}\mathscr{L}=\mathscr{L}\mathscr{F}$ (commutation property)
2. $\mathscr{F}(A_0 s)=A_0(\mathscr{F}s)$ ($A$ is independent from $x$)
3. $\mathscr{F}(A' s)=\mathscr{F}A' * \mathscr{F}s$ (convolution property)

the modal problem

$$\frac{d\hat{s}}{dn}=\hat{A}_0\hat{s}+\hat{A}' * \hat{s}+\hat{g} \tag{15}$$

$$P_2\hat{s}_0=0 \quad \text{and} \quad P_1\hat{s}_\infty=0 \tag{16}$$

is obtained. In (15) and (16) the notations

$$\hat{s}=\mathscr{F}\mathscr{L}s, \quad \hat{g}=\mathscr{F}\mathscr{L}g, \quad \hat{A}_0\hat{s}=\mathscr{L}(A_0\mathscr{F}s)$$

and

$$\hat{A}' * \hat{s}=\mathscr{L}(\mathscr{F}A' * \mathscr{F}s)$$

have been used.

## 4. Solving the Modal Problem

Propagator techniques will be used to solve the modal problem (15)–(16) in terms of Green's operator of the unperturbed problem. For a similar treatment see Saastamoinen (1970).

The propagator $\hat{U}(n, \tau)$ is defined (cf. the paper by Gilbert and Backus, 1966) as the solution of either one of the initial value problems

$$\frac{d\hat{U}(n,\tau)}{dn} = \hat{A}_0(n)\,\hat{U}(n,\tau), \qquad \hat{U}(n_0,\tau) = 1$$

or

$$\frac{d\hat{U}(n,\tau)}{d\tau} = -\hat{U}(n,\tau)\,\hat{A}_0(\tau), \qquad \hat{U}(n,\tau_0) = 1.$$

With the aid of these definitions it is possible to verify the group properties

$$\hat{U}(n,\tau)\,\hat{U}(\tau,n_0) = \hat{U}(n,n_0) \quad \text{and} \quad \hat{U}(n,\tau)^{-1} = \hat{U}(\tau,n)$$

satisfied by the propagator.

The treatment of the modal problem (15)–(16) as an initial value problem results in

$$\hat{s}(n) = \hat{U}(n,0)\,\hat{s}_0 + \int_0^n \hat{U}(n,\tau)\,\{(\hat{A}' * \hat{s})(\tau) + \hat{g}(\tau)\}\,d\tau. \tag{17}$$

The boundary conditions (16) reveal that the initial value $s_0$ is only partly known a priori. To determine the unknown part (17) is evaluated first at $n = n_\infty$. This gives

$$\hat{s}_\infty = \hat{U}(n_\infty,0)\,\hat{s}_0 + \int_0^{n_\infty} \hat{U}(n_\infty,\tau)\,\{(\hat{A}' * \hat{s})(\tau) + \hat{g}(\tau)\}\,d\tau.$$

This equation becomes

$$\hat{D}P_1\,\hat{s}_0 + \int_0^{n_\infty} P_1\,\hat{U}(n,\tau)\,\{(\hat{A}' * \hat{s})(\tau) + \hat{g}(\tau)\}\,d\tau = 0, \tag{18}$$

using the properties

$$P_1 + P_2 = 1 \quad \text{and} \quad P_1^2 = P_1$$

of the projection operators in connection with the boundary conditions (16). In (18)

$$\hat{D} = P_1\,\hat{U}(n_\infty,0)\,P_1.$$

With respect to dependence on the complex parameter $p$, the operator $\hat{D}$ possesses the unique inverse $\hat{D}^{-1}$ with the exception of the set of singularities, branch points and poles. Outside this set, (18) is solved for $P_1\,\hat{s}_0$. The result is inserted into Equation (17) giving

$$\hat{s}(n) = -\int_0^{n_\infty} \hat{U}(n,0)\,\hat{D}^{-1}P_1\,\hat{U}(n_\infty,0)\,\hat{U}(0,\tau)\,\{(\hat{A}' * \hat{s})(\tau) + \hat{g}(\tau)\}\,d\tau$$
$$+ \int_0^n \hat{U}(n,\tau)\,\{(\hat{A}' * \hat{s})(\tau) + \hat{g}(\tau)\}\,d\tau. \tag{19}$$

Finally, manipulation of (19) gives the integral equation

$$\hat{s}(n) = \int_0^{n_\infty} \hat{G}(n,\tau)\,(\hat{A}' * \hat{s})(\tau)\,d\tau + \int_0^{n_\infty} \hat{G}(n,\tau)\,\hat{g}(\tau)\,d\tau \tag{20}$$

where $\hat{G}(n, \tau)$ defined by

$$\hat{G}(n,\tau)=\begin{cases} -\hat{U}(n,0)\,\hat{D}^{-1}P_1\,\hat{U}(n_\infty,0)\,P_2\,\hat{U}(0,\tau), & \tau\leqq n\\ -\hat{U}(n,\tau)-\hat{U}(n,0)\,\hat{D}^{-1}P_1\,\hat{U}(n_\infty,0)\,P_2\,\hat{U}(0,\tau), & \tau\geqq n\end{cases}$$

is the Green's operator of the unperturbed modal problem.

For the purpose of finding a solution it appears convenient to represent (20) by the equation

$$\hat{s}=\hat{\mathscr{G}}\hat{s}+\hat{g}_0 \tag{21}$$

in some normed space $\mathscr{V}$. The operator $\hat{\mathscr{G}}$ and the vector $\hat{g}_0$ are defined as follows:

$$(\hat{\mathscr{G}}\hat{s})(n)=\int_0^{n_\infty}\hat{G}(n,\tau)\,(\hat{A}'*\hat{s})(\tau)\,d\tau$$

and

$$\hat{g}_0(n)=\int_0^{n_\infty}\hat{G}(n,\tau)\,\hat{g}(\tau)\,d\tau.$$

When the operator $\hat{\mathscr{G}}$ satisfies the condition

$$\|\hat{\mathscr{G}}\|<1$$

with respect to some norm topology of linear operators in $\mathscr{V}$, the solution of (21) can be presented by

$$\hat{s}=\hat{g}_0+\hat{\mathscr{G}}\hat{g}_0+\hat{\mathscr{G}}^2\hat{g}_0+\cdots. \tag{22}$$

For a treatment of linear operators in connection with perturbation theory see the book by Kato (1966). As to an explanation of the terms in series (22) it should be noted that $\hat{g}_0$ represents the solution of the unperturbed problem. The other terms in turn represent the perturbation effects caused by the lateral inhomogeneities.

## 5. Inversion of the Solution in Time and Space

The inversion of the solution (22) to the modal problem in time and space will be briefly dealt with. For more complete treatment in specific situations, see the references in Cochran et al. (1970).

Formally, the inversion in question is presented as

$$s(t,n,x)=\mathscr{F}^{-1}\left\{\frac{1}{2\pi i}\int_\Gamma e^{pt}\,\hat{s}(p)\,dp\right\}(x).$$

The contour $\Gamma$, in the $p$-plane, surrounds the singularities, branch points and poles of $\hat{D}^{-1}$. As shown in the references mentioned above the long time response of the system is mainly due to the residue contribution from the poles and consequently may be expressed by

$$s(t,n,x)\sim\mathscr{F}^{-1}\{\sum_j \operatorname*{Res}_{p=p_j(k)}(e^{pt}\,\hat{s}(p))\}(x).$$

It should be noted that because of the branch points the $p$-plane is divided into many Riemann sheets and the poles $\{p_j(k)\}$ should be picked up from the separate sheets.

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