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On Impulse Response Data and the Uniqueness of the Inverse Problem

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Abstract. The problem of reconstructing the density and elastic properties of a slab of infinite horizontal extent and finite thickness set in motion by a double couple excitation is considered. It is shown that the information contained in the amplitude and frequency response associated with a single horizontal wave number is sufficient to insure the uniqueness of the solution of this inverse problem.

Key words: Inverse problem – Amplitude and frequency response.

1. Introduction

This paper is concerned with the broad question of the uniqueness of the solution of the inverse problem for the internal structure of the earth.

This question may seem academic in view of the variety of data which are available for the earth and which have been used over the years to construct and refine a sequency of earth models. The data arsenal includes travel-time curves, dispersions of Love and Rayleigh waves, normal modes of vibrations. But what if we were to tackle the analogous inverse problem for the Sun for which only normal modes have been measured (Severny et al., 1976; Brookes et al., 1976)? More mundanely, what kind of data is required to infer the internal structure of a perfectly elastic, spherically symmetric ball? The truth of the matter is that we do not know how to answer this question.

Recently, I was able to answer an analogous question for a much simpler vibrating system, namely for a beam of variable density ρ and flexural rigidity EI (Barcilon, 1976a). Inspite of its glaring deficiencies, the problem which I considered had a geophysical flavor. The beam was set in motion by means of an impulsive force concentrated at one of its free ends and applied at time $t=0$: this is analogous to the occurrence of a (known) earthquake. The subsequent motion of this free end together with its slope were recorded: this is analogous to seismographic records made on the earth surface. From these data, i.e., from

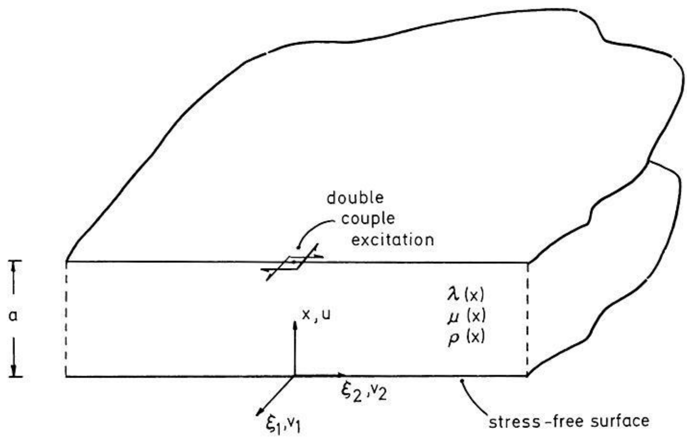

Fig. 1. Schematic diagram of the slab

the impulse response which contains both amplitude and natural frequency information, I was able to reconstruct ρ and EI *uniquely*. The idea of using this impulse response is of course not new and has been exploited masterfully by Krein (1952) in his study of the inverse Strum-Liouville problem.

For the beam the method of solution differed from that used by Krein for the string, in that the dynamical equation was not the equation used to carry out the inversion. Rather, a different set of differential equations for Wronskian-like variables analogous to propagators (see e.g., Gilbert and Backus, 1966) played a key role.

Considered as a mathematical model of the earth problem, one of the most severe shortcoming of the beam lies in the fact that it is a genuine one-dimensional problem whereas the earth problem is a three-dimensional problem with a symmetry group, namely spherical symmetry. The present paper is an attempt to examine the modifications due to three dimensionality in the context of the problem for an infinite slab.

2. Slab with Double-Couple Excitation

Let us consider a perfectly elastic slab of infinite horizontal extent and of thickness a (see Fig. 1). We shall assume that the density ρ and the Lamé parameters λ and μ are solely functions of the depth, i.e. of x. We shall also assume that the surface $x=0$ is stress-free, i.e.

$$p_{xx}=p_{x\xi_1}=p_{x\xi_2}=0 \quad \text{at } x=0, \tag{2.1}$$

where $p_{\dots}$ stands for a component of the stress tensor. The slab is set in motion at time $t=0$ by means of a double-couple applied to the upper surface $x=a$. For the sake of presentation, we consider the simplest case possible and write

$$\begin{aligned} p_{xx} &= 0 \\ p_{x\xi_1} &= -T\delta(\xi_1)\,\delta'(\xi_2)\,H(t) \qquad \text{at } x=a. \\ p_{x\xi_2} &= -T\delta'(\xi_1)\,\delta(\xi_2)\,H(t) \end{aligned} \tag{2.2}$$

In the above formula, δ stands for the Dirac delta function and H for the Heaviside step-function; a prime denotes differentiation with respect to an appropriate variable. T is related to the magnitude of the double-couple.

The slap can be looked upon as the mantle of a flat earth without liquid core and gravitational force. To pursue the analogy with the geophysical situation, we shall refer to the vertical and horizontal displacements $u(a, \boldsymbol{\xi}, t)$ and $v_\alpha(a, \boldsymbol{\xi}, t)$, $(\alpha = 1, 2)$ at the upper surface as the seismograms.

Given the excitation (2.2), the direct problem consists in finding the displacement fields in the slab for $t > 0$, i.e. in solving the equations

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[(\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v_\beta}{\partial \xi_\beta}\right] + \mu \frac{\partial^2 u}{\partial \xi_\beta \partial \xi_\beta} + \mu \frac{\partial^2 v_\beta}{\partial x \partial \xi_\beta} \tag{2.3}$$

$$\rho \frac{\partial^2 v_\alpha}{\partial t^2} = \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial \xi_\alpha} + \frac{\partial v_\alpha}{\partial x}\right)\right] + \lambda \frac{\partial^2 u}{\partial x \partial \xi_\alpha} + (\lambda + \mu)\frac{\partial^2 v_\beta}{\partial \xi_\alpha \partial \xi_\beta} + \mu \frac{\partial^2 v_\alpha}{\partial \xi_\beta \partial \xi_\beta}, \tag{2.4}$$

subject to the boundary conditions (2.1) and (2.2), where as usual

$$\begin{aligned} p_{xx} &\equiv p_{33} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v_\beta}{\partial \xi_\beta}\right) + 2\mu \frac{\partial u}{\partial x} \\ p_{x\xi_\alpha} &\equiv p_{3\alpha} = \mu\left(\frac{\partial u}{\partial \xi_\alpha} + \frac{\partial v_\alpha}{\partial x}\right) \end{aligned} \tag{2.5}$$

In (2.3), (2.4), (2.5) the summation convention holds, repeated superscripts taking the values 1, 2.

If $\rho(x)$, $\lambda(x)$ and $\mu(x)$ were known, then the above problem could be solved (in principle at least) and the seismograms $u(a, \boldsymbol{\xi}, t)$ and $v_\alpha(a, \boldsymbol{\xi}, t)$ be deduced for *all* values of $\boldsymbol{\xi}$ and t. Conversely, the inverse problem consists in deducing ρ, λ and μ from a knowledge of the seismograms. Both the direct and the inverse problem are best tackled by means of Fourier transforms in $\boldsymbol{\xi}$ and t. Therefore, denoting Fourier transforms by carets, e.g.

$$\hat{u}(x, k, \omega) = \frac{1}{(2\pi)^3} \int_0^\infty e^{i\omega t}\, dt \iint_{-\infty}^{\infty} e^{ik_\alpha \xi_\alpha} u \, \mathbf{d}\xi_1 \, \mathbf{d}\xi_2 \tag{2.6}$$

we deduce that

$$-\omega^2 \rho \hat{u} = \hat{\sigma}' - \mu l^2 \hat{u} - i\mu k_\beta \hat{v}_\beta \tag{2.7}$$

$$-\omega^2 \rho \hat{v}_\alpha = \hat{p}'_{3\alpha} - i\lambda k_\alpha \hat{u}' - (\lambda + \mu) k_\alpha k_\beta \hat{v}_\beta - \mu l^2 \hat{v}_\alpha. \tag{2.8}$$

In the above equations

$$\begin{aligned} \hat{\sigma} &= (\lambda + 2\mu)\hat{u}' - i\lambda k_\beta \hat{v}_\beta, \\ \hat{p}_{3\alpha} &= \mu(-ik_\alpha \hat{u} + \hat{v}'_\alpha), \end{aligned} \tag{2.9}$$

stand for the Fourier transforms of the stresses p_{xx} and $p_{3\alpha}$ respectively; primes for derivatives with respect to x and

$$l^2 = k_\alpha k_\alpha \tag{2.10}$$

for the square of the horizontal wave number. Finally, the boundary conditions (2.1) and (2.2) become

$$\hat{\sigma} = \hat{p}_{31} = \hat{p}_{32} = 0 \quad \text{at } x = 0 \tag{2.11}$$

and

$$\hat{\sigma} = \hat{p}_{31} + \frac{T}{8\pi^3 \omega} k_2 = \hat{p}_{32} + \frac{T}{8\pi^3 \omega} k_1 = 0 \quad \text{at } x = a \tag{2.12}$$

3. Compressive (Rayleigh) and Torsional (Love) Modes

Just as for the earth, the response of the slab to an arbitrary excitation can be synthesized by means of a superposition of 2 kinds of normal modes of oscillations which we shall refer to as the compressive and torsional modes. For a pure torsional mode, horizontal surfaces $x = const.$ remain horizontal whereas for a compressive mode they are deformed and/or displaced.

In order to separate these two modes, it is convenient to introduce the following new variables:

$$\begin{aligned} l\hat{\phi} &= -ik_\alpha \hat{v}_\alpha, \\ l\hat{\tau} &= -ik_\alpha \hat{p}_{3\alpha}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} l\hat{\psi} &= -ik_1 \hat{v}_2 + ik_2 \hat{v}_1, \\ l\hat{\chi} &= -ik_1 \hat{p}_{32} + ik_2 \hat{p}_{31}. \end{aligned} \tag{3.2}$$

These variables are analogous to those introduced by Alterman et al. (1959) in their study of the normal modes of the earth. The factor l entering in (3.1) and (3.2) has been introduced in order to facilitate the study of the degenerate case $l = 0$. Note that with these new variables

$$\begin{aligned} l\hat{v}_1 &= ik_1 \hat{\phi} - ik_2 \hat{\psi}, \\ l\hat{v}_2 &= ik_2 \hat{\phi} + ik_2 \hat{\psi}. \end{aligned} \tag{3.3}$$

As anticipated, by substituting (3.1)–(3.2) in (2.7)–(2.12) this boundary value problem splits into two independent problems. The one associated with the compressive modes is made up of the following dynamical equations

$$\begin{aligned} -\omega^2 \rho \hat{u} &= \hat{\sigma}' - \mu l^2 \hat{u} + \mu l \hat{\phi}', \\ -\omega^2 \rho l \hat{\phi} &= l\hat{\tau}' - \lambda l^2 \hat{u}' - (\lambda + 2\mu) l^3 \hat{\phi}; \end{aligned} \tag{3.4a}$$

the stress-strain relations imply that

$$\begin{aligned} \hat{\sigma} &= (\lambda + 2\mu) \hat{u} + \lambda l \hat{\phi}, \\ l\hat{\tau} &= \mu(-l^2 \hat{u} + l\hat{\phi}); \end{aligned} \tag{3.5a}$$

finally, the boundary conditions read:

$$\begin{aligned} &\hat{\sigma}=\hat{\tau}=0 \quad \text{for } x=0, \\ &\hat{\sigma}=0, \quad l\hat{\tau}=\frac{iT}{4\pi^3\omega}k_1k_2 \quad \text{for } x=a. \end{aligned} \tag{3.6a}$$

The other problem associated with the torsional modes is simpler, viz.

$$-\omega^2\rho l\hat{\psi}=l\hat{\chi}'-\mu l^3\hat{\psi}, \tag{3.4b}$$

with

$$\hat{\chi}=\mu\hat{\psi}' \tag{3.5b}$$

and

$$\begin{aligned} &\hat{\chi}=0 \quad &&\text{for } x=0, \\ &l\hat{\chi}=i\frac{T}{8\pi^3\omega}(k_2^2-k_1^2) \quad &&\text{for } x=a. \end{aligned} \tag{3.6b}$$

Note that except in the boundary conditions (3.6), k_1 and k_2 enter in the problem only through the total wave number l^2. In view of the linearity of the problem, we can therefore assert that the solutions will be of the form

$$\begin{bmatrix} \hat{u} \\ \hat{\sigma} \\ \hat{\phi} \\ \hat{\tau} \end{bmatrix}=\frac{iT}{4\pi^3\omega}\frac{k_1k_2}{l}\begin{bmatrix} \tilde{u} \\ \tilde{\sigma} \\ \tilde{\phi} \\ \tilde{\tau} \end{bmatrix}, \tag{3.7a}$$

and

$$\begin{bmatrix} \hat{\psi} \\ \hat{\chi} \end{bmatrix}=\frac{iT}{8\pi^3\omega}\frac{k_2^2-k_1^2}{l}\begin{bmatrix} \tilde{\psi} \\ \tilde{\chi} \end{bmatrix}, \tag{3.7b}$$

where all the tilde fields are *functions of* x, l *and* ω *only*. The inverse problem can now be formulated as follows: given $\tilde{u}(a,l,\omega)$, $\tilde{\phi}(a,l,\omega)$ and $\tilde{\psi}(a,l,\omega)$ find $\rho(x)$, $\lambda(x)$ and $\mu(x)$. It should be noted that aside from its role in exciting the compressive and torsional modes and in determining the factors in (3.7), the double-couple excitation as such does not enter into the inverse problem in a prominent manner.

Since from now on we shall work solely with the tilde fields we can drop the diacritical mark without any risk of confusion. These fields satisfy the following equations:

$$\begin{bmatrix} u' \\ \sigma' \\ l\phi' \\ l\tau' \end{bmatrix}=\begin{bmatrix} 0 & \dfrac{1}{\lambda+2\mu} & -\dfrac{\lambda}{\lambda+2\mu} & 0 \\ -\omega^2\rho & 0 & 0 & -1 \\ l^2 & 0 & 0 & \dfrac{1}{\mu} \\ 0 & \dfrac{\lambda l}{\lambda+2\mu} & -\omega^2\rho+4\mu\dfrac{\lambda+\mu}{\lambda+2\mu}l^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ \sigma \\ l\phi \\ l\tau \end{bmatrix} \tag{3.8a}$$

with

$$\sigma(0,l,\omega)=\tau(0,l,\omega)=\sigma(a,l,\omega)=\tau(a,l,\omega)-1=0 \tag{3.9a}$$

and

$$\begin{bmatrix}\psi'\\ \chi'\end{bmatrix}=\begin{bmatrix}0 & \frac{1}{\mu}\\ -\omega^2\rho+\mu l & 0\end{bmatrix}\begin{bmatrix}\psi\\ \chi\end{bmatrix} \tag{3.8b}$$

with

$$\chi(0,l,\omega)=\chi(a,l,\omega)-1=0. \tag{3.9b}$$

4. Information Content of Seismograms

We shall now generalize the result obtained for the beam. Namely, we shall show that the seismographs (or rather their Fourier transform) $u(a,l,\omega)$, $\phi(a,l,\omega)$ and $\psi(a,l,\omega)$ contains more information that just the dispersion curves for the Love and Rayleigh waves.

Let us first consider the problem (3.8a) (3.9a) looked upon as a direct problem. In order to solve it, we introduce two fundamental solutions of (3.8a) labelled by a superscript, such that

$$\begin{matrix} u^{(1)}=1 & & u^{(2)}=0 \\ \sigma^{(1)}=0 & \text{and} & \sigma^{(2)}=0 \\ \phi^{(1)}=0 & & \phi^{(2)}=1 \\ \tau^{(1)}=0 & & \tau^{(2)}=0 \end{matrix} \quad \text{for } x=0. \tag{4.1}$$

Consequently, the most general solution of (3.8a) which satisfies the boundary conditions at $x=0$ is:

$$\begin{bmatrix}u\\ \sigma\\ \phi\\ \tau\end{bmatrix}=A\begin{bmatrix}u^{(1)}\\ \sigma^{(1)}\\ \phi^{(1)}\\ \tau^{(1)}\end{bmatrix}+B\begin{bmatrix}u^{(2)}\\ \sigma^{(2)}\\ \phi^{(2)}\\ \tau^{(2)}\end{bmatrix} \tag{4.2}$$

The coefficients A and B are determined by the boundary conditions at $x=a$, viz.

$$\begin{aligned} A&=\frac{-\sigma^{(2)}(a,l,\omega)}{R(a,l,\omega)},\\ B&=\frac{\sigma^{(1)}(a,l,\omega)}{R(a,l\,\omega)}, \end{aligned} \tag{4.3}$$

where

$$R(x,l,\omega)=\sigma^{(1)}(x,l,\omega)\,\tau^{(2)}(x,l,\omega)-\sigma^{(2)}(x,l,\omega)\,\tau^{(1)}(x,l,\omega) \tag{4.4}$$

Note that $R(a, l, \omega)=0$ is the dispersion relation for the Rayleigh waves (compressive modes). Replacing A and B in (4.3), we see that

$$u(a, l, \omega) = -\frac{U(a, l, \omega)}{R(a, l, \omega)} \tag{4.5}$$

$$\phi(a, l, \omega) = +\frac{\Phi(a, l, \omega)}{R(a, l, \omega)} \tag{4.6}$$

where

$$U(x, l, \omega) = \{u(x, l, \omega), \sigma(x, l, \omega)\} \tag{4.7}$$

and

$$\Phi(x, l, \omega) = \{\sigma(x, l, \omega), \phi(x, l, \omega)\}. \tag{4.8}$$

In the above expressions the symbol { } is used as a shorthand notation and is defined thus:

$$\{a, b\} = a^{(1)} b^{(2)} - a^{(2)} b^{(1)}. \tag{4.9}$$

The numerators of $u(a, l, \omega)$ and $\phi(a, l, \omega)$, which corresponds to the amplitude of the response, can be interpreted as two additional dispersion relations for waves (modes) associated with the following boundary conditions at $x=a$:

$U(a, l, \omega)=0 \Leftrightarrow u=\sigma=0$
no vertical displacements
no normal stress;

$\Phi(a, l, \omega)=0 \Leftrightarrow \sigma=\phi=0$
no horizontally divergent displacements
no normal stress.

Thus, knowing the seismograms $u(a, l, \omega)$ and $\phi(a, l, \omega)$ is tantamount to knowing *three* sets of dispersion curves.

The same situation exists for the torsional (Love) modes. If $[H(x, l, \omega)\, L(x, l, \omega)]$ is a solution of (3.8a) such that

$$\begin{aligned} H(0, l, \omega) &= 1 \\ L(0, l, \omega) &= 0 \end{aligned} \tag{4.10}$$

then the solution (3.8b)–(3.9b) is

$$\begin{bmatrix} \psi \\ \chi \end{bmatrix} = \frac{1}{L(a, l, \omega)} \begin{bmatrix} H \\ L \end{bmatrix}. \tag{4.11}$$

In particular, the third "component" of the seismogram is

$$\psi(a, l, \omega) = \frac{H(a, l, \omega)}{L(a, l, \omega)}. \tag{4.12}$$

Once again, $L(a, l, \omega)=0$ is the dispersion relation for the Love waves whereas $H(a, l, \omega)=0$ corresponds to the knowledge of an additional dispersion relation associated with zero twisting displacement at the surface $x=a$.

5. A Different Version of the Inversion Problem

Following the procedure adopted for the beam (Barcilon, 1976a), we introduce additional Wronskians

$$I=\{u,\phi\},\quad J=\{u,\tau\},\quad K=\{\phi,\tau\}, \tag{5.1}$$

and derive differential equations for U, Φ, R, I, J and K, viz.

$$U'=\frac{\lambda l}{\lambda+2\mu}\Phi-lJ, \tag{5.2a}$$

$$\Phi'=-\omega^2\rho I+lK-lU+\frac{1}{\mu}R, \tag{5.2b}$$

$$R'=-\omega^2\rho J+\left[-\omega^2\rho+4\mu l^2\frac{\lambda+\mu}{\lambda+2\mu}\right]\Phi, \tag{5.2c}$$

$$I'=\frac{\Phi}{\lambda+2\mu}+\frac{1}{\mu}J, \tag{5.2d}$$

$$J'=\frac{R}{\lambda+2\mu}-\frac{\lambda l}{\lambda+2\mu}K+\frac{\lambda lU}{\lambda+2\mu}+\left[-\omega^2\rho+4\mu l^2\frac{\lambda+\mu}{\lambda+2\mu}\right]I \tag{5.2e}$$

$$K'=lJ-\frac{\lambda l}{\lambda+2\mu}\Phi. \tag{5.2f}$$

Because of (4.1), the variables U, Φ, R, I, J and K are known at $x=0$, viz.

$$U=\Phi=R=I-1=J=K=0,\qquad \text{at } x=0. \tag{5.3}$$

Now, from (5.2a) and (5.2f), we can see that

$$U'+K'=0$$

and so

$$K=-U. \tag{5.4}$$

But this is not the only relationship between these variables. Just for as for the beam we can check the validity of the following algebraic identity: $\{u,\sigma\}\{\phi,\tau\}+\{u,\phi\}\tau,\sigma\}+\{u,\tau\}\{\sigma,\phi\}=0$, or better still

$$U^2(x,l,\omega)+I(x,l,\omega)R(x,l,\omega)-J(x,l,\omega)\Phi(x,l,\omega)=0. \tag{5.5}$$

Let us recall at this stage that $U(a,l,\omega)$, $\Phi(a,l,\omega)$ and $R(a,l,\omega)$ are considered given. They are also entire functions of ω^2 and l.

The remaining half of the problem is

$$H'=\frac{1}{\mu}L, \tag{5.6a}$$

$$L'=(-\omega^2\rho+\mu l^2)H, \tag{5.6b}$$

with

$$H(0,l,\omega)-1=L(0,l,\omega)=0. \tag{5.7}$$

Also $H(a,l,\omega)$ and $L(a,l,\omega)$ are considered given.

On the face of it, the system of differential equations for U, Φ, R, I, J, K, H and L seems overdetermined in view of the fact that all of these quantities are specified at $x=0$ and most of them are also specified at $x=a$. However, we should remember that the coefficients λ, μ and ρ in these equations are unknown. A similar situation arose in the case of the beam, and I was able to exploit the apparent overdeterminancy of the problem to infer the density and flexural rigidity. We shall try to use the same technique here. To that effect, we must first discretize the problem.

6. Solution of the Discrete Inverse Problem

The three dimensionality of the slab problem, i.e. the presence of the wave number l, complicates the discretization considerably.

One of the difficulties is associated with the data. Indeed, whereas the first N eigenfrequencies of a genuinely one-dimensional problem can be considered as the eigenfrequencies of a discrete system with N degree of freedom, the counterpart for branches of dispersion relations is not true. Said differently, if $\omega=\lambda_n(l)$ are the branches of the dispersion relation $L(a,l,\omega)=0$, then the first N branches do not form an *algebraic curve* and hence they are not bona fide dispersion branches of a system with N degrees of freedom in the vertical (see e.g. Barcilon, 1976b).

Having recognized this technical difficulty, we shall assume that we know how to resolve it. For instance, we could envisage approximating the dispersion relations by algebraic curves.

We now come to the next technical difficulty that of discretizing the equations (5.2) and (5.6), or rather of deriving an analogous set suitable for a system with N-degrees of freedom. The difficulty here stems from the wide variety of seemingly acceptable discrete equations. The guiding principles for selecting an appropriate set of equations as well as the procedure we have followed is presented in an appendix. Incorporating the identity (5.4) into (5.2), we shall adopt the following discretization of these equations:

$$U_i-\frac{\lambda_i h_i l}{\lambda_i+2\mu_i}\Phi_i+h_i l J_i=U_{i-1} \tag{6.1a}$$

$$\Phi_i-\frac{h_i}{\mu_i}R_i=\Phi_{i-1}-\omega^2 m_i I_{i-1}-2lh_i U_{i-1} \tag{6.1b}$$

$$R_i=R_{i-1}-\omega^2 m_i J_{i-1}+\left[-\omega^2 m_i+4\mu_i\frac{\lambda_i+\mu_i}{\lambda_i+2\mu_i}h_i l^2\right]\Phi_{i-1} \tag{6.1c}$$

$$I_i - \frac{h_i}{\mu_i} J_i - \frac{h_i}{\lambda_i + 2\mu_i} \Phi_i = I_{i-1} \tag{6.1d}$$

$$J_i - \frac{h_i}{\lambda_i + 2\mu_i} R_i$$
$$= J_{i-1} + 2\frac{\lambda_i h_i}{\lambda_i + 2\mu_i} l\, U_{i-1} + \left[-\omega^2 m_i + 4\mu_i \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i} h_i l^2\right] I_{i-1}. \tag{6.1e}$$

In the above equations m_i stands for the mass (ρh) of the i-th segment which has been concentrated at x_i in order to make the numbers of degree of freedom finite. h_i is the separation between m_{i+1} and m_i and is *unknown*. Note that as $h_i \to 0$, the system (6.1) reduces to (5.2) with identity (5.4) included. Note also that (6.1) provides an *explicit* means for computing U_i, Φ_i, R_i, I_i and J_i. Indeed if U_{i-1}, Φ_{i-1}, R_{i-1}, I_{i-1} and J_{i-1} are known then by means of (6.1c), (6.1b), (6.1e), (6.1a) and (6.1d), in that order, we can derive U_i, Φ_i, R_i, I_i and J_i. Note also that if

$$U_0 = \Phi_0 = R_0 = I_0 - 1 = J_0 = 0, \tag{6.2}$$

then $U_i, \ldots, J_i$ are polynomials of degree i in ω^2.

Turning now our attention to the remaining half of the problem, viz. (5.6), we adopt the following discretization:

$$H_i - \frac{h_i}{\mu_i} L_i = H_{i-1}, \tag{6.3a}$$

$$L_i = L_{i-1} + [-\omega^2 m_i + \mu_i h_i l^2]\, H_{i-1}. \tag{6.3b}$$

The remarks made previously about (6.1) hold for (6.3).

We are now ready to solve the inverse problem, i.e. to find m_i, h_i, λ_i and μ_i. Recall that U_N, Φ_N, R_N, H_N and L_N are given. Then using a discrete version of (5.5), viz.

$$U_N^2 + I_N R_N - J_N \Phi_N = 0 \tag{6.4}$$

we deduce I_N and J_N. Now dividing H_N by L_N we see from (6.3a) that

$$\frac{H_N}{L_N} = \frac{h_N}{\mu_N} + \frac{H_{N-1}}{L_N}; \tag{6.5}$$

in other words we can find h_N/μ_N and H_{N-1}. We now switch to the compressive modes and more specifically to (6.1d) which we write thus:

$$\frac{I_N - \frac{h_N}{\mu_N} J_N}{\Phi_N} = \frac{h_N}{\lambda_N + 2\mu_N} + \frac{I_{N-1}}{\Phi_N}. \tag{6.4}$$

As a result $h_N/\lambda_N + 2\mu_N$ and I_{N-1} are determined. Note also that λ_N/μ_N is now known.

For the next step we turn to (6.1a) which can also be written

$$\frac{U_N}{\frac{\lambda_N}{\mu_N} \cdot \frac{h_N}{\lambda_N + 2\mu_N} \Phi_N - \frac{h_N}{\mu_N} J_N} = l\mu_N + \frac{U_{N-1}}{\frac{\lambda_N}{\mu_N} \cdot \frac{h_N}{\lambda_N + 2\mu_N} \Phi_N - \frac{h_N}{\mu_N} J_N}. \tag{6.5}$$

In so doing we have determined λ_N, μ_N, h_N, and U_{N-1}. The last quantity, viz. m_N, can be obtained from (6.1b), viz.

$$\frac{\Phi_N - \dfrac{h_N}{\mu_N} R_N + 2lh_N U_{N-1}}{I_{N-1}} = -\omega^2 m_N + \frac{\Phi_{N-1}}{I_{N-1}}. \tag{6.6}$$

Finally, by means of (6.1c), (6.1e) and (6.3b) we can deduce R_{N-1}, J_{N-1} and L_{N-1} which are needed to start the cycle over.

7. Concluding Remarks

We have solved a discrete version of the inverse problem for a slab. The excitation was assumed known and for the sake of illustration was taken to be a point double-couple. The data were extracted from the response to this excitation. More specifically both amplitude and frequency information were extracted from the seismograms. But after making measurements for all times and over all points on the upper free surface of the slab, the dependence in the wave number was not exploited at all. Assuming that this result held for the earth, it would be equivalent to dealing with a single angular number l. This leads to the following conjecture. By repeating the inversion procedure for a sequence of l's, we should be generating the same solution over and over again. If this were not the case, then it would signify that the slab/earth has some lateral structure. Perhaps a measure of this lateral inhomogeneity can be obtained by examining the scatter of the various solutions.

Eventhough the wave number l is a continuous variable for the slab problem, we have treated it as if it were a fixed parameter labeling the various modes. If we were permitted to exploit the dependence of the data on l, i.e. if we were given $R(a, l, \omega)$, $L(a, l, \omega)$, etc.... as polynomials in ω^2 and l, then the procedure could be greatly simplified. In particular, it would be possible to deduce m_i, μ_i and h_i from the Love wave *alone*. The calculations would be modified as follows. First we divide H_i by L_i, looking upon them as polynomials in ω^2, viz.

$$\frac{H_i}{L_i} = \frac{h_i}{\mu_i} + \frac{H_{i-1}}{L_i}.$$

By dividing L_i by H_{i-1}, again looked upon as polynomials in ω^2 we see that

$$\frac{L_i}{H_{i-1}} = -\omega^2 m_i + \frac{L_{i-1} + \mu_i h_i l^2 H_{i-1}}{H_{i-1}}.$$

At this stage we consider the remainder $(L_{i-1} + \mu_i h_i l^2 H_{i-1})$ as a polynomial in l and write it as well as H_{i-1} in decreasing powers of l. Forming the ratio of these polynomials, we get:

$$\frac{L_{i-1} + \mu_i h_i l^2 H_{i-1}}{H_{i-1}} = \mu_i h_i l^2 + \frac{L_{i-1}}{H_{i-1}}.$$

The missing λ_i can be easily obtained from the compressive modes.

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Appendix

Remarks on the Discretization Procedure

Let us consider the equations

$$\begin{aligned} y' &= \frac{1}{\mu} z \\ z' &= (-\omega^2 \rho + \mu l^2)\, y \end{aligned} \tag{A.1}$$

Except for a change in notation, these equations are identical with (3.8b). We shall revert to H_i and L_i after we arrive at a satisfactory discrete version of this equation.

In order to go from a continuous system to a system with N degrees of freedom, we consider the case in which

$$\rho = \sum_{i=1}^{N} m_i\, \delta(x - x_i) \tag{A.2}$$

where $x_{i+1} - x_i = h_i$. Furthermore, we assume that in the interval (x_i, x_{i+1}) μ is a constant equal to μ_i. As a result, in (x_i, x_{i+1}) the Equations (A.1) reduce to

$$\begin{aligned} y' &= \frac{1}{\mu_i} z \\ z' &= \mu_i l^2 y \end{aligned} \tag{A.3}$$

Therefore, in this same interval

$$\begin{aligned} y &= A_i \cosh l(x - x_i) + B_i \sinh l(x - x_i) \\ z &= l \mu_i A_i \sinh l(x - x_i) + l \mu_i B_i \cosh l(x - x_i) \end{aligned} \tag{A.4}$$

One way of determining A_i and B_i (but by no means the only one) consists in requiring that

$$\lim_{x \to x_i + 0} y, z = y_i, z_i. \tag{A.5}$$

As a result

$$\begin{aligned} y &= y_i \cosh l(x - x_i) + \frac{z_i}{l \mu_i} \sinh l(x - x_i) \\ z &= z_i \cosh l(x - x_i) + l \mu_i y_i \sinh l(x - x_i) \end{aligned} \tag{A.6}$$

We now make a crucial approximation: we neglect terms of order $l^2 h_i^2$ or higher and write

$$y = y_i + \frac{z_i}{\mu_i}(x - x_i)$$
$$z = z_i + \mu_i y_i l^2 (x - x_i) \tag{A.7}$$

If h_i is small, this approximation is not very restrictive. Physically, it implies that waves (or modes) must have large horizontal wave lengths to penetrate the slab.

Returning to (A.1) we integrate these equations from $x_i - \varepsilon$ to $x_i + \varepsilon$, where ε is infinitesimally small. We see that y is continuous, viz.

$$\lim_{x \to x_i - 0} y = \lim_{x \to x_i + 0} y \tag{A.8}$$

whereas z is not. In fact the jump in z at x_i is

$$[z]_i = -\omega^2 m_i y_i . \tag{A.9}$$

Making use of (A.7) we rewrite these two last equations as follows:

$$y_{i-1} + \frac{z_{i-1}}{\mu_{i-1}} h_{i-1} = y_i, \tag{A.10}$$

$$z_i - z_{i-1} - \mu_{i-1} h_{i-1} l^2 y_{i-1} = -\omega^2 m_i y_i . \tag{A.11}$$

These equations are not completely satisfactory. One of their drawbacks is that they are not associated with an acceptable energy principle. They can be corrected by replacing y_{i-1} by y_i in (A.11), viz.

$$z_i - z_{i-1} - \mu_{i-1} h_{i-1} l^2 y_i = -\omega^2 m_i y_i . \tag{A.12}$$

Note that within our approximation scheme, this step is quite legitimate. To be consistent we should also modify the second equation in (A.7) and write

$$z = z_i + \mu_i l^2 y_{i+1}(x - x_i) \tag{A.7'}$$

Now in (A.10) and (A.12) we have an explicit scheme for the computation of y_i and z_i. If $z_0 = 0$ and $y_0 = 1$, then y_i and z_i are polynomials in ω^2 but not of degree i. We shall now correct this minor flaw. We reason as follows: When we come to solve the inverse problem the amplitude and frequency data are translated in a knowledge of z and y at $N+1$, i.e. $z_N + \mu_N l^2 h_N y_{N+1}$ and y_{N+1} are given. Let us therefore define the following new variables

$$H_i = y_{i+1}$$
$$L_i = z_i + \mu_i l^2 h_i y_{i+1} \tag{A.13}$$

Substituting in (A.10) and (A.12) we get (to the usual degree of approximation)

$$H_i - \frac{h_i}{\mu_i} L_i = H_{i-1}$$
$$L_i = L_{i-1} + [-\omega^2 m_i + \mu_i h_i l^2] H_{i-1} . \tag{A.14}$$

These are the equations (6.3).

The same considerations and manipulations have led us to the set (6.1). Since the calculations are lengthy we shall not reproduce them.

References

Alterman, Z., Jarosch, H., Pekeris, C.L.: Oscillations of the earth. Proc. Roy. Soc. (London) A **252**, 80–95, 1959

Barcilon, V.: On the solution of the inverse problem with amplitude and natural frequency data. Phys. Earth Planet. Interiors **13**, 1–8, 1976a

Barcilon, V.: A discrete model of the inverse Love wave problem. Geophys. J. **44**, 61–76, 1976b

Brookes, J.R., Isaak, G.R., van der Raay, H.B.: Observation of free oscillations of the Sun. Nature **259**, 992–95, 1976

Gilbert, F., Backus, G.E.: Propagator matrices in elastic wave and vibration problems. Geophysics **31**, 326–332, 1966

Krein, M.G.: Ob obratnykh zadachakh dlya neodnorodnoi struny. Dokl. Akad. Nauk SSSR **82**, 669–672, 1952

Severny, A.B., Kotov, V.A., Tsap, T.T.: Observations of solar pulsations. Nature **259**, 87–89, 1976

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